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Tempered stable models in finance: theory and applications

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Introduction

Most of the important models in finance rest on the assumption that randomness is explained through a normal random variable. However there is ample empirical evidence against the normality assumption, since stock returns are heavy-tailed, leptokurtic and skewed, see [99, 29, 98] for example. Returns from financial assets show well-defined patterns of leptokurtosis and skewness which cannot be captured by the normality assumption. Both continuous and discrete time models can be developed by considering a non-normal infinitely divisible distribution.

Since Mandelbrot introduced the α-stable distribution to model the empirical distribution of asset prices in [81], the α-stable distribution became an alternative to the normal distribution. A conclusion of the literature is that although the empirical evidence does not support the normal distribution, it is not always consistent with an α-stable distribution. The distribution of returns for assets has heavier tails relative to the normal distribution and thinner tails than the α-stable distribution. Partly in response to those empirical inconsistencies, there is a search for suitable alternatives to the α-stable distribution. One such alternative are the families of tempered stable (TS) and tempered infinitely divisible (TID) distributions. The recent ISA Medal for Science to the authors of the model CGMY (Carr, Geman, Madan and Yor), which is a parametric example of TS distribution, has represented a revolutionary step beyond the Black and Scholes world. There is a general consensus in the literature to consider jump processes both with finite and infinite activity besides diffusion processes. The long standing problem of non-normality of returns and the issue of discontinuity of prices can be overcome:

> *At the beginning of the XXI century the CGMY model represents the model of asset price dynamics destined to substitute the diffusive processes that reigned throughout most of last century*[1].

and furthemore [114]:

> *A great deal of empirical research has stressed the importance of modeling non-Gaussian features in financial time series dynamics and option pricing frameworks. Yet, due to their simplicity, many practitioners still employ models based on the assumption of normality. Unfortunately, these approaches, although simpler, can lead to dangerous underestimation of extreme losses in risk management or badly mispriced derivative products.*

[1] http://www.isa.unibo.it/ISA/Activities/OtherEvents/2008/07/CGMYmodel.htm

Thank to the seminal work on the CGMY model [21] and the recent attention on the applications of Lévy processes and infinitely divisible distribution to finance, the theoretical and empirical literature has focused on the formalization and extension of TS models, readers are referred to [79, 60, 4, 87] and references therein.

TS distributions may have all moments finite and exponential moments of some order. The latter property is essential in the construction of TS option pricing models. The formal definition of TS processes as been proposed in the seminal work of Rosiński [107]. The KoBoL [19], the CGMY [21], the Inverse Gaussian (IG) and the TS of Tweedie [111] are only some parametric examples in this class, that have an infinite dimensional parametrization by a family of measures [115]. Further extensions or limiting cases are also given by the fractional TS framework [52], the bilateral gamma [73] and the generalized TS distribution [29, 95].

Recently, by taking into consideration the Rosiński approach, the TID framework has been developed [14]. In some cases, the characteristic function of a TID random variable is extendible to an entire function on $\mathbb{C}$, that is, it admits any exponential moment. The latter property is desirable in discrete time model with volatility clustering.

The general TS and TID formulation is difficult to use in practical applications, but it allows one to prove some interesting results regarding the calculus of the characteristic function and the random numbers generation. The infinite divisibility of these distributions allows one to construct the corresponding Lévy process and to analyze the change of measure problem and the process behavior as well. In the Chapter 5 an exponential Lévy model will be construct and some empirical results, based on the S&P 500 index, will be shown.

In general, the use of infinitely divisible distributions is obstructed by the difficulty of calibrating and simulating them. We address some numerical issues resulting from TS and TID modelling, with a view toward the density approximation and simulation [15]. Thus, we are in the position to work with the characteristic function of these distributions, estimate parameters and simulate random numbers in order to calculate option prices via Monte Carlo simulation. Even if in the CGMY case we can generate random numbers by considering a time changed Brownian motion [79], in general for TS and TID processes we do not know how one can find the time process to transform the Brownian motion into a TS process or a TID process as well. By following the approach of [105, 14], a series representation is considered to simulate TS and TID processes and distributions.

Finally, infinitely divisible distributions can be considered also for discrete time financial modelling. A typical finding concerning stock price returns and GARCH models is that they continue to exhibit a fat-tail behavior in the innovation distribution. This evidence raises a question concerning the appropriateness of conditional normality assumption [36]. For this reason, we want to consider a more flexible distributional assumption for the innovation, to allow leptokurtosis and skewness, and to study the effect on option pricing and volatility smile. By considering the Duan's GARCH model [35] where a normal distribution is taken into account, we will present in Chapter 7 an infinitely divisible GARCH framework [67]. We then construct a new GARCH model with the infinitely divisible distributed innovation and different subclasses of that GARCH model that incorporates three observed properties of asset returns: volatility clustering, fat tails, and skewness. We will present the algorithm to find the risk-neutral return processes for those GARCH models using the change of measure for TS and TID distributions. To compare the performance of these GARCH models, we report the results of the parameters estimated for the S&P 500 index and investigate the in-sample and out-of-sample performance for the S&P 500 option prices. The so called *fundamental approach* can be considered, that is option prices can be calculated by using parameters estimated by fitting the underlying asset process together with a suitable change of measure.

The remainder of this work is organized as follows. In Chapter 1 basic definitions and results on Lévy processes and infinitely divisible distributions are recall. In Chapter 2 we review the definition of TS distributions and focus our attention on same parametric examples. Then, the TID distribution is introduced in Chapter 3. Since it is by construction infinitely divisible, the corresponding Lévy process will be also considered. By following the density transformation result of Sato [109], the change of measure problem in the TS and TID class is solved in Chapter 4. The continuous TS stock returns model is presented in Chapter 5 and algorithms for the evaluation of the density function are also studied. A general random number generation method for the TS and TID class is develop in Chapter 6 and finally, it is applied, together with the change of measure argument, to the infinitely divisible (ID) GARCH framework.

Chapter 1

Preliminaries

This chapter summarizes the main results for infinitely divisible distributions and continuous-time stochastic processes, in particular Lévy processes. We focus our attention on definitions and properties, we will need in the following. More detailed introductions can be found in, for example, [109], [97] and [63].

1.1 Basic definitions and notations

In this section, we resume some basic definitions concerning probability theory. In the following, the readers are supposed to be familiar with Lebesgue's theory of integration.

Definition 1.1. *A probability space* $(\Omega, \mathcal{F}, P)$ *is a triplet of a set* Ω*, a family* $\mathcal{F}$ *of subsets of* Ω*, and a mapping* P *form* $\mathcal{F}$ *into* $\mathbb{R}$ *satisfying the followings conditions.*

(a) $\Omega \in \mathcal{F}$ *and* $\emptyset \in \mathcal{F}$*, where* $\emptyset$ *is the empty set.*

(b) If $A_n \in \mathcal{F}$ *for* $n = 1, 2, \ldots$*, then* $\bigcup_{n=1}^{\infty} A_n$ *and* $\bigcap_{n=1}^{\infty} A_n$ *are in* $\mathcal{F}$*.*

(c) If $A \in \mathcal{F}$*, then* $A^c \in \mathcal{F}$*, where* A^c *is the complement of* A $(\Omega \backslash A)$

(d) $0 \leq P(A) \leq 1$*,* $P(\Omega) = 1$*, and* $P(\emptyset) = 0$*.*

(e) If $A_n \in \mathcal{F}$ *for* $n = 1, 2, \ldots$ *and they are disjoint (that is,* $A_n \cap A_m = \emptyset$ *for* $n \neq m$*), then* $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$*.*

By following the terminology of measure theory, a probability space is a measure space with total measure 1. If $\mathcal{F}$ satisfies conditions (a), (b), and (c) of Definition 1.1, then we call $\mathcal{F}$ σ-algebra on Ω, or also tribe on Ω. The pair $(\Omega, \mathcal{F})$ will be our *measurable space*. A mapping P, which satisfies the condition (d) of Definition 1.1 is called a *probability measure*. Let $(\Omega, \mathcal{F}, P)$ be a probability space, we call any $A \in \mathcal{F}$ an *event*, and $P(A)$ the *probability* of the event A.

Definition 1.2. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. A mapping* X *on* $(\Omega, \mathcal{F})$*, which takes values in a second measurable space* $(E, \mathcal{E})$ *is a random variable if it is* $\mathcal{F}$*-measurable, that is,* $\{\omega \in \Omega : X(\omega) \in B\}$ *is in* $\mathcal{F}$ *for each element* $B \in \mathcal{E}$*.*

By the Definition above, if X is a random variable then we obtain that all sets of the following form

$$X^{-1}(B) = \{X \in B\} = \{\omega \in \Omega : X(\omega) \in B\},$$

where $B \in \mathcal{E}$, are in our σ-algebra $\mathcal{F}$.

The collection of the sets $\{X \in B\}$, with $B \in E$, is also a σ-algebra and we call it the *σ-algebra generated* by X, i.e.,

$$\sigma(X) = \{X^{-1}(B) : B \in E\}.$$

It is the smallest sigma algebra on Ω such that X is a measurable function into $(E, \mathcal{E})$. We call the measurable space $(E, \mathcal{E})$, the *state space*. For convenience, we write $P(\{\omega \in \Omega : X(\omega) \in B\})$ as $P\{X \in B\}$. The measure μ (or P_X) on $(E, \mathcal{E})$, is called distribution (or law) of X respect to P.

The law of X is the measure μ on $\mathcal{E}$, such that

$$\mu(B) = P\{X \in B\}.$$

in particular, we have

$$\mu(E) = P\{X \in E\} = P(\Omega) = 1,$$

therefore μ is a probability measure on $\mathcal{E}$.

A measure space $(E, \mathcal{E}, \mu)$ is said to be *σ-finite* (or the measure μ is σ-finite) if E is the countable union of measurable sets of finite measure. A measure space $(E, \mathcal{E}, \mu)$ is said to be *complete* (or the measure μ is complete) if every subset of a set of null measure is measurable. For example, the Lebesgue measure on $\mathbb{R}$ is σ-finite and complete.

Definition 1.3. *The collection of all Borel sets on $\mathbb{R}^d$, denoted by $\mathcal{B}(\mathbb{R}^d)$, is called the Borel σ-algebra. It can be generated by the open sets in $\mathbb{R}^d$, that is, it is the smallest σ-algebra that contains all open sets in $\mathbb{R}^d$.*

Roughly speaking, one motivation to explain, the reason why we consider the Borel σ-algebra, is that we want a σ-algebra rich enough to contain the intervals of the following form

- open intervals (a, b), with $a < b$
- closed intervals $[a, b]$, with $a \leq b$

Furthermore, it is possible to prove that the Borel σ-algebra is generated by the collection of all open intervals (or closed intervals).

Remark 1.4. *The Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ does not contain all subsets of $\mathbb{R}^d$. Readers are referred to [2].*

In general, we can consider a metric or topological space E endowed with its Borel σ-algebra $\mathcal{B}(E)$ generated by the topology (for example, the class of open subsets) in E or other σ-algebras otherwise specified. For our purposes, the state space will be the d-dimensional Euclidean space equipped with the σ-algebra of Borel sets on $\mathbb{R}^d$.

Definition 1.5. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. A mapping* X *on* $(\Omega, \mathcal{F})$*, is a real valued random variable if it is a random variable, which take values in a second measurable space* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$*. The function* F *defined by*

$$F(x) = \mu((-\infty, x]) = P\{X \leq x\}$$

is called the distribution (or law) of X*.*

If X is a real valued random variable and if the integral

$$\int_{\Omega} X(\omega) P(d\omega)$$

exists, then it is called *expectation* of X and denote by $E[X]$ or $E_P[X]$, if there is some ambiguity as to the measure P. The following equalities are fulfilled

$$E_P[X] = \int_{\Omega} X(\omega) P(d\omega) = \int_{\mathbb{R}} x\mu(dx) = \int_{\mathbb{R}} x dF(x).$$

If X is a random variable on $\mathbb{R}^d$, and $g(x)$ is a bounded measurable function on $\mathbb{R}^d$, then

$$E[g(X)] = \int_{\mathbb{R}^d} g(x)\mu(dx).$$

A random variable X is said to have a given property k almost surely (a.s.), if there is $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$ such $X(\omega)$ has the property k for every $\omega \in \Omega_0$. We consider $L^p = L^p(\Omega, \mathcal{F}, P)$ for $p \in [1, \infty)$, the space of all real valued random variables X such that $|X|^p$ is integrable, with the usual identification of any two a.s. equal random variables.

Definition 1.6. *Let* λ *and* μ *be two measure on* $(E, \mathcal{E})$*. We said that* λ *is absolutely continuous respect to* μ*,* $\lambda \ll \mu$*, if* $\forall B \in E$*, such that* $\mu(B) = 0$*, then* $\lambda(B) = 0$*. We said that* λ *and* μ *are equivalent,* $\lambda \sim \mu$*, iff* $\lambda \ll \mu$ *and* $\mu \ll \lambda$

Now, we recall a well known theorem

Theorem 1.7 (Radon-Nikodim theorem)**.** *Let* $(E, \mathcal{E}, \lambda)$ *be a* σ*-finite measure space and let* μ *be another measure on* $(E, \mathcal{E})$*. Then,* $\mu \ll \lambda$ *iff there is a non-negative function* $f \in L^1(E, \mathcal{E}, \mu)$*, such that*

$$\mu(B) = \int_B f d\lambda \tag{1.1}$$

for all $B \in \mathcal{E}$*.*

Proof. See [2]. □

In our case, we consider μ as the measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ previously defined and let λ be the Lebesgue measure on $\mathbb{R}$. If $\mu \ll \lambda$, then we can find a non-negative function $f \in L^1(E, \mathcal{E}, \mu)$, such that the equality (1.1) is satisfied. We call this function f the *density* of the random variable X with respect to the Lebesgue measure on $\mathbb{R}$, that is

$$P\{X \in B\} = \mu(B) = \int_B f(x) dx$$

for each $B \in \mathcal{B}(\mathbb{R})$. If the measure μ admits a density with respect to the Lebesgue measure on $\mathbb{R}$, by Definition 1.5, we obtain the equality

$$f(x) = F'(x) \quad a.s..$$

The n-th *moment* of a random variable X on $\mathbb{R}$ is defined by $m_n(X) = E[X^n]$. We said that a random variable has *exponential moment* if there is a $\theta \in [-a, b]$, with $a, b \in \mathbb{R}_+$ such that

$$E[e^{\theta X}] < \infty.$$

We want to recall and important definition for next applications.

Definition 1.8 (Characteristic function or Fourier transform)**.** *The characteristic function $\hat{\mu}(z)$ of a probability measure μ on $\mathbb{R}^d$ is*

$$\hat{\mu}(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle} \mu(dx), \quad z \in \mathbb{R}^d. \tag{1.2}$$

If X is a real valued random variable with probability distribution F, we indicate the characteristic function of X (or of F) as the function ϕ defined for $z \in \mathbb{R}$ by

$$\phi(z) = \hat{\mu}(z) = \int_{-\infty}^{\infty} e^{izx} F(dx) \tag{1.3}$$

to put it better,

$$\phi(z) = E[e^{izX}].$$

For distribution F with a density f,

$$\phi(z) = \int_{-\infty}^{\infty} e^{izx} f(x)dx.$$

In functional analysis, people prefer to call it *Fourier transform* and indicate it as $\hat{f}$, instead, in probability theory, it is commonly called *characteristic function* or *Fourier transform*. A detailed introduction, can be found, for example, in [43], [118], [16], [109], [2] and references therein.

The characteristic function takes value in the complex plane $\mathbb{C}$, although f is real.

Proposition 1.9. *If $E[|X|^n] < \infty$, then ϕ has n continuous derivatives at $z = 0$ and the equality*

$$m_n = E[X^n] = \left.\frac{\partial \phi(z)}{\partial z^n}\right|_{u=0}.$$

If X is a real valued random variable, then

- ϕ is continuous and $\phi(0) = 1$.

Therefore, there exists a unique continuous function ψ defined in a neighborhood of zero such that

$$\psi(0) = 0 \quad \text{and} \quad \phi(z) = e^{\psi(z)}.$$

The function ψ is called the *cumulant generating function* or *characteristic exponent* of the random variable X. When μ is a distribution on $[0,\infty)$, the *Laplace transform* of μ is defined

$$L_\mu(\lambda) = \int_0^\infty e^{-\lambda x}\mu(dx) \tag{1.4}$$

If μ is a probability distribution and L_μ its Laplace transform 1.4, then L_μ possesses derivatives of all orders given by

$$(-1)^n L_\mu^n(\lambda) = \int_0^\infty e^{-\lambda x}x^n\mu(dx)$$

with $\lambda > 0$.

Remark 1.10. *The probability measure μ admits a finite n-th moment if and only the limit*

$$\lim_{\lambda \searrow 0} L_\mu^n(\lambda)$$

is finite.

Definition 1.11. *Let X be a real valued random variable and F its distribution. The function*

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^\infty e^{tx}F(dx), \quad t \in, \mathbb{R} \tag{1.5}$$

in which the integral is assumed to converge for t in some subinterval of the reals is called the moment generating function of X.

Remark 1.12. *Contrarily to the characteristic function, which is always well-defined, the moment generating function in not always defined.*

It is not difficult to see, that the integral (1.5) may not converge for some values of t. When the integral (1.5) converges, then we obtain the following equality

$$M_X(t) = \phi(-it)$$

and if the integral converges on a neighborhood $[-\epsilon,\epsilon]$, then all n-th moments can be calculated as

$$m_n = E[X^n] = \left.\frac{\partial M_X(t)}{\partial t^n}\right|_{t=0}.$$

Another important result, we are going to use in the following is a Lemma in [109, Lemma 25.7].

Lemma 1.13. *If μ is a probability measure on $\mathbb{R}$ and $\hat{\mu}$ is extendible to an entire function on $\mathbb{C}$, then μ has finite exponential moments, that is, it has finite $e^{c|x|}$-moment for every $c > 0$.*

Now, we recall a fundamental notion of Probability theory.

Definition 1.14. *Let X be an integrable random variable on $(\Omega,\mathcal{F},P)$, $X \in L^1$, $\mathcal{A} \subset \mathcal{F}$ a sub-σ-algebra and Q the restriction of P to $\mathcal{A}$, $Q = P|_{\mathcal{A}}$. An $\mathcal{A}$-measurable and*

Q-integrable random variable V is said to be a version of the conditional expectation of X given $\mathcal{A}$, if the equality

$$\int_A X dP = \int_A V dP$$

holds for each $A \in \mathcal{A}$.

Definition 1.15. *Consider the assumptions of Definition 1.14, we call conditional expectation of X given $\mathcal{A}$, and denote it as $E[X|\mathcal{A}]$, the equivalence class in L^1 of all versions of the conditional expectation of X given $\mathcal{A}$.*

In the following proposition, we recall some elementary properties.

Proposition 1.16. *Let X and Y be a couple of real valued and integrable random variables on on $(\Omega, \mathcal{F}, P)$, and $a, b \in \mathbb{R}$. The following properties are verified*

1. *If $P\{X = a\} = 1$, then $E[X|\mathcal{A}] = E[X] = a$.*
2. *$E[aX + bY|\mathcal{A}] = aE[X|\mathcal{A}] + bE[Y|\mathcal{A}]$.*
3. *If V is a version of $E[X|\mathcal{A}]$, then $E[V] = E[X]$.*
4. *If X is $\mathcal{A}$-measurable, then X is a version of $E[X|\mathcal{A}]$.*
5. *If X is independent of $\mathcal{A}$, then $E[X]$ is a version of $E[X|\mathcal{A}]$.*
6. *If Y is $\mathcal{A}$-measurable, then $E[XY|\mathcal{A}] = YE[X|\mathcal{A}]$.*

Proof. See [58]. □

1.2 Stochastic processes

Let us fix some terminology:

Definition 1.17. *Let $(\Omega, \mathcal{F})$ be a measurable space, then we call filtration a nondecreasing family $(\mathcal{F}_t)_{t \geq 0}$ of sub-σ-algebra of $\mathcal{F}$, that is $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s < t < \infty$. We set $\mathcal{F}_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$. Furthermore, if we choose a probability measure P on $(\Omega, \mathcal{F})$, we call $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$ a filtered probability space.*

Definition 1.18. *A filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$ is said to satisfy the usual hypotheses if*

- *the measure P is complete;*
- *$(\mathcal{F}_t)_{t \geq 0}$ is right continuous, that is $\mathcal{F}_t = \bigcap_{u > t} \mathcal{F}_u$ for all t, $0 \leq t < \infty$.*
- *$\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$.*

A stochastic process is a mathematical model of a random phenomenon in time evolution. Let $(\Omega, \mathcal{F})$ be a measurable space, on which probability measure can be placed. We can consider $(\Omega, \mathcal{F})$ as the source of randomness and call it *sample space*. A stochastic process is a collection of random variables $X = \{X_t\}_{0 \leq t < \infty}$ on $(\Omega, \mathcal{F})$, which take value in a measurable space $(E, \mathcal{E})$.

Definition 1.19. *A stochastic process is a family* $(X_t)_{t\geq 0}$ *of random variables from* $(\Omega, \mathcal{F}, P)$ *to* $(E, \mathcal{E})$.

A process may be considered as a mapping from $(\Omega\times\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F})$ into $(E, \mathcal{E})$, via

$$(\omega, t) \longrightarrow X(\omega, t) = X_t(\omega).$$

Each mapping $t \to X_t(\omega)$, for a fixed $\omega \in \Omega$, is called a *path*, or a *trajectory*, of the process X.

Given a stochastic process, the simplest choice of a filtration is that generated by the process itself, i.e.

$$\mathcal{F}_t^X = \sigma(X_s; 0 \leq s \leq t).$$

Here, we recall some definitions.

Definition 1.20. *The stochastic process* X *is adapted to the filtration* $(\mathcal{F}_t)_{t\geq 0}$ *if, for each* $t \geq 0$, X_t *is an* $\mathcal{F}_t$*-measurable random variable.*

By definition, every process X is adapted to $(\mathcal{F}_t^X)_{t\geq 0}$.

Definition 1.21. *A stochastic process* X *is said to be càdlàg if it a.s. has sample paths which are right continuous, with left limits.*

Letting $X_{t_-} = \lim_{s\uparrow t X_s}$, the left limit at t, we define

$$\Delta X_t = X_t - X_{t_-}.$$

Definition 1.22. *Two stochastic processes* X *and* Y *are modifications if* $X_t = Y_t$ *a.s., each* t.

Definition 1.23. *Two processes* X *and* Y *are indistinguishable if almost all their paths agree:*

$$P(X_t = Y_t, \forall 0 \leq t < \infty) = 1.$$

The Definition 1.23 is strongest, see [63]. Now, we given a essential definition for the application of stochastic processes to mathematical finance.

Definition 1.24. *A real valued, adapted process* $(X_t)_{t\geq 0}$ *is called a martingale (resp. supermartingale, submartingale) with respect to the filtration* $(\mathcal{F}_t)_{t\geq 0}$ *if*

- $X_t \in L^1$;
- *if* $s \leq t$, *then* $E[X_t|\mathcal{F}_t] = X_s$, *a.s. (resp.* $E[X_t|\mathcal{F}_t] \leq X_s$, *resp.* $E[X_t|\mathcal{F}_t] \geq X_s$*).*

1.3 Lévy processes

In this section, we introduce Lévy processes and discuss some of their general properties. In the following we use the approach of [97] and [109]. Here we are assuming given a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ satisfying the usual hypotheses.

Definition 1.25. *An adapted process* $(X_t)_{t\geq 0}$ *with* $X_0 = 0$ *a.s. is a Lévy process if*

(a) *X has increments independent, that is,* $X_t - X_s$. *is independent of* $\mathcal{F}_s$, $0 \leq s < t < \infty$*;*

(b) *X has stationary increments, that is,* $X_t - X_s$ *has the same distribution as* X_{t-s}, $0 \leq s < t < \infty$*;*

(c) X_t *is continuous in probability, that is,* $\lim_{t \to s} X_t = X_s$*, where the limit is taken in probability.*

A particular important example of Lévy process is the Brownian motion. It is not only a basis for the theory of stochastic processes, but it is also the core of many financial models.

Definition 1.26. *An adapted process* $W = (W_t)_{t \geq 0}$ *with* $W_0 = 0$ *a.s. is a Brownian motion if*

(a) *X has increments independent, that is,* $X_t - X_s$. *is independent of* $\mathcal{F}_s$, $0 \leq s < t < \infty$*;*

(b) *X has stationary increments, that is,* $X_t - X_s$ *has the same distribution as* X_{t-s}, $0 \leq s < t < \infty$*;*

(c) *P a.s. the map* $s \to W_s(\omega)$ *is continuous.*

This definition induces the distribution of the process W_t.

Theorem 1.27. *If* $(W_t)_{t \geq 0}$ *is a Brownian motion, then* $W_t - W_0$ *is a normal random variable with mean* rt *and variance* σ^2*, where* $r, \sigma \in \mathbb{R}$*.*

Proof. See [48]. □

We recall the the density f_{nor} of a normal random variable $\mathcal{N}(\mu, \sigma^2)$, respect to the Lebesgue measure on $\mathbb{R}$, is the function so defined

$$f_{nor}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

We denote by μ^{n*} the n-fold convolution of a probability measure μ with itself, that is,

$$\mu^{n*} = \underbrace{\mu * \cdots * \mu}.$$

Definition 1.28. *A probability measure* μ *on* $\mathbb{R}^d$ *is infinitely divisible if, for any positive integer* n*, there is a probability measure* μ_n *on* $\mathbb{R}^d$ *such that* $\mu = {\mu_n}^n$*.*

If we take the Fourier transform of each X_t we get a function $\phi_t(z)$, so defined

$$\phi_t(z) = E[e^{izX_t}]$$

where $\phi_0(z) = 1$ and $\phi_{t+s}(z) = \phi_t(z)\phi_s(z)$ and $\phi_t(z) \neq 0$ for every (t, z). In particular, by using the property (c) of Definition 1.25, we obtain that if X is a Lévy process, then, for each $t > 0$, X_t has an infinitely divisible distribution. Inversely, it can be prove the following result.

Proposition 1.29. *Let μ be a infinitely divisible distribution, then there exists a Lévy process $(X_t)_{t\geq 0}$ such that μ is the distribution of X_1.*

Proof. See [109, Theorem 7.10]. □

Some books add to the Definition 1.25 an additional condition, that is the càdlàg property of the path. We prefer to prove this condition, by starting from the previous definition.

Definition 1.30. *Let X a Lévy process. There exists a unique modification Y of X which is càdlàg and which is also a Lévy process.*

Proof. See [97, Chapter I, Theorem 30]. □

In the following, we will assume that we are using the unique càdlàg version of any given Lévy process. In the following, we will call indicator function of the set A, the function so defined

$$I_A = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Theorem 1.31 (Lévy-Khintchine formula)**.** *A probability law μ of a real valued random variable is infinitely divisible with characteristic exponent ψ,*

$$\int_{\mathbb{R}} e^{i\theta x}\mu(dx) = e^{\psi(\theta)} \quad \textit{for } \theta \in \mathbb{R}$$

if and only if there exists a triple (a, σ, ν) where $a \in \mathbb{R}$, $\sigma \geq 0$ and ν is a measure on $\mathbb{R}\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}\backslash\{0\}} (1 \wedge x^2)\nu(dx) < \infty$$

such that

$$\psi(\theta) = ia\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}\backslash\{0\}} (e^{i\theta x} - 1 - i\theta x I_{\{|x|<1\}})\nu(dx) \tag{1.6}$$

for every $\theta \in \mathbb{R}$.

We say that our infinitely divisible distribution has a triplet of *Lévy characteristics or Lévy generating triplet* (or *Lévy triplet* for short) (a, σ, ν). The measure ν is called the *Lévy measure* of μ. If the Lévy measure is of the form $\nu(dx) = u(x)dx$, we call $u(x)$ the *Lévy density*. It follows that, if $(X_t)_{t\geq 0}$ is a Lévy process, there is always a Lévy triplet (a, σ, ν), such that

$$E[e^{iuX_t}] = e^{t\psi(u)}.$$

Remark 1.32. *The integrand in (1.6)*

$$e^{i\theta x} - 1 - \theta x I_{\{|x|<1\}} = \mathcal{O}(|x^2|)$$

as $|x| \to 0$ and it is integrable with respect to ν, because it is bounded outside any neighborhood of 0. More generally, if $c(x)$ is a measurable function and if

$$e^{i\theta x} - 1 - i\theta x c(x)$$

is integrable with respect to a given Lévy measure ν*, then we obtain the representation*

$$\psi(\theta) = ia_c\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}\backslash\{0\}} (e^{i\theta x} - 1 - i\theta x c(x))\nu(dx) \tag{1.7}$$

with $a_c \in \mathbb{R}$ *defined by*

$$a_c = a + \int_{\mathbb{R}} x(c(x) - I_{\{|x|<1\}})\nu(dx). \tag{1.8}$$

If ν satisfies

$$\int_{|x|>1} |x|\nu(dx) < \infty,$$

by considering $c(x) = 1$, we obtain

$$\psi(\theta) = ia_1\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}\backslash\{0\}} (e^{i\theta x} - 1 - i\theta x)\nu(dx). \tag{1.9}$$

This representation will be convenient in the following, since

$$a_1 = \int_{\mathbb{R}} x\mu(dx),$$

that is, a_1 is the mean of the distribution.

1.3.1 Poisson random measure and Lévy-Itô decomposition

Next turn briefly our attention to the analysis of the Lévy measure and its connection with jumps of a Lévy process.

Definition 1.33. *Let* $(E, \mathcal{E})$ *be a measurable space and* m *a* σ*-finite measure on this space. Then* $(M(A), A \in \mathcal{E})$ *a Poisson random measure satisfies the following conditions.*

(i) *For any* $A \in \mathcal{E}_0$*, where*

$$\mathcal{E}_0 = \{A \in \mathcal{E} : m(A) < \infty\},$$

then $M(A) = M(\omega, A)$ *is a Poisson random variable on* $(\Omega, \mathcal{F}, P)$ *such that*

$$M(A) \sim Poisson(m(A)).$$

(ii) *For any* $A \in \mathcal{E}/\mathcal{E}_0$ *we have*

$$M(A) = \infty \quad a.s.$$

(iii) *If* $A_1, \dots, A_k$ *are disjoint sets in* $\mathcal{E}_0$*, then* $M(A_1), \dots, M(A_k)$ *are independent.*

(iv) *There is an event* $\Omega_0 \in \mathcal{F}$ *with* $P(\Omega_0) = 1$*such that for every* $\omega \in \Omega_0$*,* $(M(A), A \in \mathcal{E})$ *is a measure.*

Let C be a set in $(0,\infty)\times\mathbb{R}^d$. For a Lévy process $\{X(t), t\geq 0\}$ we define the counting process

$$N_*(C) = \#\{t>0 : (t,\Delta X(t))\in C\}.$$

Theorem 1.34. *For any Lévy process $\{X(t), t\geq 0\}$, the jump counting measure N_* is a Poisson random measure on $(0,\infty)\times\mathbb{R}^d$ with mean measure $Leb\times\nu = n$, where ν is the Lévy measure of $X(1)$.*

By following the notation of [62, 105], we can also write

$$N_* = \sum_{\{t:\Delta X_t\neq 0\}} \delta_{(t,\Delta X_t)}. \tag{1.10}$$

Think of a Poisson random measure N_* as a point process on E: for each $A\in\mathcal{E}_0$, $N_*(A)$ can be regarded as the random number of points belonging to A, which is why N_* is also called a counting measure. That is, there are random elements $\{T_i\}_{i\geq 1}$ on $(\Omega,\mathcal{F},P)$ with value on E such that

$$N_*(A) = \sum_{i=1}^{\infty} I_A(T_i).$$

Now we want to underline some properties of this random measure.

Proposition 1.35. *Let N be a Poisson random measure on $(E,\mathcal{E})$ with mean measure n. Let $(\tilde{E},\tilde{\mathcal{E}})$ be another measurable space and $h: E\to\tilde{E}$ a measurable function. Then $(\tilde{N}(A), A\in\tilde{\mathcal{E}})$ is a Poisson random measure on $(\tilde{E},\tilde{\mathcal{E}})$ with σ-finite mean measure*

$$\tilde{n} = n\circ h^{-1}$$

Proof. See [102, 4.3]. □

Corollary 1.36. *We assume additionally that $\tilde{N}$ is defined on a probability space rich enough to support an independent of $\tilde{N}$ uniform random variable U on (0,1) and N can be written as*

$$N = \sum_{i=1}^{\infty} I_A(T_i)$$

for random elements $\{T_i\}_{i\geq 1}$ defined on $(\Omega,\mathcal{F},P)$ with values in $(E,\mathcal{E})$. Then there exists a sequence of random elements $\{\tilde{T}_i\}_{i\geq 1}$ defined on the same probability space as $\tilde{N}$ such that

$$\{\tilde{T}_i\}_{i\geq 1} \stackrel{d}{=} \{T_i\}_{i\geq 1}$$

and

$$\tilde{N} = \sum_{i=1}^{\infty} I_{\tilde{A}}(H(\tilde{T}_i)) \quad a.s.$$

Proof. See [62, Corollary 5.11]. □

Theorem 1.37. *If $\nu(\mathbb{R}^d)=\infty$, then, almost surely, jumping times are countable and dense in $[0,\infty)$. If $0<\nu(\mathbb{R}^d)<\infty$, then, almost surely, jumping times are infinitely many and countable in increasing order, and the first jumping time $T(\omega)$ has exponential distribution with mean $1/\nu(\mathbb{R}^d)$.*

Proof. See [109, Theorem 21.3]. □

Furthermore, also by analyzing the structure of paths of a Lévy process, it is possible to obtain the expression of the characteristic exponent (1.6).

Theorem 1.38 (Lévy-Itô decomposition)**.** *Let X be a Lévy process and ν the Lévy measure of X. Then X has a decomposition*

$$X_t = W_t + at + \int_{|x|<1} x(N_t(\cdot, dx) - t\nu(dx)) + \sum_{0<s\leq t} \Delta X_s I_{\{|\Delta X_s|\geq 1\}}$$

where $(W_t)_{t\geq 0}$ is Brownian motion; for any set Λ, $0 \notin \bar{\Lambda}$, $N_t^\Lambda = \int_\Lambda N_t(\cdot, dx)$ is a Poisson process independent of $(W_t)_{t\geq 0}$; N_t^Λ is independent of N_t^Γ if Λ and Γ are independent and N_t^Λ has parameter $\nu(\Lambda)$.

From Theorem 1.38 it follows that a Lévy process X can be write as a sum

$$X = X(1) + X(2) + X(3),$$

where $X(1)$ ia a scaled Brownian motion with drift, X(2) is a square integrable martingale with an almost surely countable number of jumps on each finite time interval which are of magnitude less than unity and $X(3)$ is a compound Poisson process. Furthermore, Theorem 1.38 allows one to give an alternative proof to Theorem 1.31.

1.3.2 Properties of Lévy processes

In the following, we want to point out some properties of Lévy processes, related to the Lévy generating triplet (a, σ, ν). Here, we give a basic classification

Definition 1.39. *Let $X = (X_t)_{t\geq 0}$ a Lévy process on $\mathbb{R}$ with generating triplet (a, σ, ν). It is said to be of*

- *type A if $\sigma = 0$ and $\nu(\mathbb{R}) < \infty$;*
- *type B if $\sigma = 0$, $\nu(\mathbb{R}) = \infty$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$;*
- *type C if $\sigma \neq 0$ or $\int_{|x|\leq 1} |x|\nu(dx) = \infty$.*

If $\sigma = 0$, then $X = (X_t)_{t\geq 0}$ is called *purely non-Gaussian* or *pure jump* Lévy process. We have also the following definitions for Lévy processes.

Definition 1.40. *Let $X = (X_t)_{t\geq 0}$ a Lévy process on $\mathbb{R}$ with generating triplet (a, σ, ν). It is said to be of*

- *if $\int_{|x|\leq 1} \nu(\mathbb{R}) < \infty$ the process is of finite activity;*
- *if $\int_{\mathbb{R}} \nu(\mathbb{R}) = \infty$ process is of infinite activity.*

An finite activity Lévy process has finitely many jumps in any finite interval. Conversely, an infinite activity Lévy process are able to capture both rare large moves and frequent small moves. High activity is accounted for by a large (in most cases infinite) number of small jumps [111].

