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Abstract

In the literature several parametric methods have been proposed to test the mean variance efficiency of a given portfolio. These tests serve to value the efficiency only in the case the underlying portfolios are uniquely determined by the mean and the variance. However, the return distributions could depend on many parameters. In addition, investors are not always risk averse and they do not necessarily follow the classical stochastic dominance rules. In this paper we propose a class of parametric, semi-parametric and non parametric methods to value the efficiency of a given portfolio with respect to a given ordering of preferences. Parametric and semi-parametric tests suggest to value the distributional distance of some parameters between the given portfolio and few other optimal portfolios. Non-parametric tests value the efficiency preference of the given portfolio with respect to all optimal portfolios. The empirical application reveals that the Fama and French market portfolio is efficient with respect to all preference orderings while the $S\&P500$ stock index is inefficient.

Key words and phrases: Nonparametric, Stochastic Ordering, Dominance Efficiency.

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1 Introduction

The portfolio problem is essentially a choice problem among random variables. The theory of the choice under uncertainty conditions has been manly developed under the assumption that investors maximize their expected utility. In particular, many studies assume that investors act as non satiable and risk averse agents and thus they should have increasing and concave utility functions. For this reason most of the criteria to verify the efficiency of a given portfolio (see, among others, Gibbons, Ross, and Shenken (1989)) are based on the first and second order stochastic dominance, see e.g. the review papers by Kroll and Levy (1980) and Levy (1992), the classified bibliography by Mosler and Scarsini (1993), and the books by Shaked and Shanthikumar (1994) and Levy (1998). Stochastic dominance theory aims at comparing random variables in the sense of stochastic orderings expressing the common preferences of rational decision-makers. Scaillet and Topaloglou (2008) develop consistent tests for stochastic dominance efficiency at any order for time-dependent data (see also Linton, Post and Wang (2005)), relying on weighted Kolmogorov-Smirnov type statistics in testing for stochastic dominance. Other stochastic dominance tests are suggested in the literature; see e.g. Anderson (1996), Beach and Davidson (1983), Davidson and Duclos (2000).

Another approach for portfolio selection, proposed by Markowitz, reduces the portfolio choice to a set of two criteria, reward and risk, with possible trade-off analysis. Usually the reward–risk model is not consistent with the utility maximization approach, even when the decision is independent from the specific form of the risk averse expected utility function, i.e. when one investment dominates another one by second order stochastic dominance. Ogryczak and Ruszczynski (1997) propose semi-variance models, where the reward–risk approach is maintained, but the choice of semi-variance instead of variance makes the model consistent with second order stochastic dominance. Moreover, Ruszczynski and Vanderbei (2003) propose mean-risk models that are solvable by linear programming and the generated optimal portfolios are not dominated in the sense of second order stochastic dominance. Other risk measures have been proposed for portfolio selection, as for example Value-at-Risk (Jorion, 1997, Duffie and Pan, 1997) or Expected-Shortfall (see Acerbi and Tasche, 2002, and Szegö (2004)), which is consistent with second order stochastic dominance. Value-at-risk is widely used in practice, but it is only consistent with respect to first order stochastic dominance. Pflug (1998) considered various classes of risk measures and gave the general properties for these classes. Generalizing the approach of Ogryczak and Ruszczynski (1997) he introduced expectation–dispersion risk measures and showed that under some conditions they are consistent with stochastic dominance, but usually not coherent. De Giorgi (2005) solves a portfolio selection problem based on reward-risk measures consistent with second order stochastic dominance. If investors have homogeneous expectations and optimally hold reward-risk efficient portfolios, then in the absence of market frictions, the portfolio of all invested wealth, or the market portfolio, will itself be a reward-risk efficient portfolio. The market portfolio should therefore be itself efficient in the sense of second order stochastic dominance according to that theory (see De Giorgi and Post (2005) for a rigorous derivation of this result).

However, the investor behavior is not known, except in some obvious circumstances. As a matter of fact, while it is obvious that investors prefer more to less, several behavioral finance analyses indicate that investors are neither risk preferring nor risk averting (see Levy and Levy (2002)). Since there exist many alternative orderings of investor preferences, we need to value the efficiency with respect to any of these orderings.

The goal of this paper is to develop and empirically compare semi-parametric, as well as non-parametric tests for valuing the efficiency of a given portfolio with respect to a given ordering of preferences. We first propose some criteria for ordering investors' preferences when all portfolios are uniquely determined by a finite number of parameters. Then, we show how to classify investors choices by the point of view of non-satiable investors with different risk aversion preferences. Using the estimation function theory (see among others, Lehmann and Casella (1998)) we describe and discuss several semi-parametric tests for the efficiency of a given portfolio with respect to different ordering preferences. Semi-parametric tests suggest to value the distributional distance of some parameters between the given portfolio and few other optimal portfolios. Non-parametric tests value the efficiency preference of the given portfolio with respect to all optimal portfolios.

The paper is organized as follows. In section 2, we recall some of the most recent classification of risk and reward measures, their properties and characteristics (uncertainty and aggressiveness), and their connection with the preference orderings. In section 3, we discuss a semi-parametric methodology, based on the estimating function theory, to value the efficiency with respect to the first and the second order stochastic dominance order when the return distributions are uniquely determined by a finite number of moments. Furthermore, we classify the ordering of portfolio distributions when these are uniquely determined by a reward measure, a deviation measure, and a finite number of other parameters. Thus, we extend the previous methodology to test the efficiency when we have this parametric dependence of the portfolios. In section 4, we discuss nonparametric methods to value the efficiency of portfolios with respect to behavioral orderings. Numerical implementation of the reward/risk measures with many parameters is difficult since we need to develop quadratic programming formulations. Nevertheless, widely available algorithms can be used to compute these models. In Section 5 we provide empirical illustrations. We analyze whether the Fama and French market portfolio can be considered as efficient according to the proposed semi-parametric tests when confronted to diversification principles made of six Fama and French benchmark portfolios formed on size and book-to-market equity ratio (Fama and French (1993)). We additionally test whether the S&P500 index could be efficient in comparison with "the best" 20 assets of the S&P500. Finally we test non parametrically the efficiency of these portfolios. We give some concluding remarks in Section 6. Proofs are gathered in an appendix.

2 Risk/reward measures and ordering derived by their use

In portfolio theory there are used several types of risk, reward, and uncertainty measures that associate a real value to a random wealth defined on a probability space $(\Omega, \Im, P)$. The use of these measures is strictly connected with an ordering of preference. Let us recall the main classification of measures and orderings and their connection (see Szegö (2004), Ortobelli et al (2006), (2007) for a detailed review).

The strongest risk ordering applied in the financial literature is the strict inequality between random variables also called monotony order. Thus, the orderings derived by the monotony order (i.e., $X > Y$ implies that X is preferred to Y) are called risk orderings. Typical examples of risk orderings are the stochastic dominance and the behavioral finance orderings. About stochastic dominance orderings, recall that X dominates Y with respect to $\alpha - th$ ($\alpha \geq 1$) order stochastic dominance (namely $X \underset{\alpha}{\geq} Y$, for some $X, Y \in L^{\alpha-1} \left\{ X / E\left(|X|^{\alpha-1}\right) < +\infty \right\}$ if $\alpha > 1$ and $X, Y \in L^0 = \{\text{all random variables}\}$ if $\alpha = 1$) if and only if the below inequality holds for every real t

$$F_X^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-y)^{\alpha-1}dF_X(y) = \frac{E\left((t-X)_+^{\alpha-1}\right)}{\Gamma(\alpha)} \leq F_Y^{(\alpha)}(t) \text{ if } \alpha{>}1 \text{ and}$$

$$F_X^{(1)}(t) := F_X(t) = \Pr(X \leq t) \leq F_Y^{(1)}(t) \text{ if } \alpha = 1,$$

where $\Gamma(\alpha) = \int_0^{+\infty} z^{\alpha-1}e^{-z}dz$. Furthermore, we observe that for any $m{>}n$, $X \underset{n}{\geq} Y$ implies $X \underset{m}{\geq} Y$ and if $X \underset{\alpha}{\geq} Y$ then $\mathrm{E}(u(X)){\geq}\mathrm{E}(u(Y))$ for every utility function $u{\in} U_\alpha$

$$U_\alpha = \Big\{u(x) = c - \textstyle\int_{x^+}^{+\infty}(y-x)^{\alpha-1}dv(y)\,|c, x \in R;\text{ where}$$
$$v \text{ is positive } \sigma\text{-finite measure} \textstyle\int_{-\infty}^{+\infty}|y|^{\alpha-1}\,dv(y) < \infty\Big\}.$$

Thus, if $X \underset{1}{\geq} Y$, any von Neuman Morgestern non satiable investor (with increasing utility function) prefers X to Y and if $X \underset{2}{\geq} Y$ then any von Neuman Morgestern non satiable risk averse investor (with increasing and concave utility function) prefer X to Y. Other examples of risk orderings are the dominance rules of behavioral finance (see Friedman and Savage (1948), Markowitz (1952), Tversky and Kahneman (1992), Levy and Levy (2002), Baucells and Heukamp (2006), Rachev et al. (2008) Edwards (1996), and the references therein). With these orderings we consider non satiable investors that are neither risk averse nor risk lover such as prospect theory type investors (see Tversky and Kahneman (1992)) and Markowitz type investors (see Markowitz (1952)). Prospect theory type investors are non satiable risk lovers at lower levels of wealth and risk averse at higher levels, while Markowitz type investors are non satiable risk averse with respect to losses and are risk seeking with respect to gains as long as the outcomes are not very extreme. Typically, given $c, d \in supp\,\{X, Y\}$, $c \geq d$ we say that X dominates Y in the sense of prospect theory ($X\ PSD\ Y$) if and only if $\forall y \in (-\infty, c]$

$$g_X(y) := \int_d^{d+c-y} F_X(u)du \leq g_Y(y) \text{ and } \tilde{g}_X(y) := \int_y^c F_X(u)du \leq \tilde{g}_Y(y)$$

if and only if $\forall(x, y) \in [0, 1] \times (-\infty, c]$,

$$g_X(x, y) := xg_X(y) + (1-x)\tilde{g}_X(y) \leq g_Y(x, y).$$

(see Ortobelli et al (2008) Levy and Levy (2002) and Baucells and Heukamp (2006)). Analogously, we say that X dominates Y in the sense of Markowitz order ($X\ MSD$

Y) if and only if $\forall y \in (-\infty, c]$

$$m_X(y) := \int_{-\infty}^{y} F_X(u)du \leq m_Y(y) \text{ and } \tilde{m}_X(y) := \int_{d+c-y}^{+\infty} F_X(u)du \leq \tilde{m}_Y(y)$$

if and only if $\forall (x, y) \in [0, 1] \times (-\infty, c]$,

$$m_X(x, y) := x m_X(y) + (1 - x)\tilde{m}_X(y) \leq m_Y(x, y).$$

Instead we call uncertainty ordering any ordering that classifies the different degree of uncertainty of the admissible choices. Typically these orderings of preference maintain the same order between the random variables and their opposite, that is, X is preferred to Y if and only if $-X$ is preferred to $-Y$. Classical example of uncertainty order is Rothschild and Stiglitz ordering also called concave order in the ordering literature (see, among others, Shaked and Shanthikumar (1993) and Müller and Stoyan (2002)). We state that X dominates Y in the sense of Rothschild and Stiglitz (X $R-S$ Y) if and only if $\mathrm{E}(u(X)) \geq \mathrm{E}(u(Y))$ for every concave utility function u, if and only if $X \underset{2}{\geq} Y$ and $\mathrm{E}(X)=\mathrm{E}(Y)$ if and only if $X \underset{2}{\geq} Y$ and $-X \underset{2}{\geq} -Y$ (see Rothschild and Stiglitz (1970)). Clearly, risk, reward, and uncertainty measures are classified with respect to their properties. However, the most important property in portfolio theory is the consistency (isotonicity) with investor's preferences $\succ$. Recall that we say that a measure ρ is consistent (isotone) with investor's preferences $\succ$, if $X \succ Y$, implies $\rho(X) \leq \rho(Y)$ ($\rho(X) \geq \rho(Y)$). This property serves to distinguish risk, reward, uncertainty and aggressive measures. An uncertainty measure is a measure consistent with an uncertainty ordering of preference. An aggressive measure is a measure isotone with an uncertainty ordering of preference. Thus an aggressive measure is the opposite of an uncertainty one. A risk (or monotone) measure is any measure consistent with monotony order, that is $\rho(X) \leq \rho(Y)$ when $X \geq Y$. A reward measure is the opposite of a risk measure, that is, when a risk measure is consistent with a given ordering, then the associated reward measure (opposite of the risk measure) is isotone with the same ordering (i.e., $\rho(X) \leq \rho(Y)$ when $X \leq Y$). Moreover, when the measure is either an uncertainty measure or an aggressive measure, we could have either an uncertainty risk measure (consistent with a risk ordering and with an uncertainty order) or an aggressive risk measure (consistent with a risk ordering and isotone with an uncertainty order). Thus when investors minimize an uncertainty risk measure obtain portfolios with minimum uncertainty and risk, while when investors minimize an aggressive risk measure obtain portfolios with minimum risk and maximum uncertainty. Observe that minimizing a risk measure is equivalent to maximize the associated reward measure. Clearly the distinction between risk and uncertainty orderings imply a different use of risk/reward and uncertainty/aggressive measures to get optimal choices (see Ortobelli et al (2005)). Generally, we say that:

1) a measure ρ is *simple* (or *invariant in law*) if ρ associates the same values $\rho(X) = \rho(Y)$ at random variables X and Y identically distributed;[1]

2) a risk measure ρ is *translation equivariant* if $\rho(X + C) = \rho(X) - C$ for any constant C (while *translation equivariant* reward measures satisfy the property $\rho(X+C) = \rho(X)+C$;

3) a measure ρ is *translation invariant* if $\rho(X+C) = \rho(X)$ for any constant C;

4) a risk measure ρ is *functional translation invariant*[2] if $\rho(X+C) \leq \rho(X)$ for any constant $C>0$ (for reward measures we have the inverted inequality);

5) a measure ρ is *positively homogeneous* (either *scale* or *scalar invariant* [3]) if $\rho(0) = 0$, and $\rho(aX) = a\ \rho(X)$, for all admissible random variables X and all $a > 0$;

6) a measure ρ is *positive* (*negative*) $\rho(X) \geq 0$ ($\rho(X) \leq 0$) for all X;

7) a measure ρ is s*ub additive* $\rho(X + Y) \leq \rho(X) + \rho(Y)$, for all X and Y

8) a measure ρ is *super additive* $\rho(X + Y) \geq \rho(X) + \rho(Y)$, for all X and Y

9) a measure ρ is convex $\rho(aX+(1\text{-}a)Y) \leq a\ \rho(X)+(1\text{-}a)\ \rho(Y)$, for all X and Y and $a \in [0, 1]$; the opposite of a convex measure is a concave measure, (i.e.,

$\rho(aX+(1\text{-}a)Y) \geq a\ \rho(X)+(1\text{-}a)\ \rho(Y)$, for all X and Y);

10) a measure ρ satisfies the *Fatou property* if for any sequence $\{X_n\}_{n\in\mathbb{N}}$ of integrable random variables that converges to the integrable random variable X with respect to the L^1 norm (namely, $X_n \to_{L^1} X$, i.e., $E(|X_n - X|) \xrightarrow{n\to\infty} 0$), then $\rho(X) \leq \liminf_{n\to\infty} \rho(X_n)$.

All these properties serve to define and classify different classes of measures. According to Artzner et al. (1999) a functional ρ is called *coherent risk measure* if it is monotone translation equivariant, sub additive and positive homogeneous. According

[1] This property is in heritage from probability metric theory (see Rachev (1991)). However this property is necessary only when we consider the consistency with a particular ordering (see Ortobelli et al (2008)) .

[2] We use the Gaivoronsky and Pflug's definition of translation invariance (conceptually right) to distinguish the translation invariance in the sense of Artzner et al. (1999) that we call translation equivariance (as suggested by Gaivoronsky and Pflug (2001)). The functional translation invariance contains both concepts and thus it is a natural generalization of them.

[3] In the paper we use the alternative definition of scale or scalar equivariant as suggested in Ortobelli (2001). However, as suggested by Prof Lucio Bertoli-Barsotti this property should be defined as scale or scalar equivariant.

to Föllmer, and Sheid (2002) a functional ρ is called *convex* (*concave*) *risk* measure if it is monotone, translation equivariant, convex (concave) measure. According to Rockafeller, Uryasev, and Zabarankin (2006) a *deviation measure* is as a positive, subadditive, positively homogeneous, translation invariant measure and an *expectation-bounded measure* is any translation equivariant, subadditive, positively homogeneous measure ρ that associates the value $\rho(X)$>-E(X) with a non-constant random variable X, whereas $\rho(X)=E(X)$ for constant X. These classifications are connected each one with the others. In particular, convex risk measures contains the class of coherent risk measures. Expectation-bounded measures that are monotone are also coherent and there is a correspondence one to one between expectation-bounded measures and deviation measures. Moreover, from Bauerle and Müller (2006) we know the links between all the previous measures and Rothshild Stiglitz order, first and second order stochastic dominance order. In particular:

a) any simple monotone measure is a risk measure consistent with first order stochastic dominance ;

b) and any simple convex measure that satisfies the Fatou property is consistent with Rothshild Stiglitz order.

Thus, simple deviation measures and simple expectation-bounded measures that satisfy the Fatou property are uncertainty measures consistent with Rothshild Stiglitz order, while all simple convex (or coherent) risk measures that satisfy the Fatou property are uncertainty risk measure consistent with first, second orders stochastic dominance and with Rothshild Stiglitz order. From Bauerle and Müller (2006) we also deduce that simple concave measures that satisfy the Fatou property are aggressive measures isotonic with Rothshild Stiglitz order. Thus all simple concave and monotone measures that satisfy the Fatou property are aggressive risk measure consistent with first order stochastic dominance and isotonic with Rothshild Stiglitz order.

Remark 1 *The above classification can be seen in terms of reward measures. Thus we call:*

1) coherent reward measure v the opposite of a coherent risk measure ρ (i.e., v(X)=-ρ(X) for all X), that is any monotone, translation equivariant, super additive and positive homogeneous measure;

2) concave reward measure v the opposite of a convex risk measure ρ (i.e., v(X)=-ρ(X) for all X), that is any monotone, translation equivariant, concave measure;

3) aggressive reward measure[4] *v the opposite of an uncertainty risk measure, that is also equal to an aggressive risk measure ρ valued on the opposite of the random variable (i.e., v(X)= ρ(-X) for all X);*

4) uncertainty reward measure v the opposite of an aggressive risk measure, that is also equal to an uncertainty risk measure ρ valued on the opposite of the random variable (i.e., v(X)= ρ(-X) for all X);

5) aggressive deviation measure v the opposite of a deviation measure, that is a negative, super additive, positively homogeneous, translation invariant measure;

6) aggressive expectation-bounded measure v the opposite of an expectation-bounded measure, that is a translation equivariant (v(X+C)=v(X)+C), super additive, positively homogeneous measure v that associates the value v(X)<E(X) with a non-constant random variable X, whereas v(X)=E(X) for constant X;

7) coherent expectation-bounded reward measure any aggressive expectation-bounded measure that is isotone with monotony order.

As observed previously all these risk/reward measures are connected to classical stochastic orderings. However, as suggested by behavioral finance, while it is credible that investors are non satiable they could be neither risk averse, nor risk lover. For this reason it has sense to consider measures that are monotone, but they are not consistent with an uncertainty/aggressive order. Typical examples are the so called aggressive coherent reward measures $c\bar{\rho}_{-X} - d\hat{\rho}_X$, where c, d>0 and $\bar{\rho}_X$ $\hat{\rho}_X$ are two simple coherent risk measures that satisfy the Fatou property (see Rachev et al (2008)). These functionals are isotone with the monotony order, but $\bar{\rho}_{-X}$ is consistent with Rothshild Stiglitz order and $-\hat{\rho}_X$ is isotone with Rothshild Stiglitz order. We refer to Ortobelli et al (2005) and to Rachev et al (2008) for further examples of the above measures. The largest class of measures consistent with preference orderings is the class of FORS measures (see Ortobelli et al (2006) (2007)). We call *FORS measure* induced by the order of preference $\succ$ any probability functional $\mu : \Lambda \times \Gamma \rightarrow R$ that is consistent with respect to the order of preferences $\succ$ defined on the space of real-valued random variables Λ (where Γ is a space of admissible benchmarks. The random variables belonging to Λ or Γ are defined on the probability space $(\Omega, \Im, P)$). Therefore if X dominates Y with respect to a given order of preferences $\succ$ on Λ ($X \succ$

[4] This definition is different from that proposed by Rachev et al. (2008) that call aggressive reward measure the opposite of an aggressive risk measure. This difference is justified since we previously define as aggressive measure any measure isotone with an uncertainty order therefore it must be the opposite of an uncertainty measure.

Y), this implies that $\mu(X,Z) \leq \mu(Y,Z)$ for a fixed benchmark Z belonging to Γ. The benchmark could not be specified and in this case the functional μ is defined from Λ to R ($\mu : \Lambda \to R$). According to Bauerle and Müller's analysis all the above measures are particular risk/reward, uncertainty/aggressive FORS measures. Moreover, given a class of FORS measures that identify the random variables belonging to Λ we can easily generate an ordering from this measure. As a matter of fact, suppose $\rho_X : [\mathbf{a},\mathbf{b}] \to \bar{R}$ (where $[\mathbf{a},\mathbf{b}] \subseteq \bar{R}^n$, $\mathbf{a} = [a_1, ..., a_n]'$, $\mathbf{b} = [b_1, ..., b_n]'$ such that $-\infty \leq a_i < b_i \leq +\infty$ i=1,. . . ,n) is a bounded variation function, for every random variable X belonging to a given class Λ and assume that the functional ρ_X is simple (i.e., for every $X, Y \in \Lambda$, $\rho_X = \rho_Y \Leftrightarrow F_X = F_Y$). If, for any fixed $\lambda \in [\mathbf{a},\mathbf{b}]$, $\rho_X(\lambda)$ is a FORS risk measure induced by a risk ordering $\succ$, then, we call FORS risk orderings induced by $\succ$ the following ordering $X \underset{\succ,1}{FORS}\ Y$ iff $\rho_X(u) \leq \rho_Y(u)$, $\forall u \in [\mathbf{a},\mathbf{b}]$. In particular, when n=1 and $[a,b] \subseteq \bar{R}$, we can easily define a new class of orderings defined for every $\alpha \geq 1$, $\forall X, Y \in \Lambda_{(\alpha)} = \left\{ X \in \Lambda \left| \left| \int_a^b |t|^{\alpha-1}\, d\rho_X(t) \right| < \infty \right. \right\}$

$$X \underset{\succ,\alpha}{FORS}\ Y \quad \text{iff } \rho_{X,\alpha}(u) \leq \rho_{Y,\alpha}(u), \quad \forall u \in [a,b]$$

where $\rho_{X,\alpha}(u) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int_a^u (u-t)^{\alpha-1} d\rho_X(t) & if \alpha > 1 \\ \rho_X(u) & if \alpha = 1 \end{cases}$. We call ρ_X the *FORS risk measure associated with the FORS ordering* of random variables belonging to class Λ. From the above definition we understand that any FORS risk measure $\rho_X : [\mathbf{a},\mathbf{b}] \to \bar{R}$ associated with the FORS ordering is identified by two properties:

a) the identity property, i.e., $\rho_X = \rho_Y \Leftrightarrow F_X = F_Y$

b) the consistency property, i.e., any time X is preferred to Y ($X \succ Y$), $\rho_X \leq \rho_Y$.

Therefore with FORS orderings we can build orderings and probability functionals consistent with the preferences of different investors' categories. In particular, even the Markowitz and prospect behavioral type orderings are some particular FORS orderings. Moreover, we can also consider many other types of behavioral orderings for satiable investors who are neither risk averse nor risk lover. As a matter of fact, many times we can identify an aggressive coherent risk FORS measure $\rho_X : [\mathbf{a},\mathbf{b}] \to \bar{R}$ associated with a behavioral FORS ordering. For example, we can consider the FORS risk ordering induced by the monotony order with the associated risk FORS measure:

$$\rho_X(\alpha, \beta) = dES_\alpha(X) - cES_\beta(-X) \quad \alpha, \beta \in [0,1], \quad c, d>0 \tag{1}$$

where $ES_\alpha(X) = \frac{-1}{\alpha}\int_0^\alpha F_X^{-1}(u)du$ is the coherent risk measure that satisfies the Fatou property called expected shortfall (or conditional value at risk, namely CVaR) defined as function of the left inverse of cumulative distribution

$$F_X^{-1}(p) = \inf\{x : \Pr(X \leq x) = F_X(x) \geq p\} \quad \forall p \in [0,1].$$

Since the function $\rho_X(\alpha,\beta)$ identify the distribution of X (even for a fixed α or β) and it is consistent with the monotony order, this FORS measure identify a particular behavioral ordering for non satiable investors who are neither risk averse nor risk lover. Clearly, we can create many other examples of behavioral orderings. However, an open question is how can we verify that a portfolio is optimal with respect to a particular ordering of preferences. In the sections that follow we deal with this problem considering either a non parametric methodology or assuming that portfolios belong to a parametric family of distributions.

3 Testing parametric preference orderings

Let us consider the optimal portfolio choice problem among n+1 assets: n of those assets are risky with gross returns $Z = [Z_1, ..., Z_n]'$ and the (n+1)th asset has risk-free gross return Z_0. When unlimited short selling is allowed, every portfolio of gross returns is a linear combination of the constant riskless gross return Z_0, and the risky gross returns Z_i i.e. $x_0 Z_0 + \sum_{i=1}^{n} x_i Z_i$ where $(x_0, x_1, ..., x_n) \in \Re^{n+1}$. However in the following we describe portfolio selection problems under institutional restrictions on the market: no short sales, limited liability, i.e. $x'Z \geq 0$ where $Z_i = \frac{P_{i,t+1}}{P_{i,t}} > 0$ and $x_i \geq 0$ $\forall i$. Under this hypothesis we can assume that the family $\wp$ of all admissible portfolios of gross returns is a scale invariant family which admits positive translations (i.e., if $X \in \wp$ then even $\alpha X \in \wp, X + t \in \wp$ for any $\alpha, t \geq 0$).