Proposition 1.41. *A Lévy process is a compound Poisson process with drift if and only if its generating triplet (a, σ, ν) satisfies*

$$\sigma = 0 \quad \textit{and} \quad \nu(\mathbb{R}) < \infty.$$

We recall the definition of *finite variation* [103]. Let Δ e a subdivision of the interval $[0, t]$ with $0 = t_0 < t_1 < \ldots < t_n = t$ and f a function $f : [a, b] \to \mathbb{R}^d$; the number $|\Delta| = \sup_i |t_{i+1} - t_i|$ is called the modulus or mesh of Δ. We consider the sum

$$S_t^{\Delta} = \sum_t |f(t_{i+1}) - f(t_i)|$$

Definition 1.42. *The function f is of finite variation if for every t*

$$S_t = \sup_{\Delta} S_t^{\Delta} < +\infty$$

The function $t \to S_t$ is call the total variation of f and S_t is the variation of f on $[0, t]$. The function S is obviously positive and increasing and if

$$\lim_{t \to \infty} S_t < +\infty,$$

the function f is said to be of bounded variation.

The same notions could be extended on any interval $[a, b]$. For example, trajectories of Brownian motion are almost surely of infinite variation. Therefore, a Lévy process with a Brownian component is of infinite variation.

Proposition 1.43. *The Brownian paths are a.s. of infinite variation on any interval.*

Proof. See [103, Corollary 2.5]. □

Consequently, it could be prove the following result.

Proposition 1.44. *A Lévy process is of finite variation if and only if its generating triplet (a, σ, ν) satisfies*

$$\sigma = 0 \quad \textit{and} \quad \int_{|x| \leq 1} |x| \nu(dx) < \infty. \tag{1.11}$$

Note that the finiteness of the integral in (1.11) also allows one to write the characteristic exponent (1.6) as

$$\psi(\theta) = id\theta + \int_{\mathbb{R} \setminus \{0\}} (e^{i\theta - 1x}) \nu(dx) \tag{1.12}$$

where the constant $d \in \mathbb{R}$ relates to the constant a and the measure ν via

$$d = \left(a - \int_{|x|<1} x \nu(dx) \right). \tag{1.13}$$

Suppose now that $\nu(-\infty, 0) = 0$, then the corresponding Lévy process has no negative jumps. If further it has $\sigma = 0$,

$$\int_0^\infty (1 \wedge x)\nu(dx) < \infty$$

and d in (1.13) is positive, it becomes clear that the Lévy process has nondecreasing paths. We call such process a *subordinator*. A subordinator is a nonnegative nondecreasing Levy process. It has no Brownian part ($\sigma = 0$), a nonnegative drift and a Lévy measure which is zero on the negative half-line (it has only positive increments). Note that a subordinator is nondecreasing and always of finite variation.

Remark 1.45. *When $X = (X_t)_{t \geq 0}$ is a subordinator, the Laplace transform of its distribution is more convenient than the characteristic function. The general form is as follows*

$$E[e^{-uX_t}] = \exp[t(\int_0^\infty (e^{-ux} - 1)\nu(dx) - du)] \tag{1.14}$$

where $u \geq 0$, see [109, Remark 21.6].

The tail behavior of the distribution of a Lévy process and, in consequence, its moments, are determined by the Lévy measure.

Proposition 1.46. *Let $X = (X_t)_{t \geq 0}$ be a Lévy process on $\mathbb{R}$ with generating triplet (a, σ, ν). The n-th absolute moment of X_t, is finite if and only if*

$$\int_{|x| \geq 0} |x|^n \nu(dx) < \infty$$

We remark, that it is not always an easy task to find the analytical behavior of tails of a infinitely divisible distribution. Now, we recall another useful property associated to the Lévy measure ν.

Proposition 1.47. *Let $X = (X_t)_{t \geq 0}$ be a Lévy process on $\mathbb{R}$ with generating triplet (a, σ, ν) and let $u \in \mathbb{R}$. The exponential moment $E[e^{uX_t}]$ is finite for some t or, equivalently, for all $t > 0$ if and only if $\int_{|x| \geq 1} e^{ux} \nu(dx) < \infty$. In this case*

$$E[e^{uX_t}] = e^{t\psi(-iu)}$$

where ψ is the characteristic exponent of the Lévy process defined by (1.6).

Proof. See [109, Theorem 25.17] or [74, Theorem 3.6] □

To model stock price dynamic, we will consider discounted processes process Y_t define as

$$Y_t = e^{\omega t + X_t}$$

where X_t is a Lévy process and ω is a *convexity correction*, defined by

$$\omega = -\psi_X(-i) \tag{1.15}$$

where ψ is the characteristic exponent of a given distribution μ as defined in (1.6). Under some measure Q, this process will be a martingale, but to prove that, we will need additional assumptions.

Proposition 1.48. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X = (X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$ with generating triplet (a, σ, ν). If for some $u \in \mathbb{R}$*

$$\int_{|x|\geq 1} e^{ux}\nu(dx) < \infty. \tag{1.16}$$

Then the process $Y = (Y_t)_{t\geq 0}$, defined as

$$Y_t = \frac{e^{uX_t}}{E[e^{uX_t}]} \qquad t \geq 0,$$

is a martingale.

Proof. By Proposition 1.24 and condition 1.16, we have $Y_t \in L^1$ and thus we can calculated the conditional expectation. Since X is a Lévy process we can write for $0 \leq s \leq t$

$$\begin{aligned} E_P\left[\frac{e^{uX_t}}{E[e^{uX_t}]}\middle|\mathcal{F}_s\right] &= E_P\left[\frac{e^{u(X_t-X_s)}}{E[e^{u(X_t-X_s)}]}\frac{e^{uX_s}}{E[e^{uX_s}]}\middle|\mathcal{F}_s\right] \\ &= X_s E_P\left[\frac{e^{u(X_t-X_s)}}{E[e^{u(X_t-X_s)}]}\middle|\mathcal{F}_s\right] = X_s. \end{aligned}$$

□

1.4 Stable distributions

There are several equivalent ways to define the class of α-stable distribution (stable paretian distribution or, shortly, stable distribution). For a complete study of stable non-gaussian distribution, we refer to [120], for the one dimensional case, [108] and [99] for a complete overview of financial applications.

Definition 1.49. *A random variable X is said to have a stable distribution, if it has a domain of attraction, i.e., if there is a sequence of i.i.d. random variables $Y_1, Y_2, \ldots$ and sequences of positive $\{d_n\}$ and real numbers $\{a_n\}$, such that*

$$\frac{Y_1 + Y_2 + \ldots + Y_n}{d_n} + a_n \xrightarrow{d} X, \tag{1.17}$$

where the notation $\xrightarrow{d}$ denotes convergence in distribution.

This definition states that stable distribution are the only distributions that can be obtained as limits of normalized sums of i.i.d. random variables. This definition can be viewed as an extension of the classical central limit theorem (CLT) [43], in fact, if we take X a gaussian distribution and the Y_is are i.i.d with finite variance, then Definition 1.49 becomes the ordinary central limit theorem.

Definition 1.50. *A random variable X is said to have a stable distribution if there are parameters $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$ such that its characteristic function has the following form:*

$$E[e^{i\theta X}] = \begin{cases} \exp\{-\sigma^\alpha|\theta|^\alpha(1 - i\beta(\text{sign}(\theta)\tan\frac{\pi\alpha}{2})) + i\mu\theta\}, & \text{if } \alpha \neq 1, \\ \exp\{-\sigma|\theta|(1 + i\beta\frac{2}{\pi}(\text{sign}(\theta)\ln|\theta|)) + i\mu\theta\}, & \text{if } \alpha = 1, \end{cases} \tag{1.18}$$

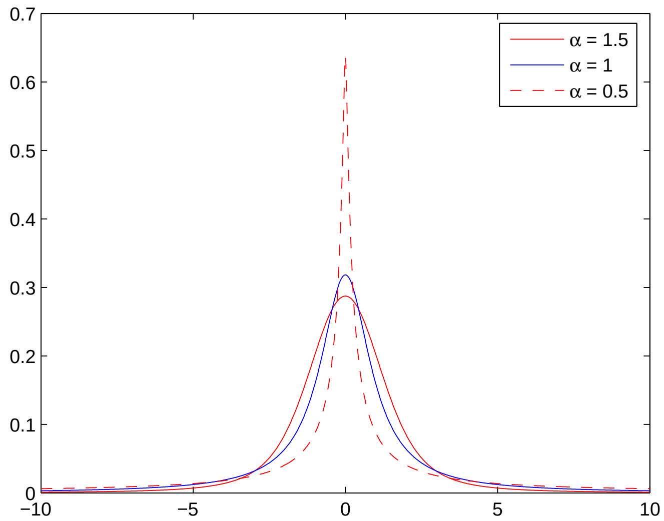

Figure 1.1: Standard symmetric α-stable densities ($\beta = 0$, $\sigma = 1$, $\mu = 0$) for varying α.

where the function sign *is so defined*

$$\mathrm{sign}(\theta) = \begin{cases} 1, & \text{if } \theta > 0, \\ 0, & \text{if } \theta = 0, \\ -1, & \text{if } \theta < 0. \end{cases}$$

We write $X \sim S_\alpha(\sigma, \beta, \mu)$.

The parameter α is the *index of stability* and can also be interpreted as a shape parameter, β is the skewness parameter, σ is a scale parameter and μ is a location parameter. It can be prove that Definitions 1.49 and 1.50 are equivalent.

Proposition 1.51. *A distribution μ on $\mathbb{R}^d$ is α-stable with $0 < \alpha < 2$ if and only if it is an infinitely divisible with characteristic triplet $(a, 0, \nu)$ and there exists a finite measure σ on S^{d-1}, the unit sphere on $\mathbb{R}^d$, such that*

$$\nu(B) = \int_{S_{d-1}} \sigma(d\xi) \int_0^\infty I_B(r\xi) \frac{dr}{r^{1+\alpha}}. \tag{1.19}$$

A distribution on $\mathbb{R}^d$ is α-stable with $\alpha = 2$ if and only if it is Gaussian.

Proof. See [109, Theorem 14.3]. □

The measure σ is called *spherical measure* and it is uniquely determined by the distribution μ of the α-stable random variable X. Therefore, if X is a real valued α-stable random variable, with $0 < \alpha < 2$ then its Lévy measure (1.19) has the following form

$$\nu(x) = \frac{a_+}{x^{\alpha+1}} I_{\{x>0\}} + \frac{a_-}{x^{\alpha+1}} I_{\{x<0\}} \tag{1.20}$$

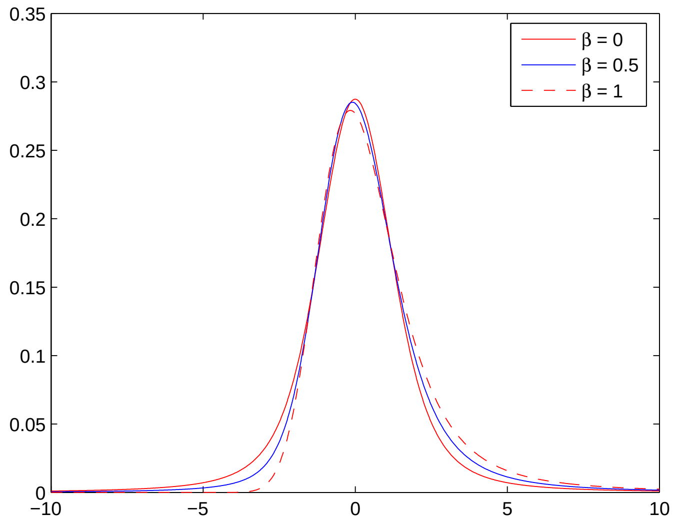

Figure 1.2: Standard symmetric α-stable densities ($\alpha = 1.5$, $\sigma = 1$, $\mu = 0$) for varying β.

with $a_+, a_- \in \mathbb{R}$. The representation of an α-stable distribution as in Proposition 1.51, will be important to understand the construction of a *tempered stable distribution*. Here we will show another property of α-stable distribution.

Definition 1.52. *Let μ be a probability measure on $\mathbb{R}^d$. It is called selfdecomposable if, for any $b > 1$, there is a probability measure ρ_b on $\mathbb{R}^d$ such that*

$$\hat{\mu}(z) = \hat{\mu}(b^{-1}z)\hat{\rho}_b(z).$$

where we indicate with $\hat{\mu}$, the Fourier transform of μ.

It can be prove that any α-stable distribution on $\mathbb{R}^d$ is selfdecomposable, see [109, Example 15.2].

By considering the Lévy measure (1.20), we can prove the following fact:

Remark 1.53. *An α-stable distributions on $\mathbb{R}$ never admit the second moment, and if $\alpha > 1$, they admit only the first moment.*

Since the distribution of an α-stable process is infinitely divisible, we can define an α-stable process.

Definition 1.54. *Let $X = (X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$. It is called a stable process if the distribution of X_t in $t = 1$ is α-stable.*

From the explicit form of the Lévy measure (1.20), we can consider $r^{-1-\alpha}dr$ as the radial part of the Lévy measure of the α-stable process. As α decrease, $r^{-1-\alpha}$ gets smaller for $0 < r < 1$ and bigger for $1 < r < \infty$. Roughly speaking, an α-stable process moves mainly by big jumps if α is close to 0, and mainly by small jumps if α is close to 2. This tendency of paths of an α-stable process can be view, for example, in the computer simulation performs in [59].

The canonical representation (1.18) has a disadvantage. The characteristic function is not a continuous function of the parameters, therefore if we want to look at the simulation problem, we prefer to consider the alternative representation [120, 117].

Definition 1.55. *A random variable X is α-stable if and only if its characteristic function is given by*

$$E[e^{i\theta X}] = \begin{cases} \exp\{-\sigma_2^\alpha|\theta|^\alpha \exp\{-i\beta_2(\mathrm{sign}(\theta)\frac{\pi}{2}K(\alpha))\} + i\mu\theta\}, & \text{if } \alpha \neq 1, \\ \exp\{-\sigma_2|\theta|(\frac{\pi}{2} + i\beta_2 \mathrm{sign}(\theta)\ln|\theta|) + i\mu\theta\}, & \text{if } \alpha = 1, \end{cases} \tag{1.21}$$

where

$$K(\alpha) = \alpha - 1 + \mathrm{sign}(1-\alpha)\begin{cases} \alpha, & \alpha < 1, \\ \alpha - 2, & \alpha > 1. \end{cases}$$

The parameters σ_2 and β_2 are related to σ and β, from the representation (1.18), as follows. For $\alpha \neq 1$, β_2 is such that

$$\tan \beta_2 \frac{\pi K(\alpha)}{2} = \beta \tan \frac{\pi\alpha}{2},$$

and the new scale parameter is

$$\sigma_2 = \sigma\left(1 + \beta^2 \tan^2 \frac{\pi\alpha}{2}\right)^{1/(2\alpha)}.$$

For $\alpha = 1$, $\beta_2 = \beta$ and $\sigma_2 = \frac{2}{\pi}\sigma$. Furthermore, we define γ_0 as

$$\gamma_0 = -\frac{\pi}{2}\beta_2 \frac{K(\alpha)}{\alpha}. \tag{1.22}$$

1.4.1 Basic properties

In the following, we recall some basis properties of stable distribution. Proofs of these facts could be found on [108].

Proposition 1.56. *Let X_1 and X_2 be independent random variable such that $X_1 \sim S_\alpha(\sigma_1, \beta_1, \mu_1)$ and $X_2 \sim S_\alpha(\sigma_2, \beta_2, \mu_2)$. Then $X_1 + X_2 \sim S_{\alpha(\sigma,\beta,\mu)}$, with*

$$\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{\frac{1}{\alpha}}, \quad \beta = \frac{\beta_1\sigma_1^\alpha + \beta_2\sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha}, \quad \mu = \mu_1 + \mu_2$$

It is not true in general, that the sum of two stable distribution with different α is stable.

Proposition 1.57. *$X \sim S_\alpha(\sigma, \beta, \mu)$ is symmetric if and only if $\beta = 0$ and $\mu = 0$.*

Proof. A random variable is symmetric, if and only if the characteristic function 1.18 is real, therefore, if and only if $\beta = 0$ and $\mu = 0$. □

We write $S\alpha S$ for a symmetric α-stable distribution. The form of its characteristic function takes the particularly simple form

$$E[e^{i\theta X}] = e^{-\sigma^\alpha|\theta|^\alpha}.$$

Now, we want to analyze the asymptotic behavior of the tail probability $P(X < -\lambda)$ and $P(X > \lambda)$ as $\lambda \to \infty$ for an α-stable distribution X. If $\alpha = 2$, the tail probabilities decrease exponentially as $\lambda \to \infty$, i.e.,

$$P(X < -\lambda) = P(X > \lambda) \sim \frac{1}{2\sigma\lambda\sqrt{\pi}} e^{\frac{-\lambda^2}{4\sigma^2}},$$

see, for example, [42]. If $\alpha < 2$, the tail behavior is totally different.

Proposition 1.58. *Let $X \sim S_\alpha(\sigma, \beta, \mu)$ with $0 < \alpha < 2$. Then*

$$\begin{cases} \lim_{\lambda\to\infty} \lambda^\alpha P(X > \lambda) = C_\alpha \frac{1+\beta}{2} \sigma^\alpha, \\ \lim_{\lambda\to\infty} \lambda^\alpha P(X < -\lambda) = C_\alpha \frac{1-\beta}{2} \sigma^\alpha, \end{cases} \tag{1.23}$$

where C_α is a constant depending on α,

$$C_\alpha = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha) cos(\frac{\pi\alpha}{2})}, & if\ \alpha \neq 1 \\ \frac{2}{\pi}, & if\ \alpha = 1 \end{cases}$$

The tail behavior (1.23) is a widely use property of stable distribution.

Proposition 1.59. *Let $X \sim S_\alpha(\sigma, \beta, \mu)$ with $0 < \alpha < 2$. Then*

$$\begin{aligned} E[|X|^p] < \infty \quad & for\ any \quad 0 < p < \alpha \\ E[|X|^p] = \infty \quad & for\ any \quad p \geq \alpha. \end{aligned}$$

There are stable distribution totally skewed on the right and for such distribution, the integral $E[e^{-\mu X}]$, with $\mu \geq 0$, exists.

Proposition 1.60. *Let $X \sim S_\alpha(\sigma, 1, 0)$, with $0 < \alpha \leq 2$, then the expectation $E[e^{-\mu X}]$, with $\mu \geq 0$, equals*

$$E[e^{-\mu X}] = \begin{cases} \exp\{-\frac{\sigma^\alpha}{\cos(\frac{\pi\alpha}{2})} \gamma^\alpha\}, & if\ \alpha \neq 1 \\ \exp\{\sigma \frac{2}{\pi} \gamma \ln(\gamma)\}, & if\ \alpha = 1. \end{cases}$$

Proof. See [108, Proposition 1.2.12]. □

1.4.2 α-stable random measure

In this section, we want to recall the definition of α-stable random measures. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $L^0(\Omega)$ the set of all real random variables defined on it. Furthermore, let us consider $(E, \mathcal{E}, m)$ a measure space and β a measurable function

$$\beta : E \to [-1, 1].$$

Define the subset $\mathcal{E}_0$ of E such that

$$\mathcal{E}_0 = \{A \in \mathcal{E} : m(A) < \infty\}.$$

Definition 1.61. *An independently scattered σ-additive set function*

$$M : \mathcal{E}_0 \to L^0(\Omega)$$

such that for each $A \in \mathcal{E}_0$,

$$M(A) \sim S_\alpha\left((m(A))^\alpha, \frac{\beta(x)m(dx)}{m(A)}, 0\right)$$

is called an α-stable random measure on $(E, \mathcal{E})$ with control measure m and skewness intensity β.

Chapter 2

Tempered stable distributions and processes

A tempered stable (TS) Lévy process combines both α-stable and gaussian trends. TS processes are well known processes in finance and statistical physics. In physics literature they appeared with the name of *truncated Lévy flight model*, in papers [72] and later in [93], by considering ideas contained in [82]. Contrarily to the stable distribution, a TS distribution may have finite variance, we may have the local behavior of the α-stable distribution, but tails are tempered. Since a TS distribution is infinitely divisible, we can construct a TS process. In [72] is shown that the convergence of the the sum of truncated Lévy flights to a normal process can be so slow that a huge number ($n \sim 10^4$) of independent events may be necessary to ensure convergence to a gaussian stochastic process. In mathematical finance, they were introduced with the name of KoBoL in [19], a four parameters subclass of TS processes was called CGMY in [21, 22], see also [64] for other applications to finance. We can find the mathematical formalization of TS processes, also called *tempering stable process*, in [107], where the entire class of TS processes is resumed, a formal and elegant definition is proposed and a view toward simulation is given. In the following, we will refer principally to [107] and [115], where some subclass previously cited are contained.

The tempering is related to an old idea of tilting a density function. Let f be a probability density function on $\mathbb{R}_+$, with finite Laplace transform $L(\theta)$, defined in (1.4). For every $\theta > 0$ define a tilted density f_θ by

$$f_\theta(x) = \frac{1}{L(\theta)} e^{-\theta x} f(x) = \exp\{-\theta x + l(\theta) + k(x)\},$$

where, taking into account the condition $f(x) > 0$, $f(x) = \exp\{k(x)\}$ and $L(\theta) = exp\{-l(\theta)\}$. The Laplace transform L_θ of f_θ is given by

$$L_\theta(\lambda) = \exp\{-l(\lambda + \theta) - l(\theta)\}. \tag{2.1}$$

Assume additionally that f is infinitely divisible, therefore we obtain from (1.14)

$$l(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\nu(dx) + \lambda d,$$

where ν is a Lévy measure on $\mathbb{R}_+$ and $d \geq 0$. From (2.1) we get

$$L_\theta(\lambda) = \exp\{\int_0^\infty (e^{-\lambda x} - 1)e^{-\theta x}\nu(dx) - \lambda d\}.$$

Therefore, tilting an infinitely divisible density $f \mapsto f_\theta$ leads to the tilting of the corresponding Lévy measure $\nu \mapsto \nu_\theta$, where $\nu_\theta(dx) = e^{-x\theta}\nu(dx)$. More generally, we take products of convolution powers $f_{\theta_i}^{*r_i}$ of f_{θ_i} with $r_i, \theta_i > 0$, where f is infinitely divisible, and then their limits, therefore we obtain distributions having the Laplace transform of the form

$$\exp\{\int_0^\infty (e^{-\lambda x} - 1)q(x)\nu(dx) - \lambda d\},$$

where q is a completely monotone function with $q(\infty)$. We call such operation on Lévy measure ν, tempering or tilting if $q(x) = e^{-x\theta}$. In the following, we focus our attention on tempering stable Lévy measure (1.19).

2.1 Definitions and properties

In this section we will review the definition and properties of the TS distributions introduced by [107]. The polar coordinates representation of a measure $\nu = \nu(dx)$ on $\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$ is the measure $\nu = \nu(dr, du)$ on $(0, \infty) \times S^{d-1}$ obtained by the bijection $x \mapsto (\|x\|, \frac{x}{\|x\|})$. By the radial representation (1.19), the Lévy measure ν_0 of an α-stable distribution on $\mathbb{R}^d$ in polar coordinates is of the form

$$\nu_0(dr, du) = r^{-\alpha-1}dr\sigma(du) \tag{2.2}$$

where $\alpha \in (0, 2)$ and σ is a finite measure on S^{d-1}. A tempered α-stable distribution is defined by tempering the radial term of ν_0 as follows:

Definition 2.1. *A probability measure μ on $\mathbb{R}^d$ is called tempered α-stable (abbreviated as TS) if is infinitely divisible without Gaussian part and has Lévy measure ν that can be written in polar coordinated*

$$\nu(dr, du) = r^{-\alpha-1}q(r, u)dr\,\sigma(du), \tag{2.3}$$

where α and σ are as above, and

$$q : (0, \infty) \times S^{d-1} \mapsto (0, \infty)$$

is a Borel function such that $q(\cdot, u)$ is completely monotone with $q(\infty, u) = 0$ for each $u \in S^{d-1}$. A TS distribution is called a proper TS distribution if

$$\lim_{r \to 0^+} q(r, u) = 1$$

for each $u \in S^{d-1}$.

The completely monotonicity of $q(\cdot, u)$ means that

$$(-1)^n \frac{d}{dr} q(r, u) > 0$$

for all $r > 0$, $u \in S^{d-1}$, and $n = 0, 1, 2, \ldots$. In particular $q(\cdot, u)$ is strictly decreasing and convex.

Remark 2.2. *(a) The class of TS distributions contains β-stable distributions with $\beta > \alpha$. Indeed, one takes $q(r,u) = r^{\alpha-\beta}$ in 2.3.*

(b) Proper TS distributions do not contain any stable distribution.

(c) TS distributions are selfdecomposable in the sense of Definition (1.52), this follows from [109, Theorem 15.10], since q is a Borel function decreasing in $r > 0$.

2.1.1 The Rosiński measure

Sometimes the only knowledge of the Lévy measure cannot be enough to obtain analytical properties of TS distributions. Therefore, the definition of Rosiński measure allows one to overcome this problem and to obtain explicit analytic formulas and more explicit calculations. More detailed, the tempering function q can be represented as the Laplace transform

$$q(r,u) = \int_0^\infty e^{-rs} Q(ds|u) \tag{2.4}$$

where $\{Q(\cdot|u)\}_{u\in S^{d-1}}$ is a measurable family of Borel measures on $(0,\infty)$.

Remark 2.3. *In the case of proper TS distribution, $Q(\cdot|u)$ is a probability measure.*

Define a measure Q on $\mathbb{R}^d$ by

$$Q(A) := \int_{S^{d-1}} \int_0^\infty I_A(vu) Q(dr|u)\sigma(du), \qquad A \in \mathcal{B}(\mathbb{R}^d), \tag{2.5}$$

we also define the measure R by

$$R(A) := \int_{\mathbb{R}^d} I_A(\frac{x}{\|x\|^2})\|x\|^\alpha Q(dx), \qquad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.6}$$

Clearly $R(\{0\}) = 0$ and $Q(\{0\}) = 0$ and Q can be expressed in terms of the measure R as follows:

$$Q(A) = \int_{\mathbb{R}^d_0} (\frac{x}{\|x\|^2})\|x\|^\alpha R(dx), \qquad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.7}$$

As we said above, the measure R is consider to obtain a most convenient description of distributional properties of TS distributions. In order to understand the relation between the Lévy measure ν, as consider in (2.3), and the measure R, we show a fundamental result, see also [107, Theorem 2.3].

Theorem 2.4. *Lévy measure ν of a TS distribution can be written in the form*

$$\nu(A) = \int_{\mathbb{R}^d_0} \int_0^\infty I_A(tx)\alpha t^{-\alpha-1}e^{-t} dt R(dx), \qquad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.8}$$

where R is a unique measure on $\mathbb{R}^d$ such that $R(\{0\}) = 0$

$$\begin{cases} \int_{\mathbb{R}^d}(\|x\|^2 \wedge \|x\|^\alpha)R(dx) < \infty, & \alpha \in (0,2) \\ \int_{\mathbb{R}^d}(\log(1+\|x\|)+1)R(dx) < \infty, & \alpha = 0 \end{cases} \tag{2.9}$$

If ν is as in (2.3) then R is given by (2.6).

Conversely, if R is a measure satisfying (2.9), then (2.3) defines the Lévy measure of a TS distribution. ν corresponds to a proper TS distribution if and only if

$$\begin{cases} \int_{\mathbb{R}^d} \|x\|^\alpha R(dx) < \infty., & \alpha \in (0,2) \\ R(\mathbb{R}^d) < \infty, & \alpha = 0 \end{cases} \tag{2.10}$$

Proof. In the following we will prove only the case $\alpha = 0$, for the case $\alpha \in (0,2)$ and the proof of existence and uniqueness of the measure R satisfying (2.9), readers are referred to Theorem 2.3 of [107]. Let us assume that $\alpha = 0$, then the condition

$$\int_{\mathbb{R}^d} (\log(1 + \|x\|) + 1) R(dx) < \infty \tag{2.11}$$

is necessary and sufficient for ν to be a Lévy measure. In order to prove the necessity, we assume that ν is a Lévy measure, i.e.

$$\int_{\|x\| \geq 1} \nu(dx) < \infty$$

By equality (2.8), it follows that

$$\begin{aligned} \infty > \int_{\|x\|>1} \nu(dx) &= \int_{\mathbb{R}^d} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1} e^{-t} dt R(dx) \\ &= \int_{\|x\| \leq 1} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1} e^{-t} dt R(dx) + \int_{\|x\|>1} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1} e^{-t} dt R(dx) \\ &\geq \int_{\|x\| \leq 1} K R(dx) + e^{-1} \int_{\|x\|>1} \log(\|x\|) R(dx) \end{aligned}$$

therefore, (2.11) holds. Conversely, if (2.11) is fulfilled, then ν is a Lévy measure, indeed

$$\begin{aligned} \int_{\|x\| \leq 1} \|x\|^2 \nu(dx) &= \int_{\mathbb{R}^d} \|x\|^2 \int_0^{\frac{1}{\|x\|}} t e^{-t} dt R(dx) \\ &\leq \int_{\mathbb{R}^d} \|x\|^2 (1 - e^{-\frac{1}{x}} (1 + \frac{1}{x})) R(dx) \\ &\leq \int_{\|x\| \leq 1} (\log(1 + \|x\|)) R(dx) + \int_{\|x\|>1} R(dx) \end{aligned}$$

and

$$\begin{aligned} \int_{\|x\|>1} \nu(dx) &= \int_{\mathbb{R}^d} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1} e^{-t} dt R(dx) \\ &\leq \int_{\|x\| \leq 1} \int_{\frac{1}{\|x\|}}^{\infty} \frac{1}{t(1 + \frac{t^2}{2})} dt R(dx) + \int_{\|x\|>1} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1} e^{-t} dt R(dx) \\ &\leq \frac{1}{2} \int_{\|x\| \leq 1} \log(1 + 2\|x\|^2) R(dx) + K'' \int_{\|x\|>1} R(dx), \\ &\leq K' \int_{\|x\| \leq 1} \log(1 + \|x\|) R(dx) + K'' \int_{\|x\|>1} R(dx), \end{aligned}$$

where K' and K'' are positive constants. □

Definition 2.5. *The unique measure R in (2.8) is called the Rosiński measure or the spectral measure of the corresponding TS distribution.*

In the following we will call the measure R, Rosiński measure, as in [115]. By considering Lévy measures of proper and general TS distributions, we explain the motivation of their differences.

Proposition 2.6. *Let ν be a given measure by (2.8). The function $s \to s^{\alpha}\nu\{\|x\| > s\}$, $s>0$ is decreasing with*

$$\lim_{s\to 0^+} s^{\alpha}\nu\{\|x\| > s\} = \alpha^{-1}\int_{\mathbb{R}^d} \|x\|^{\alpha} R(dx) \quad \textit{and} \quad \lim_{s\to\infty} s^{\alpha}\nu\{\|x\| > s\} = 0. \tag{2.12}$$

Hence ν is a Lévy measure of a proper TS distribution if and only if

$$\lim_{s\to 0^+} s^{\alpha}\nu\{\|x\| > s\} < \infty.$$

In the general case the limit (2.12) as $s \to 0^+$ is not finite. TS distributions may have moments of any order, even exponential moments of some order. The behavior of the tail depends on their Rosiński measure.

Proposition 2.7. *Let μ be a TS distribution with Lévy measure given by (2.8). Then*

(a) $\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty$ *for* $p \in (0, \alpha)$;

(b) $\int_{\mathbb{R}^d} \|x\|^{\alpha} \mu(dx) < \infty \iff \int_{\|x\|>1} \|x\|^{\alpha} \log(\|x\|) R(dx) < \infty$;

(c) $\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty \iff \int_{\|x\|>1} \|x\|^p R(dx) < \infty$ *when* $p > \alpha$;

(d) $\int_{\mathbb{R}^d} \exp\{\theta\|x\|\}\mu(dx) < \infty \iff R(\{x : \|x\| > \theta^{-1}\}) = 0$ *where* $\theta > 0$.

Proof. See [107, Proposition 2.7] □

Remark 2.8. *By (d) of the above Proposition, if the support of R is a bounded set, then some exponential moment are also finite.*

Now, we consider the characteristic function of a TS distribution. It is expressed not by using the Lévy measure ν, but by considering only the Rosińsky measure R. A feature of the Rosińsky measure, is that, allows one to write the characteristic function in an easy form.

Proposition 2.9 (Characteristic function)**.** *Let X be a random vector with TS distribution μ with Lévy measure (2.8).*

(a) If $\alpha \in (0, 2)$ and $E[\|X\|] < \infty$, then the characteristic function ϕ of X is given by

$$\phi(u) = \hat{\mu}(u) = \exp\Big(\int_{\mathbb{R}^d_0} \psi_{\alpha}(\langle u, x\rangle) R(dx) + i\langle u, b\rangle\Big) \tag{2.13}$$

where

$$\psi_{\alpha}(y) = \begin{cases} \Gamma(-\alpha)((1-iy)^{\alpha} - 1 + i\alpha y), & 0 < \alpha < 1 \textit{ and } 1 < \alpha < 2 \\ (1-iy)\log(1-iy) + iy, & \alpha = 1 \\ -\log(1-iy) - iy, & \alpha = 0 \end{cases} \tag{2.14}$$

and $b = E[X]$.

(b) If $\alpha \in (0,1)$ and

$$\int_{\|x\|\leq 1} \|x\| R(dx) < \infty, \tag{2.15}$$

holds, then $X \sim TS^0_\alpha(R, b_0)$ means that the characteristic function ϕ^0 of X is of the form

$$\hat{\mu} = \exp(\int_{\mathbb{R}^d_0} \psi^0_\alpha(\langle u, x\rangle) R(dx) + i\langle u, b_0\rangle) \tag{2.16}$$

where

$$\psi^0_\alpha(y) = \begin{cases} \Gamma(-\alpha)((1-iy)^\alpha - 1), & \alpha \in (0,1) \\ -\log(1-iy), & \alpha = 0 \end{cases} \tag{2.17}$$

and $b_0 \in \mathbb{R}^d$ is the drift vector (i.e. $b_0 = \int_{\|x\|\leq 1} \|x\|\nu(dx)$).

Before the proof, we recall some results on the limit behavior of the function ψ_α at zero and infinity, see [107].

Lemma 2.10. *The following limits are verified*

$$\begin{array}{ll} \lim_{s\to 0} s^{-2}\psi_\alpha(s) = -\frac{1}{2}\Gamma(2-\alpha), & \alpha \in [0,2), \\ \lim_{s\to\infty} s^{-1}\psi_0(s) = -i, & \alpha = 0, \\ \lim_{s\to\infty} s^{-1}\psi_\alpha(s) = -i\Gamma(1-\alpha), & \alpha \in (0,1), \\ \lim_{s\to\infty}(s^{-1}\psi_1(s) + i\log(s)) = -\frac{\pi}{2} + i, & \alpha = 1, \\ \lim_{s\to\infty} s^{-\alpha}\psi_\alpha(s) = \Gamma(-\alpha)e^{-\frac{i\alpha\pi}{2}}, & \alpha \in (1,2), \end{array} \tag{2.18}$$

Furthermore if $\alpha \in (0,1)$

$$\begin{array}{l} \lim_{s\to 0} s^{-2}\psi^0_\alpha(s) = \Gamma(1-\alpha) \\ \lim_{s\to\infty} s^{-\alpha}\psi^0_\alpha(s) = \Gamma(-\alpha)e^{-\frac{i\alpha\pi}{2}} \end{array} \tag{2.19}$$

Consequently, for each α there is a finite positive constant C_α such that for all $s \in \mathbb{R}$

$$\begin{array}{rlll} C_\alpha^{-1}(s^2 \wedge |s|^{\alpha\vee 1}) & \leq |\psi_\alpha(s)| \leq & C_\alpha(s^2 \wedge |s|^{\alpha\vee 1}), & \alpha \neq 1, \\ C_1^{-1}[s^2 \wedge |s|(1+\log^+|s|)] & \leq |\psi_1(s)| \leq & C_\alpha[s^2 \wedge |s|(1+\log^+|s|)], & \alpha = 1, \\ C_\alpha^{-1}(s^2 \wedge |s|^{\alpha}) & \leq |\psi^0_\alpha(s)| \leq & C_\alpha(s^2 \wedge |s|^{\alpha}), & \alpha \in (0,1). \\ C_0^{-1}[(1+\log(1+s))] & \leq |\psi_1(s)| \leq & C_0[|s|(1+\log(1+s))], & \alpha = 0. \end{array} \tag{2.20}$$

Proof. First, integrals (2.13) and (2.16) are well define due to conditions (2.9) and (2.20) of Lemma 2.10. It is well known that if the mean is finite, that is if the first absolute moment exists, i.e. $\int_{\mathbb{R}^d} \|x\|\mu(dx) < \infty$, then $\hat{\mu}$ can be written as

$$\hat{\mu} = \exp\left(\int_{\mathbb{R}^d} (e^{i\langle y,x\rangle} - 1 - i\langle y, x\rangle)\nu(dx) + i\langle y, b\rangle\right)$$

where $b = \int_{\mathbb{R}^d} x\mu(dx)$. By (2.8), we obtain the equality (2.13), where, if $\alpha \in [0,2)$

$$\psi_\alpha(s) = \int_0^\infty (e^{ist} - 1 - ist)t^{-\alpha-1}e^{-t}dt, \tag{2.21}$$

If $\alpha \in [0,1)$ and $\int_{\|x\|\le 1} \|x\| R(dx) < \infty$, by Proposition 2.33 $\int_{\|x\|\le 1} \|x\| \nu(dx) < \infty$, in which case $\hat{\mu}$ can be written as

$$\exp\left(\int_{\mathbb{R}^d} (e^{i\langle y,x\rangle} - 1)\nu(dx) + i\langle y, b_0\rangle \right),$$

where b_0 is the drift as defined in [109]. By (2.8), we obtain the equality (2.16), where

$$\psi_\alpha^0(s) = \int_0^\infty (e^{ist} - 1) t^{-\alpha-1} e^{-t} dt, \tag{2.22}$$

and, furthermore, the equality

$$\psi_\alpha(s) = \psi_\alpha^0(s) - is \int_0^\infty t^{-\alpha} e^{-t} dt \tag{2.23}$$

holds. Now we will prove (2.14) and (2.17). If $\alpha \in (0,2)$ and $\alpha \neq 0$, by elementary properties of the gamma function we obtain

$$\begin{aligned}
\int_0^\infty (e^{ist} - 1 - ist) t^{-\alpha-1} e^{-t} dt &= \sum_{n=2}^\infty \frac{(is)^n}{n!} \int_0^\infty t^{n-\alpha-1} e^{-t} dt \\
&= \sum_{n=2}^\infty \frac{(is)^n}{n!} \Gamma(n-\alpha) \\
&= \Gamma(-\alpha)({}_2F_1(-\alpha, b, b; is) - 1 + i\alpha s) \\
&= \Gamma(-\alpha)((1-is)^\alpha - 1 + i\alpha s)
\end{aligned}$$

where ${}_2F_1(a,b,c;z)$ is the hypergeometric function as defined in [1, 15.1.1]. With a similar calculus, if $\alpha \in (0,1)$, we obtain

$$\begin{aligned}
\int_0^\infty (e^{ist} - 1) t^{-\alpha-1} e^{-t} dt &= \sum_{n=1}^\infty \frac{(is)^n}{n!} \int_0^\infty t^{n-\alpha-1} e^{-t} dt \\
&= \sum_{n=1}^\infty \frac{(is)^n}{n!} \Gamma(n-\alpha) \\
&= \Gamma(-\alpha)({}_2F_1(-\alpha, b, b; is) - 1) \\
&= \Gamma(-\alpha)((1-is)^\alpha - 1),
\end{aligned}$$

and it is easy to check equality (2.23). Now, we consider the case $\alpha = 1$. We can write for $\alpha \in (0,2)$

$$\psi_\alpha(s) = \frac{\Gamma(2-\alpha)}{\alpha} \frac{(1-is)^\alpha - 1 + i\alpha s}{\alpha - 1}.$$

By the Dominated Convergence Theorem and l'Hôpital rule, we can calculate the limit for $\alpha \searrow 1$ and obtain

$$\psi_1(s) = (1-is)\log(1-is) + is.$$

We analyze also the case $\alpha = 0$ and write

$$\psi_\alpha(s) = \Gamma(1-\alpha)\frac{(1-is)^\alpha - 1 + i\alpha s}{-\alpha}$$

and by a similar argument, we can calculate the limit for $\alpha \searrow 0$ and obtain

$$\psi_1(s) = -\log(1-is) - is.$$

□

Remark 2.11. *Let X be a TS distributed random vector with the spectral measure R. By Proposition 2.7, see also [107, Proposition 2.7], we can say the following:*

1. *In the above definition, $E[||X||] < \infty$ if and only if $\alpha \in (1,2)$ or*

$$\alpha = 1 \text{ and } \int_{||x||>1} ||x|| \log ||x|| R(dx) < \infty, \tag{2.24}$$

or

$$\alpha \in (0,1) \text{ and } \int_{||x||>1} ||x|| R(dx) < \infty. \tag{2.25}$$

2. *If $\alpha \in (0,1)$ and $\int_{\mathbb{R}^d} ||x|| R(dx) < \infty$, then both form (2.13) and (2.16) are valid for X. Therefore $X \sim TS^0_\alpha(R, b_0)$ and $X \sim TS^0_\alpha(R, b)$, where $b = b_0 + \Gamma(1-\alpha)\int_{\mathbb{R}^d} xR(dx)$.*

A TS distribution is characterized by an index $\alpha \in (0,2)$, a Rosińsky measure R, and a shift b. A proper TS distribution is uniquely characterized by three parameters α, R and b, that is, the following result is verified.

Proposition 2.12. *The triple (α, R, b) is identifiable in the subclass of proper TS distribution.*

Proof. By Proposition (2.9), if two TS distribution have the same triple (α, R, b), are identical distributed. Conversely, let X_1 and X_2 be two proper TS distribution identical distributed, with Lévy measures ν_1 and ν_2 respectively. If $\alpha \in (1,2)$, by Proposition 2.7, the mean is finite, therefore, we can choose

$$b_1 - b_2 = \int_{\mathbb{R}^d} \|x\| d\nu(dx),$$

and by representation (2.13), the characteristic function is

$$\hat{\mu}_i(u) = \exp\left(\int_{\mathbb{R}^d_0} \psi_\alpha(\langle u, x\rangle) R_i(dx) + i\langle u, b_i\rangle\right),$$

with $i \in \{1,2\}$. X_1 and X_2 are identical distributed, therefore, they have the same characteristic function [109, Proposition 2.5] and by the Lévy-Khintchine representation (1.6), we obtain $\nu_1 = \nu_2$. Hence for every $B \in \mathcal{B}(\mathbb{R}^d_0)$, the following equality is satisfied

$$\int_{\mathbb{R}^d_0}\int_0^\infty I_B(tx)\alpha t^{-\alpha-1}e^{-t}dt(R_1 - R_2)(dx) = (\nu_1 - \nu_2)(B) = 0$$

and by the positivity of the integrand function, we get $R_1 = R_2$.

If $\alpha \in (0,1)$, by inequality (2.10)

$$\int_{\|x\|\leq 1} \|x\| R(dx) \leq \int_{\|x\|\leq 1} \|x\|^\alpha R(dx) < \infty$$

therefore we can use the representation (2.16) and the result is similar. If $\alpha = 1$, a similar result can be proved. □

Remark 2.13. *If $\alpha \in (0,1)$ and X is not a proper TS, we cannot always define the drift b.*

By Proposition 1.56 the convolution of two stable distributions, with same stability index α, is still stable. A similar result is true also for TS distribution, that is, the class of TS with same α is closed under convolution.

Proposition 2.14. *Let X_1 and X_2 be random vectors in $\mathbb{R}^d$, such that $X_i \sim TS(R_i, b_i)$ are independent, then $X_1 + X_2 \sim TS(R_1 + R_2, b_2 + b_1)$.*

In some application, it may be useful to know the form of the moment generating function, when it exists. TS distributions may have moments of any order, even exponential moments of some order. This depends on the Rosiński measure.

Proposition 2.15 (Moment generating function)**.** *Let $X \sim TS(R, 0)$ and $R(\{x : \|x\| > \theta^{-1}\}) = 0$ for some $\theta > 0$. Then for every $y\mathbb{R}^d$ with $\|y\| \leq \theta$ the moment generating function of X exists and it is equal to*

$$E[e^{\langle y,X\rangle}] = \begin{cases} \exp\{\Gamma(-\alpha)\int[(1-\langle y,x\rangle)^\alpha - 1 + \alpha\langle y,x\rangle]R(dx)\}, & \alpha \neq 1 \\ \exp\{\int[(1-\langle y,x\rangle)\log(1-\langle y,x\rangle) + \langle y,x\rangle]R(dx)\}, & \alpha = 1 \end{cases} \tag{2.26}$$

If $X \sim TS^0(R, 0)$ and $R(\{x : \|x\| > \theta^{-1}\}) = 0$, then

$$E[e^{\langle y,X\rangle}] = \exp[\Gamma(-\alpha)\int[(1-\langle y,x\rangle)^\alpha]R(dx)]. \tag{2.27}$$

The cumulants order greater than one of a TS distribution can be calculated purely in term of its Rosińsky measure R. To calculate cumulants, we want to recall a result of [115] for a multidimensional TS random variable.

Proposition 2.16. *Suppose that the Rosińsky measure R satisfies the moment condition*

$$\int_{\mathbb{R}^d_0} \|x\|^m R(dx),$$

for an $m \geq 1$ in case of $0 < \alpha < 1$, and for an $m \geq 2$, when $1 \leq \alpha < 2$. Then the m^th order cumulant of the TS random variable $X \sim TS(R, b)$ is given by

$$c_m = \Gamma(m-\alpha)\mu_{R,\otimes m},$$

where

$$\mu_{R,\otimes m} = \int_{\mathbb{R}^d_0} x^{\otimes m} R(dx),$$

where $\otimes$ stands for the Kronecker product.

Proof. See Lemma 1 of [115]. □

Remark 2.17. *The case $m = 1$ is a special one. It is possible to find a random variable $X \sim TS$ with finite first moment, but such that the moment of the measure R is infinite.*

The following Lemma shows some relations between the Rosiński measure R of the TS distribution and the Lévy measure of the α-stable distribution given by (1.51).

Proposition 2.18. *Let ν be a Lévy measure of a proper TS distribution, as in (2.3), with the Rosiński measure R. Let ν_0 be the Lévy measure of α-stable distribution given by (1.51) or equivalently (1.19). Then*

$$\nu_0(A) = \int_{\mathbb{R}^d} \int_0^\infty I_A(tx) t^{-\alpha-1} dt R(dx), \quad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.28}$$

Furthermore,

$$\sigma(B) = \int_{\mathbb{R}^d} I_B(\frac{x}{\|x\|}) \|x\|^\alpha R(dx), \quad B \in \mathcal{B}(S^{d-1}). \tag{2.29}$$

Proof. See [107, Lemma 2.14.]. □

By Proposition 2.18, we can see the relations between parameters of a proper TS distributions and stable ones.

2.2 Some TS distributions

In the literature the name *TS distribution* is usually used without taking into account the seminal work [107]. Results contained in that paper allow one to construct, not only the classical TS distribution, that is the famous *CGMY* distribution, but a potentially infinite family of distributions, i.e. the TS distributions. Additionally for each given $\alpha \in (0, 2)$, this family of distributions is closed under convolution. The only choice of a measure R, the measure we called Rosiński measure, satisfying conditions 2.9, or the choice of the tempering function q as in (2.42), give us a possible TS distribution, see [115]. We will be going to show some examples. In this examples, random variables with finite expectation are considered, i.e. $E[X] < \infty$, therefore, the characteristic function can be written in the form (2.13), that is one can use the truncation function $h(x) = x$ in the Lévy-Khinchin representation.

2.2.1 Generalized TS (GTS) distribution

As said above, we consider a tilting, that is, given a Lévy measure ν_0 of a stable distribution, we define a new Lévy measure

$$\nu(dr) = e^{-\theta r} \nu_0(dr).$$

By following the approach of [72], we consider the more general distribution, as in [29]. Let $0 < \alpha_- < 2$, $0 < \alpha_+ < 2$, c_+, c_-, λ_+ and λ_- positive constant, then

$$\nu(dr) = \frac{c_-}{|r|^{1+\alpha_-}} e^{-\lambda_- r} I_{\{r<0\}} + \frac{c_+}{|r|^{1+\alpha_+}} e^{-\lambda_+ r} I_{\{r>0\}} \tag{2.30}$$

is the Lévy measure of the *generalized TS*. The characteristic exponent for $\alpha_+, \alpha_- \neq 1$ has the form

$$\begin{aligned}\psi(u) =& iub + \Gamma(-\alpha_-)c_-((\lambda_- + iu)^{\alpha_-} - \lambda_-^{\alpha_-} - iu\lambda_-^{\alpha_- -1}\alpha_-) \\ &+ \Gamma(-\alpha_+)c_+((\lambda_+ - iu)^{\alpha_+} - \lambda_+^{\alpha_+} + iu\lambda_+^{\alpha_+ -1}\alpha_+),\end{aligned} \tag{2.31}$$

Since we are considering a parameter α_- for negative jumps and a parameter α_+ for positive jumps, we may see this law as an extension of the TS laws, in the sense that we can define a random variable X of this kind as the sum of two independent be TS random variables X_+ and X_-.

Remark 2.19. *Let X be a random variable with GTS distribution, that is X has Lévy measure given in 2.30. Then X can be written as the sum of independent TS distribution X_+ and X_-, with characteristic triplet $(\gamma_-, 0, \nu_-)$ and $(\gamma_+, 0, \nu_+)$, where*

$$\nu_-(dr) = \frac{c_-}{|r|^{1+\alpha_-}} e^{-\lambda_- r} I_{\{r<0\}} \qquad \nu_+(dr) = \frac{c_+}{|r|^{1+\alpha_+}} e^{-\lambda_+ r} I_{\{r>0\}}$$

2.2.2 KoBoL distribution

A subclass of the previous example is given by choosing $\alpha_+ = \alpha_- = \alpha$ and $\alpha \in (0, 2)$. Such distribution is a TS in the sense of Rosińsky [107], and it is appeared in the literature under the name of *KoBoL* distribution. By Proposition 2.4, to define an one dimensional TS distribution, it is sufficient to choice a completely monotone function $q(r, u)$ and a finite measure $\sigma(du)$ on $S^0 = \{\pm 1\}$. If we set

$$q(r, \pm 1) = e^{-\lambda_\pm r}, \qquad \lambda > 0, \tag{2.32}$$

and the measure

$$\sigma(\{-1\}) = c_- \quad \text{and} \quad \sigma(\{1\}) = c_+, \tag{2.33}$$

by the Definition 2.1, we get (2.30). The measures Q and R are given by formulas

$$Q = c_-\delta_{-\lambda_-} + c_+\delta_{\lambda_+} \tag{2.34}$$

and

$$R = c_-\lambda_-^{\alpha}\delta_{-\frac{1}{\lambda_-}} + c_+\lambda_+^{\alpha}\delta_{\frac{1}{\lambda_+}}, \tag{2.35}$$

where δ_λ is the Dirac measure at λ. Since $E[X] < \infty$, we can write the characteristic exponent (2.13), i.e., if $\alpha \neq 1$,

$$\begin{aligned}\psi(u) =& iub + \Gamma(-\alpha)c_-((\lambda_- + iu)^{\alpha} - \lambda_-^{\alpha} - iu\lambda_-^{\alpha-1}\alpha) \\ &+ \Gamma(-\alpha)c_+((\lambda_+ - iu)^{\alpha} - \lambda_+^{\alpha} + iu\lambda_+^{\alpha-1}\alpha),\end{aligned} \tag{2.36}$$

where we are considering the representation (1.9). If $\alpha = 1$

$$\begin{aligned}\psi(u) =& iu(b + c_+ - c_-) + c_+(\lambda_+ - iu)\log(1 - \frac{iu}{\lambda_+}) \\ &+ c_-(\lambda_- + iu)\log(1 + \frac{iu}{\lambda_-}).\end{aligned}$$

In the limiting case $\alpha = 0$, we obtain the Bilateral Gamma (BΓ) distribution [73] of parameters (c_+, c_-, λ_+, λ_-, b) defined as the convolution of two Gamma random variables $\Gamma(c_+, \lambda_+)$ and $\Gamma(c_-, \lambda_-)$ plus a shift, where the characteristic exponent has the form

$$\psi(u) = iub + c_+\left(\log\left(\frac{\lambda_+}{\lambda_+ - iu}\right) - \frac{iu}{\lambda_+}\right) + c_-\left(\log\left(\frac{\lambda_-}{\lambda_- + iu}\right) + \frac{iu}{\lambda_-}\right) \tag{2.37}$$

2.2.3 CGMY distribution

Now, by taking into account the previous definition of KoBoL distribution, we recall a well known law, widely applied in finance. Let be $\lambda_+ = M$, $\lambda_- = G$, $c_+ = c_- = C$, $\alpha = Y$ and $b = m$, we obtain the characteristic exponent, if $Y \neq 1$,

$$\begin{aligned} \psi(u) = ium &+ \Gamma(-Y)C((G+iu)^Y - G^Y + (M-iu)^Y - M^Y) \\ &- \Gamma(1-Y)C(iuM^{Y-1} - iuG^{Y-1}), \end{aligned} \tag{2.38}$$

and, if $Y = 1$

$$\psi(u) = ium + C(M-iu)\log(1 - \frac{iu}{M}) + C(G+iu)\log(1 + \frac{iu}{G}) \tag{2.39}$$

therefore, we have a $CGMY$ distribution with parameters (C, G, M, Y, m), where m is the mean of the distribution, shortly $CGMY(C, G, M, Y, m)$.

Proposition 2.20. *The Variance Gamma distribution is a special case of the CGMY distribution. If $Y \to 0$, the $CGMY$ reduces to VG, i.e.*

$$CGMY(C, G, M, Y, m) = VG(C, G, M, m).$$

Proof. Let us assume without loss of generality, $m = 0$. By equation (2.38), we can write the characteristic function of a $CGMY$ distribution with parameters $(C, G, M, Y, 0)$,

$$\begin{aligned} \phi_{CGMY}(u) = \exp\Big(&C\Gamma(-Y)((G+iu)^Y - G^Y - iuYG^{Y-1}) \\ &+ C\Gamma(-Y)((M-iu)^Y - M^Y + iuYM^{Y-1})\Big) \end{aligned}$$

and, by the L'Hospital rule, we obtain the following inequalities

$$\begin{aligned} \lim_{Y\to 0} \exp\Bigg(&-C\Gamma(1-Y)\Bigg(\frac{(G+iu)^Y - G^Y - iuG^{Y-1}}{Y} \\ &+ \frac{(M-iu)^Y - M^Y + iuM^{Y-1}}{Y}\Bigg)\Bigg) \\ &= \exp\Big(-C(\log(G+iu) - \log(G) + \log(M-iu) \\ &\quad - \log(M))\Big) + \exp(C\frac{G-M}{MG}) \\ &= \left(\frac{GM}{GM + (M-G)iu + u^2}\right)^C + \exp(C\frac{M-G}{MG}), \end{aligned}$$

that is, the characteristic function of the distribution $VG(C, G, M, 0)$, see [111]. □

2.2.4 Inverse gaussian (IG) distribution

The Lévy measure of the inverse gaussian distribution $IG(a,b)$ is

$$\nu_{IG}(dx) = \frac{a}{\sqrt{2\pi}} x^{-\frac{3}{2}} e^{-\frac{b^2}{2}x} I_{\{x>0\}} dx$$

By the Definition 2.1, we set

$$\alpha = \frac{1}{2},$$

the completely monotone function $q(r,1)$ is

$$q(r,1) = e^{-\frac{b^2}{2}r},$$

and the measure σ on $\{1\}$ is

$$\sigma(\{1\}) = \frac{a}{\sqrt{2\pi}}.$$

The measures Q and R are given by formulas

$$Q = \frac{a}{\sqrt{2\pi}} \delta_{\frac{b^2}{2}}$$

and

$$R = \frac{ab}{2\sqrt{\pi}} \delta_{\frac{2}{b^2}},$$

where δ_λ is the Dirac measure at λ. We recall well known properties of the gamma function,

$$\Gamma(1+z) = z\Gamma(z),$$

and

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

Furthermore, all positive and negative moments exist, see [111] and we can consider the characteristic function (2.13). By the following equality

$$\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}.$$

we obtain

$$\begin{aligned}\phi_{IG}(u) &= \exp(\frac{ab}{2\sqrt{\pi}} \Gamma(-\frac{1}{2})((1 - iu\frac{2}{b^2}) - 1 + \frac{iu}{b^2})) \\ &= \exp(-a(\sqrt{b^2 - 2iu} - b) - iu\frac{a}{b})\end{aligned}$$

that is, the characteristic function of $IG(a,b)$ with zero mean.

2.3 KR distribution

The class of TS distribution has an infinite dimensional parametrization by a family of measures, which makes their fitting to real data a difficult task. For this reason, we propose a more flexible parametric model, we obtain explicit analytic formulas and, therefore, a parametric statistical estimation can be developed.

Consider a TS distribution on $\mathbb{R}$ whose Lévy measure ν in polar coordinate is

$$\nu(dr, du) = s^{-\alpha-1} q(r,u) dr\, \sigma(du) \tag{2.40}$$

where

$$\sigma(A) = \frac{k_+ r_+^\alpha}{\alpha + p_+} I_A(1) + \frac{k_- r_-^\alpha}{\alpha + p_-} I_A(-1), \quad A \subset S^0, \tag{2.41}$$

and

$$\begin{aligned} q(r,1) &= (\alpha + p_+) r_+^{-\alpha-p_+} \int_0^{+\infty} e^{-rt} I_{\{t > \frac{1}{r_+}\}} t^{-\alpha-p_+-1} dt \\ q(r,-1) &= (\alpha + p_-) r_-^{-\alpha-p_-} \int_0^{+\infty} e^{-rt} I_{\{t > \frac{1}{r_-}\}} t^{-\alpha-p_--1} dt, \end{aligned} \tag{2.42}$$

with $\alpha \in (0, 2)$, $k_+, k_-, r_+, r_- > 0$ and $p_+, p_- > -\alpha$.

By Definition (2.4), it is straightforward to check that the probability measures $\{Q(\cdot|s)\}_{s \in S^0}$ corresponding to the Lévy measure ν can be deduced as

$$\begin{aligned} Q(r|1) &= ((\alpha + p_+) r_+^{-\alpha-p_+} I_{\{r > \frac{1}{r_+}\}} r^{-\alpha-p_+-1} \\ Q(r|-1) &= (\alpha + p_-) r_-^{-\alpha-p_-} I_{\{r < -\frac{1}{r_-}\}} |r|^{-\alpha-p_--1}) dx. \end{aligned} \tag{2.43}$$

Then by Definition (2.5) and equation (2.41) we can write

$$Q(A) = k_+ r_+^{-p_+} \int_{1/r_+}^{\infty} I_A(r) r^{-\alpha-p_+-1} dr + k_- r_-^{-p_+} \int_{1/r_-}^{\infty} I_A(-r) |r|^{-\alpha-p_+-1} dr$$

For a future use we define also

$$\|\sigma\| := \sigma(S^0) = \frac{k_+ r_+^\alpha}{\alpha + p_+} + \frac{k_- r_-^\alpha}{\alpha + p_-}, \tag{2.44}$$

thus the measure $Q/\|\sigma\|$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, since it is a normalized finite measure. The finiteness of the measure Q can be easily verified.

Taking into account Definition (2.6), we can write

$$
\begin{aligned}
R(B) &= \int_{\mathbb{R}} I_B\left(\frac{\operatorname{sign}(x)}{x}\right)|x|^{\alpha} Q(dx) \\
&= k_+ r_+^{-p_+} \int_0^{\infty} I_B\left(\frac{\operatorname{sign}(x)}{|x|}\right)|x|^{\alpha} I_{\{x>\frac{1}{r_+}\}} x^{-\alpha-p_+-1} dx \\
&\quad + k_- r_-^{-p_-} \int_{-\infty}^{0} I_B\left(\frac{\operatorname{sign}(x)}{|x|}\right)|x|^{\alpha} I_{\{x<-\frac{1}{r_-}\}} |x|^{-\alpha-p_--1} dx \\
&= k_+ r_+^{-p_+} \int_0^{\infty} I_B\left(\frac{1}{x}\right) I_{\{x>\frac{1}{r_+}\}} x^{-p_+-1} dx \\
&\quad + k_- r_-^{-p_-} \int_{-\infty}^{0} I_B\left(\frac{1}{x}\right) I_{\{x<-\frac{1}{r_-}\}} |x|^{-p_--1} dx \\
&= k_+ r_+^{-p_+} \int_0^{\infty} I_{\{0<y<r_+\}} y^{p_+-1} dy \\
&\quad + k_- r_-^{-p_-} \int_{-\infty}^{0} I_{\{-r_-<y<0\}} |y|^{p_--1} dy
\end{aligned}
$$

and the Rosińsky measure has the following form

$$
R(dx) = (k_+ r_+^{-p_+} I_{(0,r_+)}(x)|x|^{p_+-1} + k_- r_-^{-p_-} I_{(-r_-,0)}(x)|x|^{p_--1})\, dx. \tag{2.45}
$$

Definition 2.21. *Let $\alpha \in (0,2)$, , $k_+, k_-, r_+, r_- > 0$, $p_+, p_- \in (-\alpha, \infty) \setminus \{-1, 0\}$, and $m \in \mathbb{R}$. A TS distribution is said to be the KR tempered stable distribution (or KR distribution) with parameters (α, k_+, k_-, r_+, r_-, p_+, p_-, m) if is infinitely divisible without gaussian part and has Lévy measure ν that can be written in polar coordinates as*

$$
\nu(dr, du) = r^{-\alpha-1} q(r, u) dr \sigma(du)
$$

where the function q is given in (2.42) and the measure σ is given in (2.41) or equivalently if has Rosińsky measure of the following form

$$
R(dx) = (k_+ r_+^{-p_+} I_{(0,r_+)}(x)|x|^{p_+-1} + k_- r_-^{-p_-} I_{(-r_-,0)}(x)|x|^{p_--1})\, dx.
$$

If a random variable X follows the KR distribution then we denote

$$
X \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m).
$$

Now, we analyze some properties of KR distributions.

Proposition 2.22. *Let X be a TS distribution defined by the Rosińsky measure in (2.45), then X is a proper TS distribution with finite mean and with Lévy measure*

$$
\begin{aligned}
\nu(A) = k_+ r_+^{-p_+} &\int_0^{r_+} \int_0^{\infty} I_A(tx) t^{-\alpha-1} e^{-t} x^{p_+-1} dt\, dx \\
+ k_- r_-^{-p_-} &\int_0^{r_-} \int_0^{\infty} I_A(-tx) t^{-\alpha-1} e^{-t} x^{p_--1} dt\, dx,
\end{aligned} \tag{2.46}
$$

for each Borel set $A \in \mathcal{B}(\mathbb{R}_0)$.

Proof. By Theorem 2.4, the condition 2.10 is satisfies, that is,

$$\int_{\mathbb{R}} |x|^{\alpha} R(dx) < \infty,$$

therefore, X is a proper TS distribution. By Proposition 2.7, if $\alpha \in (1,2)$, then $E[X] < \infty$. Furthermore, if $\alpha \in (0,1)$ we have

$$\int_{\|x\|>1} \|x\|^{\gamma} R(dx) < \infty$$

where $\gamma > \alpha$ and, by Proposition 2.7, it follows that $E[X] < \infty$ for each $\alpha \in (0,2)$. By Theorem 2.4, ν can be written in the form (2.46). □

In next Proposition, we show that, unlike stable distributions, KR ones have all moments finite, including exponential moments of some order. This result follows by properties of the Rosińsky measure R. To find properties of KR distributions, we prefer to analyze Rosińsky measure R instead of the Lévy measure ν. Fundamental properties of TS distributions can be expressed directly in term of the Rosińsky measure.

Proposition 2.23 (Exponential Moments)**.** *Let X be a random variable with the proper TS distribution corresponding to the Rosińsky measure R defined in (2.45). Then $E[e^{\theta X}] < \infty$ if and only if $-r_-^{-1} \leq \theta \leq r_+^{-1}$.*

Proof. To prove this property, we consider result (d) of Proposition 2.7. Let us suppose that $\theta^{-1} \geq r_+$, by (2.45), we have

$$R(\{|x| > r_+\}) = 0$$

and $\{|x| > r_+\} \supseteq \{|x| > \theta^{-1}\}$, therefore

$$R(\{|x| > \theta^{-1}\}) = 0.$$

Conversely, let us suppose that $\theta^{-1} < r_+$, then it exists an $\varepsilon > 0$, such that $\theta^{-1} = r_+ - \varepsilon$, therefore, by (2.45)

$$R(\{|x| > \theta^{-1}\}) = R(\{r_+ - \varepsilon < |x| < r_+\}) \neq 0.$$

By Proposition 2.7, we obtain that

$$\int_{\mathbb{R}} e^{\theta |x|} \mu(dx) < \infty$$

if and only if $\theta \leq r_+^{-1}$. □

The support of the Rosińsky measure R is bounded, therefore we have some exponential moment. Now, we obtain a close form for the characteristic function of KR distributions. For convenience we exclude cases $p = 0$ and $p = -1$.

Lemma 2.24. *Let $\alpha \in (0,2)$, $p \in (-\alpha,\infty) \setminus \{-1,0\}$, $h>0$, and $u \in \mathbb{R}$. Then we have , if $\alpha \neq 1$,*

$$\int_0^h x^{p-1}(1-iux)^\alpha dx = \frac{h^p}{p} {}_2F_1(p,-\alpha;1+p;iuh) \tag{2.47}$$

and , if $\alpha = 1$,

$$\begin{aligned}&\int_0^h ((1-iux)\log(1-iux)+iux)x^{p-1}dx \\ &= h^p\Bigg(\frac{ihu}{1+p}+\frac{hu}{2+3p+p^2}\Big(hu\,{}_2F_1(2+p,1;3+p;ihu) \\ &\quad - i(2+p)\log(1-ihu)\Big)+\frac{(ihu)^{-p}}{p}\Big((p-ihu)\,{}_3F_2(1,1,1-p;2,2;1-ihu) \\ &\quad -(1-(ihu)^p)\log(1-ihu)\Big)\Bigg),\end{aligned} \tag{2.48}$$

where the hypergeometric function ${}_2F_1(a,b;c;x)$ and the generalized hypergeometric function $F_{p,q}(a_1,\cdots,a_p;b_1,\cdots,b_q;x)$.

Proof. We refer to [1] for definitions of hypergeometric function and generalized hypergeometric function. □

Using results of Lemma 2.24, we can prove the following result.

Theorem 2.25. *Let X be a random variable with the proper TS distribution corresponding to the spectral measure R defined in (2.45) with conditions $p \neq 0$ and $p \neq -1$, and let $m = E[X]$. Then the characteristic function is given as follows:*

(a) if $\alpha \neq 1$,

$$\begin{aligned}E[e^{iuX}] = \exp\Bigg[&H_\alpha(u;k_+,r_+,p_+)+H_\alpha(-u;k_-,r_-,p_-) \\ &+iu\left(m+\alpha\Gamma(-\alpha)\left(\frac{k_+r_+}{p_++1}-\frac{k_-r_-}{p_-+1}\right)\right)\Bigg],\end{aligned} \tag{2.49}$$

where

$$H_\alpha(u;a,h,p) = \frac{a\Gamma(-\alpha)}{p}\left({}_2F_1(p,-\alpha;1+p;ihu)-1\right),$$

(b) if $\alpha = 1$,

$$\begin{aligned}E[e^{iuX}] = \exp\Bigg[&G_\alpha(u;k_+,r_+,p_+)+G_\alpha(-u;k_-,r_-,p_-) \\ &+iu\left(m+\left(\frac{k_+r_+}{p_++1}-\frac{k_-r_-}{p_-+1}\right)\right)\Bigg],\end{aligned} \tag{2.50}$$

where

$$\begin{aligned}G_\alpha(u;a,h,p) = &\frac{ahu}{2+3p+p^2}\Big(hu_2F_1(2+p,1;3+p;ihu)\\&-i(2+p)\log(1-ihu)\Big)\\&+\frac{a(ihu)^{-p}}{p}\Big((p-ihu)_3F_2(1,1,1-p;2,2;1-ihu)\\&-(1-(ihu)^p)\log(1-ihu)\Big).\end{aligned}$$

Proof. By Proposition 2.22, $E[X]<\infty$. By Definition 2.13, we have

$$\log E[e^{iuX}] = \begin{cases}\int_{\mathbb{R}}\Gamma(-\alpha)((1-iux)^\alpha-1+i\alpha ux)R(dx)+imu & \text{if} \quad \alpha\neq 1\\ \int_{\mathbb{R}}((1-iux)\log(1-iux)+iux)R(dx)+imu & \text{if} \quad \alpha=1\end{cases}$$

In case $\alpha\neq 1$, we have

$$\begin{aligned}&\int_{\mathbb{R}}\Gamma(-\alpha)((1-iux)^\alpha-1+i\alpha ux)R(dx)+imu\\&=k_+r_+^{-p_+}\Gamma(-\alpha)\int_0^{r_+}((1-iux)^\alpha-1-i\alpha ux)x^{p_+-1}dx\\&\quad+k_-r_-^{-p_-}\Gamma(-\alpha)\int_0^{r_-}((1+iux)^\alpha-1+i\alpha ux)x^{p_--1}dx+imu.\end{aligned}$$

By (2.47), (2.49) is obtained. Similarly, In case $\alpha=1$, we have

$$\begin{aligned}&\int_{\mathbb{R}}((1-iux)\log(1-iux)+iux)R(dx)+imu\\&=k_+r_+^{-p_+}\int_0^{r_+}((1-iux)\log(1-iux)+iux)x^{p_+-1}dx\\&\quad+k_-r_-^{-p_-}\int_0^{r_-}((1+iux)\log(1+iux)-iux)x^{p_--1}dx+imu,\end{aligned}$$

and by (2.48), (2.50) is obtained. □

Now, we consider Proposition 2.16 in order to find characteristics of the KR distribution.

Proposition 2.26. *Let* $X\sim KR(\alpha,k_+,k_-,r_+,r_-,p_+,p_-,m)$ *with* $\alpha\neq 1$. *Then the cumulants* $c_k(X)$ *are given by* $c_1(X)=m$ *and*

$$c_k(X)=\Gamma(k-\alpha)\left(\frac{k_+r_+^k}{p_++k}+(-1)^k\frac{k_-r_-^k}{p_-+k}\right) \tag{2.51}$$

where $k\geq 2$.

Proof. By the form of the characteristic function (2.13), we obtain the mean m. By Proposition 2.16 and Definition 2.21, we have to calculate the following integral

$$c_k(X) = \Gamma(m-\alpha)\int_{\mathbb{R}_0} x^k R(dx)$$

in order to find (2.51). □

Remark 2.27. *Let $X \sim KR(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$ with $\alpha \neq 1$. By Proposition 2.26, we obtain the mean, variance, skewness and excess kurtosis of X which are given as follows:*

(a) $E[X] = c_1(X) = m$,

(b) $\mathrm{Var}(X) = c_2(X) = \Gamma(2-\alpha)\left(\frac{k_+r_+^2}{p_++2} + \frac{k_-r_-^2}{p_-+2}\right)$,

(c) $s(X) = \frac{c_3(X)}{c_2(X)^{3/2}} = \frac{\Gamma(3-\alpha)\left(\frac{k_+r_+^3}{p_++3} - \frac{k_-r_-^3}{p_-+3}\right)}{\Gamma(2-\alpha)^{3/2}\left(\frac{k_+r_+^2}{p_++2} + \frac{k_-r_-^2}{p_-+2}\right)^{3/2}}$,

(d) $k(X) = \frac{c_4(X)}{c_2(X)^2} = \frac{\Gamma(4-\alpha)\left(\frac{k_+r_+^4}{p_++4} + \frac{k_-r_-^4}{p_-+4}\right)}{\Gamma(2-\alpha)^2\left(\frac{k_+r_+^2}{p_++2} + \frac{k_-r_-^2}{p_-+2}\right)^2}$.

The CGMY distribution is a particular case of the KR distribution.