3.1 Semi-parametric tests for stochastic orderings depending on the first moments

In the literature several parametric methods have been proposed to test the mean variance efficiency of a given portfolio (see, among others, Gibbons, Ross, and Shenken (1989)). These tests serve to value the efficiency only in the case the underline portfolios are uniquely determined by the mean and the variance. However return distributions could depend on more parameters. For example, let us assume that all portfolios of gross returns belong to a scale invariant family of positive random variables which admits positive translations and it is uniquely determined by the first four moments. Suppose we have a portfolio with vector weights, mean, standard deviation, skewness and kurtosis respectively given by $x^p = [x_1^p, ..., x_n^p]'$, m_p, σ_p, s, k then as proved by Ortobelli (2001) all non-satiable investors will prefer the portfolios solution of the following optimization problems

$$\begin{array}{ll} \text{Problem (1)} \ \max_x x'Qx \ \text{ subject to} & \text{Problem (2)} \ \max_x \dfrac{x'E(Z)}{\sqrt{x'Qx}} \text{ subject to} \\ \dfrac{x'E(Z)}{\sqrt{x'Qx}} \geq \dfrac{m_p}{\sigma_p};\ x'e = 1;\ x_i \geq 0,\ i = 1,..,n & \sqrt{x'Qx} \geq \sigma_p;\ x'e = 1; x_i \geq 0,\ i = 1,..,n \\ \frac{E\left((x'Z - E(x'Z))^3\right)}{(x'Qx)^{3/2}} = s;\ \frac{E\left((x'Z - E(x'Z))^4\right)}{(x'Qx)^2} = k & \frac{E\left((x'Z - E(x'Z))^3\right)}{(x'Qx)^{3/2}} = s;\ \frac{E\left((x'Z - E(x'Z))^4\right)}{(x'Qx)^2} = k \end{array} \tag{2}$$

where Q is the variance covariance matrix of the n assets. Similarly, all non-satiable risk averse investors will prefer the portfolios solution of the following optimization problem

$$\begin{array}{ll} \text{Problem (3)} \ \max_x E(x'Z) \text{ subject to} & \text{Problem (4)} \ \max_x \dfrac{x'E(Z)}{\sqrt{x'Qx}} \ \text{ subject to} \\ \dfrac{x'E(Z)}{\sqrt{x'Qx}} \geq \dfrac{m_p}{\sigma_p};\ x'e = 1;\ x_i \geq 0;\ i = 1,..,n & E(x'Z) \geq m_p;\ x'e = 1;\ x_i \geq 0;\ i = 1,..,n \\ \frac{E\left((x'Z - E(x'Z))^3\right)}{(x'Qx)^{3/2}} = s;\ \frac{E\left((x'Z - E(x'Z))^4\right)}{(x'Qx)^2} = k & \frac{E\left((x'Z - E(x'Z))^3\right)}{(x'Qx)^{3/2}} = s;\ \frac{E\left((x'Z - E(x'Z))^4\right)}{(x'Qx)^2} = k \end{array} \tag{3}$$

Let $x^{(1)} = [x_1^{(1)}, ..., x_n^{(1)}]'$, m_1, σ_1, $x^{(2)}$, m_2, σ_2, $x^{(3)}$, m_3, σ_3, and $x^{(4)}$, m_4, σ_4 be the vector of weights, the mean and the standard deviation respectively of the portfolio solutions of the above problems. Thus, theoretically non satiable investors prefer the portfolio with mean, standard deviation, skewness and kurtosis m_i, σ_i, s, k, i=1,2, to the portfolio with parameters m_p, σ_p, s, k. Similarly non-satiable risk averse investors prefer the portfolio with parameters m_i, σ_i, s, k, i=3,4, respect to portfolio with parameters m_p, σ_p, s, k. In order to test the first (second) order stochastic dominance efficiency of the portfolio with parameters m_p, σ_p, s, k we consider the null hypothesis H_0 that the portfolio with parameters m_p, σ_p, s, k is first (second) order stochastic dominance efficient (non dominated) against the hypothesis H_1 that portfolio with parameters m_i, σ_i, s, k, i=1,2, (m_i, σ_i, s, k, i=3,4) first (second) order stochastically dominates it. In the recent literature have been proposed several tests based on the Kolmogorov Smirnov statistic to compare stochastic ordering preferences (see, among others, Scaillet and Topaloglou (2008)). Alternatively, we discuss a semi parametric statistic obtained by estimating functions theory (see Lehmann and Casella (1998)).

Then, suppose that $R = (R_1, ..., R_T)$ is a random vector on a probability space and the distribution family of this vector is parameterized by $\xi = (\xi_1, ..., \xi_p)$. An estimating function $h(R_s, \xi)$ is called unbiased if $E(h(R, \xi)) = 0$ for all admissible ξ. Generally, the number of EFs is set equal to the number of parameters ξ_k (k=1,. . . ,p) considering the linear combinations $l_{\xi,k} = \sum_{s=1}^{T} \sum_{i=1}^{n} a_{k,i,s} h_i(R_s, \xi)$ k=1,. . . ,p of unbiased Efs $h_i(R, \xi)$ (i=1,. . . ,n). The unbiased estimating functions $h_i(R, \xi)$ are also mutually orthogonal, i.e., for every $i \neq j$; i,j=1,. . . ,n, $E(h_i(R, \xi)h_j(R, \xi)) = 0$. In particular, among all the linear combination $l_{\xi,k} = \sum_{s=1}^{T} \sum_{i=1}^{n} a_{k,i,s} h_i(R_s, \xi)$ of unbiased mutually orthogonal estimating functions the functions $l^*_{\xi,k} = \sum_{s=1}^{T} \sum_{i=1}^{n} \frac{E\left(\frac{\partial h_i(R_s,\xi)}{\partial \xi_k}\right)}{E(h_i^2(R_s, \xi))} h_i(R_s, \xi)$ k=1,. . . ,p are optimal estimating functions. Then, an estimate $\hat{\xi}$ of ξ is obtained by solving the system of estimating equations $l^*_{\xi,k} = 0$, k=1,. . . ,p. According to the estimating function theory the optimal Efs obtained as consistent solution of equations $l^*_{\xi,k} = 0$, after orthogonalization, standardization and optimal combination have the property

$$\sqrt{T}\left(\hat{\xi} - \xi\right) \rightarrow MVN(0, V_{EF}^{-1})$$

where $V_{EF} = [v_{i,j}]_{i,j=1,...,p}$ and $v_{i,j} = E(\frac{\partial l^*_{\xi,i}}{\partial \xi_j})$ i,j=1,...,p. In estimating function theory the estimators are implicitly defined. On the other hand not all the solutions of estimating equations can be considered optimal estimators. Thus, using some convergent methods to compute the roots of equations it is important to start by an approximate solution to get optimal estimates (see Crowder (1986)). However, in some cases we can easily obtain optimal solutions. Typical examples of optimal estimating functions are those proposed by Godambe and Thompson (1989) based on the first four central moments of a given statistic. In Godambe and Thompson's model we have two unbiased and mutually orthogonal estimating functions:

$$h_1(R_t, \xi) = f(R_t) - m(\xi)$$

and

$$h_2(R_t, \xi) = (f(R_t) - m(\xi))^2 - \sigma^2(\xi) - s(\xi)\sigma(\xi)\left(f(R_t) - m(\xi)\right)$$

where f is a measurable real function $E(f(R_t)) = m(\xi)$, $E\left((f(R_t) - m(\xi))^2\right) = \sigma^2(\xi)$, and $s(\xi) = \frac{E\left((f(R_t) - m(\xi))^3\right)}{\sigma^3(\xi)}$. Therefore the optimal estimating functions are given by

$$l^*_{\xi,k} = \sum_{s=1}^{T} \left(a_{k,1,s} h_1(R_s, \xi) + a_{k,2,s} h_2(R_s, \xi)\right)$$

where $a_{k,i,s} = \dfrac{E\left(\frac{\partial h_i(R_s,\xi)}{\partial \xi_k}\right)}{E(h_i^2(R_s,\xi))}$ i=1,2, k=1,. . . ,p. We call this class of estimating functions *GT estimating functions*. Under regularity assumptions the following proposition determines the class of consistent solutions of $m(\xi)$ root of equations $l^*_{\xi,k} = 0$.

Proposition 1. *Suppose we have a sample $R = (R_1, ..., R_T)$ of i.i.d. observations. The consistent estimates of $m(\xi)$ of GT estimating equations $l^*_{\xi,k} = 0$ are given by the values $\hat{\xi}_k$ solutions of the equations for k=1,...,p,:*

$$\widehat{m}(\xi) = \begin{cases} \frac{1}{T}\sum_{t=1}^{T} f(R_t) + c_k - \left(c_k^2 - \frac{1}{T}\sum_{t=1}^{T}\left(f(R_t) - \frac{1}{T}\sum_{t=1}^{T} f(R_t)\right)^2 + \sigma^2(\xi)\right)^{1/2} & \textit{if } c_k > 0 \\ \frac{1}{T}\sum_{t=1}^{T} f(R_t) + c_k + \left(c_k^2 - \frac{1}{T}\sum_{t=1}^{T}\left(f(R_t) - \frac{1}{T}\sum_{t=1}^{T} f(R_t)\right)^2 + \sigma^2(\xi)\right)^{1/2} & \textit{if } c_k \le 0 \end{cases}$$

where $c_k = \dfrac{a_{k,1} - a_{k,2}s(\xi)\sigma(\xi)}{2a_{k,2}}$, $a_{k,1} = \dfrac{E\left(\frac{\partial h_1(R_s,\xi)}{\partial \xi_k}\right)}{E(h_1^2(R_s,\xi))}$ and $a_{k,2} = \dfrac{E\left(\frac{\partial h_2(R_s,\xi)}{\partial \xi_k}\right)}{E(h_2^2(R_s,\xi))}$. Moreover, we get optimal GT estimates of $\hat{\xi}_k$ as solutions of the above equations when the regularity conditions of implicit function theorem are satisfied and we can determine the estimates in all the domain of its definition.

The solutions of GT estimating functions defined in the above proposition could serve to identify quickly the optimal estimators. In our context, the above equations permits us to define optimal estimators. As a matter of fact, let us suppose we have the historical observations $R = (R_1, ..., R_T)$ of a portfolio with parameters $m, \sigma, s = \frac{E\left((R-E(R))^3\right)}{E\left((R-E(R))^2\right)^{3/2}}$, $k = \frac{E\left((R-E(R))^4\right)}{E\left((R-E(R))^2\right)^2}$. Then we can consider the following GT estimating functions

$$l_m = \sum_{t=1}^{T} \left(a_{1,t}h_1(R_t, m, \sigma) + b_{1,t}h_2(R_t, m, \sigma)\right)$$

$$l_\sigma = \sum_{t=1}^{T} b_{2,t}h_2(R_t, m, \sigma)$$

where $h_1(R_t, m, \sigma) = R_t - m$, $h_2(R_t, m, \sigma) = (R_t - m)^2 - \sigma^2 - s\sigma\,(R_t - m)$, $a_{1,t} = \frac{E\left(\frac{\partial h_1(R,m,\sigma)}{\partial m}\right)}{E(h_1^2(R,m,\sigma))} = -\frac{1}{\sigma^2}$, $b_{1,t} = \frac{E\left(\frac{\partial h_2(R,m,\sigma)}{\partial m}\right)}{E(h_2^2(R,m,\sigma))} = \frac{s}{\sigma^3(k-1-s^2)}$, $a_{2,t} = \frac{E\left(\frac{\partial h_1(R,m,\sigma)}{\partial \sigma}\right)}{E(h_1^2(R,m,\sigma))} = 0$, $b_{2,t} = \frac{E\left(\frac{\partial h_2(R,m,\sigma)}{\partial \sigma}\right)}{E(h_2^2(R,m,\sigma))} = \frac{-2}{\sigma^3(k-1-s^2)}$. By equaling to zero the function l_m we get the consistent estimating function of the mean

$$\hat{m} = \begin{cases} \bar{R} - \frac{(k-1)\sigma}{2s} + \frac{1}{2}\left(\left(\frac{(k-1)\sigma}{s}\right)^2 - 4\left(\dfrac{1}{T}\sum_{t=1}^{T}\left(R_t - \bar{R}\right)^2 - \sigma^2\right)\right)^{1/2} & \text{if } s > 0 \\ \bar{R} - \frac{(k-1)\sigma}{2s} - \frac{1}{2}\left(\left(\frac{(k-1)\sigma}{s}\right)^2 - 4\left(\dfrac{1}{T}\sum_{t=1}^{T}\left(R_t - \bar{R}\right)^2 - \sigma^2\right)\right)^{1/2} & \text{if } s < 0 \end{cases} \tag{4}$$

where $\bar{R} = \dfrac{1}{T}\sum_{t=1}^{T} R_t$. By equaling to zero the function l_σ we get the consistent estimating function of the standard deviation

$$\hat{\sigma} = \begin{cases} \dfrac{s}{2}\left(\bar{R} - m\right) - \frac{1}{2}\left(s^2\left(\bar{R} - m\right)^2 + \dfrac{4}{T}\sum_{t=1}^{T}\left(R_t - m\right)^2\right)^{1/2} & \text{if } s > 0 \\ \dfrac{s}{2}\left(\bar{R} - m\right) + \frac{1}{2}\left(s^2\left(\bar{R} - m\right)^2 + \dfrac{4}{T}\sum_{t=1}^{T}\left(R_t - m\right)^2\right)^{1/2} & \text{if } s < 0 \end{cases}$$

However, imposing the vector (l_m, l_σ) equal to zero we get the joint estimates of the mean and of the standard deviation that are respectively the sample mean $\bar{R} = \hat{m} = \dfrac{1}{T}\sum_{t=1}^{T} R_t$ and the sample standard deviation $\hat{\sigma} = \left(\dfrac{1}{T}\sum_{t=1}^{T}\left(R_t - \bar{R}\right)^2\right)^{1/2}$. The estimates $\left(\dfrac{1}{\sqrt{T}}\sum_{t=1}^{T} R_t, \left(\sum_{t=1}^{T}\left(R_t - \bar{R}\right)^2\right)^{1/2}\right)'$ must converge in distribution to a bivariate Gaussian vector with mean $\left(m\sqrt{T}, \sigma\sqrt{T}\right)'$ and variance covariance matrix given by V^{-1} where $V = [v_{i,j}]$ i,j=1,2 $v_{1,1} = E\left(\frac{\partial l_m}{\partial m}\right) = \frac{k-1}{\sigma^2(k-1-s^2)}$; $v_{1,2} = v_{2,1} = E\left(\frac{\partial l_m}{\partial \sigma}\right) = \frac{-2s}{\sigma^2(k-1-s^2)}$; $v_{2,2} = E\left(\frac{\partial l_\sigma}{\partial \sigma}\right) = \frac{4}{\sigma^2(k-1-s^2)}$, that is $V^{-1} = \dfrac{\sigma^2}{4}\begin{bmatrix} 4 & 2s \\ 2s & k-1 \end{bmatrix}$.
When the portfolio with weights x^p is not dominated the above optimization problems should give as solution the portfolio x^p. The null hypothesis (H_0^i i=1,2) is that the portfolio x^p is not first order dominated from the *i-th* portfolio. A simple test for first order efficiency should based on the decision rule

$$Reject\ H_0^i\ if \begin{cases} \sqrt{T}\left(\hat{m}_i - \hat{m}_p\right) \geq c_{1,\alpha}^i \\ \sqrt{T}\left(\hat{\sigma}_i - \hat{\sigma}_p\right) \geq c_{2,\alpha}^i \end{cases} \quad for \quad i = 1, 2, \tag{5}$$

where $\hat{m}_i, \hat{m}_p$ and $\hat{\sigma}_i, \hat{\sigma}_p$ are respectively the sample means and the sample standard deviations of portfolios with weights $x^{(i)}, x^p$. Similarly, if portfolio x^p is not first order dominated, then the null hypothesis H_0^i i=3,4 is that the portfolio with weights x^p is not second order dominated from the i-th portfolio (i=3,4). Then a simple test for second order efficiency should based on the decision rule

$$Reject\ H_0^i\ if \begin{cases} \sqrt{T}\left(\hat{m}_i - \hat{m}_p\right) \geq c_{1,\alpha}^i \\ \sqrt{T}\left(\hat{\sigma}_i - \hat{\sigma}_p\right) \leq c_{2,\alpha}^i \end{cases} \quad for \quad i = 3, 4, \tag{6}$$

As confirmed by the following proposition the intuitive ideas of the above tests are justified by a theoretical point of view.