Proposition 2.28. *The KR distribution with parameters (α, k_+, k_-, r_+, r_-, p_+, p_-, m) converges weakly to the CGMY distribution as $p_\pm \to \infty$ provided that $\alpha \neq 1$ and $k_\pm = c(\alpha + p_\pm)r_\pm^{-\alpha}$ for $c > 0$.*

Proof. By the Lévy theorem, see Theorem 15.4 in [76], it suffices to prove the pointwise convergence of the characteristic function. We have

$$\begin{aligned}
&\lim_{p_+\to\infty} \frac{k_+\Gamma(-\alpha)}{p_+}\left({}_2F_1(p_+, -\alpha; 1+p_+; ir_+u) - 1\right)\\
&= c\Gamma(-\alpha)r_+^{-\alpha}\lim_{p_+\to\infty}\frac{\alpha+p_+}{p_+}\sum_{n=1}^{\infty}\frac{(p_+)_n(-\alpha)_n}{(1+p_+)_n}\frac{(iur_+)^n}{n!}\\
&= c\Gamma(-\alpha)r_+^{-\alpha}\lim_{p_+\to\infty}\sum_{n=1}^{\infty}\frac{(\alpha+p_+)(-\alpha)_n}{p_++n}\frac{(iur_+)^n}{n!}\\
&= c\Gamma(-\alpha)r_+^{-\alpha}\sum_{n=1}^{\infty}(-\alpha)_n\frac{(iur_+)^n}{n!}\\
&= c\Gamma(-\alpha)r_+^{-\alpha}\sum_{n=1}^{\infty}\binom{\alpha}{n}(-iur_+)^n\\
&= c\Gamma(-\alpha)r_+^{-\alpha}\left((1-iur_+)^\alpha - 1\right)\\
&= c\Gamma(-\alpha)\left((r_+^{-1}-iu)^\alpha - r_+^{-\alpha}\right).
\end{aligned}$$

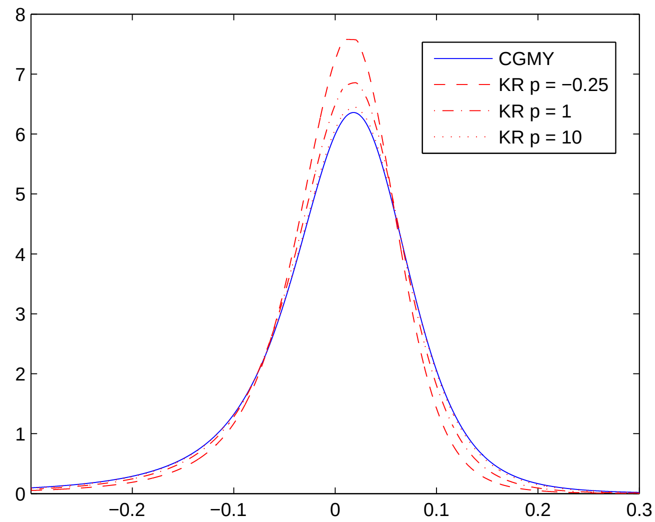

Figure 2.1: Probability density of the CGMY distribution with parameters $C = 0.01$, $G = 2$, $M = 10$, $Y = 1.25$, and the KR distributions with $\alpha = Y$, $k_\pm = C(Y+p)r_\pm^{-\alpha}$, $r_+ = 1/M$, $r_- = 1/G$, where $p = p_+ = p_- \in \{ -0.25, 1, 10 \}$.

Similarly, we have

$$\lim_{p_-\to\infty} \frac{k_-\Gamma(-\alpha)}{p_-} \left({}_2F_1(p_-, -\alpha; 1+p_-; -ir_-u) - 1\right) = c\Gamma(-\alpha)\left((r_-^{-1}+iu)^\alpha - r_-^{-\alpha}\right).$$

Moreover, we have

$$\begin{aligned}
\mu &\equiv m + \lim_{p_+\to\infty} \alpha\Gamma(-\alpha)\frac{k_+r_+}{p_++1} - \lim_{p_-\to\infty} \alpha\Gamma(-\alpha)\frac{k_-r_-}{p_-+1} \\
&= m + \lim_{p_+\to\infty} \alpha\Gamma(-\alpha)\frac{c(\alpha+p_+)r_+^{1-\alpha}}{p_++1} - \lim_{p_-\to\infty} \alpha\Gamma(-\alpha)\frac{c(\alpha+p_-)r_-^{1-\alpha}}{p_-+1} \\
&= m + c\alpha\Gamma(-\alpha)(r_+^{1-\alpha} - r_-^{1-\alpha}).
\end{aligned}$$

In all, we have

$$\begin{aligned}
&\lim_{p_+,p_-\to\infty} E[e^{iuX}] \\
&= \exp\left(i\mu u + c\Gamma(-\alpha)\left(\left((r_+^{-1}-iu)^\alpha - r_+^{-\alpha}\right) + \left((r_-^{-1}+iu)^\alpha - r_-^{-\alpha}\right)\right)\right).
\end{aligned}$$

where $X \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$. That completes the proof. □

Figure 2.1 shows that the KR distributions converge to the CGMY distribution when parameter $p = p_+ = p_-$ increases.

Tail Behavior

In this section, we will discuss the probability tails of the KR distribution. Although the exact asymptotic behavior of its tails is difficult to obtain unlike those of the stable distribution, it is possible to calculate the upper and lower bounds.

In the following, we provide an upper bound for the probability tails by mean of the well-known Chebyshev's Inequality.

Proposition 2.29. *Let be X a random variable with KR TS distribution, $X \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$ with $\alpha \neq 1$. Then the following inequality is fulfilled*

$$\mathrm{P}(|X - m| \geq \lambda) \leq \frac{C}{\lambda^2}$$

where C does not depend on λ.

Proof. By Remark 2.27, X has mean and variance, therefore we consider the Chebyshev's Inequality

$$\mathrm{P}(|X - m| \geq \lambda) \leq \frac{1}{\lambda^2}\mathrm{Var}(X).$$

We obtain

$$\mathrm{P}(|X - m| \geq \lambda) \leq \frac{1}{\lambda^2}\Gamma(2-\alpha)\left(\frac{k_+ r_+^2}{p_+ + 2} + \frac{k_- r_-^2}{p_- + 2}\right)$$

and the result is proved. □

A natural further interest is in a lower bound of the probability tails. By following the approach of [64], below we will give a lower bound. We consider the following result:

Proposition 2.30. *Let X be an infinitely divisible random variable in $\mathbb{R}$, with Lévy triplet $(b, 0, M(dx))$. Then we have*

$$\mathrm{P}(|X - m| \geq \lambda) \geq \frac{1}{4}(1 - \exp(-M(u \in \mathbb{R} : |u| \geq 2\lambda))), \quad \lambda > 0. \tag{2.52}$$

for all $m \in \mathbb{R}$.

Proof. See Lemma 5.4 of [20]. □

For further analysis, we need an auxiliary result.

Lemma 2.31. *For $a \in \mathbb{R}_+$, the following equality holds*

$$\int_\beta^\infty s^{-a-1}e^{-s}ds = \beta^{-a-1}e^{-\beta} + o(\beta^{-a-1}e^{-\lambda})$$

as $\beta \to \infty$.

Proof. By integration by parts, if $\beta > 0$, we obtain

$$\int_\beta^\infty s^{-a-1}e^{-s}ds = \beta^{-a-1}e^{-\beta} - (a+1)\int_\beta^\infty s^{-a-2}e^{-s}ds \leq \beta^{-a-1}e^{-\beta}$$

and

$$\begin{aligned}\int_\beta^\infty s^{-a-1}e^{-s}ds &= \beta^{-a-1}e^{-\beta} - (a+1)\beta^{-a-2}e^{-\beta} + (a+1)(a+2)\int_\beta^\infty s^{-a-3}e^{-s}ds \\ &\geq \beta^{-a-1}e^{-\beta} - (a+1)\beta^{-a-2}e^{-\beta},\end{aligned}$$

when $\beta \to \infty$, the result is proved. □

Taking into account Proposition 2.30 and Lemma 2.31, we can prove the following result.

Proposition 2.32. *Let be X a random variable with KR distribution,*

$$X \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$$

with $\alpha \neq 1$. Then the following inequality is fulfilled

$$\mathrm{P}(|X-m| \geq \lambda) \geq C\frac{e^{-\frac{2\lambda}{\bar{r}}}}{\lambda^{\alpha+2}}$$

as $\lambda \to \infty$, where C does not depend on λ and $\bar{r} = \max(r_+, r_-)$.

Proof. Applying the following elementary fact

$$1 - \exp(-z) \sim z, \quad z \to 0$$

and according to (2.52) and Lemma 2.31, we obtain

$$\mathrm{P}(|X-m| \geq \lambda) \geq \frac{1}{4}\left(1 - \exp\left[-\int_{\mathbb{R}_0}\int_{\frac{2\lambda}{|x|}}^{\infty} s^{-\alpha-1}e^{-s}dsR(dx)\right]\right) \tag{2.53}$$

$$\sim \frac{\lambda^{-\alpha-1}}{2^{\alpha+3}}\int_{\mathbb{R}_0}|x|^{\alpha+1}e^{-\frac{2\lambda}{|x|}}R(dx), \tag{2.54}$$

as $\lambda \to \infty$. By using equality (2.45) and Lemma 2.31, the integral can be written as

$$\begin{aligned}
\int_{\mathbb{R}_0}|x|^{\alpha+1}e^{-\frac{2\lambda}{|x|}}R(dx) &= k_+r_+^{-p_+}\int_0^{r_+}x^{\alpha+p_+}e^{-\frac{2\lambda}{x}}dx + k_-r_-^{-p_-}\int_0^{r_-}x^{\alpha+p_-}e^{-\frac{2\lambda}{x}}dx \\
&= (2\lambda)^{\alpha+p_++1}k_+r_+^{-p_+}\int_{\frac{2\lambda}{r_+}}^{\infty}t^{-\alpha-p_+-2}e^{-t}dt \\
&\quad + (2\lambda)^{\alpha+p_-+1}k_-r_-^{-p_-}\int_{\frac{2\lambda}{r_-}}^{\infty}t^{-\alpha-p_--2}e^{-t}dt \\
&\sim (2\lambda)^{-1}\frac{k_+}{r_+^{\alpha+2p_++2}}e^{-\frac{2\lambda}{r_+}} + (2\lambda)^{-1}\frac{k_-}{r_-^{\alpha+2p_-+2}}e^{-\frac{2\lambda}{r_-}} \\
&\sim \bar{C}(2\lambda)^{-1}e^{-\frac{2\lambda}{\bar{r}}}
\end{aligned}$$

as $\lambda \to \infty$, where $\bar{r} = \max(r_+, r_-)$. Combining this with (2.53), we get

$$\mathrm{P}(|X-m| \geq \lambda) \geq C\frac{e^{-\frac{2\lambda}{\bar{r}}}}{\lambda^{\alpha+2}}.$$

□

2.4 Power tails TS distribution

In this section, we will show another possible choice for the measure R. Let us define the measure R as

$$R(x) = c_-(1-x)^{-(\alpha+\gamma_+)}I_{x<0} + c_+(1+x)^{-(\alpha+\gamma_-)}I_{x>0} \tag{2.55}$$

where $\gamma_+, \gamma_- > 2$ and $c_+, c_- \geq 0$. Let X be a TS distribution with Rosińsky measure defined as in (2.55), then by Proposition 2.7, the following moment condition holds,

$$E[X^p] < \infty \qquad 0 < p < \alpha + \min(\gamma_+, \gamma_-).$$

By the inequality above, we have not exponential moments, therefore this is not a useful distribution to develop an option pricing theory.

2.5 Some TS processes

In this section, we resume some properties of TS processes. By Definition 2.1, if μ is a TS distribution, it is infinitely divisible, therefore, there exists a Lévy process $(X_t)_{t\geq 0}$ such that μ is the distribution of X_1, therefore in general a Lévy processes can be define. Now, we briefly recall some properties.

As in Proposition (1.44), we want to find which conditions have to be satisfied in order to obtain a process of finite variation. By Definition 2.1, a TS distribution is infinitely divisible without Gaussian part, therefore, we can consider a Lévy process associated to a TS distribution. We recall the following result of [107].

Proposition 2.33. *Let ν and R be related by (2.8), where R satisfies (2.9). Then*

$$\int_{\|x\|\leq 1} \|x\|\nu(dx) < \infty \Longleftrightarrow \alpha \in (0,1) \text{ and } \int_{\|x\|\leq 1} \|x\|R(dx) < \infty.$$

If we consider a Lévy process associated with a TS distribution, in order to obtain a finite variation process, it is sufficient to analyze only the Rosińsky measure. To the purpose to analyze pathwise properties of such process, we prove the following additional result.

Proposition 2.34. *Let ν and R be related by (2.8), where R satisfies (2.9). Then*

$$\int_{\|x\|\leq 1} \nu(dx) = \infty.$$

Proof. Let ν and R two measure as in (2.8). The following equality is verified

$$\int_{\|x\|\leq 1} \nu(dx) = \int_{\|x\|\leq 1} \int_0^{\|x\|^{-1}} t^{-\alpha-1} e^{-t} dt R(dx).$$

For each choice of the measure R, this integral diverges. □

By Definition 1.40, a TS process has always infinite activity and, according to Definition 1.39, it cannot be of *type A*.

2.5.1 KoBoL process

The TS process is obtained by taking a one-dimensional stable process and multiplying the Lévy measure with a decreasing exponential on each half of the real axis. After this exponential softening, the small jumps keep their initial stable-like behavior whereas the large jumps become much less violent. A TS process is thus a Lévy process on $\mathbb{R}$ with no Gaussian component and a Lévy density of the form (2.30). Excluding negative α is not too strict, since we exclude only process of compound Poisson type, and by Definition 2.1, we have to exclude also the case $\alpha_+ \neq \alpha_-$, in order to define a TS process.

By Proposition 2.33 and the Rosinńsky measure (2.35), the process has trajectories of finite variation if and only if $\alpha < 1$. The limiting case $\alpha = 0$ corresponds to an infinite activity process. If in addition we have also the condition $c_+ = c_-$, we obtain the variance gamma process. Of practical interest will be the *CGMY* process

Definition 2.35. *A Lévy process $X = (X_t)_{t\geq 0}$ is said to be a* CGMY TS process *(or shortly, a* CGMY process*) with parameters $(C,\ G,\ M,\ Y,\ m)$ if*

$$X_1 \sim \text{CGMY}(C, G, M, Y, m).$$

2.5.2 KR process

Starting from the definition of KR distribution, we can define a Lévy process.

Definition 2.36. *A Lévy process $X = (X_t)_{t\geq 0}$ is said to be a* KR TS process *(or shortly, a* KR process*) with parameters $(\alpha,\ k_+,\ k_-,\ r_+,\ r_-,\ p_+,\ p_-,\ m)$ if $X_1 \sim \text{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$.*

In order to analyze the path properties of the KR process, we focus our attention on the Rosińsky measure.

Proposition 2.37. *The process $(X_t)_{t\geq 0} \sim \text{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$ has finite variation if $\alpha \in (0, 1)$ and infinite variation if $\alpha \in [1, 2)$.*

Proof. By Proposition 1.44, to prove that a Lévy process without gaussian part has finite variation, it is sufficient to verify that the inequality (1.11) is fulfilled. By Proposition 2.33, if $\alpha \in (0, 1)$, the process has finite variation if the integral

$$\int_{|x|<1} |x| R(dx) \tag{2.56}$$

converges. It is straightforward to verify the convergence of 2.56, by Definition (2.45) and inequalities $p_+, p_- > -1$. Furthermore, by Proposition 2.33, if $\alpha \in [1, 2)$ it follows that the process has infinite variation. Thus

$$\int_{|x|<1} |x| \nu(dx) < \infty$$

if and only if $\alpha \in (0, 1)$.

□

Chapter 3

Tempered infinitely divisible distributions and processes

The formal and elegant definition of tempered stable distributions and processes has been proposed in the work of Rosiński [107] where a completely monotone function is chosen to transform the Lévy measure of a stable distribution. Tempered stable distributions may have all moments finite and exponential moments of some order. The idea of selecting a different tempering function has been already considered in the literature, see [53]. In this chapter, by following the approach of Rosiński [107] and considering a particular family of tempering functions, a new class of distributions is introduced with the same suitable properties of the tempered stable class, but with the advantage that it may admit exponential moments of any order. By multiplying the Lévy measure of a stable distribution with a positive definite radial function, see [110], instead of with a completely monotone function as in [107], we obtain the class of tempered infinitely divisible (TID) [14] distributions. In some cases, the characteristic function of a TID random variable is extendible to an entire function on $\mathbb{C}$, that is, it admits any exponential moment.

Some practical problems in the field of mathematical finance have motivated our studies. Furthermore, we want to fill a gap in the literature. The modified tempered stable (MTS) distribution is not a TS distribution of the Rosiński type [71] even though its properties are very close to that class. We will prove that the MTS distribution is in the TID class.

Although this distributional family is constructed by tempering the Lévy measure of a stable distribution, any stability property is lost. We will proceed as following. In Section 3.1, basic definitions and distributional properties are given. Working with the Lévy measure may be a difficult task, therefore a spectral measure R is needed to figure out all characteristics of this class. This measure describes all distributional properties and allows one to obtain a close formula for the characteristic function. Since TID distributions are by construction infinitely divisible, a TID Lévy process can be considered. In Section 3.2, TID processes are analyzed. If the time scale increases, the TID process looks like a Gaussian process; conversely, if the time scale decreases, it looks like a stable process. Furthermore, under some condition on the tempering function, the change of measure problem between stable and TID processes can be solved. In Section 6.1.10, a view toward simulation is given. Taking into consideration [105, 107], a series representation is derived in terms of a measure Q, as already proved for the TS class.

Similar to the TS framework, this class of distribution has an infinite dimensional parametrization by a family of measures [115], making it difficult to use. For this reason, in Section 3.3 some parametric examples in one dimension are proposed, and characteristic functions are derived.

3.1 Tempered infinitely divisible distribution

It is well known that the Lévy measure ν_0 of an α-stable distribution on $\mathbb{R}^d$ can be written in polar coordinates in the form

$$\nu_0(dr, du) = r^{-\alpha-1} dr\, \sigma(du), \tag{3.1}$$

where $\alpha \in (0, 2)$ and σ is a finite measure on the unit sphere S^{d-1}.

Theorem 3.1. *If μ_0 is an α-stable distribution, then its characteristic function has the form*

$$\hat{\mu}_0(y) = \begin{cases} \exp\left\{-c_\alpha \int_{S^{d-1}} |\langle y,u\rangle|^\alpha (1 - i \tan\frac{\pi\alpha}{2} sgn\langle y,u\rangle)\sigma(du) + i\langle y,a\rangle\right\}, & \alpha \neq 1, \\ \exp\left\{-c_1 \int_{S^{d-1}} (|\langle y,u\rangle| + i\frac{2}{\pi}\langle y,u\rangle \log|\langle y,u\rangle|)\sigma(du) + i\langle y,a\rangle\right\}, & \alpha = 1, \end{cases} \tag{3.2}$$

where $a \in \mathbb{R}^d$ and

$$c_\alpha = \begin{cases} |\Gamma(-\alpha)\cos(\frac{\pi\alpha}{2})|, & \alpha \neq 1, \\ \frac{\pi}{2}, & \alpha = 1. \end{cases}$$

Proof. See [109, Theorem 14.10]. □

Definition 3.2. *If Y is an α-stable random vector with characteristic function (3.2), we will write $Y \sim S_\alpha(\sigma, a)$.*

Taking into account the approach of [107], we want to modify the radial component of ν_0 and obtain a probability distribution with lighter tails than stable ones. A TID distribution is defined by tempering the radial term of ν_0 as follows.

Definition 3.3. *Let μ be a infinitely divisible probability measure on $\mathbb{R}^d$ without gaussian part. We call μ tempered infinitely divisible (TID) if its Lévy measure ν can be written in polar coordinates as*

$$\nu(dr, du) = r^{-\alpha-1} q(r,u) dr\, \sigma(du) \tag{3.3}$$

where α is a real number $\alpha \in [0, 2)$, σ a finite measure on the unit sphere S^{d-1} and $q : (0,\infty) \times S^{d-1} \mapsto (0,\infty)$ is a Borel function defined by

$$q(r,u) := \int_0^\infty e^{-r^2 s^2/2} Q(ds|u), \tag{3.4}$$

with $\{Q(\cdot|u)\}_{u\in S^{d-1}}$ a measurable family of Borel measure on $(0,\infty)$. If $q(0+,u) = 1$ for each $u \in S^{d-1}$, μ is referred to as a proper TID. The function q is called a tempering function.

In the case where $\{Q(\cdot|u)\}_{u\in S^{d-1}}$ are finite non-negative Borel measures on $(0,\infty)$, $q(\cdot,u)$ are positive definite radial functions on $\mathbb{R}^d$. By [110], the following results holds.

Theorem 3.4. *A continuous function* $\varphi:(0,\infty)\to\mathbb{R}$ *is positive definite and radial on* $\mathbb{R}^d$ *for all* d *if and only if it is of the form*

$$\varphi(r)=\int_0^\infty e^{-r^2s^2}\mu(ds),$$

where μ *is a finite non-negative Borel measure on* $(0,\infty)$*.*

Define a measure Q on $\mathbb{R}^d$ by

$$Q(A):=\int_{S^{d-1}}\int_0^\infty I_A(ru)Q(dr|u)\sigma(du),\qquad A\in\mathcal{B}(\mathbb{R}^d). \tag{3.5}$$

It is easy to check that $Q(\{0\})=0$. We also define a measure R on $\mathbb{R}^d$ by

$$R(A):=\int_{\mathbb{R}^d}I_A(\frac{x}{\|x\|^2})\|x\|^\alpha Q(dx),\qquad A\in\mathcal{B}(\mathbb{R}^d). \tag{3.6}$$

The measure R is equivalent to the measure Q and clearly $R(\{0\})=0$. By definition of R, for each Borel function F, the following equality is satisfied

$$\int_{\mathbb{R}^d}F(x)R(dx)=\int_{\mathbb{R}^d}F(\frac{x}{\|x\|^2})\|x\|^\alpha Q(dx), \tag{3.7}$$

in the sense that when one sides exists then the other exists and are equal. By choosing

$$F(x)=I_A(\frac{x}{\|x\|^2})\|x\|^\alpha,$$

then Q can be written as

$$Q(A):=\int_{\mathbb{R}^d}I_A(\frac{x}{\|x\|^2})\|x\|^\alpha R(dx),\qquad A\in\mathcal{B}(\mathbb{R}^d). \tag{3.8}$$

Sometimes the only knowledge of the Lévy measure cannot be enough to obtain analytical properties of tempered infinitely divisible distributions. Therefore, the definitions of measures Q and R allow one to overcome this problem and to obtain explicit analytic formulas and more explicit calculations. The following result allows one to figure out relations between the Lévy measure ν and the measure R above defined.

Proposition 3.5. *Let* μ *be a TID distribution, the corresponding Lévy measure* ν *be defined as in (3.3) and* R *as in (3.6). Then* ν *can be written in the form*

$$\nu(A)=\int_{\mathbb{R}^d}\int_0^\infty I_A(tx)t^{-\alpha-1}e^{-t^2/2}dtR(dx),\qquad A\in\mathcal{B}(\mathbb{R}^d), \tag{3.9}$$

if and only if the measure R *on* $\mathbb{R}^d$ *satisfies the following conditions,* $R(\{0\})=0$ *and*

$$\begin{cases}\int_{\mathbb{R}^d}(\|x\|^2\wedge\|x\|^\alpha)R(dx)<\infty, & 0<\alpha<2,\\ \int_{\mathbb{R}^d}(\log(1+\|x\|)+1)R(dx)<\infty, & \alpha=0.\end{cases} \tag{3.10}$$

Proof. Let μ be a TID distribution with Lévy measure ν. First we will prove that there exists at least one measure R, defined in (3.6), such that ν, defined in (3.3), can be written in the form (3.9). To show this result, we take the measure R as in (3.6) and for each $A \in \mathcal{B}(\mathbb{R}^d)$, by considering (3.4), (3.5), (3.6), (3.7) and Fubini theorem, we have

$$\begin{aligned}
\nu(A) &= \int_{S^{d-1}} \int_0^\infty I_A(ru) r^{-\alpha-1} q(r,u) dr \sigma(du) \\
&= \int_{S^{d-1}} \int_0^\infty \int_0^\infty I_A(ru) r^{-\alpha-1} e^{-r^2 s^2/2} Q(ds|u) dr \sigma(du) \\
&= \int_{S^{d-1}} \int_0^\infty \left(\int_0^\infty I_A(ru) r^{-\alpha-1} e^{-r^2 s^2/2} dr \right) Q(ds|u)\sigma(du) \\
&= \int_{S^{d-1}} \int_0^\infty \left(\int_0^\infty I_A(\frac{t}{s}u) t^{-\alpha-1} e^{-t^2/2} dt \right) s^\alpha Q(ds|u)\sigma(du) \\
&= \int_0^\infty \left(\int_{S^{d-1}} \int_0^\infty I_A(t\frac{u}{s}) s^\alpha Q(ds|u)\sigma(du) \right) t^{-\alpha-1} e^{-t^2/2} dt \\
&= \int_0^\infty \left(\int_{\mathbb{R}^d} I_A(t\frac{x}{\|x\|^2}) \|x\|^\alpha Q(dx) \right) t^{-\alpha-1} e^{-t^2/2} dt \\
&= \int_0^\infty \left(\int_{\mathbb{R}^d} I_A(ty) R(dy) \right) t^{-\alpha-1} e^{-t^2/2} dt \\
&= \int_{\mathbb{R}^d} \int_0^\infty I_A(ty) t^{-\alpha-1} e^{-t^2/2} dt R(dy)
\end{aligned} \tag{3.11}$$

Conversely, given a measure R, let Q be the measure defined by (3.8) and let us consider the decomposition $Q(dr, du) = Q(dr|u)\sigma(du)$, where σ is a finite measure on S^{d-1}. Thus, we can define $q(r,u)$ by (3.4) and the computation (3.11) proves that ν can be written in the form (3.3).

Now we want to prove that ν is a Lévy measure if and only if (3.10) holds. Suppose ν a Lévy measure, then we have

$$\int_{\|x\|\leq 1} \|x\|^2 \nu(dx) < \infty$$

and by considering (3.9) and $\alpha \neq 0$, we obtain

$$\begin{aligned}
\infty > \int_{\|x\|\leq 1} \|x\|^2 \nu(dx) &= \int_{\mathbb{R}^d} \|x\|^2 \int_0^{1/\|x\|} t^{1-\alpha} e^{-t^2/2} dt R(dx) \\
&\geq \int_{\|x\|\leq 1} \|x\|^2 \int_0^1 t^{1-\alpha} e^{-t^2/2} dt R(dx) + \int_{\|x\|>1} \|x\|^2 \int_0^{1/\|x\|} t^{1-\alpha} e^{-t^2/2} dt R(dx) \\
&\geq e^{-1/2}(2-\alpha)^{-1} \int_{\|x\|\leq 1} \|x\|^2 R(dx) + e^{-1/2}(2-\alpha)^{-1} \int_{\|x\|>1} \|x\|^\alpha R(dx)
\end{aligned}$$

thus, we obtain the desired inequality

$$\int_{\mathbb{R}^d} (\|x\|^2 \wedge \|x\|^\alpha) R(dx) < \infty.$$

Now, let us consider the case $\alpha = 0$. By definition of the Lévy measure we have

$$\int_{\|x\|\geq 1} \nu(dx) < \infty$$

and by considering (3.9) with $\alpha = 0$, the following inequalities are satisfied

$$\begin{aligned}
\infty > \int_{\|x\|>1} \nu(dx) &= \int_{\mathbb{R}^d} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1} e^{-t^2/2} dt R(dx) \\
&= \int_{\|x\|\leq 1} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1} e^{-t^2/2} dt R(dx) + \int_{\|x\|>1} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1} e^{-t^2/2} dt R(dx) \\
&\geq \int_{\|x\|\leq 1} K R(dx) + e^{-1/2} \int_{\|x\|>1} \log(\|x\|) R(dx)
\end{aligned}$$

where K is a finite constant. Then, also when $\alpha = 0$, condition (3.10) is a necessary condition. Conversely, now we prove that (3.10) is also sufficient. Suppose that there is a measure R satisfying (3.10). Then the measure ν can be written in the form (3.9). If $\alpha \neq 0$, we can write

$$\begin{aligned}
&\int_{\|x\|\leq 1} \|x\|^2 \nu(dx) = \\
&\quad = \int_{\mathbb{R}^d} \|x\|^2 \int_0^{\frac{1}{\|x\|}} t^{1-\alpha} e^{-t^2/2} dt R(dx) \\
&\quad \leq \int_{\|x\|\leq 1} \|x\|^2 \int_0^{\infty} t^{1-\alpha} e^{-t^2/2} dt R(dx) + \frac{1}{2-\alpha} \int_{\|x\|>1} \|x\|^\alpha R(dx) \\
&\quad = 2^{-\frac{\alpha}{2}} \Gamma(1 - \frac{\alpha}{2}) \int_{\|x\|\leq 1} \|x\|^2 R(dx) + \frac{1}{2-\alpha} \int_{\|x\|>1} \|x\|^\alpha R(dx) < \infty
\end{aligned}$$

and

$$\begin{aligned}
\int_{\|x\|>1} \nu(dx) &= \int_{\mathbb{R}^d} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1-\alpha} e^{-t^2/2} dt R(dx) \\
&\leq C \int_{\|x\|\leq 1} \int_{\frac{1}{\|x\|}}^{\infty} t^{-3} dt R(dx) + \int_{\|x\|>1} \int_{\frac{1}{\|x\|}}^{\infty} t^{-\alpha-1} dt R(dx) \\
&= \frac{C}{2} \int_{\|x\|\leq 1} \|x\|^2 R(dx) + \frac{1}{\alpha} \int_{\|x\|>1} \|x\|^\alpha R(dx),
\end{aligned}$$

where $C := \sup_{t\geq 1} t^{2-\alpha} e^{-t^2/2}$. Thus ν is a Lévy measure.

Considering the case where $\alpha = 0$, we obtain

$$\begin{aligned}
\int_{\|x\|\leq 1} \|x\|^2 \nu(dx) &= \int_{\mathbb{R}^d} \|x\|^2 \int_0^{\frac{1}{\|x\|}} t e^{-t^2/2} dt R(dx) \\
&\leq \int_{\mathbb{R}^d} \|x\|^2 (1 - e^{-\frac{1}{2\|x\|^2}}) R(dx) \\
&\leq \int_{\|x\|\leq 1} \|x\|^2 R(dx) + \int_{\|x\|>1} R(dx)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\|x\|>1} \nu(dx) = \\
&= \int_{\mathbb{R}^d} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1} e^{-t^2/2} dt R(dx) \\
&\leq \int_{\|x\|\leq 1} \int_{\frac{1}{\|x\|}}^{\infty} \frac{1}{t(1+t^2/2)} dt R(dx) + \int_{\|x\|>1} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1} e^{-t^2/2} dt R(dx) \\
&\leq \int_{\|x\|\leq 1} \|x\|^2 R(dx) + \int_{\|x\|>1} (\log(\|x\|) + e^{-1/2}) R(dx).
\end{aligned}$$

Thus, ν is a Lévy measure.

Now, in order to show that (3.9) is well defined, we want to show that R is uniquely determined. We will prove it by contradiction. Let R_1 and R_2 be two measures on $\mathbb{R}^d$ satisfying (3.9). Then, by previous argument, (3.10) has to be satisified also. By contradiction, we suppose that there exists a Borel set A such that $R_1(A) \neq R_2(A)$. By equation (3.8), we can define Q_1 and Q_2 from R_1 and R_2 and consider the polar representation

$$Q_i(dr, du) = Q_i(dr|u)\sigma(du)$$

where σ is a probability measure on S^{d-1} and $\{Q_i(\cdot|u)\}_{u\in S^{d-1}}$ are measurable families of Borel measure on $(0, \infty)$. Without any loss of generality, we assume that σ is not the null measure on S^{d-1}. If $\alpha \neq 0$, by definition (3.8) and conditions (3.10), the inequality

$$\begin{aligned}
\infty > \int_{\mathbb{R}^d} (\|x\|^2 \wedge \|x\|^\alpha) R_i(dx) = \int_{\mathbb{R}^d} (\|x\|^{-2} \wedge \|x\|^{-\alpha}) \|x\|^\alpha Q_i(dx) \\
= \int_{S^{d-1}} \int_0^\infty (s^{\alpha-2} \wedge 1) Q_i(ds|u).
\end{aligned}$$

holds. Therefore, the tempering function

$$q_i(r, u) = \int_0^\infty e^{-r^2 s^2/2} Q_i(ds|u)$$

is well defined. Since $R_1(A) \neq R_2(A)$ also $Q_1(A) \neq Q_2(A)$. By assumption, R_i verifies (3.10). Then, by using the same calculus done to obtain (3.11), we can find $\nu(A)$ and write

$$\int_{S^{d-1}} \int_0^\infty I_A(ru) r^{-\alpha-1} (q_1(r, u) - q_2(r, u)) dr \sigma(du) = 0$$

and we find the contradiction. A similar argument shows the uniqueness of the measure R also in the case $\alpha = 0$. □

Remark 3.6. *The case $\alpha = 0$ is consider only for completeness and the theory will be not completely extended to this limiting case. It may be an interesting case in some applications.*

Remark 3.7. *If alpha* $\in (0,2)$ *and* R *satisfies the following additional inequality*

$$\int_{\mathbb{R}^d} \|x\|^\alpha R(dx) < \infty, \quad 0 < \alpha < 2, \tag{3.12}$$

we will call μ *a proper TID distribution. The measure* Q *has the form*

$$Q(\mathbb{R}^d) = \int_{\mathbb{R}^d} \|x\|^\alpha R(dx), \quad 0 < \alpha < 2, \tag{3.13}$$

In this case Q *is a finite measure and it can be represented in polar coordinates as* $Q(dr, du) = Q(dr|u)\sigma(du)$*, where* $Q(\cdot|u)$ *are finite measures and* σ *is a finite measure on* S^{d-1}.

Definition 3.8. *The unique measure* R *in (3.9) is called a spectral measure of the corresponding TID distribution. We will call* R *the Rosiński measure [115].*

We focus on the following result.

Remark 3.9. *If* μ *is a proper TID distribution, then* Q *is a finite measure and* $\{Q(\cdot|u)\}_{u\in S^{d-1}}$ *is a measurable family of finite Borel measures on* $(0,\infty)$*. Since the equation* $q(0+,u) = 1$ *holds, they are probability measures. Furthermore, for any fixed* $u \in S^{d-1}$*, function* $q(\cdot,u)$ *are positive definite radial functions.*

Taking into consideration Lemma 2.14 of [107], we want to figure out the relation between parameters of the proper TID distribution and stable ones.

Proposition 3.10. *Let* ν *be a Lévy measure of a proper TID distribution, as in (3.3), with corresponding spectral measure* R*. Then, the Lévy measure* ν_0 *of an* α*-stable distribution, given in (3.1), can be written in the following form*

$$\nu_0(A) = \int_{\mathbb{R}^d}\int_0^\infty I_A(tx)t^{-\alpha-1}dtR(dx), \qquad A \in \mathcal{B}(\mathbb{R}^d) \tag{3.14}$$

and additionally

$$\sigma(B) = \int_{\mathbb{R}^d} I_B\left(\frac{x}{\|x\|}\right)\|x\|^\alpha R(dx), \qquad B \in \mathcal{B}(S^{d-1}). \tag{3.15}$$

Proof. By definitions (3.6) and (3.5) we obtain for $A \in \mathcal{B}(\mathbb{R}^d)$

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_0^\infty I_A(tx)t^{-\alpha-1}dtR(dx) &= \int_{\mathbb{R}^d}\int_0^\infty I_A(t\frac{x}{\|x\|^2})t^{-\alpha-1}\|x\|^\alpha dtQ(dx)\\
&= \int_{\mathbb{R}^d}\int_0^\infty I_A(s\frac{x}{\|x\|^2})s^{-\alpha-1}\|x\|^\alpha dsQ(dx)\\
&= \int_{S^{d-1}}\int_0^\infty I_A(su)s^{-\alpha-1}ds\sigma(du)\\
&= \nu_0(A).
\end{aligned}$$

Since we are considering a proper TID distribution, then, by remark 3.9, $Q(\cdot|u)$ are finite measures on $(0,\infty)$. Therefore we can write

$$\begin{aligned}
\int_{\mathbb{R}^d} I_B\left(\frac{x}{\|x\|}\right)\|x\|^\alpha R(dx) &= \int_{\mathbb{R}^d} I_B\left(\frac{x}{\|x\|}\right)Q(dx)\\
&= \int_B\int_0^\infty Q(ds|u)\sigma(du) = c\sigma(B).
\end{aligned}$$

Since $Q(ds|u)$ are probability measures, then $c = 1$ and (3.15) holds. □

3.1.1 Distributional properties

A TID distribution may have moments and also exponential moments of any order. The behavior of the tails depends on the measure R.

Proposition 3.11. *Let μ be a TID distribution with Lévy measure ν given by (3.9) and $\alpha \in (0, 2)$. Then*

(a) For $p \in (0, \alpha)$

$$\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty;$$

(b) $\int_{\mathbb{R}^d} \|x\|^\alpha \mu(dx) < \infty \iff \int_{\|x\|>1} \|x\|^\alpha \log(\|x\|) R(dx) < \infty$;

(c) If $p > \alpha$, then

$$\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty \iff \int_{\|x\|>1} \|x\|^p R(dx) < \infty;$$

(d) For each $\theta > 0$, we have

$$\int_{\mathbb{R}^d} e^{\theta\|x\|} \mu(dx) < \infty \iff \int_{\|x\|>1} \|x\|^{-(\alpha+1)} e^{\frac{\theta^2\|x\|^2}{2}} R(dx) < \infty.$$

(e) If $\alpha = 0$ and $p > 0$, then

$$\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty \iff \int_{\|x\|>1} \|x\|^p R(dx) < \infty;$$

Proof. It is well known that moments conditions for μ are related to the corresponding conditions for $\nu_{|\{\|x\|>1\}}$, see [109].

Let we consider $p > 0$. Then we obtain

$$\begin{aligned} \int_{\|x\|>1} \|x\|^p \nu(dx) &= \int_{\|x\|\le 1} \|x\|^p \int_{\frac{1}{\|x\|}}^{\infty} t^{p-\alpha-1} e^{-t^2/2} dt R(dx) \\ &\quad + \int_{\|x\|>1} \|x\|^p \int_{\frac{1}{\|x\|}}^{\infty} t^{p-\alpha-1} e^{-t^2/2} dt R(dx) \\ &= I^{(1)}(x) + I^{(2)}(x). \end{aligned}$$

By (3.10), the following inequality holds

$$I^{(1)}(x) \le C \int_{\|x\|\le 1} \int_{\frac{1}{\|x\|}}^{\infty} t^{-3} dt R(dx) \le \frac{C}{2} \int_{\|x\|\le 1} \|x\|^2 R(dx) < \infty \tag{3.16}$$

where $C := \sup_{t\ge 1} t^{p+2-\alpha} e^{-t^2/2}$. The inequality (3.16) shows that the integral $I^{(1)}(x)$ is always finite.

If $p < \alpha$, then

$$I^{(2)}(x) \le \int_{\|x\|>1} \|x\|^p \int_{\frac{1}{\|x\|}}^{\infty} t^{p-\alpha-1} dt R(dx) = \frac{1}{\alpha - p} \int_{\|x\|>1} \|x\|^\alpha R(dx) < \infty,$$

by inequality (3.10), condition (a) is fulfilled. If $p = \alpha$, we have

$$I^{(2)}(x) \leq \int_{\|x\|>1} \|x\|^\alpha \int_{\frac{1}{\|x\|}}^1 t^{-1} dt R(dx) + \int_{\|x\|>1} \|x\|^\alpha \int_1^\infty e^{-t^2/2} dt R(dx)$$
$$= \int_{\|x\|>1} \|x\|^\alpha (\log(\|x\|) + C_1) R(dx)$$

where C_1 is a finite constant, and from the other side,

$$I^{(2)}(x) \geq e^{-1/2} \int_{\|x\|>1} \|x\|^\alpha \log(\|x\|) R(dx).$$

Therefore condition (b) is satisfied. Now, we suppose $p > \alpha$. Let us define

$$\bar{C} = \int_1^\infty t^{p-\alpha-1} e^{-t^2/2} dt.$$

Then, the following inequality holds

$$I^{(2)}(x) \geq \bar{C} \int_{\|x\|>1} \|x\|^p R(dx)$$

and furthermore,

$$I^{(2)}(x) \leq \int_{\|x\|>1} \|x\|^p \int_0^\infty t^{p-\alpha-1} e^{-t^2/2} dt R(dx).$$

By changing variables in the integral, we have

$$\int_0^\infty t^{p-\alpha-1} e^{-t^2/2} dt = 2^{(p-\alpha)/2-1} \int_0^\infty z^{(p-\alpha)/2-1} e^{-z} dz = 2^{(p-\alpha)/2-1} \Gamma\Big(\frac{p-\alpha}{2}\Big),$$

thus

$$I^{(2)}(x) \leq 2^{(p-\alpha)/2-1} \Gamma\Big(\frac{p-\alpha}{2}\Big) \int_{\|x\|>1} \|x\|^p R(dx).$$

This proves (c).

In order to prove (d), we consider the integral

$$\int_{\|x\|>1} e^{\theta\|x\|} \nu(dx) = \int_{\mathbb{R}^d} \int_{\frac{1}{\|x\|}}^\infty e^{\theta t\|x\|} t^{-(\alpha+1)} e^{-t^2/2} dt R(dx)$$

and we define

$$I_\theta(x) := \int_{\frac{1}{\|x\|}}^\infty e^{\theta t\|x\|} t^{-(\alpha+1)} e^{-t^2/2} dt.$$

It is easy to check that as $\|x\| \to 0$, then $I_\theta(x)$ goes to 0 exponentially fast. Now, let us consider the case $\|x\| \to \infty$. We have

$$I_\theta(x) = e^{\theta^2\|x\|^2/2} \int_{\frac{1}{\|x\|}}^\infty t^{-(\alpha+1)} e^{-(t-\theta\|x\|)^2/2} dt.$$

Define $K_\theta(x)$ by

$$K_\theta(x) := \int_{\frac{1}{\|x\|}}^{\infty} t^{-(\alpha+1)} e^{-(t-\theta\|x\|)^2/2} dt,$$

by changing variables in the integral we obtain

$$\begin{aligned} K_\theta(x) &= \int_{\frac{1}{\|x\|}-\theta\|x\|}^{\infty} (t+\theta\|x\|)^{-(\alpha+1)} e^{-t^2/2} dt \\ &= \int_{\frac{1}{\|x\|}-\theta\|x\|}^{-\frac{\theta\|x\|}{2}} (t+\theta\|x\|)^{-(\alpha+1)} e^{-t^2/2} dt + \int_{-\frac{\theta\|x\|}{2}}^{\infty} (t+\theta\|x\|)^{-(\alpha+1)} e^{-t^2/2} dt \\ &= K_\theta^{(1)}(x) + K_\theta^{(2)}(x). \end{aligned}$$

Furthermore, the following inequality is satisfied

$$\begin{aligned} K_\theta^{(1)}(x) &\leq e^{-\theta^2\|x\|^2/8} \int_{\frac{1}{\|x\|}-\theta\|x\|}^{-\frac{\theta\|x\|}{2}} (t+\theta\|x\|)^{-(\alpha+1)} dt \\ &\leq C\|x\|^{\alpha+2} e^{-\theta^2\|x\|^2/8}. \end{aligned}$$

and for $\|x\| \to \infty$,

$$\begin{aligned} K_\theta^{(2)}(x) &\leq \int_{-\frac{\theta\|x\|}{2}}^{\infty} (t+\theta\|x\|)^{-(\alpha+1)} e^{-t^2/2} dt \\ &\sim (\theta\|x\|)^{-(\alpha+1)} \int_{-\infty}^{\infty} e^{-t^2/2} dt = \sqrt{2\pi}(\theta\|x\|)^{-(\alpha+1)}, \end{aligned}$$

It follows that, for $\|x\| \to \infty$,

$$K_\theta(x) \sim \sqrt{2\pi}(\theta\|x\|)^{-(\alpha+1)},$$

and

$$I_\theta(x) \sim \sqrt{2\pi}(\theta\|x\|)^{-(\alpha+1)} e^{\theta^2\|x\|^2/2},$$

therefore condition (d) holds. Part (e) can be proved with a similar argument to (c). □

Remark 3.12. *If the measure R has a bounded support, then $E(e^{\theta\|X\|}) < \infty$ for all $\theta > 0$. We have exponential moments of any order.*

As we said before, sometimes it is more convenient to work with the measure R; in order to find some distributional property of a TID distribution. Taking into account Proposition 2.8 of [107], we will show a result about finite variation.

Proposition 3.13. *Let ν be the Lévy measure of a TID distribution, and R a measure as in (3.9) and (3.6). These conditions are equivalent*

(i) $\int_{\|x\|\leq 1} \|x\| \nu(dx) < \infty$

(ii) $\int_{\|x\|\leq 1} \|x\| R(dx) < \infty$ *and* $\alpha \in (0, 1)$

Proof. Suppose condition (i) is fulfilled. Choose $r \geq 1$ such that $R(\{\|x\| \leq r\}) \neq 0$. We can write the following relations

$$\begin{aligned}
\int_{\|x\|\leq 1} \|x\|\nu(dx) &= \int_{\mathbb{R}^d} \|x\| \int_0^{1/\|x\|} t^{-\alpha} e^{-t^2/2} dt R(dx) \\
&\geq \int_{\|x\|\leq r} \|x\| \int_0^{1/\|x\|} t^{-\alpha} e^{-t^2/2} dt R(dx) \\
&= 2^{-\frac{1}{2}(\alpha+1)} \int_{\|x\|\leq r} \|x\| \int_0^{1/(2\|x\|^2)} z^{-\frac{1}{2}(\alpha+1)} e^{-z} dz R(dx) \\
&\geq 2^{-\frac{1}{2}(\alpha+1)} \int_{\|x\|\leq r} \|x\| R(dx) \int_0^{1/(2r^2)} z^{-\frac{1}{2}(\alpha+1)} e^{-z} dz.
\end{aligned}$$

By condition (i), we obtain $\alpha < 1$ and

$$\int_{\|x\|\leq 1} \|x\| R(dx) < \infty.$$

Conversely, if (ii) holds, then

$$\begin{aligned}
&\int_{\|x\|\leq 1} \|x\|\nu(dx) = \\
&= \int_{\|x\|\leq 1} \|x\| \int_0^{1/\|x\|} t^{-\alpha} e^{-t^2/2} dt R(dx) + \int_{\|x\|>1} \|x\| \int_0^{1/\|x\|} t^{-\alpha} e^{-t^2/2} dt R(dx) \\
&\leq \int_{\|x\|\leq 1} \|x\| \int_0^{\infty} t^{-\alpha} e^{-t^2/2} dt R(dx) + \frac{1}{1-\alpha} \int_{\|x\|>1} \|x\|^{\alpha} R(dx) \\
&= 2^{-\frac{\alpha}{2}-\frac{1}{2}} \Gamma(\frac{1}{2} - \frac{\alpha}{2}) \int_{\|x\|\leq 1} \|x\| R(dx) + \frac{1}{1-\alpha} \int_{\|x\|>1} \|x\|^{\alpha} R(dx) < \infty,
\end{aligned}$$

which proves the converse. □

3.1.2 Characteristic function of a TID distribution

It is well known that given a Lévy measure of a infinitely divisible distribution, we have an explicit formula for the characteristic function, see [109]. Sometimes, working with a Lévy measure of the form (3.3) may be difficult and, as a consequence, we will provide an expression for the characteristic function of a TID distribution with respect to the measure R. The measure R allows one to obtain explicit analytic formulas and more explicit calculations. Let us now define functions

$$\psi_\alpha(s) = \int_0^\infty (e^{ist} - 1 - ist) t^{-\alpha-1} e^{-t^2/2} dt, \tag{3.17}$$

and

$$\psi_\alpha^0(s) = \int_0^\infty (e^{ist} - 1) t^{-\alpha-1} e^{-t^2/2} dt. \tag{3.18}$$

In order to find a more useful form for the characteristic function of a TID distribution, we will need the following results.

Lemma 3.14. *The following limits are verified*

$$\begin{array}{ll} \lim_{s\to 0} s^{-2}\psi_\alpha(s) = -2^{-\frac{\alpha}{2}-1}\Gamma(1-\frac{\alpha}{2}), & \alpha\in(0,2) \\ \lim_{s\to\infty} s^{-1}\psi_0(s) = -i\sqrt{\frac{\pi}{2}}, & \alpha = 0 \\ \lim_{s\to\infty} s^{-1}\psi_\alpha(s) = -2^{-\frac{\alpha}{2}-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{\alpha}{2})i, & \alpha\in(0,1) \\ \lim_{s\to\infty}(s^{-1}\psi_1(s) + i\log s) = -\frac{\pi}{2}+i, & \alpha = 1 \\ \lim_{s\to\infty} s^{-\alpha}\psi_\alpha(s) = \Gamma(-\alpha)e^{-i\alpha\frac{\pi}{2}}, & \alpha\in(1,2) \end{array} \tag{3.19}$$

Furthermore, if $\alpha\in(0,1)$ we have

$$\begin{array}{l} \lim_{s\to\infty} s^{-1}\psi^0_\alpha(s) = 2^{-\frac{\alpha}{2}-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{\alpha}{2})i, \\ \lim_{s\to\infty} s^{-\alpha}\psi^0_\alpha(s) = \Gamma(-\alpha)e^{-i\alpha\frac{\pi}{2}}, \end{array} \tag{3.20}$$

Then, there exists for each α a finite positive constant C_α such that for all $s\in\mathbb{R}$ the following inequalities are fulfilled

$$\begin{array}{rcll} C_\alpha^{-1}(s^2\wedge|s|^{\alpha\vee 1}) & \le |\psi_\alpha(s)| \le & C_\alpha(s^2\wedge|s|^{\alpha\vee 0,1}), & \alpha\ne 1, \\ C_1^{-1}[s^2\wedge|s|(1+\log^+|s|)] & \le |\psi_1(s)| \le & C_1[s^2\wedge|s|(1+\log^+|s|)], & \alpha = 1, \\ C_\alpha^{-1}(s^2\wedge|s|^{\alpha}) & \le |\psi^0_\alpha(s)| \le & C_\alpha(s^2\wedge|s|^{\alpha}), & \alpha\in(0,1). \\ C_0^{-1}[1+\log(1+s)] & \le |\psi_0(s)| \le & C_0[1+\log(1+s)], & \alpha = 0. \end{array} \tag{3.21}$$

Proof. By solving the limit and using [109, Lemma 14.11], (3.19) and (3.20) are verified. □

Lemma 3.15. *Let us consider the confluent equation*

$$x\frac{d^2y}{dx^2} + (c-x)\frac{dy}{dx} - ax = 0. \tag{3.22}$$

Then the solution of this differential equation is

$$y = AM(a,c;z) + BU(a,c;z)$$

where A and B are constant and $M(a,c;z)$ is the Kummer's or confluent hypergeometric function of first kind [1, 13.1.2] and $U(a,c;z)$ is the confluent hypergeometric function of second kind [1, 13.1.3].

Proof. For a complete overview on confluent hypergeometric function see [116] or [1]. □

Lemma 3.16. *Let $\alpha\in(0,2)$, $\alpha\ne 1$. Then we have*

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{z^n}{n!}\Gamma\left(\frac{1}{2}(n-\alpha)\right) &= \Gamma\Big(-\frac{\alpha}{2}\Big)M\left(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{z}{2}\Big)^2\right) \\ &\quad + z\Gamma\Big(\frac{1-\alpha}{2}\Big)M\left(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{z}{2}\Big)^2\right) \end{aligned} \tag{3.23}$$

Proof. Since the series converges, we can split it in two parts

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}\Gamma\left(\frac{1}{2}(n-\alpha)\right) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}\Gamma\left(n-\frac{\alpha}{2}\right) + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}\Gamma\left(n+\frac{1-\alpha}{2}\right).$$

By the Legendre duplication formula [1, 6.1.18], we obtain the following equalities

$$\begin{aligned}(2n)! = &\ n!2^{2n}\left(\frac{1}{2}\right)_n,\\ (2n+1)! = &\ n!2^{2n}\left(\frac{3}{2}\right)_n,\end{aligned}$$

and we define the Pochhammer's symbols as $(a)_n = \Gamma(n+a)/\Gamma(a)$ [1, 6.1.22]. Thus, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{z^n}{n!}\Gamma\left(\frac{n-\alpha}{2}\right) =\\ &= \Gamma\left(-\frac{\alpha}{2}\right)\sum_{n=0}^{\infty}\frac{z^{2n}}{n!2^{2n}}\frac{\left(-\frac{\alpha}{2}\right)_n}{\left(\frac{1}{2}\right)_n} + z\Gamma\left(\frac{1-\alpha}{2}\right)\sum_{n=0}^{\infty}\frac{z^{2n}}{n!2^{2n}}\frac{\left(\frac{1-\alpha}{2}\right)_n}{\left(\frac{3}{2}\right)_n}\\ &= \Gamma\left(-\frac{\alpha}{2}\right)M\left(-\frac{\alpha}{2},\frac{1}{2};\left(\frac{z}{2}\right)^2\right) + z\Gamma\left(\frac{1-\alpha}{2}\right)M\left(\frac{1-\alpha}{2},\frac{3}{2};\left(\frac{z}{2}\right)^2\right).\end{aligned}$$

□

Theorem 3.17. *(Characteristic function) Let μ be a TID distribution with Lévy measure given by (3.9), $\alpha \in [0,2)$ and $\alpha \neq 1$. If the distribution has finite mean, i.e. $\int_{\mathbb{R}^d}\|x\|\mu(dx) < \infty$, then*

$$\hat{\mu}(y) = \exp\left\{\int_{\mathbb{R}^d}\psi_\alpha(\langle y,x\rangle)R(dx) + i\langle y,m\rangle\right\} \tag{3.24}$$

where

$$\begin{aligned}\psi_\alpha(s) &= 2^{-\frac{\alpha}{2}-1}\Bigg(\Gamma\Big(-\frac{\alpha}{2}\Big)M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{i\sqrt{2}s}{2}\Big)^2\Big)\\ &\quad + i\sqrt{2}s\Gamma\Big(\frac{1-\alpha}{2}\Big)M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{i\sqrt{2}s}{2}\Big)^2\Big)\\ &\quad - i\sqrt{2}s\Gamma\Big(\frac{1}{2}-\frac{\alpha}{2}\Big) - \Gamma\Big(-\frac{\alpha}{2}\Big)\Bigg).\end{aligned} \tag{3.25}$$

and $m = \int_{\mathbb{R}^d} x\mu(dx)$. Furthermore, if $0 < \alpha < 1$, the characteristic function can be written in an alternative form

$$\hat{\mu}(y) = \exp\left\{\int_{\mathbb{R}^d}\psi_\alpha^0(\langle y,x\rangle)R(dx) + i\langle y,m_0\rangle\right\} \tag{3.26}$$

where

$$\begin{aligned}\psi_\alpha(s) &= 2^{-\frac{\alpha}{2}-1}\Bigg(\Gamma\Big(-\frac{\alpha}{2}\Big)M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{i\sqrt{2}s}{2}\Big)^2\Big)\\ &\quad+i\sqrt{2}s\Gamma\Big(\frac{1-\alpha}{2}\Big)M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{i\sqrt{2}s}{2}\Big)^2\Big)-\Gamma\Big(-\frac{\alpha}{2}\Big)\Bigg).\end{aligned} \tag{3.27}$$

Proof. First, integrals (3.24) and (3.26) are well defined due to conditions (3.10) and (3.21) of Lemma 3.14. It is well known that if the mean is finite, that is if the first absolute moment exists, i.e. $\int_{\mathbb{R}^d}\|x\|\mu(dx)<\infty$, then $\hat{\mu}$ can be written as

$$\hat{\mu}=\exp\Big(\int_{\mathbb{R}^d}(e^{i\langle y,x\rangle}-1-i\langle y,x\rangle)\nu(dx)+i\langle y,m\rangle\Big)$$

where $m=\int_{\mathbb{R}^d}x\mu(dx)$. By (3.9), we obtain the equality (3.24), where, if $\alpha\in[0,2)$

$$\psi_\alpha(s)=\int_0^\infty(e^{ist}-1-ist)t^{-\alpha-1}e^{-t^2/2}dt, \tag{3.28}$$

If $\alpha\in[0,1)$ and $\int_{\|x\|\le1}\|x\|R(dx)<\infty$, by Proposition 3.13 $\int_{\|x\|\le1}\|x\|\nu(dx)<\infty$, in which case $\hat{\mu}$ can be written as

$$\exp\Big(\int_{\mathbb{R}^d}(e^{i\langle y,x\rangle}-1)\nu(dx)+i\langle y,m_0\rangle\Big),$$

where m_0 is the drift as defined in [109]. By (3.9), we obtain the equality (3.26), where

$$\psi_\alpha^0(s)=\int_0^\infty(e^{ist}-1)t^{-\alpha-1}e^{-t^2/2}dt, \tag{3.29}$$

and, furthermore, the equality

$$\psi_\alpha(s)=\psi_\alpha^0(s)-is\int_0^\infty t^{-\alpha}e^{-t^2/2}dt \tag{3.30}$$

holds. Now we will prove (3.25) and (3.27). If $\alpha\in(0,1)$, we obtain by equality (3.23)

$$\begin{aligned}&\int_0^\infty(e^{ist}-1)t^{-\alpha-1}e^{-t^2/2}dt=\\ &\quad=\sum_{n=1}^\infty\frac{(is)^n}{n!}\int_0^\infty t^{n-\alpha-1}e^{-t^2/2}dt\\ &\quad=\sum_{n=1}^\infty\frac{(is)^n}{n!}2^{\frac{1}{2}(n-\alpha-2)}\Gamma\Big(\frac{1}{2}(n-\alpha)\Big)\\ &\quad=2^{-\frac{\alpha}{2}-1}\sum_{n=1}^\infty\frac{(i\sqrt{2}s)^n}{n!}\Gamma\Big(\frac{1}{2}(n-\alpha)\Big)\\ &\quad=2^{-\frac{\alpha}{2}-1}\Bigg(\Gamma\Big(-\frac{\alpha}{2}\Big)M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{i\sqrt{2}s}{2}\Big)^2\Big)\\ &\qquad+i\sqrt{2}s\Gamma\Big(\frac{1-\alpha}{2}\Big)M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{i\sqrt{2}s}{2}\Big)^2\Big)-\Gamma\Big(-\frac{\alpha}{2}\Big)\Bigg).\end{aligned}$$

With a similar calculus, if $\alpha \in (0, 2)$ and $\alpha \neq 1$, we obtain

$$\begin{aligned}
&\int_0^\infty (e^{ist} - 1 - ist)t^{-\alpha-1}e^{-t^2/2}dt = \\
&= \sum_{n=2}^\infty \frac{(is)^n}{n!} \int_0^\infty t^{n-\alpha-1}e^{-t^2/2}dt \\
&= \sum_{n=2}^\infty \frac{(is)^n}{n!} 2^{\frac{1}{2}(n-\alpha-2)}\Gamma\left(\frac{1}{2}(n-\alpha)\right) \\
&= 2^{-\frac{\alpha}{2}-1} \sum_{n=2}^\infty \frac{(i\sqrt{2}s)^n}{n!}\Gamma\left(\frac{1}{2}(n-\alpha)\right) \\
&= 2^{-\frac{\alpha}{2}-1}\Bigg(\Gamma\left(-\frac{\alpha}{2}\right)M\left(-\frac{\alpha}{2}, \frac{1}{2}; \left(\frac{i\sqrt{2}s}{2}\right)^2\right) \\
&\quad + i\sqrt{2}s\Gamma\left(\frac{1-\alpha}{2}\right)M\left(\frac{1}{2}-\frac{\alpha}{2}, \frac{3}{2}; \left(\frac{i\sqrt{2}s}{2}\right)^2\right) \\
&\quad - \sqrt{2}is\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right) - \Gamma\left(-\frac{\alpha}{2}\right)\Bigg)
\end{aligned}$$

□

Remark 3.18. *With a similar technique, the characteristic exponent also can be calculated also for both cases $\alpha = 0$ and $\alpha = 1$.*

Definition 3.19. *We will write $X \sim TID_\alpha(R, m)$ to indicate that X is a TID random variable with characteristic function (3.24) and $X \sim TID^0_\alpha(R, m_0)$ to indicate that X is a TID random variable with characteristic function (3.26). The constant m is exactly the mean $m = E[X]$.*

3.2 TID processes

In this section, we will introduce TID processes. By Definition 3.3, if μ is a TID distribution, it is infinitely divisible and therefore there exists a Lévy process $\{X(t) : t \geq 0\}$ such that μ is the distribution of $X(1)$.

3.2.1 Short and long time behavior

The following theorems will show the different behavior of a TID process for different time scale. If one decreases the time scale, a TID process looks like a stable process; otherwise, if one increases the time scale, it looks like a Brownian motion. To figure out this different time behavior, we consider the time rescaled process

$$\{X_h(t) : t \geq 0\} = \{X(ht) : t \geq 0\}, \tag{3.31}$$

where $h > 0$.

Theorem 3.20. *(Short time behavior) Let $\{X(t) : t \geq 0\}$ be a TID Lévy process in $\mathbb{R}^d$ such that the distribution of $X(1)$ has spectral measure R.*

(a) *Let us consider a TID process with $X(1) \sim TID^0_\alpha(R,0)$, if $\alpha \in (0,1)$ and with $X(1) \sim TID_\alpha(R,0)$, if $\alpha \in (1,2)$. Assume that*

$$\int_{\mathbb{R}^d} \|x\|^\alpha R(dx) < \infty \tag{3.32}$$

and let σ be the finite measure on S^{d-1} defined in (3.15). Then

$$h^{-1/\alpha} X_h \xrightarrow{d} Y, \tag{3.33}$$

as $h \to 0$, where $\{Y(t) : t \geq 0\}$ is a strictly α-stable Lévy process with $Y(1) = S_\alpha(\sigma, 0)$.

(b) *Let us consider a TID process with $X(1) \sim TID_\alpha(R,0)$, if $\alpha = 1$. Assume that*

$$\int_{\mathbb{R}^d} \|x\| \,|\log \|x\|| \, R(dx) < \infty. \tag{3.34}$$

Then

$$h^{-1/\alpha} X_h - a_h \xrightarrow{d} Y,$$

where

$$a_h(t) = t \log h \int_{\mathbb{R}^d} x R(dx),$$

and $\{Y(t) : t \geq 0\}$ is an α-stable Lévy process with $Y(1) \sim S_1(\sigma, b)$ with

$$b = \int_{\mathbb{R}^d} x(1 - \log \|x\|) R(dx).$$

Proof. Since $\{h^{-1/\alpha} X_h(t) : t \geq 0\}$ is a Lévy process, by [62, Theorem 13.17], it is enough to show the convergence in distribution of $h^{-1/\alpha} X_h(1)$ to $Y(1)$. By a Paul Lévy theorem (also called the continuity theorem) [43, Theorem 2, p.508], the convergence in distribution can be proved by considering the pointwise convergence of the respective characteristic functions.

First, we want to prove (a). If $\alpha(0,1)$, we obtain

$$\begin{aligned} E[e^{i\langle y, h^{-1/\alpha} X_h(1)\rangle}] &= E[e^{i\langle h^{-1/\alpha} y, X(h)\rangle}] \\ &= \exp\left\{ \int_{\mathbb{R}^d} h \psi^0_\alpha (h^{-1/\alpha} \langle y, x\rangle) R(dx) \right\}, \end{aligned} \tag{3.35}$$

The upper bounds (3.21) of Lemma 3.14 and condition (3.32) allow one to apply the dominated convergence theorem to the above integral. By definitions (3.25) and (3.27) it easy to check that $\psi^0_\alpha(-s) = \overline{\psi^0_\alpha(s)}$ and $\psi_\alpha(-s) = \overline{\psi_\alpha(s)}$. Now, by (3.20) and [109, Theorem 14.10], we calculate the limit $h \to 0$ under the integral (3.35)

$$\begin{aligned} \lim_{h \to 0} h \psi^0_\alpha (h^{-1/\alpha} \langle y, x\rangle) &= \Gamma(-\alpha) |\langle y, x\rangle|^\alpha \exp\left\{ -i \frac{\alpha\pi}{2} sgn\langle y, x\rangle \right\} \\ &= \Gamma(-\alpha) \cos\frac{\alpha\pi}{2} |\langle y, x\rangle|^\alpha \left(1 - i \tan\frac{\alpha\pi}{2} sgn\langle y, x\rangle \right). \end{aligned}$$

Therefore, under the assumption $\alpha \in (0,1)$, (3.33) holds. A similar argument proves (a) also in the case $\alpha \in (1,2)$. Let us consider the case $\alpha = 1$. By definition of a_h, the equality

$$E[\exp\{i\langle y, h^{-1}X_h(1) - a_h(1)\rangle\}] = \exp\left\{\int_{\mathbb{R}^d}(h\psi_1(h^{-1}\langle y,x\rangle) - i\langle y,x\rangle \log h)R(dx)\right\}. \tag{3.36}$$

is fulfilled, then, by assumption (3.34) and [107, Theorem 3.1], (b) holds. □

Now, we will prove that if one increases the time scale, a TID process looks like a Brownian motion.

Theorem 3.21. *(Long time behavior) Let $\{X(t) : t \geq 0\}$ be a TID Lévy process in $\mathbb{R}^d$ such that the distribution of $X(1) \sim TID_\alpha(R, 0)$ and $\alpha \in (0,2)$. Assume that*

$$\int_{\mathbb{R}^d} \|x\|^2 R(dx) < \infty. \tag{3.37}$$

Then

$$h^{-\frac{1}{2}}X_h \xrightarrow{d} B,$$

as $h \to \infty$, where $\{B(t) : t \geq 0\}$ is a Brownian motion with characteristic function

$$E[e^{i\langle y, B(t)\rangle}] = \exp\left\{-t2^{-\frac{\alpha}{2}-1}\Gamma(1-\frac{\alpha}{2})\int_{\mathbb{R}^d}\langle y,x\rangle^2 R(dx)\right\}. \tag{3.38}$$

Proof. Since $\{h^{-\frac{1}{2}}X_h(t) : t \geq 0\}$ is a Lévy process, by [62, Theorem 13.17], it is enough to show the convergence in distribution of $h^{-\frac{1}{2}}X_h(1)$ to $B(1)$. By the continuity theorem [43, Theorem 2, p.508], the convergence in distribution can be proved by considering the pointwise convergence of the respective characteristic functions. By considering equality (3.24), we can write

$$\begin{aligned} E[e^{i\langle y, h^{-\frac{1}{2}}X_h(1)\rangle}] &= E[e^{i\langle h^{-\frac{1}{2}}y, X(h)\rangle}] \\ &= \exp\left\{\int_{\mathbb{R}^d} h\psi(h^{-\frac{1}{2}}\langle y,x\rangle)R(dx)\right\}. \end{aligned}$$

The upper bounds (3.21) and condition (3.37) allow one to apply the dominated convergence theorem to the above integral and by considering (3.19) we obtain

$$\lim_{h\to\infty} h\psi(h^{-\frac{1}{2}}\langle y,x\rangle) = -2^{-\frac{\alpha}{2}-1}\Gamma(1-\frac{\alpha}{2})\langle y,x\rangle^2,$$

which verifies (3.38). □

3.3 Examples

A real TID law can be defined by fixing a positive definite radial function q with a measure σ on S^1 or alternatively by defining its spectral measure R. We are going to show in the following three parametric examples of TID laws in one dimension. In the first example, the measure R is the sum of two Dirac measures multiplied for opportune constants. The spectral measure R of the second example has a non-trivial bounded support and the derived TID distribution has exponential moments of any order. In the last example, the MTS distribution is considered, see [71, 67], the spectral measure is defined on an unbounded support and there exist exponential moments of some order.

3.3.1 Example 1: RDTS

The Lévy measure of the *rapidly decreasing tempered stable* (RDTS) distribution has the form

$$\nu(dx) = (c_+ e^{-\lambda_+^2 x^2/2} 1_{x>0} + c_- e^{-\lambda_-^2 |x|^2/2} 1_{x<0}) \frac{dx}{|x|^{\alpha+1}}, \tag{3.39}$$

and can be written in polar coordinates as

$$\nu(dr, du) = r^{-\alpha-1} q(r,u) dr\, \sigma(du)$$

where

$$q(r,1) = e^{-\lambda_+^2 r^2/2}, \quad q(r,-1) = e^{-\lambda_-^2 r^2/2}, \tag{3.40}$$

and

$$\sigma(1) = c_+, \quad \sigma(-1) = c_-. \tag{3.41}$$

The positive definite radial function q, by Theorem 3.4, has the form

$$q(r,u) = \int_0^\infty e^{-r^2 s^2/2} Q(ds|u)$$

where

$$Q(ds|1) = \delta_{\lambda_+}(s) ds,$$

and we have

$$Q(A) = c_+ \int_A \delta_{\lambda_+}(x) dx + c_- \int_A \delta_{-\lambda_-}(x) dx, \tag{3.42}$$

and hence the spectral measure R can be defined

$$R(A) = c_+ \int_A \lambda_+^\alpha \delta_{1/\lambda_+}(x) dx + c_- \int_A \lambda_-^\alpha \delta_{-1/\lambda_-}(x) dx. \tag{3.43}$$

Definition 3.22. *Let c_+, c_-, λ_+, λ_- strictly positive constants, $\alpha \in (0,2)$, $\alpha \neq 1$ and $\mu \in \mathbb{R}$. An infinitely divisible distribution is called the simple TID distribution with parameter (α, c_+, c_-, λ_+, λ_-) and mean μ, if its Lévy triplet is given by (0, μ, ν) where*

$$\nu(dx) = (c_+ e^{-\lambda_+^2 x^2/2} 1_{x>0} + c_- e^{-\lambda_-^2 |x|^2/2} 1_{x<0}) \frac{dx}{|x|^{\alpha+1}}.$$

Proposition 3.23. *The characteristic function of the simple TID distribution with parameter (α, c_+, c_-, λ_+, λ_-, μ) becomes*

$$\phi(u) = \exp\left(iu\mu + G(iu; \alpha, c_+, \lambda_+) + G(-iu; \alpha, c_-, \lambda_-)\right) \tag{3.44}$$

where

$$\begin{aligned} G(x;\alpha,C,\lambda) &= 2^{-\alpha/2-1}C\lambda^{\alpha}\Bigg(\Gamma\Big(-\frac{\alpha}{2}\Big)M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{\sqrt{2}x}{2\lambda}\Big)^2\Big) \\ &+\frac{\sqrt{2}x}{\lambda}\Gamma\Big(\frac{1-\alpha}{2}\Big)M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{\sqrt{2}x}{2\lambda}\Big)^2\Big) \\ &-\frac{\sqrt{2}x}{\lambda}\Gamma\Big(\frac{1}{2}-\frac{\alpha}{2}\Big)-\Gamma\Big(-\frac{\alpha}{2}\Big)\Bigg). \end{aligned} \tag{3.45}$$

Proof. It follows by Theorem 3.17. □

Remark 3.24. *Let $X \sim RDTS(c_+, c_-, \lambda_+, \lambda_-, \alpha, m)$ with $\alpha \neq 1$. The mean, variance, skewness, and excess kurtosis can be easily calculated using the cumulants [69], and are given as follows:*

(a) $E[X] = c_1(X) = m$,

(b) $\mathrm{Var}(X) = c_2(X) = 2^{-\frac{\alpha}{2}}\Gamma\Big(1-\frac{\alpha}{2}\Big)(c_+\lambda_+^{\alpha-2}+c_-\lambda_-^{\alpha-2})$,

(c) $s(X) = \dfrac{c_3(X)}{c_2(X)^{3/2}} = 2^{\frac{1}{2}+\frac{\alpha}{4}}\dfrac{\Gamma(\frac{3-\alpha}{2})\,(c_+\lambda_+^{\alpha-3}-c_-\lambda_-^{\alpha-3})}{\big(\Gamma(1-\frac{\alpha}{2})(c_+\lambda_+^{\alpha-2}+c_-\lambda_-^{\alpha-2})\big)^{3/2}}$,

(d) $k(X) = \dfrac{c_4(X)}{c_2(X)^2} = 2^{\frac{\alpha}{2}+1}\dfrac{\Gamma(\frac{4-\alpha}{2})\,(c_+\lambda_+^{\alpha-4}+c_-\lambda_-^{\alpha-4})}{\big(\Gamma(1-\frac{\alpha}{2})(c_+\lambda_+^{\alpha-2}+c_-\lambda_-^{\alpha-2})\big)^{2}}$.

Definition 3.25. *A Lévy process $X = (X_t)_{t\geq 0}$ is said to be a* RDTS TID process *(or shortly, a* CGMY process*) with parameters (c_+, c_-, λ_+, λ_-, α, m) if*

$$X_1 \sim \mathrm{RDTS}(c_+, c_-, \lambda_+, \lambda_-, \alpha, m).$$

3.3.2 Example 2: non trivial spectral measure

In the first example, the spectral measure R has no zero mass only at two points. Now we will consider a spectral measure with power decay defined on a bounded support of $\mathbb{R}$. By taking into consideration the construction of the KR tempered stable distribution in [68], we can consider the same spectral measure R. Indeed we have

$$R(dx) = (c_+r_+^{-p_+}I_{(0,r_+)}(x)|x|^{p_+-1} + c_-r_-^{-p_-}I_{(-r_-,0)}(x)|x|^{p_--1})\,dx. \tag{3.46}$$

By Theorem 3.17, the characteristic function of this distribution can be written in close form.