Proposition 2. *When the portfolios are uniquely determined by the first four moments we can guarantee that there exist opportune values $c^i_{j,\alpha}$ (see the appendix), i=1,2,3,4, j=1,2, such that*

$$\lim_{T\to\infty} P\left(reject\ H_0^i \middle| \ H_0^i\ is\ true\right) \leq \alpha \ and \ \lim_{T\to\infty} P\left(reject\ H_0^i \middle| \ H_0^i\ is\ false\right) = 1.$$

Using estimating function theory we can also value the efficiency when we use different reward, risk measures. However, in these cases we need to know the ordering of portfolio distributions with respect to the new parameters. In particular we can generalize some results proved by Ortobelli (2001).

3.2 Stochastic orderings depending on reward and risk measures

Under institutional restrictions on the market we can assume that the portfolios of gross returns are positive random variables belonging to scale family, denoted with $\sigma\tau_k^+(\bar{a})$, that admits positive translations and it has the following characteristics[5] m_X is a positive, positive homogeneous reward measure consistent with monotone order that satisfies the relation $m_{X+t} = m_X + t$ for every real $t \geq 0$.:

1. Every distribution F_X belonging to $\sigma\tau_k^+(\bar{a})$ is associated to a positive random variable X and is identified by k parameters $(m_X, \sigma_X, a_{1,X}..., a_{k-2,X}) \in A \subseteq R^k$, where m_X and σ_X are respectively a simple positive reward measure isotone with the monotony order (i.e., $X \geq Y$ implies $m_X \geq m_Y$), and a positive scale parameter associated to the random variable X. We assume that the class $\sigma\tau_k^+(\bar{a})$ is weakly determined from its parameterization. That is, the equality $(m_X, \sigma_X, a_{1,X}..., a_{k-2,X}) = (m_Y, \sigma_Y, a_{1,Y}..., a_{k-2,Y})$ implies that $F_X \overset{d}{=} F_Y$, but the converse is not necessarily true [6].

2. For every admissible real $t \geq 0$ and for every $F_X \in \sigma\tau_k^+(\bar{a})$, the distribution function F_X has the same parameters of the distribution $F_{X+t} \in \sigma\tau_k^+(\bar{a})$, except the

[5] In this classification we consider reward measures instead of the mean and this represents the main difference with the classification proposed by Ortobelli (2001).

[6] We use the same notation of Ortobelli (2001). However in all the examples dealt in the paper we use only distribution families uniquely determined from the parameters.

reward measure and the scale parameter. In particular, the reward measure is translation equivariant (i.e., $m_{X+t} = m_X + t$ for every real $t \geq 0$) and the application $f_X(t) = \sigma_{X+t}$ is a non increasing continuous function.

3. For every admissible positive α and for every $F_X \in \sigma\tau_k^+(\bar{a})$, the distribution function F_X has the same parameters of the distribution $F_{\alpha X}$ except the reward measure and the scale parameter which are positive homogeneous, i.e., $m_{\alpha X} = \alpha m_X$ and $\sigma_{\alpha X} = \alpha \sigma_X$.

Remark 2 *Observe that for any positive random variable a reward measure isotone with the monotony order is positive. In particular, any coherent reward measure and any aggressive-coherent reward measure applied to random variables belonging to a $\sigma\tau_k^+(\bar{a})$ is consistent with monotony order, positive, positive homogenous and translation equivariant (see Rachev et al. 2008).*

Remark 3 *Requiring that all the other parameters except the reward and risk measures must be translation and scale invariant is not a very strong assumption. As a matter of fact, as observed by Ortobelli 2001, we can always assume that the family depends on some given parameters and then we can find another parametrization of the family that satisfies the above conditions. Typical example is the case of portfolios depending on a finite number of moments (see Ortobelli 2001).*

When portfolios belong to a $\sigma\tau_k^+(\bar{a})$ class, we can identify stochastic dominance relations among portfolios.

Theorem 1 *Assume all random admissible portfolios of gross returns belonging to a $\sigma\tau_k^+(\bar{a})$ class. Let $w'Z$ and $y'Z$ be a couple of portfolios respectively determined by the parameters $(m_{w'Z}, \sigma_{w'Z}, a_{1,p}, ..., a_{k-2,p})$ and $(m_{y'Z}, \sigma_{y'Z}, a_{1,p}, ..., a_{k-2,p})$. Then, the following implications hold:*

1. $\frac{m_{w'Z}}{\sigma_{w'Z}} \geq \frac{m_{y'Z}}{\sigma_{y'Z}}$ *and* $\sigma_{w'Z} \geq \sigma_{y'Z}$ *(with at least one inequality strict) implies* $w'Z$ *FSD* $y'Z$.

2. *Suppose* $\dfrac{m_{w'Z}}{\sigma_{w'Z}} = \dfrac{m_{y'Z}}{\sigma_{y'Z}}$. *Then* $\sigma_{w'Z} > \sigma_{y'Z}$ *if and only if* $w'ZFSDy'Z$

3. $w'Z$ *FSD* $y'Z$ *implies that* $m_{w'Z} \geq m_{y'Z}$ *and* $w'Z\underset{\succ}{FORS}y'Z$ *for any simple FORS risky ordering.*

From the previous theorem we deduce that most of the results relative to the first order stochastic dominance proposed in the Ortobelli's classification are still valid when we consider reward measures different from the mean. Thus, as follows by the next example we can easily find optimal portfolios assuming that the portfolio depends on a reward measure and finite number of parameters.

Example 1 Let us assume that all portfolios of gross returns belong to a scale invariant family of positive random variables $\sigma\tau_4^+(\bar{a})$which is uniquely determined by parameters $(m_X, \sigma_X, a_{1,X}, a_{2,X})$. Suppose we have a portfolio with vector weights, and parameters respectively given by $x^p = [x_1^p, ..., x_n^p]'$, m_p, σ_p, s, k then all non-satiable investors will prefer the portfolios solution of the following optimization problems

$$\begin{array}{c} \max\limits_x \sigma_{x'Z} \quad \text{subject to} \\ \dfrac{m_{x'Z}}{\sigma_{x'Z}} \geq \dfrac{m_p}{\sigma_p};\ x'e = 1;\ a_{1,x'Z} = s; \\ a_{2,x'Z} = k;\ x_i \geq 0,\ i = 1,..,n \end{array} \qquad \begin{array}{c} \max\limits_x \dfrac{m_{x'Z}}{\sigma_{x'Z}} \quad \text{subject to} \\ \sigma_{x'Z} \geq \sigma_p;\ x'e = 1;\ a_{1,x'Z} = s; \\ a_{2,x'Z} = k;\ x_i \geq 0,\ i = 1,..,n \end{array} \tag{7}$$

However in the previous analysis we do not analyze the relations of different reward measures.

Proposition 3. *Suppose the two parametric family $\sigma\tau_2^+(\bar{a})$ admits two possible parameterizations $(m_{1,X}, \sigma_X)$ and $(m_{2,X}, \sigma_X)$ with the same translation invariant scale parameter (i.e. $\sigma_{X+t} = \sigma_X$ for any real t). Then for any random variable X belonging to $\sigma\tau_2^+(\bar{a})$ we have that $(m_{1,X} - m_{2,X})/\sigma_X$ is a constant. That is, if there exist a random variable $X \in \sigma\tau_2^+(\bar{a})$ such that $m_{1,X} > m_{2,X}$ then $m_{1,Y} > m_{2,Y} \forall Y \in \sigma\tau_2^+(\bar{a})$.*

The same proposition is substantially valid for scale and translation invariant families $\sigma\tau_2(\bar{a})$ of (non necessarily positive) random variables. Moreover considering that it is theoretically indifferent using one or any other existing translation invariant scale parameter for a $\sigma\tau_2(\bar{u})$ (or $\sigma\tau_2^+(\bar{a})$) class. Then the above proposition tells us that the ranking among different portfolios given by the ratio between any reward measure and a deviation measure is practically the same if we assume that all portfolios are uniquely determined by two parameters. Observe that when we fix all parameters except the reward measure and the scale parameter of a given $\sigma\tau_k^+(\bar{a})$ family, we obtain a particular $\sigma\tau_2^+(\bar{a})$ class. Therefore, every $\sigma\tau_k^+(\bar{a})$ class weakly determined by the parameters (m, σ, y) (where m is a reward measure, σ is the scale parameter and $y \in B \subseteq R^{k-2}$ is the vector of the other parameters) can be seen as the union of $\sigma\tau_2^+(m, \sigma, \bar{y}) =: V(\bar{y})$ families (i.e., $\sigma\tau_k^+(\bar{a}) = \bigcup\limits_{\bar{y} \in B} V(\bar{y})$). Therefore, the above proposition can be applied to the respective components of the union. However, we cannot guarantee that the constant ratio between the difference of two admissible reward measures and the dispersion measure of the same component $V(\bar{y})$ remains equal for every other component of the union (i.e. it is not necessarily true that the ratio remains constant for all the distributions of the family $\sigma\tau_k^+(\bar{a})$). In fact, it could be that different ranking measures penalize the asymmetry or the kurtosis parameters in a different way. The above Proposition can be used even to prove second order stochastic relations using reward measures different from the mean.

Theorem 2 *Assume that all random admissible portfolios of gross returns belonging to a $\sigma\tau_k^+(\bar{a})$ class. The family $\sigma\tau_k^+(\bar{a})$ admits two possible parameterizations with the same parameter except the reward measure. Assume that the scale parameter is translation invariant (i.e., $\sigma_{X+t} = \sigma_X$ for any real t) and the mean is one of the two possible reward measures. Let $w'Z$ and $y'Z$ be a couple of portfolios respectively determined by the parameters $(m_{w'Z}, \sigma_{w'Z}, a_{1,p}, ..., a_{k-2,p})$ and $(m_{y'Z}, \sigma_{y'Z}, a_{1,p}, ..., a_{k-2,p})$ (or $(E(w'Z), \sigma_{w'Z}, a_{1,p}, ..., a_{k-2,p})$ and $(E(y'Z), \sigma_{y'Z}, a_{1,p}, ..., a_{k-2,p})$). Then, we distinguish two cases.*

1) *Suppose $m_{w'Z} > E(w'Z)$ (or equivalently $m_{y'Z} > E(y'Z)$), then the following implications hold:*

1a) $\frac{m_{w'Z}}{\sigma_{w'Z}} \geq \frac{m_{y'Z}}{\sigma_{y'Z}}$ *and* $m_{w'Z} \geq m_{y'Z}$ *(with at least one inequality strict) implies* $w'Z\ SSD\ y'Z$.

1b) $m_{w'Z} \geq m_{y'Z}$ *and* $\sigma_{w'Z} \leq \sigma_{y'Z}$ *(with at least one inequality strict) implies* $w'Z\ SSD\ y'Z$.

1c) $w'Z\ SSD\ y'Z$ *and* $E(w'Z) = E(y'Z)$ *(i.e.* $w'ZR - S\ y'Z$*) implies* $m_{w'Z} < m_{y'Z}$ *and* $\sigma_{w'Z} < \sigma_{y'Z}$.

2) *Suppose $m_{w'Z} < E(w'Z)$ (or equivalently $m_{y'Z} < E(y'Z)$), then the following implications hold:*

2a) $w'Z\ SSD\ y'Z$ *implies* $m_{w'Z} \geq m_{y'Z}$.

2b) $w'Z\ SSD\ y'Z$ *but* $w'Z$ *does not dominates at first order* $y'Z$ *(this assumption includes the case* $w'ZR - S\ y'Z$*) implies* $m_{w'Z} \geq m_{y'Z}$ *and* $\sigma_{w'Z} < \sigma_{y'Z}$.

The previous theorem classifies reward measures with respect to the consistency with second order stochastic dominance and Rothshild Stiglitz uncertainty order:

1) those bigger than the mean are consistent with Rothshild Stiglitz uncertainty order;

2) those smaller than the mean are isotonic with second stochastic order and Rothshild Stiglitz order.

Therefore, when the aggressive-coherent reward measures maintaining their value bigger than the mean and performance ratio bigger than a fixed one we find portfolios that cannot be dominated by non satiable risk lover investors (investors with increasing and convex utility function). This portfolio is SSD dominant, but it could be further dominated at the second order by portfolios that have the same mean and lower dispersion. While if we maximize aggressive-coherent reward measures maintaining their value lower than the mean, we get portfolios that are not SSD dominated. Moreover, in all cases, maximizing uncertainty or coherent or aggressive-coherent reward measures we get portfolios that are not FSD dominated.

Example 2 Let us assume that all portfolios of gross returns belong to a scale invariant family of positive random variables $\sigma\tau_4^+(\bar{a})$ which is uniquely determined by parameters $(m_X, \sigma_X, a_{1,X}, a_{2,X})$. Suppose we have a portfolio with vector weights, and parameters respectively given by $x^p = [x_1^p, ..., x_n^p]'$, m_p, σ_p, s, k. Assume that the dispersion parameter σ_X is translation invariant (i.e., $\sigma_{X+t} = \sigma_X$ for any real t). If $m_p > E\left((x^p)' Z\right)$ then $m_X > E(X)$ for any portfolio X with the same parameters s, k and all non-satiable risk averse investors will prefer the portfolios solution of the following optimization problem

$$\begin{array}{cc} \max\limits_x \ m_{x'Z} \quad \text{subject to} & \max\limits_x \ \dfrac{m_{x'Z}}{\sigma_{x'Z}} \quad \text{subject to} \\ \dfrac{m_{x'Z}}{\sigma_{x'Z}} \geq \dfrac{m_p}{\sigma_p};\ x'e = 1; a_{1,x'Z} = s; & m_{x'Z} \geq m_p;\ x'e = 1;\ a_{1,x'Z} = s; \\ a_{2,x'Z} = k;\ x_i \geq 0,\ i = 1,..,n & a_{2,x'Z} = k;\ x_i \geq 0,\ i = 1,..,n \end{array} \tag{8}$$

The solution of these problems give us:

a) dominating portfolios w.r.t. SSD order if $m_p > E\left((x^p)' Z\right)$. However it could exist a portfolio that is still preferred to the solution by all non-satiable risk averse investors, but certainly it does not exist a portfolio that is preferred by all non satiable risk lover investors.

b) non-dominated portfolios w.r.t. SSD order among those portfolios with the same parameters s, k, if $m_p \leq E\left((x^p)' Z\right)$.

Moreover, for all the solutions of optimization problems (7) and (8) we get non dominated FSD portfolios (among those portfolios with the same parameters s, k). About risk FORS orderings we can determine the following results.

Theorem 3 *Assume ρ_X is a FORS risk measure associated to a FORS ordering defined on the real interval [a,b]. Then, for any $\alpha > 1$ the following implications*

hold:
1) If $\rho_X(t) \leq \rho_Y(t)$ *for every* $t \leq t_0 \in (a,b)$ *and* $\rho_X(t) \geq \rho_Y(t)$ *for* $t \geq t_0 \in (a,b)$ *and the inequalities are strict for some t, then* $-\infty < \lim_{x \to b} \rho_{X,\alpha}(x) - \rho_{Y,\alpha}(x) \leq 0$ *if and only if* $X \underset{\succ,\alpha}{FORS} Y$.
2) Assume all random admissible portfolios of gross returns belong to a $\sigma\tau_k^+(\bar{a})$ *class where the translation parameter is* $m_Y = -\dfrac{\Gamma(\alpha)}{(b-a)^\alpha}\rho_{Y,\alpha}(b)$ *for* $\alpha > 1$ *and* $|a|, |b| < \infty$. *Suppose* ρ_X *is a monotone decreasing FORS risk measure that is negative, positive homogeneous and translation equivariant for the random variables belonging to* $\sigma\tau_k^+(\bar{a})$ *(i.e.,* $\rho_{cX+d} = c\rho_X - d \forall c, d \in R^+$, $\forall X \in \sigma\tau_k^+(\bar{a})$*). Let* $w'Z$ *and* $y'Z$ *be a couple of portfolios respectively determined by the parameters* $(m_{w'Z}, \sigma_{w'Z}, a_{1,p}, ..., a_{k-2,p})$ *and* $(m_{y'Z}, \sigma_{y'Z}, a_{1,p}, ..., a_{k-2,p})$. *Then,* $\frac{m_{w'Z}}{\sigma_{w'Z}} \geq \frac{m_{y'Z}}{\sigma_{y'Z}}$ *and* $m_{w'Z} > m_{y'Z}$ *(with at least one inequality strict) implies* $w'Z \underset{\succ,\alpha}{FORS} y'Z$, *while* $w'Z \underset{\succ,\alpha}{FORS} y'Z$ *implies* $m_{w'Z} \geq m_{y'Z}$.

The above theorem suggests a methodology to find optimal FORS portfolios for any $\alpha > 1$.

Example 3 Assume as FORS ordering the α ($\alpha > 1$) bounded inverse stochastic dominance order, i.e., X dominates Y in the sense of the α inverse stochastic dominance order (namely $X \underset{-\alpha}{\geq} Y$), if and only if $\rho_{X,\alpha}(p) = -F_X^{(-\alpha)}(p) \leq -F_Y^{(-\alpha)}(p) \forall p \in [0,1]$ (see Ortobelli et al. (2008)) where $F_X^{(-1)}(p) = F_X^{-1}(p)$ $\forall p \in [0,1]$

$$F_X^{(-\alpha)}(p) = \frac{1}{\Gamma(\alpha)} \int_0^p (p-u)^{\alpha-1} dF_X^{(-1)}(u) = \frac{1}{\Gamma(\alpha-1)} \int_0^p (p-u)^{\alpha-2} F_X^{(-1)}(u) du \quad \forall p \in [0,1].$$

Assume that all random admissible portfolios of gross returns belong to a $\sigma\tau_k^+(\bar{a})$ class where the translation parameter is $m_X = (\alpha-1)\int_0^1 (1-u)^{\alpha-2} F_X^{(-1)}(u) du$ for $\alpha > 1$. $\rho_X = -F_X^{(-1)}$ is a monotone decreasing FORS risk measure that is negative, positive homogeneous and translation equivariant for any positive random variable. Suppose we have a portfolio with parameters $(m_p, \sigma_p, a_{1,p}, ..., a_{k-2,p})$. Then all non-satiable investors with α inverse stochastic dominance preference will prefer the portfolios solutions of the following optimization problems (8), where $m_{w'Z} = (\alpha-1)\int_0^1 (1-u)^{\alpha-2} F_{w'Z}^{-1}(u) du$ can be approximated considering the consistent estimator (for i.i.d. observations) $\hat{m}_X = \dfrac{(\alpha-1)}{N} \sum_{i=1}^{N} \left(\dfrac{N-i}{N}\right)^{\alpha-2} X_{i:N}$, and $X_{i:N}$ is the i-th ordered observation. In particular, when α=2, we get $m_X = E(X)$ and the solutions of problems (8) are optimal for non satiable risk averse investors (as a matter of fact $X \underset{-2}{\geq} Y$ iff X *SSD* Y).

3.3 Semi-parametric tests for orderings depending on reward and risk measures

Next, we discuss semi-parametric tests for the stochastic dominance in the case we use different reward and risk measures as in the previous examples. Thus, let us assume that all portfolios of gross returns belong to a scale invariant family of positive continuous random variables $\sigma\tau_4^+(\bar{a})$ which is uniquely determined by parameters $(m_X, \sigma_X, a_{1,X}, a_{2,X})$. In particular, we assume the aggressive-coherent reward measure

$$m_X = ES_\beta(-X) - ES_\alpha(X) = E\left(\frac{I_{[X \geq t_\beta(X)]}X}{(1-\beta)} + \frac{I_{[X \leq t_\alpha(X)]}X}{\alpha}\right) \tag{9}$$

with $t_\alpha(X) = F_X^{-1}(\alpha)$, $\alpha < \beta$ and the parameters

$$\sigma_X = E\left(\left(\frac{I_{[X \geq t_\beta(X)]}X}{(1-\beta)} + \frac{I_{[X \leq t_\alpha(X)]}X}{\alpha} - m_X\right)^2\right)^{0.5},$$

$$a_{1,X} = \frac{E\left(\left(\frac{I_{[X \geq t_\beta(X)]}X}{(1-\beta)} + \frac{I_{[X \leq t_\alpha(X)]}X}{\alpha} - m_X\right)^3\right)}{\sigma_X^3}$$

and

$$a_{2,X} = \frac{E\left(\left(\frac{I_{[X \geq t_\beta(X)]}X}{(1-\beta)} + \frac{I_{[X \leq t_\alpha(X)]}X}{\alpha} - m_X\right)^4\right)}{\sigma_X^4}.$$

where $I_{[X \in A]} = \begin{cases} 1 & \text{if } X \in A \\ 0 & \text{otherwise} \end{cases}$. Suppose we have T observations of the vector of gross returns, then we can assume $\alpha = \frac{q}{T}$, $\beta = \frac{r}{T}$ with $q, r \in \mathbb{N}$ and $q < r < T$ such that the α, β percentiles are respectively the q-th and r-th ordered observations, i.e., $t_\alpha(X) = X_{q:T}$ and $t_\beta(X) = X_{r:T}$. Hence, given a portfolio of gross returns with vector weights, and parameters respectively given by $x^p = [x_1^p, ..., x_n^p]'$, m_p, σ_p, s, k, (here $s = a_{1,(x^p)'Z}$, $k = a_{2,(x^p)'Z}$) we can propose an estimating function test to verify the efficiency of the portfolio. Let $x^{(1)} = [x_1^{(1)}, ..., x_n^{(1)}]'$, m_1, σ_1, $x^{(2)}$, m_2, σ_2, be the vector of weights, the reward measures and the dispersions of the portfolios respectively solutions of problems (7). In particular we know from the previous subsection that all non-satiable investors with the behavioral finance ordering (1) will prefer the portfolios with parameters m_i, σ_i, s, k, i=1,2, to the portfolio with parameters m_p, σ_p, s, k of the optimization problems (7). Consider the historical observations $R = (R_1, ..., R_T)$ of the portfolio. As for the moments estimators, we can consider the following estimating function statistics

$$l_m = \sum_{t=1}^{T} \left(a_{1,t} h_1(R_t, m, \sigma) + b_{1,t} h_2(R_t, m, \sigma) \right)$$

$$l_\sigma = \sum_{t=1}^{T} b_{2,t} h_2(R_t, m, \sigma)$$

where

$$h_1(R_t, m, \sigma) = \frac{I_{[R_t \geq t_\beta(R_t)]} R_t}{(1-\beta)} + \frac{I_{[R_t \leq t_\alpha(R_t)]} R_t}{\alpha} - m_R,$$

$$h_2(R_t, m, \sigma) = \left(\frac{I_{[R_t \geq t_\beta(R_t)]} R_t}{(1-\beta)} + \frac{I_{[R_t \leq t_\alpha(R_t)]} R_t}{\alpha} - m_R \right)^2 - \sigma_R^2 + \\ - s\sigma_R \left(\frac{I_{[R_t \geq t_\beta(R_t)]} R_t}{(1-\beta)} + \frac{I_{[R_t \leq t_\alpha(R_t)]} R_t}{\alpha} - m_R \right),$$

$a_{1,t} = \frac{E\left(\frac{\partial h_1(R,m,\sigma)}{\partial m}\right)}{E(h_1^2(R,m,\sigma))} = -\frac{1}{\sigma_R^2}$, $b_{1,t} = \frac{E\left(\frac{\partial h_2(R,m,\sigma)}{\partial m}\right)}{E(h_2^2(R,m,\sigma))} = \frac{s}{\sigma_R^2(k-1-s^2)}$, $a_{2,t} = \frac{E\left(\frac{\partial h_1(R,m,\sigma)}{\partial \sigma}\right)}{E(h_1^2(R,m,\sigma))} = 0$, $b_{2,t} = \frac{E\left(\frac{\partial h_2(R,m,\sigma)}{\partial \sigma}\right)}{E(h_2^2(R,m,\sigma))} = \frac{-2}{\sigma_R^2(k-1-s^2)}$. By equaling to zero the functions l_m and l_σ we get the estimating functions of the aggressive-coherent reward measure $ES_\beta(-X) - ES_\alpha(X)$ and its standard deviation. The estimating functions formulas are the same of the previous subsection (formula (4) and subsequent), where instead of R_t there is $\left(\frac{I_{[R_t \geq t_\beta(R_t)]} R_t}{(1-\beta)} + \frac{I_{[R_t \leq t_\alpha(R_t)]} R_t}{\alpha} \right)$. Thus by setting equal to zero the vector (l_m, l_σ) we get the joint estimates:

$$\bar{R} = \hat{m} = \frac{1}{T} \sum_{t=1}^{T} \left(\frac{I_{[R_t \geq t_\beta(R_t)]} R_t}{(1-\beta)} + \frac{I_{[R_t \leq t_\alpha(R_t)]} R_t}{\alpha} \right)$$

and

$$\bar{\sigma} = \hat{\sigma} = \left(\frac{1}{T} \sum_{t=1}^{T} \left(\frac{I_{[R_t \geq t_\beta(R_t)]} R_t}{(1-\beta)} + \frac{I_{[R_t \leq t_\alpha(R_t)]} R_t}{\alpha} - \bar{R} \right)^2 \right)^{1/2}.$$

The estimates $\sqrt{T}\left(\bar{R} - m_R, \bar{\sigma} - \sigma_R\right)'$ converge in distribution to a bivariate Gaussian vector with null mean and variance covariance matrix given by $V^{-1} = \frac{\sigma^2}{4} \begin{bmatrix} 4 & 2s \\ 2s & k-1 \end{bmatrix}$. According to the previous analysis, when the portfolio with weights x^p is not dominated a simple test for first order stochastic dominance and behavioral finance ordering (1) should based on the decision rule

$$Reject \quad H_0^i \text{ if } \begin{cases} \sqrt{T}\left(\hat{m}_i - \hat{m}_p\right) \geq c_{1,\alpha}^i \\ \sqrt{T}\left(\hat{\sigma}_i - \hat{\sigma}_p\right) \geq c_{2,\alpha}^i \end{cases} \text{ for } \quad i = 1, 2. \tag{10}$$

As for the moment case, when the i-th portfolio has the same distribution of portfolio with weights x^p and we assume that all its parameters are known (i.e., $\hat{m}_p = m_p = m_i; \hat{\sigma}_p = \sigma_p = \sigma_i$) we can choose $(c^i_{1,\alpha}, c^i_{2,\alpha}) \in \mathbb{R}^2_+$ such that

$$\frac{1}{\pi\sigma_p^2\sqrt{k-1-s^2}} \int_{c^i_{1,\alpha}}^{+\infty} \int_{c^i_{2,\alpha}}^{+\infty} e^{\frac{-(k-1)}{2(k-1-s^2)}\left[\frac{x^2}{\sigma_p^2} - \frac{4sxy}{\sigma_p^2(k-1)} + \frac{4y^2}{\sigma_p^2(k-1)}\right]} dxdy = \alpha$$