Lemma 3.26. *Let $\alpha \in (0,2)$ and $\alpha \neq 1$. Then, the following equality holds*

$$\begin{aligned} &\int \Big(\Gamma\Big(-\frac{\alpha}{2}\Big) M\Big(-\frac{\alpha}{2}, \frac{1}{2}; \Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big) \\ &\qquad + i\sqrt{2}ux\Gamma\Big(\frac{1-\alpha}{2}\Big) M\Big(\frac{1}{2}-\frac{\alpha}{2}, \frac{3}{2}; \Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big)\Big) x^{p-1} dx \\ &\qquad = \frac{x^p}{p}\Gamma\Big(-\frac{\alpha}{2}\Big) {}_2F_2\Big(\frac{p}{2}, -\frac{\alpha}{2}; 1+\frac{p}{2}, \frac{1}{2}; \Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big) \\ &\qquad + i\sqrt{2}u\frac{x^{p+1}}{p+1}\Gamma\Big(\frac{1-\alpha}{2}\Big) {}_2F_2\Big(\frac{1}{2}+\frac{p}{2}, \frac{1}{2}-\frac{\alpha}{2}; \frac{3}{2}+\frac{p}{2}, \frac{3}{2}; \Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big) \end{aligned} \tag{3.47}$$

Proof. By equation (3.23), we can write

$$\begin{aligned} &\int \Big(\Gamma\Big(-\frac{\alpha}{2}\Big) M\Big(-\frac{\alpha}{2}, \frac{1}{2}; \Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big) \\ &\qquad + i\sqrt{2}ux\Gamma\Big(\frac{1-\alpha}{2}\Big) M\Big(\frac{1}{2}-\frac{\alpha}{2}, \frac{3}{2}; \Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big)\Big) x^{p-1} dx \\ &\qquad = \int \sum_{n=0}^{\infty} \frac{(i\sqrt{2}u)^n x^{n+p-1}}{n!} \Gamma\left(\frac{1}{2}(n-\alpha)\right) dx \end{aligned} \tag{3.48}$$

Since the series converges on each bounded interval on $\mathbb{R}$, we obtain

$$\int \sum_{n=0}^{\infty} \frac{(i\sqrt{2}u)^n x^{n+p-1}}{n!} \Gamma\left(\frac{1}{2}(n-\alpha)\right) dx = x^p \sum_{n=0}^{\infty} \frac{(i\sqrt{2}ux)^n}{n!(n+p)} \Gamma\left(\frac{1}{2}(n-\alpha)\right).$$

Furthermore, the following equalities are fulfilled

$$\begin{aligned} \frac{p}{2n+p} &= \frac{\left(\frac{p}{2}\right)_n}{\left(1+\frac{p}{2}\right)_n} \\ \frac{p+1}{2n+1+p} &= \frac{\left(\frac{1}{2}+\frac{p}{2}\right)_n}{\left(\frac{3}{2}+\frac{p}{2}\right)_n} \end{aligned}$$

and by a similar argument of Lemma 3.16, equation (3.47) is verified. □

Proposition 3.27. *The characteristic function of the TID distribution with parameter $(\alpha, c_+, c_-, \lambda_+, \lambda_-, p_+, p_-)$, mean m and with spectral measure (3.46) is*

$$\phi(u) = \exp\left(ium + c_+ B(iu; \alpha, r_+, p_+) + c_- B(-iu; \alpha, r_-, p_-)\right) \tag{3.49}$$

where

$$\begin{aligned} &B(iu; \alpha, r, p) = \\ &= 2^{-\alpha/2-1}\frac{1}{p}\Gamma\Big(-\frac{\alpha}{2}\Big)\Big({}_2F_2\Big(\frac{p}{2}, -\frac{\alpha}{2}; 1+\frac{p}{2}, \frac{1}{2}; \Big(\frac{i\sqrt{2}ur}{2}\Big)^2\Big) - 1\Big) \\ &\quad + 2^{-\alpha/2-1}\frac{i\sqrt{2}ur}{p+1}\Gamma\Big(\frac{1-\alpha}{2}\Big)\Big({}_2F_2\Big(\frac{1}{2}+\frac{p}{2}, \frac{1}{2}-\frac{\alpha}{2}; \frac{3}{2}+\frac{p}{2}, \frac{3}{2}; \Big(\frac{i\sqrt{2}ur}{2}\Big)^2\Big) - 1\Big) \end{aligned}$$

Proof. Since the support of the measure R is bounded, by Proposition 3.11 the distribution has exponential moments of any order and in particular finite mean. By Theorem 3.17, we can consider the representation (3.25) and by Lemma 3.26 the characteristic exponent can be computed. □

3.3.3 Example 3 : MTS distribution

A parametric example of TID distributions has been already considered in the literature, the MTS distribution, see [71, 70, 67]. The Lévy measure of a MTS distribution is defined as

$$M(dx) = \left(C_+ \frac{\lambda_+^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ x)}{x^{\frac{\alpha+1}{2}}} 1_{x>0} + C_- \frac{\lambda_-^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_- |x|)}{|x|^{\frac{\alpha+1}{2}}} 1_{x<0} \right) dx,$$

where $\lambda_+, \lambda_-, C_+, C_- > 0$, $\alpha \in (0,2)$, and $\alpha \neq 1$. The tempering function q is of the form

$$q(r,u) = \begin{cases} (\lambda_+ r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ r), & u = 1 \\ (\lambda_- r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_- r), & u = -1, \end{cases} \tag{3.50}$$

and the measure σ is

$$\sigma(1) = C_+, \qquad \sigma(-1) = C_-.$$

Lemma 3.28. *Let $z > 0$ and $K_\nu(x)$ the modified Bessel function of second kind, then the equality*

$$2z^{\frac{\nu}{2}} K_\nu(2\sqrt{z}) = \int_0^\infty e^{-zt-\frac{1}{t}} t^{-\nu-1} dt \tag{3.51}$$

is satisfied.

Proof. By equality [50, 8.432(7)], we have

$$K_\nu(xp) = \frac{p^\nu}{2} \int_0^\infty e^{-\frac{xt}{2}-\frac{xp^2}{2t}} t^{-\nu-1} dt. \tag{3.52}$$

By setting $x = 2z$ and $p = 1/\sqrt{z}$, then we can write

$$K_\nu(2\sqrt{z}) = \frac{z^{-\frac{\nu}{2}}}{2} \int_0^\infty e^{-zt-\frac{1}{t}} t^{-\nu-1} dt$$

hence the equality (3.51) holds. □

Lemma 3.29. *Let μ be a MTS distribution, then*

$$Q(ds|\pm 1) = e^{-\lambda_\pm^2/2s^2} s^{-\alpha-2} \lambda_\pm^{\alpha+1} ds, \tag{3.53}$$

and

$$R(dx) = \left(C_+ \lambda_+^{\alpha+1} e^{-\frac{\lambda_+^2 x^2}{2}} I_{x>0} + C_- \lambda_-^{\alpha+1} e^{-\frac{\lambda_-^2 x^2}{2}} I_{x<0} \right) dx. \tag{3.54}$$

Proof. By setting $\nu = \frac{\alpha+1}{2}$ and $z = (\lambda r)^2/4$ into (3.51) and changing variable $t = 2s^2/\lambda^2$, we have

$$\begin{aligned}(\lambda r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda r) &= 2^{\frac{\alpha}{2}-\frac{1}{2}} \int_0^\infty e^{-\frac{r^2 s^2}{2}} e^{-\frac{\lambda^2}{2s^2}} \left(\frac{2s^2}{\lambda^2}\right)^{-\frac{\alpha+3}{2}} 4s\lambda^{-2} ds \\ &= 2^{\frac{\alpha}{2}-\frac{1}{2}} \int_0^\infty e^{-\frac{r^2 s^2}{2}} e^{-\frac{\lambda^2}{2s^2}} s^{-\alpha-2} \lambda^{\alpha+1} 2^{-\frac{\alpha}{2}+\frac{1}{2}} ds \\ &= \int_0^\infty e^{-\frac{r^2 s^2}{2}} e^{-\frac{\lambda^2}{2s^2}} s^{-\alpha-2} \lambda^{\alpha+1} ds\end{aligned}$$

By applying this result into (3.50), we have

$$q(r, \pm 1) = \int_0^\infty e^{-\frac{r^2 s^2}{2}} \left(e^{-\frac{\lambda^2}{2s^2}} s^{-\alpha-2} \lambda^{\alpha+1}\right) ds.$$

and obtain the equation (3.53) by the definition of $Q(ds|u)$. Moreover, for $A \in \mathcal{B}(\mathbb{R})$, we have

$$\begin{aligned}Q(A) &= \int_{S^0} \int_0^\infty I_A(ru) Q(dr|u) \sigma(du) \\ &= \int_0^\infty I_A(r) Q(dr|1)\sigma(1) + \int_0^\infty I_A(-r) Q(dr|-1)\sigma(-1) \\ &= C_+ \lambda_+^{\alpha+1} \int_0^\infty I_A(r) e^{-\lambda_+^2/2r^2} r^{-\alpha-2} dr \\ &\quad + C_- \lambda_-^{\alpha+1} \int_0^\infty I_A(-r) e^{-\lambda_-^2/2r^2} r^{-\alpha-2} dr.\end{aligned}$$

Hence,

$$\begin{aligned}R(A) &= \int_{\mathbb{R}} I_A\Big(\frac{x}{|x|^2}\Big) |x|^\alpha Q(dx) \\ &= \int_{S^0} \int_0^\infty I_A\Big(\frac{ru}{r^2}\Big) r^\alpha Q(dr|u)\sigma(du) \\ &= C_+ \lambda_+^{\alpha+1} \int_0^\infty I_A\Big(\frac{1}{r}\Big) r^\alpha e^{-\lambda_+^2/2r^2} r^{-\alpha-2} dr \\ &\quad + C_- \lambda_-^{\alpha+1} \int_0^\infty I_A\Big(-\frac{1}{r}\Big) r^\alpha e^{-\lambda_-^2/2r^2} r^{-\alpha-2} dr \\ &= C_+ \lambda_+^{\alpha+1} \int_0^\infty I_A(x) e^{-\frac{\lambda_+^2 x^2}{2}} dx \\ &\quad + C_- \lambda_-^{\alpha+1} \int_0^\infty I_A(-x) e^{-\frac{\lambda_-^2 x^2}{2}} dx \\ &= \int_A \Big(C_+ \lambda_+^{\alpha+1} e^{-\frac{\lambda_+^2 x^2}{2}} I_{x>0} + C_- \lambda_-^{\alpha+1} e^{-\frac{\lambda_-^2 x^2}{2}} I_{x<0}\Big) dx.\end{aligned}$$

□

Lemma 3.30. *Let $\alpha \in (0,2)$ and $\alpha \neq 1$. Then, the following equality holds*

$$\int_0^\infty \Big(\Gamma\Big(-\frac{\alpha}{2}\Big)M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big) + i\sqrt{2}ux\Gamma\Big(\frac{1-\alpha}{2}\Big)M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big)\Big)e^{-x^2\lambda^2/2}dx = \frac{1}{\lambda}\sqrt{\frac{\pi}{2}}\Gamma\Big(-\frac{\alpha}{2}\Big)\Big(1+\frac{u^2}{\lambda^2}\Big)^{\frac{\alpha}{2}} + \frac{i\sqrt{2}u}{\lambda^2}\Gamma\Big(\frac{1}{2}-\frac{\alpha}{2}\Big){}_2F_1\Big(1,\frac{1}{2}-\frac{\alpha}{2};\frac{3}{2};-\frac{u^2}{\lambda^2}\Big). \tag{3.55}$$

Proof. By equation (3.23), we can write

$$\int_0^\infty \Big(\Gamma\Big(-\frac{\alpha}{2}\Big)M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big) + i\sqrt{2}ux\Gamma\Big(\frac{1-\alpha}{2}\Big)M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big)\Big)e^{-x^2\lambda^2/2}dx \tag{3.56}$$

$$= \int_0^\infty \sum_{n=0}^\infty \frac{(i\sqrt{2}u)^n}{n!}\Gamma\Big(\frac{1}{2}(n-\alpha)\Big)x^n e^{-x^2\lambda^2/2}dx$$

Since the series converges on each bounded interval on $\mathbb{R}$, by a similar argument of Lemma 3.16, we can write

$$\int_0^\infty \sum_{n=0}^\infty \frac{(i\sqrt{2}u)^n}{n!}\Gamma\Big(\frac{1}{2}(n-\alpha)\Big)x^n e^{-x^2\lambda^2/2}dx =$$

$$= \sum_{n=0}^\infty \frac{(i\sqrt{2}u)^n}{n!}\Gamma\Big(\frac{1}{2}(n-\alpha)\Big)2^{-\frac{1}{2}+\frac{n}{2}}\lambda^{-n-1}\Gamma\Big(\frac{1}{2}(n+1)\Big)$$

$$= \frac{1}{\sqrt{2}\lambda}\sum_{n=0}^\infty \Big(\frac{2iu}{\lambda}\Big)^n \frac{1}{n!}\Gamma\Big(\frac{1}{2}(n-\alpha)\Big)\Gamma\Big(\frac{1}{2}(n+1)\Big)$$

$$= \frac{1}{\sqrt{2}\lambda}\sum_{n=0}^\infty \Big(\frac{2iu}{\lambda}\Big)^{2n} \frac{1}{2n!}\Gamma\Big(\frac{1}{2}(2n-\alpha)\Big)\Gamma\Big(\frac{1}{2}(2n+1)\Big) + \frac{1}{\sqrt{2}\lambda}\sum_{n=0}^\infty \Big(\frac{2iu}{\lambda}\Big)^{2n+1} \frac{1}{(2n+1)!}\Gamma\Big(\frac{1}{2}(2n+1-\alpha)\Big)\Gamma\Big(\frac{1}{2}(2n+2)\Big)$$

$$= \frac{1}{\sqrt{2}\lambda}\Gamma\Big(-\frac{\alpha}{2}\Big)\Gamma\Big(\frac{1}{2}\Big)\sum_{n=0}^\infty \Big(-\frac{u^2}{\lambda^2}\Big)^n \frac{1}{n!}\frac{\big(-\frac{\alpha}{2}\big)_n\big(\frac{1}{2}\big)_n}{\big(\frac{1}{2}\big)_n} + \frac{i\sqrt{2}u}{\lambda^2}\Gamma\Big(\frac{1}{2}-\frac{\alpha}{2}\Big)\sum_{n=0}^\infty \Big(-\frac{u^2}{\lambda^2}\Big)^n \frac{1}{n!}\frac{\big(\frac{1}{2}-\frac{\alpha}{2}\big)_n(1)_n}{\big(\frac{3}{2}\big)_n}$$

$$= \frac{1}{\lambda}\sqrt{\frac{\pi}{2}}\Gamma\Big(-\frac{\alpha}{2}\Big)\Big(1+\frac{u^2}{\lambda^2}\Big)^{\frac{\alpha}{2}} + \frac{i\sqrt{2}u}{\lambda^2}\Gamma\Big(\frac{1}{2}-\frac{\alpha}{2}\Big){}_2F_1\Big(1,\frac{1}{2}-\frac{\alpha}{2};\frac{3}{2};-\frac{u^2}{\lambda^2}\Big).$$

□

Proposition 3.31. *The characteristic function of the MTS distribution with parameter $(\alpha, C_+, C_-, \lambda_+, \lambda_-)$, mean m and with spectral measure (3.54) is*

$$\phi(u) = \exp\left(ium + C_+H(iu;\alpha,\lambda_+) + C_-H(-iu;\alpha,\lambda_-)\right) \tag{3.57}$$

where

$$\begin{aligned} H(iu;\alpha,\lambda) \; &= \frac{\lambda^\alpha\sqrt{\pi}}{2^{\frac{\alpha}{2}+\frac{3}{2}}}\Gamma\Big(-\frac{\alpha}{2}\Big)\bigg(\Big(1+\frac{u^2}{\lambda^2}\Big)^{\frac{\alpha}{2}}-1\bigg) \\ &+ \frac{i\lambda^{\alpha-1}u}{2^{\frac{\alpha}{2}+\frac{1}{2}}}\Gamma\Big(\frac{1}{2}-\frac{\alpha}{2}\Big)\bigg({}_2F_1\Big(1,\frac{1}{2}-\frac{\alpha}{2};\frac{3}{2};-\frac{u^2}{\lambda^2}\Big)-1\bigg). \end{aligned} \tag{3.58}$$

Proof. By definition of the measure R, by Proposition 3.11, the distribution has finite mean. By Theorem 3.17, we can consider the representation (3.25) and by Lemma 3.30 the characteristic exponent can be computed. □

Chapter 4

The change of measure problem

A basic result in mathematical finance, sometimes called the fundamental theorem of asset pricing (see [40]), is that for a stochastic process $(\tilde{S}_t)_{t\geq 0}$, the existence of an equivalent martingale measure is essentially equivalent to the absence of arbitrage opportunities. In finance the process $(\tilde{S}_t)_{t\geq 0}$ describes the random evolution of the discounted price of one or several financial assets. The equivalence of no-arbitrage with the existence of an equivalent probability martingale measure is at the basis of the entire theory of pricing by arbitrage. Starting from the economically meaningful assumption that $\tilde{S}$ does not allow arbitrage profits, the theorem allows the probability P on the underlying probability space $(\Omega, \mathcal{F}, P)$ to be replaced by an equivalent measure Q such that the process $\tilde{S}$ becomes a martingale under the new measure. This makes it possible to use the rich machinery of martingale theory. In particular the problem of fair pricing of contingent claims is reduced to taking expected values with respect to the measure Q. This method of pricing contingent claims is known to actuaries since the introduction of actuarial skills, centuries ago and known by the name of equivalence principle.

The above paragraph is the initial part of the seminal work [32] one of the most important paper in mathematical finance, see also [33, 34] for further details. Now, the problem is how to find an EMM in our framework. In the exponential Lévy model, the equivalent martingale measure (EMM) of a given market measure is not unique in general. For this reason, we have to find a method to select one of them.

One classical method to choose an EMM is the Esscher transform; another reasonable method is finding the *minimal entropy martingale measure*, as presented by [44]. However, while these methods are mathematically elegant and have a financial meaning in a utility maximization problem, the model prices obtained from the EMM did not match the market prices observed for options.

Now, we want to find conditions under which the Lévy process $(X_t)_{t\geq 0}$ under the measure P is still a Lévy process under a new measure Q. In order to find an equivalent measure, we will consider the general result of density transformation between Lévy processes in [109]. Even if we restrict our attention to structure preserving measures, the class of probabilities equivalent to a given one is surprisingly large. A construction of the Radon-Nikodim derivative is also given and it will be used in the following also for simulation algorithms.

Let $D = ([0, \infty), \mathbb{R}^d)$ be the space of mappings ω from $[0, \infty)$ into $\mathbb{R}^d$ right continuous with left limits and define the process $X_t(\omega) = x(t, \omega) = \omega(t)$. Consider the σ-algebra $\mathcal{F}_t$ generated from the process x_t and $\mathcal{F}_D$ the σ-algebra that makes x_t measurable. Any Lévy process $((x_t)_{t\geq t}, P^0)$ induces a probability measure P^D on $(D, \mathcal{F}^D)$ such that $((X_t)_{t\geq t}, P^D)$ is a Lévy process identical in law with $((x_t)_{t\geq t}, P^0)$. Processes we will be going to consider are of the form $((X_t)_{t\geq t}, P)$ and $((X_t)_{t\geq t}, \tilde{P})$ where P and $\tilde{P}$ are a probability measure on $(D, \mathcal{F}^D)$. Now we will show the condition for the mutual absolutely continuity of $P_{|\mathcal{F}_t}$ and $\tilde{P}_{|\mathcal{F}_t}$ for every t and for one dimensional processes.

Theorem 4.1. *Let $((X_t)_{t\geq 0}, P)$ and $((X_t)_{t\geq 0}, \tilde{P})$ be Lévy processes on $\mathbb{R}$ with generating triplets (a, σ, ν) and $(\tilde{a}, \tilde{\sigma}, \tilde{\nu})$, respectively. Then the following two statements are equivalent.*

(1) $P_{|\mathcal{F}_t} \approx \tilde{P}_{|\mathcal{F}_t}$

(2) *The generating triplets satisfy*

$$\sigma = \tilde{\sigma}$$
$$\nu \approx \tilde{\nu}$$

with the function $\varphi(x)$ defined by

$$\frac{d\tilde{\nu}}{d\nu} = e^{\varphi(x)}$$

and satisfying

$$\int_{\mathbb{R}^d} (e^{\varphi(x)/2} - 1)^2 \nu(dx) < \infty,$$

and

$$\tilde{a} - a - \int_{|x|\leq 1} x(\tilde{\nu} - \nu)(dx) \in \mathcal{R}(\sigma) \in \mathbb{R}^d.$$

where with $\mathcal{R}(L)$ we indicate the range of a linear operator L on $\mathbb{R}^d$.

Proof. See [109, Theorem 33.1]. □

Another important result allow us to construct a process U_t, such that

$$U_t = \frac{d\tilde{P}_{|\mathcal{F}_t}}{dP_{|\mathcal{F}_t}},$$

and

$$A \in \mathcal{F}_t \qquad \tilde{P} = E^P[Z_t I_A].$$

This result will be very important also for some simulation techniques.

Theorem 4.2. *Let $((X_t)_{t\geq 0}, P)$ and $((X_t)_{t\geq 0}, \tilde{P})$ be Lévy processes on $\mathbb{R}$ with generating triplets (a, σ, ν) and $(\tilde{a}, \tilde{\sigma}, \tilde{\nu})$, respectively. Suppose that the equivalent conditions (1) and (2) in the Theorem 4.1 are satisfied. Choose $\eta \in \mathbb{R}^d$ such that*

$$\tilde{a} - a - \int_{\|x\|\leq 1} x(\tilde{\nu} - \nu)(dx) = \sigma\eta.$$

Then we can define P-a.s.,

$$\begin{aligned} U_t &= \langle \eta, (X_t - X_t^{\nu}) \rangle - \frac{t}{2} \langle \eta, \sigma\eta \rangle - t\langle \gamma, \eta \rangle \\ &+ \lim_{\varepsilon \downarrow 0} \left(\sum_{(s, \Delta X_s) \in (0,t] \times \{|x| > \varepsilon\}} \varphi(\Delta X_s) - t \int_{|x|>\varepsilon} (e^{\varphi(x)-1}) \nu(dx) \right), \end{aligned} \tag{4.1}$$

where φ is the function in (2) *of Theorem 4.1 and $(X_t - X_t^{\nu})_{t\geq 0}$ is the continuous part of the process $(X_t)_{t\geq 0}$ under the measure P. The convergence in the right hand side of (4.1) is uniform in t on any bounded interval, P-a.s..We have for every $t \in [0, \infty)$,*

$$E^P[e^{U_t}] = E^{\tilde{P}}[e^{-U_t}] = 1$$

and

$$\frac{d\tilde{P}_{|\mathcal{F}_t}}{dP_{|\mathcal{F}_t}} = U_t \quad P - a.s.$$

The process $(U_t)_{t\geq 0}$ under the measure P is a Lévy process on $\mathbb{R}$ with generating triplet (a_U, σ_U, ν_U) expressed by

$$\begin{aligned} \sigma_U &= \langle \eta, \sigma\eta \rangle, \\ \nu_U &= (\nu\varphi^{-1})_{\mathbb{R}\setminus\{0\}}, \\ a_U &= -\frac{1}{2}\langle \eta, \sigma\eta \rangle - \int_{\mathbb{R}^d} (e^y - 1 - yI_{0<|y|\leq 1}(y))(\nu\varphi^{-1})(dy). \end{aligned} \tag{4.2}$$

An application of the above result, can be find in [107], where the author prove, under some conditions, the absolutely continuity of TS processes with respect to α-stable processes. By applying the result of [107] we can also obtain an alternative proof of the change of measure result of [95], where a particular TS distribution is chosen.

Theorem 4.3. *In the above setting consider two probability measures P and $\tilde{P}$ on $(\Omega, \mathcal{F})$ such that the process $(X_t)_{t\geq 0}$ under P is a Lévy α-stable process while under $\tilde{P}$ it is a proper TS process. Let us assume that under P, $X_1 \sim S_\alpha(\sigma, a)$ where σ is related to R by (2.29) and $\alpha \in (0, 2)$, while under $\tilde{P}$, $X_1 \sim TS^0(R, b)$ when $\alpha \in (0, 1)$ and $X(1) \sim TS$ when $\alpha \in [1, 2)$. Let ν, the Lévy measure corresponding to R, be as in 2.3, where $q(0^+, u) = 1$ for all $u \in S^{d-1}$. Then*

(i) *$P_{|\mathcal{F}_t}$ and $\tilde{P}_{|\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$ if and only if*

$$\int_{S^d} \int_0^1 (1 - q(r, u))^2 r^{-1-\alpha} dr\sigma(du) < \infty \tag{4.3}$$

and

$$b - a = \begin{cases} 0, & 0 < \alpha < 1 \\ \int_{\mathbb{R}^d} x(\log \|x\| - 1)R(dx), & \alpha = 1 \\ \Gamma(1 - \alpha) \int_{\mathbb{R}^d} xR(dx), & 1 < \alpha < 2. \end{cases} \tag{4.4}$$

Condition (4.3) implies that the integrals in (4.4) exist. Furthermore, if either (4.3) and (4.4) fails, then $P_{|\mathcal{F}_t}$ and $\tilde{P}_{|\mathcal{F}_t}$are singular for all $t > 0$.

(ii) *If (4.3) and (4.4) hold, then for each $t > 0$*

$$\frac{d\tilde{P}}{dP}_{|\mathcal{F}_t} = e^{U_t}, \tag{4.5}$$

where $(U_t)_{t\geq 0}$ is a Lévy process on $(\Omega, \mathcal{F}, P)$ given by

$$\begin{aligned} U_t = \lim_{\delta\downarrow 0} \Bigg\{ & \sum_{\{s\leq t:\|\Delta X_s\|>\delta\}} \log q\left(\|\Delta X_s\|, \frac{\Delta X_s}{\|\Delta X_s\|}\right) \\ & + t\int_{S^{d-1}}\int_{\delta}^{\infty}(1-q(r,u))r^{-\alpha-1}dr\sigma(du)\Bigg\}. \end{aligned} \tag{4.6}$$

The convergence is uniform in t on any bounded interval, P_0-a.s.. Furthermore, the Lévy measure ν_U of U_1 is concentrated on $(-\infty, 0)$ and determined by

$$\int_{-\infty}^{0} F(s)\nu_U(ds) = \int_{S^{d-1}}\int_0^{\infty} F(\log(q(r,u)))r^{-\alpha-1}dr\sigma(du)$$

for every Borel function F. The characteristic function of U_1 if of the form

$$E_P[e^{i\theta U_1}] = \exp\left\{i\theta a_0 + \int_{-\infty}^{0}(e^{i\theta v} - 1 - i\theta v I_{[-1,0)}(v))\nu_U(dv)\right\}, \tag{4.7}$$

where

$$a_0 = -\int_{-\infty}^{0}(e^{i\theta v} - 1 - i\theta v I_{[-1,0)})\nu_U(dv).$$

Proof. See [107, Theorem 4.1]. □

We will show now an application of above results. Even if the following result has been already proved in [95] we will like to show it as an consequence of the Theorem 4.3.

Proposition 4.4. *Let $(X_t)_{t\geq 0}$ a tempered stable process on the probability space $(\Omega, \mathcal{F}, P)$ with characteristic triplet $(0, a, \nu)$, where the Lévy density is of the form*

$$\nu(x) = \frac{c_+ e^{-\lambda_+ x}}{x^{1+\alpha_+}} I_{\{x>0\}}$$

Then there exists an equivalent measure Q such that the process $(X_t)_{t\geq 0}$ is stable with Lévy triplet $(0, \tilde{a}, \tilde{\nu})$ where

$$\tilde{\nu}(x) = \frac{c_+}{x^{1+\alpha_+}} I_{\{x>0\}},$$

and

$$\tilde{a} - a = \begin{cases} 0, & 0 < \alpha < 1 \\ c_+\lambda_+^{\alpha-1}(\log\frac{1}{\lambda_+} - 1), & \alpha = 1 \\ c_+\lambda_+^{\alpha-1}\Gamma(1-\alpha), & 1 < \alpha < 2. \end{cases} \tag{4.8}$$

Furthermore, the change of measure is defined as

$$\frac{dQ}{dP} = e^{Z_t}.$$

where the process Z_t is of the form

$$Z_t = \lambda X_t + ct,$$

with

$$c = \begin{cases} -\lambda_+ a_0 + c_+ \lambda_+^\alpha \Gamma(-\alpha), & 0 < \alpha < 1 \\ -\lambda_+ a_1 + c_+ \lambda_+^\alpha, & \alpha = 1 \\ -\lambda_+ a_1 + c_+ \lambda_+^\alpha (1-\alpha)\Gamma(-\alpha), & 1 < \alpha < 2. \end{cases}$$

Proof. By equations (2.35) and (2.32), under the measure P, the Rosińsky measure is defined as

$$R = c_+ \lambda_+^\alpha \delta_{\frac{1}{\lambda_+}}$$

and the function q is

$$q(r,1) = e^{-\lambda_+ r}, \qquad \lambda_+ > 0,$$

then the condition

$$\int_0^1 (1 - e^{-\lambda_+ r})^2 r^{-1-\alpha} < \infty$$

is satisfied, since $\alpha > 0$. Taking into account equation (2.28) with $\sigma(du) = \delta_1(u)$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} \int_0^\infty I_A(tx) t^{-\alpha-1} dt R(dx) &= c_+ \lambda_+^\alpha \int_0^\infty I_A(\frac{t}{\lambda_+}) t^{-\alpha-1} dt \\ &= c_+ \int_0^\infty I_A(s) s^{-\alpha-1} dt = \tilde{\nu}(A). \end{aligned}$$

If we set the measure R in equation (4.4), then (4.8) is fulfilled.

Now, we will prove the second statement. Under the above assumptions, we have

$$q(r,1) = e^{-\lambda_+ r}, \qquad \lambda_+ > 0$$

and by equation (4.6) with $U_t = -Z_t$, we can write

$$\begin{aligned} Z_t &= \lim_{\delta \downarrow 0} \left\{ \lambda_+ \sum_{\{s \le t : |\Delta X_s| > \delta\}} |\Delta X_s| - t c_+ \int_\delta^\infty (1 - e^{-\lambda_+ r}) r^{-\alpha-1} dr \right\} \\ &= \lim_{\delta \downarrow 0} \left\{ \lambda_+ \left(\sum_{\{s \le t : |\Delta X_s| > 1\}} |\Delta X_s| + \sum_{\{s \le t : \delta < |\Delta X_s| \le 1\}} |\Delta X_s| \right) \right. \\ &\qquad \left. - t c_+ \int_\delta^\infty (1 - e^{-\lambda_+ r}) r^{-\alpha-1} dr \right\} \end{aligned}$$

Now, by the Lévy-Itô decomposition (Theorem 1.38), for a general Lévy process having a generating triplet $(0, a, \nu)$, the process is equal to the sum of the jumps of magnitude greater than 1, plus the sum of the jumps of magnitude less than 1 compensated by its mean plus a drift term. This result allows us to calculate the limit above. Indeed, we can write

$$\begin{aligned} Z_t &= \lim_{\delta\downarrow 0}\Bigg\{\lambda_+\Bigg(\sum_{\{s\le t:|\Delta X_s|>1\}}|\Delta X_s| + \sum_{\{s\le t:\delta<|\Delta X_s|\le 1\}}|\Delta X_s| \\ &\quad - t\int_\delta^\infty r(c(r)+I_{\{|r|\le 1\}})\nu(dr)\Bigg) + \lambda_+ t\int_\delta^\infty r(c(r)+I_{\{|r|\le 1\}})\nu(dr) \\ &\quad - tc_+\int_\delta^\infty (1-e^{-\lambda_+ r})r^{-\alpha-1}dr\Bigg\}. \end{aligned} \tag{4.9}$$

Taking into account equality (1.8), the equality

$$a_0 = a - \int_{\mathbb{R}} rI_{\{|r|\le 1\}}(r)\nu(dr)$$

holds and if $0<\alpha<1$, the last integral in (4.9) converges, therefore by setting $c\equiv 0$ we obtain

$$Z_t = \lambda_+X_t + t(-\lambda_+a_0 + c_+\lambda_+^\alpha\Gamma(-\alpha)).$$

By integrating by parts and elementary properties of the gamma function, we can see that the equality

$$\int_0^\infty (1-e^{-\lambda_+ r}-\lambda_+re^{-\lambda_+ r})r^{-1-\alpha}dr = (\alpha-1)\lambda_+^\alpha\Gamma(-\alpha) \tag{4.10}$$

holds. Taking into account equality (1.8), the equality

$$a_1 = a + \int_{\mathbb{R}} r(c(r) - I_{\{|r|\le 1\}}(r))\nu(dr)$$

holds and if $1<\alpha<2$, the last integral in (4.9) does not converge and we have to compensate small jumps, therefore by setting $c\equiv 1$ and by using (4.10) we obtain

$$Z_t = \lambda_+X_t + t(-\lambda_+a_1 + c_+\lambda_+^\alpha(1-\alpha)\Gamma(-\alpha)).$$

Finally, let us consider the limiting case $\alpha\searrow 1$ and set $c\equiv 1$ as in the previous case. By a similar argument, we obtain

$$Z_t = \lambda_+X_t + t(-\lambda_+a + c_+\lambda_+^\alpha).$$

Hence (4.8) is fulfilled. □

Remark 4.5. *If we consider the process $X_t = X_t^+ + X_t^-$ where X_t^+ has Lévy measure*

$$\nu(x) = \frac{c_+e^{-\lambda_+x}}{x^{1+\alpha_+}}I_{\{x>0\}}$$

and X_t^- has Lévy measure

$$\nu(x) = \frac{c_-e^{-\lambda_-x}}{x^{1+\alpha_-}}I_{\{x<0\}}$$

we obtain the result of [95, Theorem 3.1].

4.1 Change of measure between TS processes

In this section, we are going to talk about the change of measure problem between two tempered stable processes. Roughly speaking, taking into consideration Theorem 4.3, we will consider two probability measures P_1 and P_2 on $(\Omega, \mathcal{F})$ such that the processes $(X_t^1)_{t\geq 0}$ under P_1 $(X_t^2)_{t\geq 0}$ under P_2 are proper tempered stable processes with the same tail index α and a probability measure $\tilde{P}$ on $(\Omega, \mathcal{F})$ such that the process $(X_t)_{t\geq 0}$ under P is a Lévy α-stable process. Furthermore, if we suppose that P is mutually absolutely continuous respect to P_1 and P_2, then under some additional assumptions, also P_1 and P_2 are mutually absolutely continuous.

Proposition 4.6. *Let us suppose the setting of the Theorem 4.3 and consider two probability measures P_1 and P_2 on $(\Omega, \mathcal{F})$ such that the processes $(X_t^1)_{t\geq 0}$ under P_1 and $(X_t^2)_{t\geq 0}$ under P_2 are proper tempered stable processes with the same tail index α and a probability measure P on $(\Omega, \mathcal{F})$ such that the process $(X_t)_{t\geq 0}$ under P is a Lévy α-stable process.*

Let us assume that under P, $X_1 \sim S_\alpha(\sigma, a)$ where σ is related to R by (2.29) and $\alpha \in (0,2)$, while under P_i, $X_1 \sim TS^0(R_i, b_i)$ when $\alpha \in (0,1)$ and $X(1) \sim TS(R_i, b_i)$ when $\alpha \in [1,2)$, for $i = 1,2$. Let ν, the Lévy measure corresponding to R, be as in 2.3, where $q(0^+, u) = 1$ for all $u \in S^{d-1}$. Furthermore, let us assume that $P_{|\mathcal{F}_t}$ and $P_{1|\mathcal{F}_t}$ (respectively $P_{2|\mathcal{F}_t}$) are mutually absolutely continuous for every $t > 0$. Then, $P_{1|\mathcal{F}_t}$ and $P_{2|\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$ if and only if

$$\int_{S^d} \int_0^1 (1 - q_i(r,u))^2 r^{-1-\alpha} dr \sigma(du) < \infty \tag{4.11}$$

for $i = 1,2$, and

$$b_1 - b_2 = \begin{cases} 0, & 0 < \alpha < 1 \\ \int_{\mathbb{R}^d} x(\log\|x\| - 1) R_1(dx) - \int_{\mathbb{R}^d} x(\log\|x\| - 1) R_2(dx), & \alpha = 0 \\ \Gamma(1-\alpha)\{\int_{\mathbb{R}^d} x R_1(dx) - \int_{\mathbb{R}^d} x R_2(dx)\}, & 1 < \alpha < 2. \end{cases} \tag{4.12}$$

Conditions (4.11) implies that the integrals in (4.12) exist.

Proof. By using the transitive property of the mutually absolutely continuity and Theorem 4.3, we obtain (4.11) and (4.12). □

The advantage of the above theorem is that we can find conditions for the equivalence of probability measures in a easier way than directly applying the Theorem 4.1. As already noted in [107, Example 4], if we take

$$q(r,u) = e^{-r^\beta},$$

where $0 < \beta \leq \frac{\alpha}{2}$, then condition (4.3) fails.

The direct application of the Theorem 4.1 gives us another possible density transformation in the tempered stable Lévy processes framework, see also [64].