Clearly, in this case the parameters have a different interpretation. Moreover, using the example 3, we can propose tests to value the efficiency with respect to an α ($\alpha > 1$) inverse stochastic dominance order. Thus, assume all random admissible portfolios of gross returns belong to a $\sigma\tau_4^+(\bar{a})$ class where the parameters are:

$$m_X = (\alpha-1)E\left(X(1-F_X(X))^{\alpha-2}\right),$$

$$\sigma_X = \left((\alpha-1)^2 E\left(X^2(1-F_X(X))^{2\alpha-4}\right) - m_X^2\right)^{0.5}$$

$$a_{1,X} = \frac{E\left(\left((\alpha-1)X(1-F_X(X))^{\alpha-2} - m_X\right)^3\right)}{\sigma_X^3}$$

and

$$a_{2,X} = \frac{E\left(\left((\alpha-1)X(1-F_X(X))^{\alpha-2} - m_X\right)^4\right)}{\sigma_X^4}.$$

and suppose we have a portfolio with parameters (m_p, σ_p, s, k) (where $s = a_{1,(x^p)'Z}$, $k = a_{2,(x^p)'Z}$). In particular, we can assume that the distribution function F_R is the empirical one, i.e., $F_R(x) = \frac{1}{T}\sum_{t=1}^{T} I_{[R_t \leq x]}$. Then all non-satiable investors with α inverse stochastic dominance preference will prefer the portfolios solutions of the following optimization problems. Let $x^{(1)} = [x_1^{(1)}, ..., x_n^{(1)}]'$, m_1, σ_1, $x^{(2)}$, m_2, σ_2, be the vector of weights, the reward measures and the dispersions of the portfolios respectively solutions of problems (8). So when we propose the estimating function estimators imposing the vector (l_m, l_σ) equal to zero we get the joint estimates:

$$\bar{R} = \hat{m} = \frac{(\alpha-1)}{T}\sum_{i=1}^{T}\left(\frac{T-i}{T}\right)^{\alpha-2} R_{i:T},$$

and

$$\bar{\sigma} = \hat{\sigma} = \left(\frac{1}{T}\sum_{t=1}^{T}\left((\alpha-1)\left(\frac{T-t}{T}\right)^{\alpha-2} R_{t:T} - \bar{R}\right)^2\right)^{1/2}.$$

When the portfolio with weights x^p is not dominated a simple test for α inverse stochastic dominance efficiency should based on the decision rule

$$\textit{Reject}\quad H_0^i \text{ if } \begin{cases} \sqrt{T}\,(\hat{m}_i - \hat{m}_p) \geq c_{1,\alpha}^i \\ \sqrt{T}\,(\hat{\sigma}_i - \hat{\sigma}_p) \leq c_{2,\alpha}^i \end{cases} \text{for } i = 1, 2,$$

These examples have shown how to use estimating function theory to test preference orderings when we assume that portfolios are uniquely determined by a finite number of parameters. Next we will discuss non parametric tests for behavioral finance orderings.

4 Non parametric tests for behavioral finance orderings

Let us consider a FORS risk measure $\rho_X : [\mathbf{a},\mathbf{b}] \to \bar{R}$ (where $[\mathbf{a},\mathbf{b}] \subseteq \bar{R}^n$) associated to a FORS behavioral ordering of portfolios X belonging to a class of admissible portfolios. Assume that $\forall u \in [\mathbf{a},\mathbf{b}]$ $\rho_X(u) = E(f(X,u))$ for a given measurable function $f : R \times [\mathbf{a},\mathbf{b}] \to R$ such that $\sup\limits_{X\in\Lambda, u\in[\mathbf{a},\mathbf{b}]} E(f^2(X,u)) < +\infty$. Then we know that given a sample $(Z_1, ..., Z_T)$ of i.i.d. return vectors the empirical estimator $\hat{\rho}_{x'Z}(u) = \dfrac{1}{T}\sum\limits_{i=1}^{T} f(x'Z_i, u)$ converges to the mean $E(f(x'Z,u))$ and from central limit theorem follows that $\sqrt{T}\,(\hat{\rho}_{x'Z}(u) - \rho_{x'Z}(u))$ converge to a Gaussian random variable with null mean and variance $E(f^2(x'Z,u)) - (\rho_{x'Z}(u))^2$ for any $u \in [\mathbf{a},\mathbf{b}]$ and for any admissible portfolio $x'Z$. Now suppose we have a portfolio with vector weights $x^p = [x_1^p, ..., x_n^p]'$. According to our previous definition the portfolio $(x^p)'Z$ dominates in the sense of the FORS ordering another admissible portfolio $y'Z$ (i.e., $(x^p)'Z\underset{\succ,1}{FORS}y'Z$) if and only if $\rho_{(x^p)'Z}(u) \leq \rho_{y'Z}(u)$ $\forall u \in [\mathbf{a},\mathbf{b}]$. Thus, we assume that the null hypothesis H_0, $\rho_{(x^p)'Z}(u) \leq \rho_{y'Z}(u)$ $\forall u \in [\mathbf{a},\mathbf{b}]$ and for any portfolio $y'Z$. So, similarly to stochastic dominance tests suggested by Scailet and Topaloglu, a nonparametric test for FORS order efficiency should based on the following decision rule for some positive c_α

$$\textit{reject}\quad H_0 \text{ if } \hat{S}_\rho = \sqrt{T} \sup_{u\in[\mathbf{a},\mathbf{b}]; y: \sum_{i=1}^{n} y_i = 1; y_i \geq 0} \left[\hat{\rho}_{(x^p)'Z}(u) - \hat{\rho}_{y'Z}(u)\right] \geq c_\alpha \qquad (11)$$

where $\Lambda = \left\{ y \in \mathbb{R}^n \,\middle|\, y_i \geq 0; \sum\limits_{i=1}^{n} y_i = 1 \right\}$. Let $u^* \in [\mathbf{a},\mathbf{b}]$ and $y^* \in \Lambda$ such that $\hat{S}_\rho = \hat{S}_\rho^* := \sqrt{T}\left(\hat{\rho}_{(x^p)'Z}(u^*) - \hat{\rho}_{(y^*)'Z}(u^*)\right)$. Then under the H_0 hypothesis, we have that

$\left(\rho_{(x^p)'Z}(u^*) - \rho_{(y^*)'Z}(u^*)\right) \leq 0$ and

$$\begin{aligned}\hat{S}_\rho &= \hat{S}^*_\rho - \sqrt{T}\left(\rho_{(x^p)'Z}(u^*) - \rho_{(y^*)'Z}(u^*)\right) + \sqrt{T}\left(\rho_{(x^p)'Z}(u^*) - \rho_{(y^*)'Z}(u^*)\right) \\ &\leq \sqrt{T}\left(\hat{\rho}_{(x^p)'Z}(u^*) - \rho_{(x^p)'Z}(u^*)\right) - \sqrt{T}\left(\hat{\rho}_{(y^*)'Z}(u^*) - \rho_{(y^*)'Z}(u^*)\right)\end{aligned}$$

and we have the equality if $(x^p)'Z$ has the same distribution of $(y^*)'\,Z$. Thus

$$\lim_{T\to\infty} \Pr\left(\hat{S}_\rho \geq c_\alpha \mid H_0 \text{ } is \text{ } true\right) \leq \lim_{T\to\infty} \Pr\left(X \geq c_\alpha\right) = \alpha(c)$$

where X is Gaussian distributed with null mean and variance:

$$\begin{aligned}\tilde{\sigma} = E\left(f^2((x^p)'Z)\right) - \left(\rho_{(x^p)'Z}(u^*)\right)^2 + E\left(f^2((y^*)'Z)\right) - \left(\rho_{(y^*)'Z}(u^*)\right)^2 + \\ -2E\left(f((y^*)'Z)f((x^p)Z)\right) + 2\rho_{(x^p)'Z}(u^*)\rho_{(y^*)'Z}(u^*).\end{aligned}$$

Thus with test (11) we provide a test that never rejects more often than $\alpha(c)$ for any portfolio $(x^p)'Z$ that satisfies the null hypothesis. Moreover when H_0 hypothesis is false there exists a $\bar{u} \in [\mathbf{a},\mathbf{b}]$ and a portfolio $y \in \Lambda = \left\{ y \in \mathbb{R}^n \,\middle|\, y_i \geq 0; \sum_{i=1}^n y_i = 1 \right\}$ such that $\rho_{(x^p)'Z}(\bar{u}) - \rho_{y'Z}(\bar{u}) = \gamma > 0$. Since, $\hat{S}_\rho \geq \sqrt{T}\left(\hat{\rho}_{(x^p)'Z}(\bar{u}) - \hat{\rho}_{y'Z}(\bar{u})\right)$ then $\lim_{T\to\infty} \Pr\left(\hat{S}_\rho \geq c \,|H_0 \text{ } is \text{ } false\right) = 1$. Let us consider the following behavioral finance examples.

Example 4 Consider the FORS measure $\rho_X(\alpha,\beta) = ES_\alpha(X) - ES_\beta(-X)$ where $(\alpha,\beta) \in [0,1] \times [0,1]$ associated to the FORS behavioral ordering. In this case we have that

$$\hat{S}_\rho = \sqrt{T} \sup_{(\alpha,\beta)\in[0,1]^2; y\in\Lambda} \left[\hat{\rho}_{(x^p)'Z}(\alpha,\beta) - \hat{\rho}_{y'Z}(\alpha,\beta)\right]$$

where $\hat{\rho}_R(\alpha,\beta) = \dfrac{-1}{T}\sum_{t=1}^T \dfrac{I_{[R_t \geq t_\beta(R_t)]}R_t}{(1-\beta)} + \dfrac{I_{[R_t \leq t_\alpha(R_t)]}R_t}{\alpha}$ for a given portfolio R with historical observations $(R_1,...,R_T)$. Observe that we still obtain a FORS behavioral finance ordering if we fix the parameter α (either β) in the above definition of $\rho_X(\alpha,\beta)$. In this case it is like if we fix the coherency (aggressiveness) of investors.

Example 5 Let us consider the FORS measure $m_X(z,u)$ where $(z,u) \in [0,1] \times (-\infty,0])$. That is, we assume the Markowitz behavioral ordering described by Levy and Levy (2002). In this case we obtain:

$$\hat{S}_\rho = \sqrt{T} \sup_{(z,u)\in[0,1]\times(-\infty,0]; y\in\Lambda} \left[\hat{\rho}_{(x^p)'Z}(z,u) - \hat{\rho}_{y'Z}(z,u)\right]$$

where $\hat{\rho}_R(z,u) = \widehat{m}_R(z,u) = \dfrac{1}{T}\sum_{t=1}^T \left(zI_{[u-R_t\geq 0]}\left(u - R_t\right) + (1-z)I_{[R_t+u\geq 0]}\left(R_t + u\right)\right)$ for a given portfolio R with historical observations $(R_1,...,R_T)$. Similar considerations can be easily verified for prospect behavioral orderings.

5 An empirical application

In this section we present the results of an empirical application. To illustrate the potential of the proposed semi-parametric and non-parametric test statistics, we test whether different reward/risk measures rationalize the market portfolio. Thus, we test for the efficiency of the market portfolio with respect to all possible portfolios constructed from a set of assets. Although we focus the analysis on non-parametric tests, we additionally test parametrically the efficiency of the market portfolio to compare our results with previous studies.

We use two different data sets. The first one is the six Fama and French benchmark portfolios. They are constructed at the end of each June, and correspond to the intersections of two portfolios formed on size (market equity, ME) and three portfolios formed on the ratio of book equity to market equity (BE/ME). The size breakpoint for year t is the median NYSE market equity at the end of June of year t. BE/ME for June of year t is the book equity for the last fiscal year end in $t-1$ divided by ME for December of $t-1$. Firms with negative BE are not included in any portfolio. The annual returns are from January to December. We use data on monthly excess returns (month-end to month-end) from July 1963 to October 2001 (460 monthly observations) obtained from the data library on the home page of Kenneth French (http://mba.turc.dartmouth.edu/pages/faculty/ken.french). The test portfolio is the Fama and French market portfolio, which is the value-weighted average of all non-financial common stocks listed on NYSE, AMEX, and Nasdaq, and covered by CRSP and COMPUSTAT. Moreover, we apply our tests to a data-set of daily returns on the 20 best (in terms of returns) $S\&P500$ stock returns from 12 March 1999 to 12 March 2008, a total of 2348 return observations. As the market portfolio, we use the $S\&P500$ stock index.

5.1 Description of the data

We use two different data sets for the empirical application. In the first one, we use the six Fama and French benchmark portfolios as our set of risky assets. They are constructed at the end of each June, and correspond to the intersections of two portfolios formed on size (market equity, ME) and three portfolios formed on the ratio of book equity to market equity (BE/ME). The size breakpoint for year t is the median NYSE market equity at the end of June of year t. BE/ME for June of year t is the book equity for the last fiscal year end in $t-1$ divided by ME for December of $t-1$. Firms with negative BE are not included in any portfolio. The annual returns are from January to December. We use data on monthly excess returns (month-end to month-end) from July 1963 to October 2001 (460 monthly observations) obtained from the data library on the homepage of Kenneth French (http://mba.turc.dartmouth.edu/pages/faculty/ken.french). The test portfolio is the Fama and French market portfolio, which is the value-weighted average of all non-financial common stocks listed on NYSE, AMEX, and Nasdaq, and covered by CRSP and COMPUSTAT. In the second data set, we use the 20 best assets (in terms of historical returns) of the $S\&P500$.

First we analyze the statistical characteristics of the data that are used in the test statistics. Table 1, exhibits the first four moments of the Fama and French MArket portfolio and the 6 benchmark portfolios covering the period from July 1963 to October 2001 (460 monthly observations).

Descriptive Statistics

No.	Mean	Std. Dev.	Skewness	Kurtosis
Fama and French Market portfolio	0.462	4.461	-0.498	2.176
Benchmark portfolios				
1	0.316	7.07	-0.337	-1.033
2	0.726	5.378	-0.512	0.570
3	0.885	5.385	-0.298	1.628
4	0.323	4.812	-0.291	-1.135
5	0.399	4.269	-0.247	-0.706
6	0.581	4.382	-0.069	-0.929

Table 1: Descriptive statistics of monthly returns in % from July 1963 to October 2001 (460 monthly observations) for the Fama and French market portfolio and the six Fama and French benchmark portfolios formed on size and book-to-market equity ratio. Portfolio 1 has low BE/ME and small size, portfolio 2 has medium BE/ME and small Size, portfolio 3 has high BE/ME and small size, ..., portfolio 6 has high BE/ME and large size.