Proposition 4.7. *Let us suppose the setting of the Theorem 4.3 and consider two probability measures P_1 and P_2 on $(\Omega, \mathcal{F})$ such that the processes $(X_t^1)_{t\geq 0}$ under P_1 and $(X_t^2)_{t\geq 0}$ under P_2 are proper tempered stable processes with the same tail* *index α. Let us assume that under P_i, $X_1 \sim TS^0(R_i, b_i)$ when $\alpha \in (0,1)$ and $X(1) \sim TS(R_i, b_i)$ when $\alpha \in [1,2)$, for $i = 1,2$. Then, $P_{1|\mathcal{F}_t}$ and $P_{2|\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$ if and only if*

$$\int_{S^d}\int_0^1 (\sqrt{q_2(r,u)} - \sqrt{q_1(r,u)})^2 r^{-1-\alpha} dr\sigma(du) < \infty \tag{4.13}$$

and

$$b_1 - b_2 \overset{?}{=} \begin{cases} 0, & 0 < \alpha < 1 \\ \int_{\mathbb{R}^d} x(\log\|x\| - 1)R_1(dx) - \int_{\mathbb{R}^d} x(\log\|x\| - 1)R_2(dx), & \alpha = 1 \\ \Gamma(1-\alpha)\{\int_{\mathbb{R}^d} xR_1(dx) - \int_{\mathbb{R}^d} xR_2(dx)\}, & 1 < \alpha < 2. \end{cases} \tag{4.14}$$

Conditions (4.11) implies that the integrals in (4.12) exist.

As far as we know, it is not yet clear if there exists not trivial examples of completely monotone functions q_1 and q_2 such that conditions (4.11) fail but condition (4.13) is fulfilled.

4.1.1 Change of measure for KR processes

A flexible distribution is needed in order to find an equivalent change of measure and, at the same time, take into account the historical estimates. To this end, we focus our attention on the KR tempered stable distribution. The risk-neutral process can be fitted by matching model prices to market prices of options using nonlinear least squares. The easy form of the characteristic function of the KR distribution allows one to obtain a suitable solution to the calibration problem. To demonstrate the advantages of the exponential KR model, we will present some empirical results. First, we have to show a result about the change of measure problem for KR tempered stable distribution.

Theorem 4.8. *Consider two probability measures P_1, P_2 and the canonical process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ given above. Let us suppose $(X_t)_{t\geq 0}$ is the KR tempered stable process under P_i with parameters (α_i, k_{i+}, k_{i-}, r_{i+}, r_{i-}, p_{i+}, p_{i-}, m_i) with $i = 1,2$ and*

$$\begin{cases} p_{i\pm} > \frac{1}{2} - \alpha_i, & \alpha_i \in (0,1) \\ p_{i\pm} > 1 - \alpha_i, & \alpha_i \in [1,2) \end{cases}. \tag{4.15}$$

Then $P_1|_{\mathcal{F}_t}$ and $P_2|_{\mathcal{F}_t}$ are equivalent for every $t > 0$ if and only if

$$\alpha := \alpha_1 = \alpha_2, \tag{4.16}$$

$$\frac{k_{1+}r_{1+}^{\alpha}}{\alpha + p_{1+}} = \frac{k_{2+}r_{2+}^{\alpha}}{\alpha + p_{2+}}, \qquad \frac{k_{1-}r_{1-}^{\alpha}}{\alpha + p_{1-}} = \frac{k_{2-}r_{2-}^{\alpha}}{\alpha + p_{2-}}, \tag{4.17}$$

if $\alpha \neq 1$ then

$$m_2 - m_1 = \Gamma(1-\alpha)\left(\left(\frac{k_{2+}r_{2+}}{p_{2+}+1} - \frac{k_{2-}r_{2-}}{p_{2-}+1}\right) - \left(\frac{k_{1+}r_{1+}}{p_{1+}+1} - \frac{k_{1-}r_{1-}}{p_{1-}+1}\right)\right) \tag{4.18}$$

and if $\alpha = 1$ *then*

$$m_2 - m_1 = \frac{k_{i+}r_{i+}}{p_{i+}+1}\left(\log r_{i+} - \frac{p_{i+}+2}{p_{i+}+1}\right) - \frac{k_{i-}r_{i-}}{p_{i-}+1}\left(\log r_{i-} - \frac{p_{i-}+2}{p_{i-}+1}\right) - \frac{k_{i+}r_{i+}}{p_{i+}+1}\left(\log r_{i+} - \frac{p_{i+}+2}{p_{i+}+1}\right) - \frac{k_{i-}r_{i-}}{p_{i-}+1}\left(\log r_{i-} - \frac{p_{i-}+2}{p_{i-}+1}\right) \tag{4.19}$$

Proof. By equation (2.45), the spectral measure R_i of a distribution

$$KR(\alpha_i, k_{i,+}, k_{i,-}, r_{i,+}, r_{i,-}, p_{i,+}, p_{i,-}, m_i),$$

the is equal to

$$R_i(dx) = (k_{i+}r_{i+}^{-p_{i+}} I_{x\in(0,r_{i+})}|x|^{p_{i+}-1} + k_{i-}r_{i-}^{-p_{i-}} I_{x\in(-r_{i-},0)}|x|^{p_{i-}-1})dx \tag{4.20}$$

and by equation (2.40) we have

$$\sigma_i(A) = \frac{k_{i,+}r_{i,+}^{\alpha_i}}{\alpha_i + p_{i,+}} I_A(1) + \frac{k_{i,-}r_{i,-}^{\alpha_i}}{\alpha_i + p_{i,-}} I_A(-1), \qquad A \subset S^0$$

and

$$q_i(r, \pm 1) = (\alpha + p_{i\pm})r_{i\pm}^{-\alpha_i - p_{i\pm}} \int_0^{+\infty} e^{-rt} I_{\{t > \frac{1}{r_{i\pm}}\}} t^{-\alpha_i - p_{i\pm} - 1} dt.$$

First, we will show that condition (4.15) implies the finiteness of integrals (4.11). To prove this fact, let us consider $\alpha_i \in (0,1)$ and if $p_{i,\pm} > \frac{1}{2} - \alpha_i$ then we have

$$\begin{aligned}
\frac{d}{dr} q_i(r, \pm 1) &= -(\alpha_i + p_{i\pm})r_{i\pm}^{-\alpha_i - p_{i\pm}} \int_{1/r_{i\pm}}^{\infty} e^{-rt} t^{-\alpha_i - p_{i\pm}} dt \\
&\geq -(\alpha_i + p_{i\pm})r_{i\pm}^{-\alpha_i - p_{i\pm}} \int_{1/r_{i\pm}}^{\infty} \frac{1}{\sqrt{rt}} t^{-\alpha_i - p_{i\pm}} dt \\
&= -\frac{\alpha_i + p_{i\pm}}{\sqrt{r_{i\pm}}(\alpha_i + p_{i\pm} - \frac{1}{2})} r^{-\frac{1}{2}}.
\end{aligned}$$

If we consider $\alpha \in [1,2)$ and if $p_{i,\pm} > 1 - \alpha_i$, then we have

$$\begin{aligned}
\frac{d}{dr} q_i(r, \pm 1) &= -(\alpha_i + p_{i\pm})r_{i\pm}^{-\alpha_i - p_{i\pm}} \int_0^{r_{i\pm}} e^{-r/s} s^{\alpha_i + p_{i\pm} - 2} ds \\
&\geq= -(\alpha_i + p_{i\pm})r_{i\pm}^{-\alpha_i - p_{i\pm}} \int_0^{r_{i\pm}} s^{\alpha_i + p_{i\pm} - 2} ds \\
&= -\frac{\alpha_i + p_{i\pm}}{r_{i,\pm}(\alpha_i + p_{i\pm} - 1)}.
\end{aligned}$$

Let

$$K_i = \begin{cases} \min\left\{-\frac{\alpha_i + p_{i+}}{\sqrt{r_{i+}}(\alpha_i + p_{i+} - 1/2)}, -\frac{\alpha_i + p_{i-}}{\sqrt{r_{i-}}(\alpha_i + p_{i-} - 1/2)}\right\}, & \alpha_i \in (0,1) \\ \min\left\{-\frac{\alpha_i + p_{i+}}{r_{i+}(\alpha_i + p_{i+} - 1)}, -\frac{\alpha_i + p_{i-}}{r_{i-}(\alpha_i + p_{i-} - 1)}\right\}, & \alpha_i \in [1,2) \end{cases}$$

then

$$0 > \frac{d}{dr} q_i(r, \pm 1) \geq \begin{cases} K_i r^{-1/2}, & \alpha_i \in (0,1) \\ K_i, & \alpha_i \in [1,2) \end{cases}.$$

By the integration of the last inequality on the interval $(0,r)$, we obtain

$$0 \geq q_i(r,\pm 1) - 1 = q_i(r,\pm 1) - q_i(0,\pm 1) \geq \begin{cases} 2K_i r^{1/2}, & \alpha_i \in (0,1) \\ K_i r, & \alpha_i \in [1,2) \end{cases}.$$

Hence,

$$\begin{aligned} &\int_{S^0}\int_0^1 (1-q_i(r,u))^2 r^{-\alpha_i-1} dv\,\sigma(du) \\ &\leq \begin{cases} \int_{S^0}\int_0^1 4K_i^2 r^{-\alpha_i} dr\,\sigma(du), & \alpha_i \in (0,1) \\ \int_{S^0}\int_0^1 K_i^2 r^{-\alpha_i+1} dv\,\sigma(du), & \alpha_i \in [1,2) \end{cases} \\ &= \begin{cases} \frac{4K_i^2}{1-\alpha_i}\int_{S^0}\sigma(du), & \alpha_i \in (0,1) \\ \frac{K_i^2}{2-\alpha_i}\int_{S^0}\sigma(du), & \alpha_i \in [1,2) \end{cases} \\ &< \infty, \end{aligned}$$

therefore (4.11) holds. Since

$$\int_{\mathbb{R}} |x| R_i(dx) < \infty$$

the mean of the random variable X_1 under the measure P_i is finite. By Remark 2.11, if $\alpha \in (0,1)$ we can write

$$m_i = b_{i0} + \Gamma(1-\alpha)\int_{\mathbb{R}} x R_i(dx).$$

where m_i is the mean of the random variable X_1 that is

$$m_i = \int_{\mathbb{R}} x\mu_i(dx).$$

The finiteness of the mean allows us to consider only the representation (2.13) for all $\alpha \in (0,2)$. By condition (4.4), we obtain

$$m_2 - m_1 = \begin{cases} \int_{\mathbb{R}^d} x(\log\|x\|-1)R_2(dx) - \int_{\mathbb{R}^d} x(\log\|x\|-1)R_1(dx), & \alpha = 1 \\ \Gamma(1-\alpha)\{\int_{\mathbb{R}^d} xR_2(dx) - \int_{\mathbb{R}^d} xR_1(dx)\}, & \alpha \neq 1. \end{cases}$$

To complete the proof, we will calculate integrals above. If $\alpha = 1$, then $p_{j\pm} > 0$ and integrating by parts we have

$$\begin{aligned} &\int_{\mathbb{R}} x(\log|x|-1)R_i(dx) \\ &= k_{i+} r_{i+}^{-p_{i+}} \int_0^{r_{i+}} (\log x - 1) x^{p_{i+}} dx - k_{i-} r_{i-}^{-p_{i-}} \int_0^{r_{i-}} (\log x - 1) x^{p_{i-}} dx \\ &= \left(\frac{k_{i+} r_{i+}}{p_{i+}+1} \left(\log r_{i+} - \frac{p_{i+}+2}{p_{i+}+1} \right) - \frac{k_{i-} r_{i-}}{p_{i-}+1} \left(\log r_{i-} - \frac{p_{i-}+2}{p_{i-}+1} \right) \right), \end{aligned}$$

and if $\alpha \neq 1$, then

$$\begin{aligned} \int_{\mathbb{R}} xR_i(dx) &= k_{i+} r_{i+}^{-p_{i+}} \int_0^{r_{i+}} x^{p_{i+}} dx - k_{i-} r_{i-}^{-p_{i-}} \int_0^{r_{i-}} x^{p_{i-}} dx \\ &= \left(\frac{k_{i+} r_{i+}}{p_{i-}+1} - \frac{k_{i-} r_{i-}}{p_{i-}+1} \right) \end{aligned}$$

By Theorem 4.6, $P_{1|\mathcal{F}_t}$ and $P_{2|\mathcal{F}_t}$ are equivalent for every $t > 0$ if and only if (4.11) and (4.12) are verified, then we obtain the result that $P_{1|\mathcal{F}_t}$ and $P_{1|\mathcal{F}_t}$ are equivalent for every $t > 0$ if and only if the parameters satisfy (4.16), (4.17) and condition (4.18), if $\alpha \neq, 1$ or condition (4.19), if $\alpha = 1$. □

The family of tempered stable process is flexible under change of measure and we can also prove absolutely continuity between a KoBoL process and a KR process. The idea comes from the fact that we can easily find estimates based on historical data by using a KoBoL process in the real world and then, under some conditions on parameters, the risk-neutral process can be fitted by matching model prices to market prices of options using nonlinear least squares and a KR dynamic.

Proposition 4.9. *Consider two probability measures P_1, P_2 and the canonical process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ given above. Let us suppose $(X_t)_{t\geq 0}$ is a KoBoL process under P_1 with parameters (α_1, c_+, c_-, λ_+, λ_-, m_1) and a KR tempered stable process under P_2 with parameters (α_2, k_+, k_-, r_+, r_-, p_+, p_-, m_2)*

$$\begin{cases} p_\pm > \frac{1}{2} - \alpha, & \alpha \in (0,1) \\ p_\pm > 1 - \alpha, & \alpha \in [1,2). \end{cases} \tag{4.21}$$

Then $P_1|_{\mathcal{F}_t}$ and $P_2|_{\mathcal{F}_t}$ are equivalent for every $t > 0$ if and only if

$$\alpha := \alpha_1 = \alpha_2, \tag{4.22}$$

$$c_+ = \frac{k_+ r_+^\alpha}{\alpha + p_+}, \qquad c_- = \frac{k_- r_-^\alpha}{\alpha + p_-}, \tag{4.23}$$

if $\alpha \neq 1$ then

$$m_2 - m_1 = \Gamma(1-\alpha)\left(\left(\frac{k_+ r_+}{p_+ + 1} - \frac{k_- r_{2-}}{p_- + 1}\right) - (c_+\lambda_+^{\alpha-1} - c_-\lambda_-^{\alpha-1})\right) \tag{4.24}$$

and if $\alpha = 1$ then

$$\begin{aligned} m_2 - m_1 = {} & \frac{k_{i+} r_{i+}}{p_{i+}+1}\left(\log r_{i+} - \frac{p_{i+}+2}{p_{i+}+1}\right) - \frac{k_{i-} r_{i-}}{p_{i-}+1}\left(\log r_{i-} - \frac{p_{i-}+2}{p_{i-}+1}\right) \\ & - \left(\frac{c_+}{\lambda_+}\left(\log\frac{1}{\lambda_+} - 1\right) - \frac{c_-}{\lambda_-}\left(\log\frac{1}{\lambda_-} - 1\right)\right) \end{aligned} \tag{4.25}$$

Proof. See the proof of Theorem 4.8. □

Now, it is well understood how we can find a density transformation in the tempered stable processes framework. Furthermore, by using Theorem 4.6 where measure change between two CGMY processes is considered, we obtain the same result of [66].

4.1.2 Change of measure for GTS processes

If we consider a GTS distribution with parameters $\alpha_+ \neq \alpha_-$, it is not a tempered stable distribution in the Rosiński sense, anyway, the change of measure problem can be solved, by using similar arguments of previous sections.

Proposition 4.10. *Consider two probability measures P, $\tilde{P}$ and the canonical process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ given above. Let us suppose $(X_t)_{t\geq 0}$ is a GTS process under P with parameters $(\alpha_+,\ \alpha_-\ c_+,\ c_-,\ \lambda_+,\ \lambda_-,\ m)$ and a GTS tempered stable process under $\tilde{P}$ with parameters $(\tilde{\alpha}_+,\ \tilde{\alpha}_-,\ \tilde{c}_+,\ \tilde{c}_-,\ \tilde{\lambda}_+,\ \tilde{\lambda}_-,\ \tilde{m})$. Then $P|_{\mathcal{F}_t}$ and $\tilde{P}|_{\mathcal{F}_t}$ are equivalent for every $t > 0$ if and only if*

$$\alpha_+ = \tilde{\alpha}_+ \qquad \alpha_- = \tilde{\alpha}_-, \tag{4.26}$$

$$c_+ = \tilde{c}_+, \qquad c_- = \tilde{c}_-, \tag{4.27}$$

if $\alpha_+, \alpha_- \neq 1$ then

$$\tilde{m} - m = \Gamma(1-\alpha_+)c_+(\tilde{\lambda}_+^{\alpha_+-1} - \lambda_+^{\alpha_+-1}) - \Gamma(1-\alpha_-)c_-(\tilde{\lambda}_-^{\alpha_--1} - \lambda_-^{\alpha_--1}) \tag{4.28}$$

and if $\alpha_+ = \alpha_- = 1$ then

$$\begin{aligned} \tilde{m} - m = &\left(\frac{c_+}{\tilde{\lambda}_+}\left(\log\frac{1}{\tilde{\lambda}_+} - 1\right) - \frac{c_-}{\tilde{\lambda}_-}\left(\log\frac{1}{\tilde{\lambda}_-} - 1\right)\right) \\ &- \left(\frac{c_+}{\lambda_+}\left(\log\frac{1}{\lambda_+} - 1\right) - \frac{c_-}{\lambda_-}\left(\log\frac{1}{\lambda_-} - 1\right)\right) \end{aligned} \tag{4.29}$$

4.2 The Esscher transform

The procedure of tilting to obtain a density transformation is also related to the Esscher transform. Having in mind the definition of TS distributions, let ν be a Lévy measure of a TS distribution X such that

$$\int_{|x|\geq 1} e^{\theta x}\nu(dx) < \infty,$$

that is X has exponential moment of order θ, then the measure $\tilde{\nu}(dx) = e^{\theta x}\nu(dx)$ is also a Lévy measure. Of course this measure change is only a particular case of Theorem 4.1, but it is widely used in financial as well as in insurance mathematics [47]. Due to the structure of the Lévy measure, in some case, if the initial random variable is TS, also the transformed one is TS. To prove this fact, we will be going to consider the IG and the BΓ distribution.

Proposition 4.11. *Let $\phi(u)$ and $\tilde{\phi}(u)$ be the characteristic functions for the infinitely divisible distributions with Lévy triples $(a,\ 0,\ \nu)$ and $(\tilde{a},\ 0,\ \tilde{\nu})$ respectively, where $\tilde{\nu}(dx) = e^{\theta x}\nu(dx)$ and condition*

$$\int_{|x|\geq 1} e^{\theta x}\nu(dx) < \infty,$$

then, assuming a truncation function $h(x) = x$ in the Lévy-Khinchin representation, the following relation holds

$$\log\tilde{\phi}(u) = \log\phi(u - i\theta) - \log\phi(-i\theta) + iu\Big(\tilde{a} - a - \int_{\mathbb{R}} x(e^{\theta x} - 1)\nu(dx)\Big) \tag{4.30}$$

Proof. By Theorem 4.1, the result is proved. □

The result above gives a easy procedure to find the characteristic function of the transformed random variable, having the characteristic function of the initial random variable.

Proposition 4.12. *Let X a random variable with distribution $X \sim IG(a,b,m)$ and $\tilde{X}$ the random variable obtained by the Esscher transform of X. Then $\tilde{X}$ is a random variable with distribution $X \sim IG(a, \sqrt{b^2-2\theta}, m)$.*

Proof. A $X \sim IG(a,b,m)$ random variable has Lévy measure

$$\nu(dx) = \frac{a}{\sqrt{2\pi}} x^{-\frac{3}{2}} e^{-\frac{b^2}{2}x} I_{\{x>0\}} dx$$

and characteristic function

$$\phi(u) = \exp(-a(\sqrt{b^2-2iu}-b) - iu\frac{a}{b} + ium).$$

First, we evaluate the integral of the last term of equation 4.30, that is

$$\begin{aligned}
\int_{\mathbb{R}} x(e^{\theta x}-1)\nu(dx) &= \int_{\mathbb{R}} \frac{a}{\sqrt{2\pi}} x^{-\frac{3}{2}} e^{-\frac{b^2}{2}x} x(e^{\theta x}-1) I_{\{x>0\}} dx \\
&= \frac{a}{\sqrt{2\pi}} \int_0^\infty \sum_1^\infty x^{-\frac{1}{2}} e^{-\frac{b^2}{2}x} \frac{(\theta x)^n}{n!} dx \\
&= \frac{a}{\sqrt{2\pi}} \sum_1^\infty \frac{\theta^n}{n!} \int_0^\infty x^{n-\frac{1}{2}} e^{-\frac{b^2}{2}x} dx \\
&= \frac{a}{\sqrt{2\pi}} \sum_1^\infty \frac{\theta^n}{n!} \Big(\frac{b^2}{2}\Big)^{-(n+\frac{1}{2})} \Gamma\Big(n+\frac{1}{2}\Big) \qquad (4.31) \\
&= \frac{a}{\sqrt{2\pi}} \Big(\frac{b^2}{2}\Big)^{-\frac{1}{2}} \sum_1^\infty \frac{1}{n!} \Big(\frac{2\theta}{b^2}\Big)^n \Gamma\Big(n+\frac{1}{2}\Big) \\
&= \frac{a}{\sqrt{2\pi}} \Big(\frac{b^2}{2}\Big)^{-\frac{1}{2}} \sqrt{\pi}\bigg(\Big(1-\frac{2\theta}{b^2}\Big)^{-\frac{1}{2}} - 1\bigg) \\
&= \frac{a}{\sqrt{b^2-2\theta}} - \frac{a}{b}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\log\tilde{\phi}(u) &= -a(\sqrt{b^2-2i(u-i\theta)}-b) - i(u-i\theta)\frac{a}{b} + i(u-i\theta)m \\
&\quad + a(\sqrt{b^2-2\theta}-b) + \theta\frac{a}{b} - \theta m - iu\Big(\frac{a}{\sqrt{b^2-2\theta}} - \frac{a}{b}\Big) \\
&= -a(\sqrt{b^2-2\theta-2iu} - \sqrt{b^2-2\theta}) - iu\frac{a}{\sqrt{b^2-2\theta}} + ium,
\end{aligned}$$

which proves the result. □

Proposition 4.13. *Let X a random variable $X \sim \mathrm{B}\Gamma(a_+, b_+, a_-, b_-, m)$ and $\tilde{X}$ the random variable obtained by the Esscher transform of X. If $\theta < b_+$, then $\tilde{X}$ is a random variable $X \sim \mathrm{B}\Gamma(a_+, b_+-\theta, a_-, b_-+\theta, m)$.*

Proof. A $X \sim B\Gamma(a_+, b_+, a_-, b_-, m)$ random variable has Lévy measure

$$\nu(dx) = a_+ x^{-1} e^{-b_+ x} I_{\{x>0\}} + a_- |x|^{-1} e^{-b_- |x|} I_{\{x<0\}} dx$$

and the characteristic exponent is given by

$$\psi(u) = ium + a_+ \Big(\log \Big(\frac{b_+}{b_+ - iu} \Big) - \frac{iu}{b_+} \Big) + a_- \Big(\log \Big(\frac{b_-}{b_- + iu} \Big) + \frac{iu}{b_-} \Big)$$

We evaluate the integral of the last term of equation 4.30,

$$\begin{aligned}
\int_{\mathbb{R}} x(e^{\theta x} - 1)\nu(dx) &= a_+ \int_0^{+\infty} (e^{(\theta - b_+)x} - e^{-b_+ x}) dx - a_- \int_{-\infty}^0 e^{\theta x - b_- |x|} - e^{-b_- |x|}) dx \\
&= a_+ \int_0^{+\infty} (e^{(\theta - b_+)x} - e^{-b_+ x}) dx - a_- \int_{-\infty}^0 e^{\theta x - b_- |x|} - e^{-b_- |x|}) dx \\
&= a_+ \int_0^{+\infty} (e^{(\theta - b_+)x} - e^{-b_+ x}) dx - a_- \int_0^{+\infty} (e^{-(\theta + b_-)x} - e^{-b_- x}) dx \\
&= \frac{a_+}{b_+ - \theta} - \frac{a_+}{b_+} - \frac{a_-}{b_- + \theta} + \frac{a_-}{b_-},
\end{aligned}$$

Therefore by considering the integral above, the evaluation of (4.30) gives the desired result. □

4.3 Change of measure for TID processes

In this section, a result on density transformations between stable and TID processes is considered.

Theorem 4.14. *Let P_0 and P be probability measures on $(\Omega, \mathcal{F})$ such that the canonical process $\{X(t) : t \geq 0\}$ is a Lévy α-stable process $S_\alpha(\sigma, a)$ under P_0, while it is a proper TID process $TID_\alpha(R, b)$ under P, where σ is given by equation (3.15). Then*

(i) *$P_{0|\mathcal{F}_t}$ and $P_{|\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$ if and only if*

$$\int_{S^{d-1}} \int_0^1 \Big(1 - q(r,u)\Big)^2 r^{-\alpha-1} dr \sigma(du) < \infty, \tag{4.32}$$

and

$$b - a = \begin{cases} 0, & 0 < \alpha < 1 \\ \int_{\mathbb{R}^d} x(\log \|x\| + \frac{\log 2}{2} - 1 + \frac{\gamma}{2}) R(dx), & \alpha = 1 \\ 2^{-\frac{1}{2} - \frac{\alpha}{2}} \Gamma(\frac{1}{2} - \frac{\alpha}{2}) \int_{\mathbb{R}^d} x R(dx), & 1 < \alpha < 2 \end{cases} \tag{4.33}$$

Condition (4.32) implies that the integral exist. Furthermore, if either (4.32) or (4.33) fails, then $P_{0|\mathcal{F}_t}$ and $P_{|\mathcal{F}_t}$ are singular for all $t > 0$.

(ii) *If (4.32) and (4.33) hold, then for each $t > 0$*

$$\frac{dP}{dP_0}_{|\mathcal{F}_t} = e^{Z_t}, \tag{4.34}$$

where $\{Z_t : t \geq 0\}$ *is a Lévy process on* $(\Omega, \mathcal{F}, P_0)$ *given by*

$$Z_t = \lim_{\delta \downarrow 0} \Bigg\{ \sum_{\{s \leq t : \|\Delta X_s\| > \delta\}} \log q\Big(\|\Delta X_s\|, \frac{\Delta X_s}{\|\Delta X_s\|}\Big) + t \int_{S^{d-1}} \int_{\delta}^{\infty} (1 - q(r,u)) r^{-\alpha-1} dr \sigma(du) \Bigg\}.$$

The convergence is uniform in t *on any bounded interval,* P_0*-a.s..*

Proof. First, we will prove part (i). By equalities (3.1) and (3.3), we have

$$\frac{d\nu}{d\nu_0}(x) = q\Big(\|x\|, \frac{x}{\|x\|}\Big), \quad x \in \mathbb{R}^d \setminus \{0\}. \tag{4.35}$$

and for each $A \in \mathcal{B}(\mathbb{R}^d)$

$$\int_A q\Big(\|x\|, \frac{x}{\|x\|}\Big) \nu_0(dx) = \int_{S^{d-1}} \int_0^{\infty} I_A(ru) q(r,u) r^{-\alpha-1} dr \sigma(du) = \nu(A).$$

By Theorem 33.1 in [109], define the function $\phi(x)$ by

$$\frac{d\nu}{d\nu_0}(x) = e^{\phi(x)}.$$

Thus, we have

$$\phi(x) = \log q\Big(\|x\|, \frac{x}{\|x\|}\Big),$$

and $P_{0|\mathcal{F}_t}$ and $P_{\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$ if and only if

$$\int_{\mathbb{R}^d} \Big(e^{\frac{\phi(x)}{2} - 1} - 1\Big)^2 \nu_0(dx) < \infty \tag{4.36}$$

and

$$B_\alpha = 0, \tag{4.37}$$

where B_α is defined by

$$B_\alpha = \begin{cases} b + \int_{\|x\| \leq 1} x\nu(dx) - (a + \int_{\|x\| \leq 1} x\nu_0(dx)) - \int_{\|x\| \leq 1} x(\nu - \nu_0)(dx), & 0 < \alpha < 1, \\ b - \int_{\|x\| > 1} x\nu(dx) - (a - c\int_{S^{d-1}} u\sigma(du)) - \int_{\|x\| \leq 1} x(\nu - \nu_0)(dx), & \alpha = 1, \\ b - \int_{\|x\| > 1} x\nu(dx) - (a - \int_{\|x\| > 1} x\nu_0(dx)) - \int_{\|x\| \leq 1} x(\nu - \nu_0)(dx), & 1 < \alpha < 2. \end{cases}$$

In the case $\alpha = 1$, $c = 1 - \gamma$, where γ is the Euler constant. The inequality (4.36) can be written as

$$\int_{\mathbb{R}^d} \Big(1 - q^{1/2}\Big(\|x\|, \frac{x}{\|x\|}\Big)\Big)^2 \nu_0(dx) < \infty \tag{4.38}$$

Since the integrand is bounded and ν_0 is a Lévy measure, we may focus our attention only on integration over $\{\|x\| \leq 1\}$. Applying elementary inequalities

$$\frac{1}{4}(1-y)^2 \leq (1 - \sqrt{y})^2 \leq (1-y)^2$$

for $y \in [0,1]$, inequality (4.38) becomes

$$\int_{\{\|x\|\le 1\}} \left(1 - q\Big(\|x\|, \frac{x}{\|x\|}\Big)\right)^2 \nu_0(dx),$$

and writing the above integral in polar coordinates, we obtain (4.32). Now, we will prove the equivalence between conditions (4.33) and (4.37). By finiteness of the integral above and Hölder inequality, we have

$$\begin{aligned} &\int_{\|x\|\le 1} \|x\|(\nu_0 - \nu)(dx) = \int_{\|x\|\le 1} \|x\|\Big(1 - q\Big(\|x\|, \frac{x}{\|x\|}\Big)\Big)\nu_0(dx) \\ &\le \Big(\int_{\|x\|\le 1} \|x\|^2 \nu_0(dx)\Big)^{1/2} \Big(\int_{\|x\|\le 1} \Big(1 - q\Big(\|x\|, \frac{x}{\|x\|}\Big)\Big)^2 \nu_0(dx)\Big)^{1/2} < \infty \end{aligned} \tag{4.39}$$

If $0 < \alpha < 1$, then $B_\alpha = b - a = 0$ by (4.33). Suppose $1 < \alpha < 2$, then we have

$$\int_{\|x\|>1} \|x\|\nu(dx) = \int_{\|x\|>1} \|x\| q\Big(\|x\|, \frac{x}{\|x\|}\Big)\nu_0(dx).$$

and, since we are considering a proper TID process

$$q\Big(\|x\|, \frac{x}{\|x\|}\Big) \le 1,$$

and ν_0 is the Lévy measure of an α-stable distribution with $1 < \alpha < 2$, we obtain

$$\int_{\|x\|>1} \|x\|\nu(dx) \le \int_{\|x\|>1} \|x\|\nu_0(dx) < \infty.$$

Furthermore, by (4.39)

$$\int_{\mathbb{R}^d} \|x\|(\nu_0 - \nu)(dx) < \infty.$$

By using (3.9) and (3.14) and integrating by parts, the following result is obtained

$$\begin{aligned} \int_{\mathbb{R}^d} \|x\|(\nu_0 - \nu)(dx) &= \int_{\mathbb{R}^d} \int_0^\infty \|x\| t^{-\alpha}(1 - e^{-t^2/2}) dt R(dx) \\ &= -2^{-(1+\alpha)/2}\Gamma\Big(\frac{1}{2} - \frac{\alpha}{2}\Big) \int_{\mathbb{R}^d} \|x\| R(dx) < \infty. \end{aligned}$$

With a similar calculus, we can write

$$B_\alpha = \int_{\mathbb{R}^d} x(\nu_0 - \nu)(dx) + b - a = -2^{-(1+\alpha)/2}\Gamma(\frac{1}{2} - \frac{\alpha}{2}) \int_{\mathbb{R}^d} x R(dx) + b - a = 0,$$

where the last equality follows by (4.33), proving (4.37). It remains to verify (4.37) in the case $\alpha = 1$. By taking into account (4.39),

$$\begin{aligned} \infty > \int_{\|x\|\le 1} \|x\|(\nu_0 - \nu)(dx) &= \int_{\mathbb{R}^d} \|x\| \int_0^{1/\|x\|} t^{-1}(1 - e^{-t^2/2}) dt R(dx) \\ \ge \frac{1}{4} \int_{\|x\|\le 1} \|x\| \int_1^{\frac{1}{\|x\|}} t^{-1} dt R(dx) &= \frac{1}{4} \int_{\|x\|\le 1} \|x\|| \log \|x\| R(dx). \end{aligned}$$

Since the last integral is finite, the integral in (4.33) is well defined and we can calculate

$$\begin{aligned}\int_{\|x\|\leq 1} x(\nu_0-\nu)(dx) &= \int_{\mathbb{R}^d} x\int_0^{1/\|x\|} t^{-1}(1-e^{-t^2/2})dtR(dx)\\ &= \frac{1}{2}\int_{\mathbb{R}^d} x\Big(E_1\Big(\frac{1}{2\|x\|^2}\Big)-2\log\|x\|-\log 2+\gamma\Big)R(dx)\end{aligned}$$

by changing variable and equation 5.1.39 in [1], where the function $E_1(x)$ is the exponential integral function defines by

$$E_1(x)=\int_x^\infty t^{-1}e^{-t}.$$

By part (b) of Proposition 3.11, the first moment is finite, thus

$$\int_{\|x\|>1} x\nu(dx)<\infty$$

and

$$\begin{aligned}\int_{\|x\|>1} x\nu(dx) &= \int_{\mathbb{R}^d} x\int_{1/\|x\|}^\infty t^{-1}e^{-t^2/2}dtR(dx)\\ &= \frac{1}{2}\int_{\mathbb{R}^d} xE_1\Big(\frac{1}{2\|x\|^2}\Big)R(dx).\end{aligned}$$

Adding together the above results, we have

$$\begin{aligned}B_1 &= b-\frac{1}{2}\int_{\mathbb{R}^d} xE_1\Big(\frac{1}{2\|x\|^2}\Big)R(dx)-a+(1-\gamma)\int_{S^{d-1}} u\sigma(du)\\ &\quad+\frac{1}{2}\int_{\mathbb{R}^d} x\Big(E_1\Big(\frac{1}{2\|x\|^2}\Big)-2\log\|x\|-\log 2+\gamma\Big)R(dx)\\ &= b-a+(1-\gamma)\int_{\mathbb{R}^d} xR(dx)-\int_{\mathbb{R}^d} x\Big(\log\|x\|+\frac{\log 2}{2}\Big)R(dx)+\frac{\gamma}{2}\int_{\mathbb{R}^d} xR(dx)=0\end{aligned}$$

By considering the remark in [109, Notes page 236], we can complete the proof of part (i). Indeed, since ν and ν_0 are mutually absolutely continuous by (4.35), $P_{0|\mathcal{F}_t}$ and $P_{|\mathcal{F}_t}$ are mutually absolutely continuous or singular for all $t>0$.

Part (ii) is an application of Theorem 33.2 of [109], where the form of Radon-Nikodym derivative is specified for two mutually absolutely continuous Lévy processes. □

4.3.1 Change of measure for RDTS processes

If we consider a RDTS distribution as defined by Equation (3.44) with parameters the change of measure problem can be solved, by using similar arguments of previous sections.

Proposition 4.15. *Consider two probability measures P, $\tilde{P}$ and the canonical process $(X_t)_{t\geq 0}$ on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0})$ given above. Let us suppose $(X_t)_{t\geq 0}$ is a RDTS* *process under P with parameters $(\alpha, c_+, c_-, \lambda_+, \lambda_-, m)$ and a RDTS process under $\tilde{P}$ with parameters $(\tilde{\alpha}, \tilde{c}_+, \tilde{c}_-, \tilde{\lambda}_+, \tilde{\lambda}_-, \tilde{m})$. Then $P|_{\mathcal{F}_t}$ and $\tilde{P}|_{\mathcal{F}_t}$ are equivalent for every $t > 0$ if and only if*

$$\alpha = \tilde{\alpha}, \tag{4.40}$$

$$c_+ = \tilde{c}_+, \qquad c_- = \tilde{c}_-, \tag{4.41}$$

if $\alpha \neq 1$ then

$$\tilde{m} - m = 2^{-\frac{1+\alpha}{2}} \Gamma\left(\frac{1-\alpha}{2}\right) \Big(c_+(\tilde{\lambda}_+^{\alpha-1} - \lambda_+^{\alpha-1}) - c_-(\tilde{\lambda}_-^{\alpha-1} - \lambda_-^{\alpha-1})\Big). \tag{4.42}$$

Proof. By Theorem 4.14 and by considering a similar argument of Theorem 4.6, the result follows. □

Chapter 5

TS exponential Lévy processes in stock price modeling

Most of the concepts in theoretical and quantitative finance that have been developed over the past decades rest upon the assumption that asset returns follow a normal distribution. By now, there is, however, ample empirical evidence that many financial return series are heavy tailed and skewed. Since Mandelbrot introduced the α-stable distribution to model the empirical distribution of asset prices in [81], the α-stable distribution became the most popular alternative to the normal distribution. While the empirical evidence does not support the normal distribution, it is also not consistent with an α-stable distribution. The distribution of returns for assets has heavier tails relative to the normal distribution and thinner tails than the α-stable distribution. Therefore TS distributions can partially help us to model financial return series.