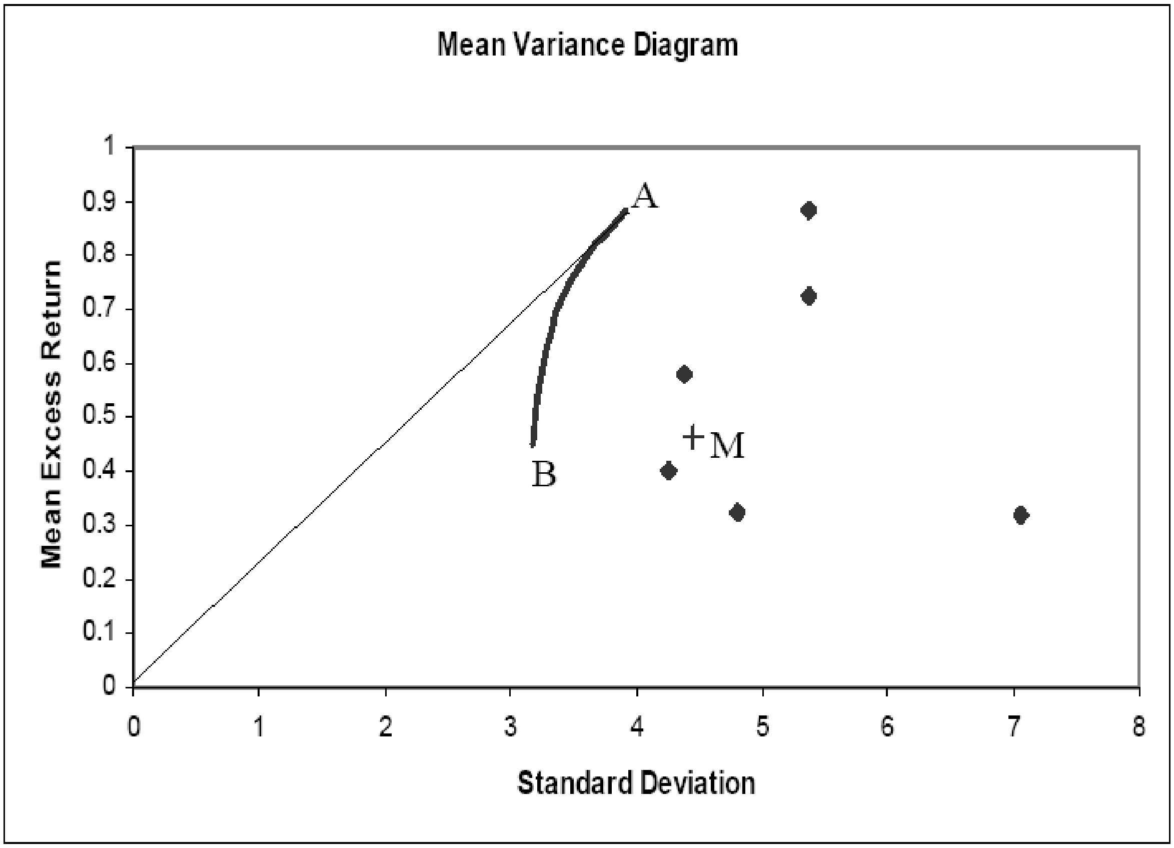

Figure 1: Mean-standard deviation efficient frontier of six Fama and French benchmark portfolios. The plot also shows the mean and standard deviation of the individual benchmark portfolio returns and of the Fama and French market (M) portfolio return, which is the test portfolio.

Table 2 exhibits the first four moments of the $S\&P500$ index and the 20 assets covering the period from 12 March 1999 to 12 March 2008, a total of 2348 daily return observations.

We observe from both Tables that asset returns exhibit considerable variance in comparison to their mean. Moreover, the skewness and kurtosis indicate that normality cannot be accepted for the majority of them.

One interesting feature is the comparison of the behavior of the market portfolio with that of the individual portfolios. Figure 1 shows the mean-standard deviation efficient frontier of the six Fama and French benchmark portfolios. The plot also shows the mean and standard deviation of the individual benchmark portfolio returns and of the Fama and French market (M) portfolio return. We observe that the test portfolio

(M) is mean-standard deviation inefficient. It is clear that we can construct portfolios that achieve a higher expected return for the same level of standard deviation, and a lower standard deviation for the same level of expected return. If the investor utility function is not quadratic, then the risk profile of the benchmark portfolios cannot be totally captured by the variance of these portfolios. Generally, the variance is not a satisfactory measure. It is a symmetric measure that penalizes gains and losses in the same way. Moreover, the variance is inappropriate to describe the risk of low probability events. Figure 1 is silent on return moments other than mean-variance (such as higher-order central moments and lower partial moments). This motivates us to test whether the market portfolio is efficient when higher moments are taken into account.

5.2 Semi-Parametric Tests

First, we parametrically test for the efficiency of the Fama and French portfolio with respect to the 6 benchmark portfolios. We solve the optimization models (2) and (3) to test for the first and second order stochastic dominance efficiency of the market portfolio. Both models are infeasible, meaning that we cannot construct a portfolio that dominates the market portfolio under the first and second order stochastic dominance criteria. Although Figure 1 shows that the Market portfolio is inefficient in the Mean-Variance framework, the infeasibility of the models proves the opposite. Remember that these models take into account the first four moments and not only the mean and the variance. Scaillet and Topaloglou (2002) show that although the market portfolio is inefficient compared to the benchmark portfolios in the mean-variance scheme, the first and second order stochastic dominance efficiency of the market portfolio prove the opposite under more general schemes. These results indicate that the whole distribution rather than the mean and the variance plays an important role in comparing portfolios. When a different reward/risk scheme is used (9) the model is again infeasible, indicating the efficiency of the market portfolio in these semi-parametric tests.

We observe different results when we test for the efficiency of the $S\&P500$ index with respect to the 20 best assets. In order to test for the first and second order stochastic dominance efficiency we solve the optimization models (2) and (3). To test for different reward/efficiency preference ordering, we solve the optimization model (8). Then, we use the test statistics (5) and (6) and (10), where $\hat{m}_i$ and $\hat{\sigma}_i$

are the mean and the standard deviation of the optimal portfolios for models (2), (3) and (8) respectively. In this empirical experiment we first set the critical level c_i of the test statistics to the conventional 0.05 (5%) and this will be the significance level for composite null hypotheses. The results are summarized in Table 3. We observe that we reject the null hypothesis H_0^i, for each model i we solve. In all cases, the test statistic is greater than the critical value c_i. This indicates that the efficiency of the $S\&P500$ is rejected. We can construct optimal portfolios that dominate the market in all the presented reward/risk models.

Although the Fama and French market portfolio is efficient compared to the benchmark portfolios in all the different reward/risk models, the $S\&P500$ index found to be inefficient. This inefficiency of the market portfolio is interesting for investors. If the market portfolio is not efficient, individual investors could diversify across diverse asset portfolios and outperform the market.

6 Concluding remarks

In this paper we discuss several methodologies to value the portfolio efficiency with respect to a given preference ordering. Moreover we introduce some stochastic dominance rules when all portfolio returns are uniquely determined by a reward measure, a deviation measure and others scale and translation invariant parameters. Several extension are possible. First most of the previous discussion can be extended to stationary series that are not necessarily independent. As a matter of fact, estimating function theory has been introduced in a very general context and all the asymptotic results are still true under some regularity assumptions (see, among others, Li and Turtle (2000), Godambe and Thompson (1989), Crowder (1986)). Second we can obtain analogous results adding a further estimating equation as proposed by Godambe and Thompson (1989), and Iaquinta et al (2003). In this case we determine the estimating parameters as functions of the first six central moments. Clearly this choice should be motivated by some observed anomalies of the return distributions. Moreover, even the proposed classification of parametric choices under uncertainty conditions can be further extended to random variables which are not necessarily positive. Finally the tests here introduced can be used and applied to many possible orderings and for many risk/reward measures. So, for obvious space reasons, we should analyze and discuss several examples in a separate empirical work.

7 APPENDIX: Proofs

Proof of Proposition 1: If we solve the estimating equations $l^*_{\xi,k}=0$ in terms of $m(\xi)$ we get the two solutions

$$m(\xi)=\frac{1}{T}\sum_{t=1}^{T}f(R_t)+c_k\pm\left(c_k^2-\frac{1}{T}\sum_{t=1}^{T}\left(f(R_t)-\frac{1}{T}\sum_{t=1}^{T}f(R_t)\right)^2+\sigma^2(\xi)\right)^{1/2}.$$

However when $T\to+\infty$ the unique consistent admissible equations are:
$m(\xi)=\frac{1}{T}\sum_{t=1}^{T}f(R_t)+c_k-\left(c_k^2-\frac{1}{T}\sum_{t=1}^{T}\left(f(R_t)-\frac{1}{T}\sum_{t=1}^{T}f(R_t)\right)^2+\sigma^2(\xi)\right)^{1/2}$ when $c_k>0$ and
$m(\xi)=\frac{1}{T}\sum_{t=1}^{T}f(R_t)+c_k+\left(c_k^2-\frac{1}{T}\sum_{t=1}^{T}\left(f(R_t)-\frac{1}{T}\sum_{t=1}^{T}f(R_t)\right)^2+\sigma^2(\xi)\right)^{1/2}$ if $c_k\le 0$.

Proof of Proposition 2: Under the H_0^i hypothesis, when the portfolios are uniquely determined by the first four moments and the portfolios present the same skewness and kurtosis we have that $(m_i-m_p)\le 0$ $(\sigma_i-\sigma_p)\le 0$ for i=1,2 and $(m_i-m_p)\le 0$, $(\sigma_i-\sigma_p)\ge 0$ for i=3,4. Therefore

$$\begin{aligned}\sqrt{T}(\hat m_i-\hat m_p)&=\sqrt{T}(\hat m_i-m_i-\hat m_p+m_p)+\sqrt{T}(m_i-m_p)\le\\ &\le\sqrt{T}(\hat m_i-m_i)-\sqrt{T}(\hat m_p-m_p)\quad\text{for } i=1,2,3,4\end{aligned}$$

$$\begin{aligned}\sqrt{T}(\hat\sigma_i-\hat\sigma_p)&=\sqrt{T}(\hat\sigma_i-\sigma_i-\hat\sigma_p+\sigma_p)+\sqrt{T}(\sigma_i-\sigma_p)\le\\ &\le\sqrt{T}(\hat\sigma_i-\sigma_i)-\sqrt{T}(\hat\sigma_p-\sigma_p)\quad\text{for } i=1,2\end{aligned}$$

$$\text{and }\sqrt{T}(\hat\sigma_i-\hat\sigma_p)\ge\sqrt{T}(\hat\sigma_i-\sigma_i)-\sqrt{T}(\hat\sigma_p-\sigma_p)\quad\text{for } i=3,4$$

Thus, choosing opportunely the values $c^i_{j,\alpha}$, i=1,2,3,4, j=1,2, we can guarantee that $\lim_{T\to\infty}P(\textit{reject } H_0^i \mid H_0^i \textit{ is true})\le\alpha$. Similarly when H_0^i hypothesis is false then $(m_i-m_p)>0$, $(\sigma_i-\sigma_p)>0$ for i=1,2 and $(m_i-m_p)>0$, $(\sigma_i-\sigma_p)<0$ for i=3,4 then

$$\lim_{T\to\infty}P\left(\textit{reject } H_0^i \,\middle|\, H_0^i \textit{ is false}\right)=1.$$

When H_0^i hypothesis is verified, then, in the worst case, the i-th portfolio has the same distribution of portfolio with weights x^p. So, if we assume that all parameters of the portfolio with weights $x^p$are known i.e., $\hat m_p=m_p=m_i;\hat\sigma_p=\sigma_p=\sigma_i$ the asymptotic density of the vector $\left[\sqrt{T}(\hat m_i-m_p);\sqrt{T}(\hat\sigma_i-\sigma_p)\right]$ is given by

$$f(x,y)=\frac{e^{\frac{-(k-1)}{2(k-1-s^2)}\left[\frac{x^2}{\sigma_p^2}-\frac{4sxy}{\sigma_p^2(k-1)}+\frac{4y^2}{\sigma_p^2(k-1)}\right]}}{\pi\sigma_p^2\sqrt{k-1-s^2}}.$$

Thus, for any $\alpha \leq 0.5$, and for i=1,2, we can choose $(c_{1,\alpha}^i, c_{2,\alpha}^i) \in \mathbb{R}_+^2$ such that

$$\frac{1}{\pi\sigma_p^2\sqrt{k-1-s^2}}\int_{c_{1,\alpha}^i}^{+\infty}\int_{c_{2,\alpha}^i}^{+\infty} e^{\frac{-(k-1)}{2(k-1-s^2)}\left[\frac{x^2}{\sigma_p^2}-\frac{4sxy}{\sigma_p^2(k-1)}+\frac{4y^2}{\sigma_p^2(k-1)}\right]}dxdy=\alpha$$

so that

$\lim_{T\to\infty} \mathrm{P}\left(rejectH_0^i\,|H_0^i istrue\right) \leq \lim_{T\to\infty} P\left(\sqrt{T}\left(\hat{m}_i - m_p\right) \geq c_{1,\alpha}^i; \sqrt{T}\left(\hat{\sigma}_i - \sigma_p\right) \leq c_{2,\alpha}^i\right) = \alpha.$

Analogously, we can choose $(c_{1,\alpha}^i, c_{2,\alpha}^i) \in \mathbb{R}_+^2$ for i=3,4 such that

$$\frac{1}{\pi\sigma_p^2\sqrt{k-1-s^2}}\int_{c_{1,\alpha}^i}^{+\infty}\int_{-\infty}^{c_{2,\alpha}^i} e^{\frac{-(k-1)}{2(k-1-s^2)}\left[\frac{x^2}{\sigma_p^2}-\frac{4sxy}{\sigma_p^2(k-1)}+\frac{4y^2}{\sigma_p^2(k-1)}\right]}dydx=\alpha$$

Proof of Theorem 1: *Implication 1:* As a consequence of the assumptions follows

$$q = m_{w'Z} - m_{y'Z} \geq 0 \text{ and } \sigma_{w'Z} \geq \sigma_{y'Z} \geq \sigma_{y'Z+t}$$

for every $t \geq 0$. Moreover, for every $t \geq 0$ the function $g(t) \equiv \dfrac{m_{y'Z}+t}{\sigma_{y'Z+t}}$ is an increasing continuous positive function that tends to infinity for big values of t. As a consequence of definition of $\sigma\tau_k^+(\bar{a})$ family there exist $t \leq q$ such that the random variable $\frac{w'Z}{\sigma_{w'Z}}$ has the same parameters of $\frac{y'Z+t}{\sigma_{y'Z+t}}$ and hence $\frac{w'Z}{\sigma_{w'Z}} \overset{d}{=} \frac{y'Z+t}{\sigma_{y'Z+t}}$. Then, for every $\lambda \geq 0$:

$$P\left(w'Z \leq \lambda\right) \leq P\left(\frac{y'Z+t}{\sigma_{y'Z+t}} \leq \frac{\lambda}{\sigma_{y'Z+t}}\right) \leq P\left(y'Z \leq \lambda\right)$$

Observe that at least one of the two inequalities $\sigma_{w'Z} \geq \sigma_{y'Z}$ and $q \geq 0$ is strict by hypothesis. Then, at least one of the previous inequalities is strict for some real $\lambda \geq 0$. Therefore, $w'ZFSD\ y'Z$.

Implication 2: According to definition of $\sigma\tau_k^+(\bar{a})$ family, it follows

$$\frac{w'Z}{\sigma_{w'Z}} \overset{d}{=} \frac{y'Z}{\sigma_{y'Z}}$$

because the two random variables have the same parameters. If $\sigma_{w'Z} > \sigma_{y'Z}$, then for every $t \geq 0$

$$P\left(w'Z \leq t\right) = P\left(\frac{w'Z}{\sigma_{w'Z}} \leq \frac{t}{\sigma_{w'Z}}\right) \leq P\left(\frac{w'Z}{\sigma_{w'Z}} \leq \frac{t}{\sigma_{y'Z}}\right) = P\left(y'Z \leq t\right)$$

and the above inequality is strict for some t. Conversely, if $w'ZFSD\ y'Z$, then there exists a probability space $(\Omega, \Im, P)$ and two random variables X and Y defined on this space such that $X{>}Y$ and X , Y have the same distributions of $w'Z$ and $y'Z$. Since m_X is a simple reward measure $m_{w'Z} \geq m_{y'Z}$ and must be $\sigma_{w'Z} > \sigma_{y'Z}$.

Implication 3: If $w'ZFSD\ y'Z$, then there exists a probability space $(\Omega, \Im, P)$ and two random variables X and Y defined on this space such that $X{>}Y$ and X,Y have the same distributions of $w'Z$ and $y'Z$. Since the reward measure is law invariant and it is isotonic with the monotone order then $m_{w'Z} \geq m_{y'Z}$. Any risky FORS ordering is isotonic with monotone order if $X{>}Y$, $X \underset{\succ}{FORS} Y$. Moreover the ordering is simple then it is uniquely determined by distributions $w'Z \underset{\succ}{FORS} y'Z$.

Proof of Proposition 3: Let $(m_{1,X}, \sigma_X)$ and $(m_{2,X}, \sigma_X)$ be two parameterizations of the class. Observe that for every distribution functions $F_U, F_Y \in \sigma\tau_2^+(a)$, $F_{V_1} := F_{\frac{U-m_{1,U}}{\sigma_U}} = F_{\frac{Y-m_{1,Y}}{\sigma_Y}}$ and $F_{V_2} := F_{\frac{U-m_{2,U}}{\sigma_U}} = F_{\frac{Y-m_{2,Y}}{\sigma_Y}}$. Then for every $F_X \in \sigma\tau_2^+(\bar{a})$ identified by the parameters $(m_{i,X}, \sigma_X)$, i=1,2 we get

$$F_X = F_{\sigma_X V_1 + m_{1,X}} = F_{\sigma_X V_2 + m_{2,X}}.$$

Thus, $V_1 + (m_{1,X} - m_{2,X})/\sigma_X \overset{d}{=} V_2$ and, $(m_{1,X} - m_{2,X})/\sigma_X$ is constant for every $F_X \in \sigma\tau_2^+(\bar{a})$.

Proof Theorem 2: *Case 1* Suppose $\frac{m_{w'Z}}{\sigma_{w'Z}} \geq \frac{m_{y'Z}}{\sigma_{y'Z}}$ and $m_{w'Z} \geq m_{y'Z}$. From theorem 1 if $\sigma_{w'Z} \geq \sigma_{y'Z}$ implies $w'Z\ FSD\ y'Z$ that implies $w'Z\ SSD\ y'Z$. Thus assume $\sigma_{w'Z} < \sigma_{y'Z}$. From proposition 1 we know that $\dfrac{m_{w'Z}}{\sigma_{w'Z}} - \dfrac{m_{y'Z}}{\sigma_{y'Z}} = \dfrac{E(w'Z)}{\sigma_{w'Z}} - \dfrac{E(y'Z)}{\sigma_{y'Z}} \geq 0$. Since $m_{w'Z} > E(w'Z)$ then $\dfrac{m_{w'Z} - E(w'Z)}{\sigma_{w'Z}} = k > 0$ that implies $0 \leq m_{w'Z} - m_{y'Z} = E(w'Z) - E(y'Z) + k\,(\sigma_{w'Z} - \sigma_{y'Z})$, i.e., $0 < k\,(\sigma_{y'Z} - \sigma_{w'Z}) \leq E(w'Z) - E(y'Z)$. Then by Ortobelli (2001) we know that $w'Z\ SSD\ y'Z$ .Similar considerations follow when we assume $m_{w'Z} \geq m_{y'Z}$ and $\sigma_{w'Z} \leq \sigma_{y'Z}$. Moreover if $w'Z\ R-S\ y'Z$ it is not possible that $\sigma_{w'Z} \geq \sigma_{y'Z}$ since $w'Z\ SSD\ y'Z$ implies $w'Z\ FSD\ y'Z$ against the hypothesis $E(w'Z) = E(y'Z)$. Thus it should be $\sigma_{w'Z} < \sigma_{y'Z}$. From proposition 1

$$m_{w'Z} - m_{y'Z} = E(w'Z) - E(y'Z) + k\,(\sigma_{w'Z} - \sigma_{y'Z}) = k\,(\sigma_{w'Z} - \sigma_{y'Z}) < 0$$

Case 2 We know that $w'Z\ SSD\ y'Z$ implies that $E(w'Z) - E(y'Z) \geq 0$. If $w'Z\ SSD\ y'Z$ and $\sigma_{w'Z} \geq \sigma_{y'Z}$ we know by Ortobelli (2001) that implies $w'Z\ FSD\ y'Z$ that implies $m_{w'Z} \geq m_{y'Z}$ for Theorem 1. If $w'Z\ SSD\ y'Z$ and $\sigma_{w'Z} < \sigma_{y'Z}$ then $\dfrac{m_{w'Z} - E(w'Z)}{\sigma_{w'Z}} = k < 0$ because $m_{w'Z} < E(w'Z)$. Then using the same arguments of case 1 we find $0 \leq m_{w'Z} - m_{y'Z} = E(w'Z) - E(y'Z) + k\left(\sigma_{w'Z} - \sigma_{y'Z}\right)$ that explains case 2a) That is $w'Z\ SSD\ y'Z$ implies $m_{w'Z} \geq m_{y'Z}$ that $m_{w'Z} \geq m_{y'Z}$. If $w'Z\ SSD\ y'Z$ and $\sigma_{w'Z} \geq \sigma_{y'Z}$ we know by Ortobelli (2001) that implies $w'Z\ FSD\ y'Z$. Thus must be $\sigma_{w'Z} < \sigma_{y'Z}$.

Proof Theorem 3: *Implication 1* If $X \underset{\succ,\alpha}{FORS} Y$ then

$$\rho_{X,\alpha}(x) := \frac{1}{\Gamma(\alpha-1)} \int_a^x (x-u)^{\alpha-2} \rho_X(u) du \leq \rho_{Y,\alpha}(x)$$

$\forall x \in (a,b)$. Conversely, if

$$-\infty < \lim_{x \to b} \rho_{X,\alpha}(x) - \rho_{Y,\alpha}(x) =$$
$$= \lim_{x \to b} \frac{1}{\Gamma(\alpha-1)} \left(\int_a^{t_0} (x-u)^{\alpha-2} \left(\rho_X(u) - \rho_Y(u)\right) du + \int_{t_0}^{x} (x-u)^{\alpha-2} \left(\rho_X(u) - \rho_Y(u)\right) du \right) \leq 0$$

then $0 \leq \int_{t_0}^x (x-u)^{\alpha-2} \left(\rho_X(u) - \rho_Y(u)\right) du \leq \int_a^{t_0} (x-u)^{\alpha-2} \left(\rho_Y(u) - \rho_X(u)\right) du$ for any $x \in (t_0, b)$. that is $\rho_{X,\alpha}(x) \leq \rho_{Y,\alpha}(x)$, $\forall x \in [a,b]$. Thus $X \underset{\succ,\alpha}{FORS} Y$.

Implication 2 Observe that if $m_Y = -\dfrac{\Gamma(\alpha)}{(b-a)^{\alpha}} \rho_{Y,\alpha}(b)$, then m_X is a positive, positive homogeneous translation equivariant reward measure that is consistent with monotone order. From theorem 2, $\frac{m_{w'Z}}{\sigma_{w'Z}} \geq \frac{m_{y'Z}}{\sigma_{y'Z}}$, $\sigma_{w'Z} \geq \sigma_{y'Z}$implies $w'Z FSD\ y'Z$, which implies $w'Z \underset{\succ}{FORS} y'Z$ and thus $w'Z \underset{\succ,\alpha}{FORS} y'Z$. Next, assume $\sigma_{w'Z} < \sigma_{y'Z}$. Therefore, there exists $t \geq 0$ such that $\frac{m_{w'Z}}{\sigma_{w'Z}} = \frac{m_{y'Z+t}}{\sigma_{y'Z+t}}$ and $w'Z \overset{d}{=} \dfrac{y'Z+t}{\beta}$ where $\beta = \dfrac{\sigma_{y'Z+t}}{\sigma_{w'Z}}$. Thus, we can distinguish two cases:

1. $m_{w'Z} \geq m_{y'Z}+t$ and $\sigma_{w'Z} \geq \sigma_{y'Z+t}$. As a consequence of Theorem 2 $w'Z FSD\ y'Z$.
2. $m_{y'Z} \leq m_{w'Z} < m_{y'Z} + t$ and $\sigma_{w'Z} < \sigma_{y'Z+t}$. Hence, there exists $\lambda_0 \in [a,b]$ such that for every $\lambda \in [a, \lambda_0]$, $\rho_{w'Z}(\lambda) \geq M := \dfrac{-t}{\beta - 1}$,

$$\rho_{w'Z}(\lambda) = \rho_{(y'Z+t)/\beta}(\lambda) = \frac{\rho_{y'Z}(\lambda) - t}{\beta} \leq \rho_{y'Z}(\lambda)$$

and for every $\lambda \in (\lambda_0, b]$ such that $\rho_{w'Z}(\lambda) \leq M$, it holds

$$\rho_{w'Z}(\lambda) \geq \rho_{y'Z}(\lambda).$$

By hypothesis $m_{w'Z} > m_{y'Z}$, thus we cannot have $y'Z\ FORS\ w'Z$ (otherwise $y'Z \underset{\succ,\alpha}{FORS}\ w'Z$, and $m_{w'Z} = -\dfrac{\Gamma(\alpha)}{(b-a)^{\alpha}} \rho_{w'Z,\alpha}(b) \leq m_{y'Z}$). Hence, there exists $\lambda \in [a,b]$

such that $\rho_{w'Z}(\lambda) \leq \rho_{y'Z}(\lambda)$. If for all $w'ZFORS\ y'Z$ clearly $w'Z\underset{\succ,\alpha}{FORS}y'Z$, otherwise there exists $\lambda \in [a,b]$ such that $\rho_{w'Z}(\lambda) \geq \rho_{y'Z}(\lambda)$. Therefore, from implication 1 we get $w'Z\underset{\succ,\alpha}{FORS}\ y'Z$.

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Descriptive Statistics				
No.	Mean	Std. Dev.	Skewness	Kurtosis
S&P 500 index	0.007	1.215	0.109	2.468
20 best assets				
1	0.250	31.444	3.790	90.910
2	0.244	15.182	0.321	4.661
3	0.211	22.253	0.746	9.669
4	0.203	11.285	0.815	6.259
5	0.199	9.232	0.921	6.832
6	0.197	19.330	0.261	15.486
7	0.193	14.092	0.779	8.679
8	0.175	11.781	0.337	4.975
9	0.171	10.564	-1.591	29.454
10	0.155	5.931	-1.228	28.366
11	0.155	23.699	0.686	7.789
12	0.154	13.117	1.078	10.169
13	0.153	14.081	-0.233	12.545
14	0.152	11.435	-0.136	11.802
15	0.149	13.434	0.323	10.721
16	0.149	24.148	0.826	6.789
17	0.144	8.548	0.145	2.751
18	0.143	9.690	0.311	10.413
19	0.143	8.450	0.412	3.677
20	0.140	21.935	0.750	7.465

Table 2: Descriptive statistics of daily returns in % from 12 March 1999 to 12 March 2008, a total of 2348 daily return observations for the $S\&P500$ stock index and the best twenty assets.

Test statistics for preference ordering efficiency		
Model	$\sqrt{T}(\hat{m}_i - \hat{m}_p)$	$\sqrt{T}(\hat{\sigma}_i - \hat{\sigma}_p)$
$\max_x x'Qx$ (*model* 2)	0.09837	0.93093
$\max_x E(x'Z)$ (*model* 3)	0.10709	0.76940
$\max_x ES_\beta(-x'Z) - ES_\alpha(x'Z)$ (*model* 8)	0.453404	0.271258

Table 3: Semi-Parametric Test statistics of various preference ordering models for the $S\&P500$.

Redazione

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