In this chapter, we will discuss a parametric approach to risk-neutral density extraction from option prices based on the knowledge of the estimated historical density. A continuous time model will be considered.

There is enough empirical evidence that many financial return series are heavy-tailed and exhibit variances that change through time, indeed gaussian hypothesis as well as exponential Lévy model does not describe the statistical properties of financial time series very well. In this chapter we do not consider time dependent variance, it will be done in Chapter 7.

It has been observed, [45] and [111], that while price processes for financial assets must have a jump component they need not have a diffusion component. Jumps are necessary in order to capture the large moves that occasionally occur. The explanation usually given for the use of a diffusion component is that it captures the small moves which occur much more frequently. However, TS processes have infinite activity, i.e. with $\int_{|x|<1} \nu(dx) = \infty$, and they are able to capture both rare large moves and frequent small moves. High activity is accounted for by a large (in most cases infinite) number of small jumps. It is well know that if we consider pure jump Lévy processes models, the empirical performance of these models is typically not improved by adding a diffusion component. Thus, it is not a restrictive hypothesis if we consider only pure jump processes, i.e. with no Brownian component ($\sigma = 0$).

5.1 A model for the stock price process

Let $(X_t)_{t\geq 0}$ be a canonical process, that is the process of the form $X_t(\omega) = w(t)$ with $t \geq 0$ and $w \in \Omega$[1], where $\Omega = D([0,\infty), \mathbb{R}^d$. Furthermore let Ω be equipped with the σ-field $\mathcal{F} = \sigma\{X_s : s \geq 0\}$ and the right-continuous natural filtration $\mathcal{F}_t = \bigcap_{s\geq t} \sigma\{X_u : u \geq s\}$ with $t \geq 0$. The canonical process is completely described by a probability measure P on the measurable space $(\Omega, \mathcal{F})$.

The exponential Lévy model assumes that the logarithmic returns of the stock price process are given by a Lévy process. Hence, the stock price dynamic $(S_t)_{t\geq 0}$ under the a market measure P is assumed to be given by

$$S_t = S_0 e^{(\mu+\omega)t+X_t} \tag{5.1}$$

where μ is the mean rate of return on the stock and ω is a *convexity correction*, defined by Equation (1.15)

$$\omega = -\psi(-i) \tag{5.2}$$

where ψ is the characteristic exponent of a given distribution as defined in (1.6).

By the Fundamental Theorem of Asset Pricing, we have to find an equivalent measure Q such that the discounted stock price

$$\tilde{S}_t = e^{-rt} S_t$$

is a martingale. Q is commonly called *equivalent martingale measure* (EMM). Recall that a contingent claim is a non-negative $\mathcal{F}$-measurable random variable C, $C \in L_0^+(\mathcal{F}_T, P)$ representing a contract that pays out $\Pi(T, C(w))$ dollars at time T if $w \in \Omega$ occurs. At the time 0 its value or current price $\Pi(0, C)$ is then the value that the parties to the contract would deem a *fair price* for entering into this contract. It is well know, that the price $\Pi(t, C)$ ad time t of an European contingent claim can be calculated as the expected value of the discounted value of its payout, that is

$$\Pi(t, C) = E_Q[e^{-r(T-t)}\Pi(T, C)|\mathcal{F}_t]$$

where r is the risk free rate, for more details see [41, 13]. Under this setting, if we find an equivalent martingale measure, we are able to find a fair price for our contingent claims, indeed we can calculate option prices. By our construction, it is easy to see that under a risk neutral measure the process $(S_t)_{t\geq 0}$ is

$$S_t = S_0 e^{(r+\omega)t+X_t}. \tag{5.3}$$

Under the assumption of Proposition 1.48,

$$\int_{|x|\geq 1} e^x \nu(dx) < \infty,$$

we obtain that the discounted stock price process

$$\tilde{S}_t = S_0 e^{\omega t + X_t}$$

[1] In the literature ω is used instead of w, but ω has been already used in the previous sections for the convexity correction. We use w to avoid confusion.

is a martingale. In order to find an EMM we have to find a density transformation such that the dynamic of the stock price process is of the form (5.3). We will discuss a parametric approach to risk-neutral density extraction from option prices based on the knowledge of the estimated historical density. In order to reach this goal, we need do find relations between market and risk neutral parameters. The purpose of this section is to bridge two strands of the literature, one pertaining to the objective or physical measure used to model the underlying asset and the other to the risk-neutral measure used to price derivatives. Numerous papers have confronted empirical evidence obtained from derivative security markets with results from the underlying and vice versa. In particular, issues related to the informational content of option prices have been examined extensively in the literature [25]. Indeed the common procedure is to estimate all parameters from the cross-section of observed option prices [11, 7, 80], without considering a direct connection between historical information and information coming from option prices. The choice of a model implies also an out-of-the-sample performance analysis, which tests predictive capabilities of the model. This latter point will be analyzed as well.

5.2 Estimation

The stock price model previously defined becomes the classical Black-Scholes model if the driving process is a Brownian motion. In this particular well known case, the market and the risk-neutral distributions are both lognormal with the same shape parameter and different location parameter. In the general case, if the driving process X_t is a Lévy process, the market is incomplete and the transformation between market and risk neutral distribution is nontrivial. There are various techniques for performing such a transformation, including the PDE approach, the general equilibrium approach and the changing measure approach [61].

In the Chapter 4, the change of measure problem for TS processes has been widely analyzed. The goal of this section is to apply the theory explained so far.

By applying this method, we do not need to assume any economical motivation for our model. The risk aversion of investor is not necessary and, furthermore, no assumption on the pricing kernel is made. Working with the TS distribution allow us to estimate objective and risk neutral measures together. First, market parameters are estimated by fitting the exponential TS Lévy model for the stock price, then, by considering market parameters, option prices are fitted by using a suitable change of measure.

5.2.1 TS model

In section 5.1, we have introduced a model for the stock price by considering a given processes X_t. Since TS distributions are infinitely divisible, then TS processes can be considered as driving processes for the stock price.

By Proposition 4.6, one can consider a TS driving process to model the stock price under the statistical measure and then, by the change of measure properties, find a suitable risk neutral measure such that conditions 4.11 and 4.12 are satisfied.

Proposition 5.1. *Under the above notation, assume that* $(S_t)_{t\in[0,T]}$ *has a* TS *driving process* $(X_t^1)_{t\in[0,T]} \sim TS(R_1, b_1)$ *with a given set of parameters* θ *under the* *market measure* P*, and a driving process* $(X_t^2)_{t\in[0,T]} \sim TS(R_2, b_2)$ *with a given set of parameters* $\tilde{\theta}$ *under a measure* Q*. Furthermore, assume that (4.11) is fulfilled. Then* Q *is an* EMM *of* P *if and only if*

$$\alpha = \tilde{\alpha}, \tag{5.4}$$

$$\sigma_\theta(A) = \sigma_{\tilde{\theta}}(A), \qquad \forall A \in S_0 \tag{5.5}$$

and

$$\mu - \psi_1(-i;\theta) - (r-d) + \psi_2(-i;\tilde{\theta}) = \Gamma(1-\alpha)\left(\int_{\mathbb{R}} xR_1(dx) - \int_{\mathbb{R}} xR_2(dx)\right) \tag{5.6}$$

Proof. The proof follow by definition (5.1) and Proposition 4.6. □

Now, we intend to define the KR model. For convenience, we exclude the case $\alpha = 1$ and define a function

$$\begin{aligned}\psi^{KR}(u;k_+,k_-,r_+,r_-,p_+,p_-,\alpha,m) &= H_\alpha(u;k_+,r_+,p_+) + H_\alpha(-u;k_-,r_-,p_-)\\ &\quad + iu\left(m + \alpha\Gamma(-\alpha)\left(\frac{k_+r_+}{p_++1} - \frac{k_-r_-}{p_-+1}\right)\right),\end{aligned} \tag{5.7}$$

on $u \in \{z \in \mathbb{C} \mid -\mathrm{Im}(z) \in (-r_-^{-1}, r_+^{-1})\}$, which is same as the exponent of (2.49).

Definition 5.2. *In the above setting, if* $(X_t)_{t\in[0,T]}$ *is the KR process with parameters* $(\alpha,\ k_+,\ k_-,\ r_+,\ r_-,\ p_+,\ p_-,\ m)$ *where*

$$\begin{aligned}&\alpha \in (0,1)\cup(1,2),\\ &k_+,k_-,r_- \in (0,\infty),\\ &r_+ \in (0,1),\\ &p_+,p_- \in (1/2-\alpha,\infty)\setminus\{0\},\ \text{if } \alpha\in(0,1),\\ &p_+,p_- \in (1-\alpha,\infty)\setminus\{0\},\ \text{if } \alpha\in(1,2),\end{aligned}$$

and $m = \mu - \psi_\alpha(-i;k_+,k_-,r_+,r_-,p_+,p_-,0)$ *for some* $\mu \in \mathbb{R}$*, then the process* $(S_t)_{t\in[0,T]}$ *is called the KR price process with parameters* $(\alpha,\ k_+,\ k_-,\ r_+,\ r_-,\ p_+,\ p_-,\ \mu)$ *and we say that the stock price process follows the exponential KR model.*

Remark 5.3.

1. *We have the condition* $r_+ \in (0,1)$ *for* $\psi^{KR}(-i;k_+,k_-,r_+,r_-p_+,p_-,\alpha,0)$ *and* $E[e^{X_t}]$ *to be well defined.*

2. *By the condition* $\begin{cases} p_+,p_- \in (1/2-\alpha,\infty)\setminus\{0\}, & \text{if } \alpha\in(0,1)\\ p_+,p_- \in (1-\alpha,\infty)\setminus\{0\}, & \text{if } \alpha\in(1,2)\end{cases}$*, we are able to use Theorem 4.8 for finding an equivalent measure.*

3. *Since* $m = \mu - \psi_\alpha(-i;k_+,k_-,r_+,r_-,p_+,p_-,0)$*, we have*

$$E[S_t] = S_0E[e^{X_t}] = S_0e^{\mu t}.$$

Corollary 5.4. *Assume that* $(S_t)_{t\in[0,T]}$ *is the the KR price process with parameters* $(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, \mu)$ *under the market measure* P*, and with parameters* $(\tilde{\alpha}, \tilde{a}_+, \tilde{a}_-, \tilde{r}_+, \tilde{r}_-, \tilde{p}_+, \tilde{p}_-, r-d)$ *under a measure* Q*. Then* Q *is an* EMM *of* P *if and only if*

$$\alpha = \tilde{\alpha}, \tag{5.8}$$

$$\frac{k_+ r_+^\alpha}{\alpha + p_+} = \frac{\tilde{k}_+ \tilde{r}_+^\alpha}{\alpha + \tilde{p}_+}, \qquad \frac{k_- r_-^\alpha}{\alpha + p_-} = \frac{\tilde{k}_- \tilde{r}_-^\alpha}{\alpha + \tilde{p}_-} \tag{5.9}$$

and

$$\begin{aligned} \mu - (r-d) = & H_\alpha(-i; k_+, r_+, p_+) + H_\alpha(i; k_-, r_-, p_-) \\ & - H_\alpha(-i; \tilde{k}_+, \tilde{r}_+, \tilde{p}_+) - H_\alpha(i; \tilde{k}_-, \tilde{r}_-, \tilde{p}_-). \end{aligned} \tag{5.10}$$

Proof. By Proposition 5.1, the result holds. □

In the same way we can define the KoBoL model. Without losing generality, also in this case we exclude the case $\alpha = 1$ and define a function

$$\begin{aligned} \psi^{KoBoL}(u; c_+, c_-, \lambda_+, \lambda_-, \alpha, m) &= iub + \Gamma(-\alpha)c_-((\lambda_- + iu)^\alpha - \lambda_-^\alpha - iu\lambda_-^{\alpha-1}\alpha) \\ &\quad + \Gamma(-\alpha)c_+((\lambda_+ - iu)^\alpha - \lambda_+^\alpha + iu\lambda_+^{\alpha-1}\alpha), \end{aligned} \tag{5.11}$$

on $u \in \{z \in \mathbb{C} \mid -\mathrm{Im}(z) \in (-\lambda_-, \lambda_+)\}$, which is same as the exponent of (2.36).

Definition 5.5. *In the above setting, if* $(X_t)_{t\in[0,T]}$ *is the KoBoL process with parameters* $(c_+,\ c_-,\ \lambda_+,\ \lambda_-,\ \alpha,\ m)$ *where*

$$\begin{aligned} &\alpha \in (0,1) \cup (1,2), \\ &c_+, c_-, \lambda_-, \lambda_- \in (0, \infty), \\ &r_+ \in (0,1), \end{aligned}$$

and $b = \mu - \psi^{KoBoL}(-i; c_+, c_-, \lambda_+, \lambda_-, \alpha, 0)$ *for some* $\mu \in \mathbb{R}$*, then the process* $(S_t)_{t\in[0,T]}$ *is called the KoBoL price process with parameters* $(\alpha,\ c_+,\ c_-,\ \lambda_+,\ \lambda_-,\ \mu)$ *and we say that the stock price process follows the exponential KoBoL model.*

Corollary 5.6. *Assume that* $(S_t)_{t\in[0,T]}$ *is the KoBoL price process with parameters* $(\alpha,\ c_+,\ c_-,\ \lambda_+,\ \lambda_-,\ \mu)$ *under the market measure* P*, and with parameters* $(\tilde{\alpha},\ \tilde{c}_+,\ \tilde{c}_-,\ \tilde{\lambda}_+,\ \tilde{\lambda}_-,\ r-d)$ *under a measure* Q*. Then* Q *is an* EMM *of* P *if and only if*

$$\alpha = \tilde{\alpha}, \tag{5.12}$$

$$c_+ = \tilde{c}_+ \qquad c_- = \tilde{c}_- \tag{5.13}$$

and

$$\begin{aligned} \mu - (r-d) &= \Gamma(-\alpha)[c_-((\lambda_- + iu)^\alpha - \lambda_-^\alpha) + c_+((\lambda_+ - iu)^\alpha - \lambda_+^\alpha] \\ &\quad - \Gamma(-\alpha)[\tilde{c}_-((\tilde{\lambda}_- + iu)^\alpha - \tilde{\lambda}_-^\alpha) + \tilde{c}_+((\tilde{\lambda}_+ - iu)^\alpha - \tilde{\lambda}_+^\alpha] \end{aligned} \tag{5.14}$$

Proof. By Proposition 5.1, the result holds. □

If we set $Y = \alpha$, $C = c_+ = c_-$, $\lambda_+ = M$ and $\lambda_- = G$, we obtain the well known CGMY model [21]. By Proposition 4.9, we can also construct a model such that, first we consider the KoBoL model to estimate market parameters and then by using the market estimation we can estimate parameters under the risk neutral measure. Relations between parameters follows by Proposition 5.1.

5.2.2 Evaluating the density function

In order to calibrate asset returns models through exponential Lévy process or TS GARCH model [68, 67], one needs a correct evaluation of both the pdf and cdf functions. With the pdf function it is possible to construct a maximum likelihood estimator (MLE), while the cdf function allows one to assess the goodness of fit. Even if the MLE method may lead to local maximum rather than to a global one due to the multi dimensionality of the optimization problem, the results obtained seem to be satisfactory from the point of view of goodness of fit tests. Actually, an analysis on estimation methods for this kind of distributions would be interesting, but it is far from the purposes of this work.

Numerical methods are needed to evaluate the pdf function. By the definition of the characteristic function as the Fourier transform of the density function [43], we consider the inverse Fourier transform that is

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iux}E[e^{iuX}]du \tag{5.15}$$

where $f(x)$ is the density function. If the density function has to be calculated for a large number of x values, the fast Fourier Transform (FFT) algorithm can be employed as described in [113]. The use of the FFT algorithm largely improves the speed of the numerical integration above and the function f is evaluated on a discrete and finite grid, consequently a numerical interpolation is necessary for x values out of the grid. Since a personal computer cannot deal with infinite numbers, the integral bounds $(-\infty,\infty)$ in equation (5.15) are replaced with $[-M, M]$, where M is large value. We take $M \sim 2^{16}$ or 2^{15} in our study and we have also noted that smaller values of M generate large errors in the density evaluation given by a wave effect in both density tails. We have to point out that the numerical integration as well as the interpolation may causes some numerical errors. The method above is a general method that can be used if the density function is not known in closed form.

While the calculus of the characteristic function in the CGMY case involves only elementary functions, more interesting is the evaluation of the characteristic function in the KR case that is connected with the Gaussian hypergeometric function. Equation (2.49) implies the evaluation of the hypergeometric ${}_2F_1(a, b; c; z)$ function only on the straight line represented by the subset $I = \{iy \mid y \in \mathbb{R}\}$ of the complex plane $\mathbb{C}$. We do not need a general algorithm to evaluate the function on the entire complex plane $\mathbb{C}$, but just on a subset of it. This can be done by means of the analytic continuation, without having recourse neither to numerical integration nor to numerical solution of a differential equation [96] (for a complete table of the analytic continuation formulas for arbitrary values of $z \in \mathbb{C}$ and of the parameters a, b, c, see [12] or [49]). The hypergeometric function belongs to the special function class and often occurs in many practical computational problems. It is defined by the power series

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \quad |z| < 1. \tag{5.16}$$

where $(a)_n := \Gamma(a+n)/\Gamma(n)$ is the Ponchhammer symbol. By [1] the following relations are fulfilled[2]

$$
\begin{aligned}
{}_2F_1(a,b,c;z) &= (1-z)^{-b}{}_2F_1\left(b,c-a,c,\frac{z}{z-1}\right) \quad \text{if } \left|\frac{z}{z-1}\right|<1 \\
{}_2F_1(a,b,c;z) &= (-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}{}_2F_1\left(a,a-c+1,a-b+1,\frac{1}{z}\right) \\
&\quad + (-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}{}_2F_1\left(b,b-c+1,b-a+1,\frac{1}{z}\right) \quad \text{if } \left|\frac{1}{z}\right|<1 \\
{}_2F_1(a,b,c;-iy) &= \overline{{}_2F_1(a,b,c;iy)} \quad \text{if } y\in\mathbb{R}.
\end{aligned}
\tag{5.17}
$$

First by the last equality of (5.17), one can determine the values of ${}_2F_1(a,b,c;z)$ only for the subset $I_+ = \{iy \mid y \in \mathbb{R}_+\}$ and then simply consider the conjugate for the set $I_- = \{iy \mid y \in \mathbb{R}_-\}$, remembering that ${}_2F_1(a,b,c;0) = 1$. Second, we split the positive real line $\mathbb{R}_+$ in three subsets without intersection,

$$
\begin{aligned}
I_+^1 &= \{iy \mid 0 < y \le 0.5\} \\
I_+^2 &= \{iy \mid 0.5 < y \le 1.5\} \\
I_+^3 &= \{iy \mid y > 1.5\},
\end{aligned}
$$

then we use (5.16) to evaluate ${}_2F_1(a,b,c;z)$ in I_+^1. Then, the first and the second equalities of (5.17) together with (5.16) are enough to evaluate ${}_2F_1(a,b,c;z)$ in I_+^2 and I_+^3 respectively. This subdivision allows one to truncate the series (5.16) to the integer $N = 500$ and obtain the same results as Mathematica. We point out that the value of y ranges in the interval $[-M, M]$ previously defined. This method together with the Matlab vector calculus increase considerably the speed with respect to algorithms based on the numerical solution of differential equation [96]. Our method is grounded only on basic summations and multiplication. As a result the computational effort in the KR density evaluation is comparable to that of CGMY one. The KR characteristic function is necessary also to price options, not only for MLE estimation. Indeed, by using the approach of Carr and Madan [23] and the same analytic continuation as above, risk-neutral parameters may be directly estimated from option prices, without calibrate the underlining.

5.2.3 Estimation of market parameters

We will test these continuous time models on the S&P 500 index. We will consider adjusted closing prices of the S&P 500 index from Monday 12 April 1996 to Wednesday 12 April 2006 provided by Datastream for a total of 2501 observations. The size of this data set, 2501 observations, is large enough for standard model fitting. The dividend yield will be not used, since adjusted closing pricing are taken into account, that is $d_t = 0$, for each t in our sample. For the daily interest rate process we take the time series of the above time window of the 3-months Treasury rate and the 1-year zero rate is calculated by using the bootstrap method [55].

[2] See [1] or [12] for a complete overview on the analytic continuation of the Gaussian hypergeometric function.

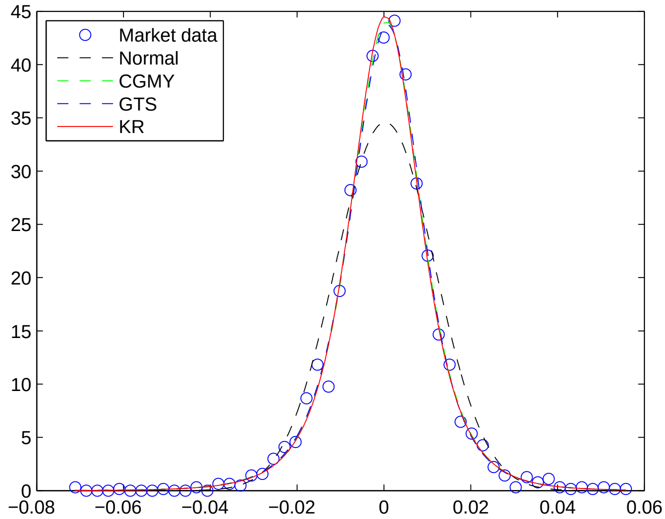

Figure 5.1: S&P 500 market parameters estimated by MLE approach on Wednesday 12 April 2006.

We first estimate the market parameters of each model from 10 years of time-series data. Our estimation procedure follows the classical maximum likelihood estimation (MLE) method. The discrete Fourier transform (DFT) is used to invert the characteristic function and evaluate the likelihood function in the all cases.

Let $(\Omega, \mathcal{A}, \mathrm{P})$ be a probability space and $\{X_i\}_{1\le i\le n}$ a given set of independent and identically distributed real random variables. In the following, let us consider $X_i(\omega) = x_i$, for each $i = 1, \dots, n$. Let F be the distribution of X_i, and $x_1 \le x_2 \le \dots \le x_n$. The empirical cumulative distribution function $\hat{F}_n(x)$ is defined by

$$\hat{F}_n(x) = \frac{\text{no. observations} \le x}{n} = \begin{cases} 0, & x < x_1 \\ \frac{i}{n}, & x_i \le x \le x_{i+1},\ i = 1, \dots, n-1 \\ 1, & x_n \le x. \end{cases}$$

Table 5.1: S&P 500 market parameters estimated by MLE approach on Wednesday 12 April 2006.

	σ	μ					
Normal	0.1824	0.0871					
	C	G	M	Y	m		
CGMY	0.8613	60.0000	67.7897	1.0457	0.0841		
	C_+	C_-	G	M	Y_+	Y_-	m
GTS	0.7119	0.5412	59.9427	59.9427	1.0457	1.1521	0.0805
	k_+	k_-	r_+	r_-	$p_+ = p_-$	α	m
KR	960.7840	1.9115e+3	0.0158	0.0125	13.4336	0.9000	0.0873

Table 5.2: Historical estimation results from 12 April 1996 to 12 April 2006.

	KS	AD	AD^2	AD^2_{up}	χ^2(p-value)
Normal	0.0519	37.9525	13.4591	2.2580e+03	2.5659e+02(1.065e-14)
CGMY	0.0151	0.0389	3.1659	10.1990	98.9847(0.5936)
GTS	0.0157	0.0409	3.2428	10.4760	97.2380(0.6148)
KR	0.0138	0.0683	3.1370	9.5037	96.7591(0.5731)

A statistic measuring the difference between $\hat{F}_n(x)$ and $F(x)$ is called the empirical distribution function (EDF) statistic [30]. These statistics include the Kolmogorov-Smirnov (KS) statistic [30, 86, 112]. Our goal is to test if the empirical distribution function of an observed data sample belongs to a family of hypothesized distributions, i.e.

$$H_0 : F = F_0 \ \text{ vs } \ H_1 : F \neq F_0 \tag{5.18}$$

Suppose a test statistic D takes the value d, the p-value of the statistic will then be the value

$$p\text{-value} = P(D \geq d).$$

We reject the hypothesis H_0 if the p-value is less than a given level of significance, which we take to be equal to 0.05. Let us consider a test for hypotheses of the type (5.18) concerning continuous cumulative distribution function, the Kolmogorov-Smirnov test. The KS statistic D_n measures the absolute value of the maximum distance between the empirical distribution function $\hat{F}$ and the theoretical distribution function F, putting equal weight on each observation,

$$D_n = \sup_{x_i} |F(x_i) - \hat{F}_n(x_i)| \tag{5.19}$$

where $\{x_i\}_{1\leq i\leq n}$ is a given set of observations. Using the procedure of [86], we can easily evaluate the distribution of D_n and find the p-value for our test.

Furthermore, to assess the goodness of fit, we consider some other classical statistical tests. It might be of interest to test the ability to model to forecast extreme events. To this end, we also provide the AD statistics. We consider different versions of the AD statistic. In its simplest version, it is a variance-weighted KS statistic

$$AD_n = \sup_{x_i} \frac{|F(x_i) - \hat{F}(x_i)|}{\sqrt{F(x_i)(1 - F(x_i))}}. \tag{5.20}$$

A more generally used version of this statistic belongs to the quadratic class defined by the Cramér-von Mises family [30], i.e.

$$AD_n^2 = n \int_{-\infty}^{\infty} \frac{(\hat{F}_n(x) - F(x))^2}{F(x)(1 - F(x))} dF(x) \tag{5.21}$$

and by the Probability Integral Transformation (PIT) formula [30], we obtain the computing formula for the AD_n^2 statistic

$$AD_n^2 = -n + \frac{1}{n}\sum_{i=1}^{n}(1 - 2i)\log(z_i) - \frac{1}{n}\sum_{i=1}^{n}(1 + 2(n - i))\log(1 - z_i)$$

where z_i is $z_i = F(x_i)$, with $i = 1, \dots, n$. To evaluate the distribution of the AD_n^2 statistic, we use the procedure described in [84]. The AD_{up}^2 statistic is also provided, see [30]. Furthermore we follow the parametric procedure for testing the goodness of fit given in [111], that is the χ^2-test. In this first part, we analyze the goodness of fit of our time series model respect to historical data.

5.2.4 Estimation of risk neutral parameters

In this section, we will discuss a parametric approach to risk-neutral density extraction from option prices based on knowledge of the estimated historical density. Therefore, taking into account the estimation results of Section 5.2.3 under the market probability measure, we want to estimate parameters under a risk-neutral measure. Data were supplied by Option Metrics's IvyDB in the Wharton Research Data Services. The market option prices are computed by using the Black-Scholes formula with the implied volatilities and dividends given by IvyDB. Option prices of European call option on Wednesday 12 April 2006 with different maturities (9, 37, 65, 156 and 247 days) will be considered in the optimization procedure. Option with time to maturity more than 100 days, implied volatility more than 0.7, price less than \$0.05 and such that $|S_0/K - 1| > 0.10$, where S_0 is the initial underlying price and K is the strike price, are discarded. The change of measure relations of Chapter 4 together with the market estimation of the previous section allow one to obtain a risk neutral estimation flexible enough to fit both underling stock and option prices. Furthermore, in order to test the forecasting performance of our continuous time models, estimated risk neutral parameters are used to calculate European call options prices one week ahead (with maturities 2, 30, 58, 149 and 240 days), by using asset prices, time to maturities and interest rate on Wednesday 19 April 2006.

Let us consider a given market model and observed prices $\hat{C}_i$ of call options with maturities T_i and strikes K_i, $i \in \{1, \dots, N\}$, where N is the number of options on a fixed day. The risk-neutral process is fitted by matching model prices to market prices using nonlinear least squares. Hence, to obtain a practical solution to the calibration problem, our purpose is to find a parameter set $\tilde{\theta}$, such that the optimization problem

$$\min_{\tilde{\theta}} \sum_{i=1}^{N} (\hat{C}_i - C^{\tilde{\theta}}(T_i, K_i))^2 \tag{5.22}$$

is solved, where by $\hat{C}_i$ we denote the price of an option as observed in the market and by $C_i^{\tilde{\theta}}$ the price computed according to a pricing formula in a chosen model with a parameter set $\tilde{\theta}$.

The class of TS distribution is flexible enough to allow a joint market and risk neutral estimation. By using market estimation, we can find risk neutral parameters which verify conditions of Proposition 5.1.

If we consider the CGMY model, by Corollary 5.6, we can consider the historical estimation for parameters $\tilde{Y}$ and $\tilde{C}$ and find a solution to the minimization problem (5.22) which satisfies condition (5.12), (5.13) and (5.14). Therefore, we can estimate parameters $\tilde{M}$ and $\tilde{G}$ under a risk-neutral measure. The optimization procedure involves 4 parameters except r and 3 equality constraints. Consequently we have only one free parameter to solve (5.22). A similar argument holds in the GTS case as well.

Table 5.3: Risk neutral parameters calibrated by cross sectional S&P 500 European call option data on Wednesday 12 April 2006. The annual risk free interest rate is $r = 0.0455$

	σ						
Normal	0.1824						
	C	G	M	Y			
CGMY	0.8613	183.1590	210.7473	1.0457			
	C_+	C_-	G	M	Y_+	Y_-	
GTS	0.7119	0.5412	188.9524	287.1487	1.0457	1.1521	
	k_+	k_-	r_+	r_-	p_+	p_-	α
KR	1.3833e+4	85.7494	0.0026	0.0145	40	-0.1650	0.9000

Table 5.4: Option pricing errors results on Wednesday 12 April 2006 for different Lévy models.

	APE	AAE	RMSE	ARPE
Normal	0.2553	10.1630	13.3233	1.9704
CGMY	0.0187	0.7450	0.9026	0.1600
GTS	0.0234	0.9327	1.2219	0.1495
KR	0.0158	0.6288	0.7624	0.1566

If we consider the KR exponential model, according to Definition 5.2 and Corollary 5.6, we can find parameters $\tilde{k}_+$, $\tilde{k}_-$, $\tilde{r}_+$ and $\tilde{r}_-$, such that conditions (5.8), (5.9), and (5.10) are satisfied and (5.22) is solved. We have 7 parameters except r and 4 equality constraints, namely 3 free parameters to minimize (5.22), i.e.

$$\alpha = \tilde{\alpha},$$

$$\begin{aligned} \tilde{p}_+ &= \frac{\tilde{k}_+\tilde{r}_+^\alpha}{k_+r_+^\alpha}(\alpha + p_+) - \alpha, \\ \tilde{p}_- &= \frac{\tilde{k}_-\tilde{r}_-^\alpha}{k_-r_-^\alpha}(\alpha + p_-) - \alpha \end{aligned}$$

and

$$\begin{aligned} \mu - (r - d) = &H_\alpha(-i; k_+, r_+, p_+) + H_\alpha(i; k_-, r_-, p_-) \\ &- H_\alpha(-i; \tilde{k}_+, \tilde{r}_+, \tilde{p}_+) - H_\alpha(i; \tilde{k}_-, \tilde{r}_-, \tilde{p}_-). \end{aligned}$$

By following Proposition 4.9 and Proposition 5.1, if we consider in the market the GTS distribution and the KR distribution in the risk neutral world, conditions on parameters becomes

$$\alpha = \tilde{\alpha},$$

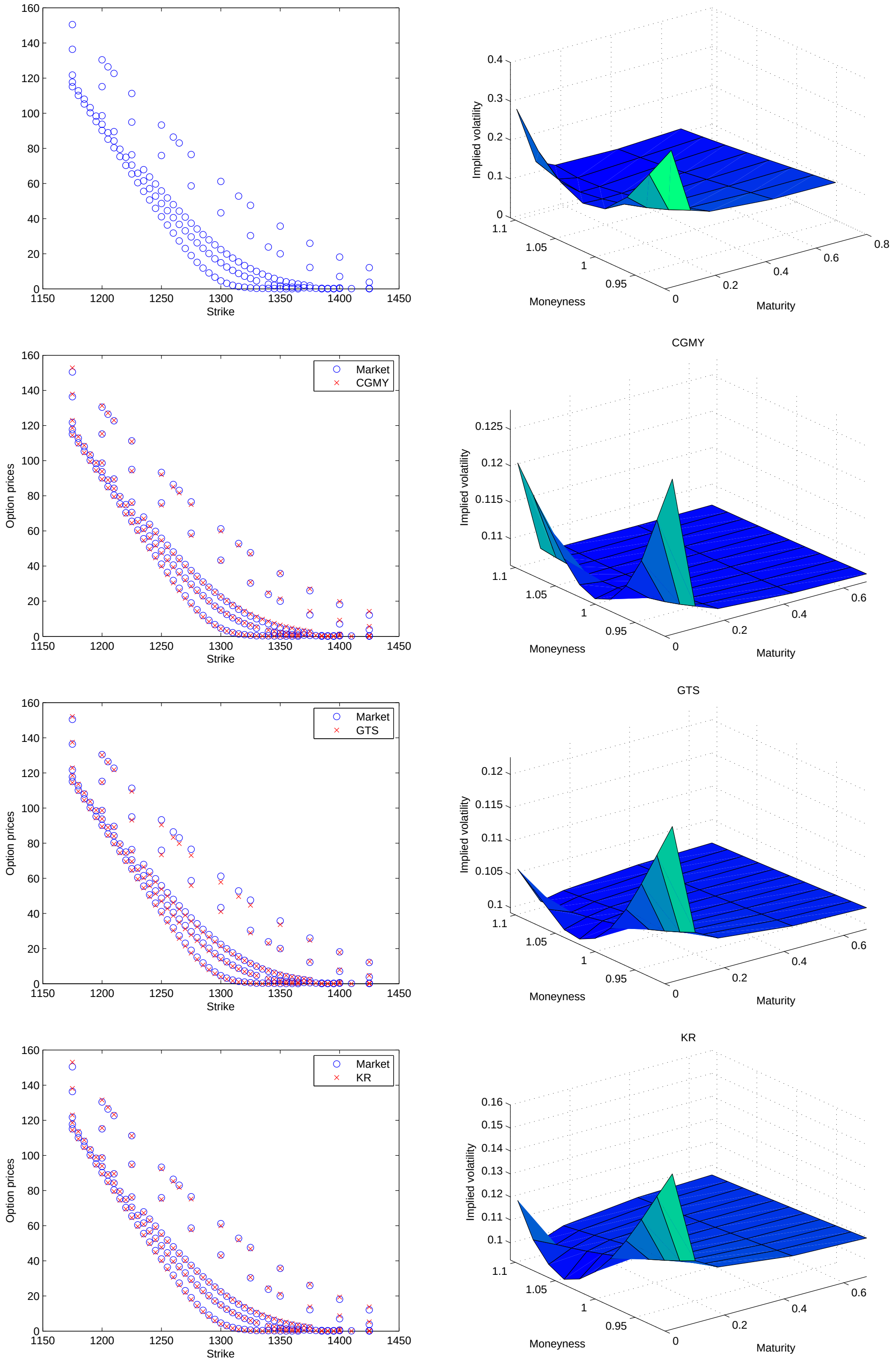

Figure 5.2: S&P 500 European Call Option prices on Wednesday 12 April 2006 and related implied volatility surface. CGMY, GTS and KR model fitting and related implied volatility surface.

Table 5.5: Out-of-sample option pricing errors results on Wednesday 19 April 2006 for different Lévy models. Estimated parameters on Wednesday 12 March 2006 are used to evaluate European call options prices one week ahead, by using asset prices, time to maturities and interest rate on Wednesday 19 March 2006.

	APE	AAE	RMSE	ARPE
Normal	0.3484	12.5187	15.8007	2.6869
CGMY	0.0519	1.8644	2.1568	0.4260
GTS	0.0352	1.2658	1.4979	0.32955
KR	0.0530	1.9026	2.1800	0.3659

$$\begin{aligned} \tilde{p}_+ &= \frac{\tilde{k}_+ \tilde{r}_+^\alpha}{c_+} - \alpha, \\ \tilde{p}_- &= \frac{\tilde{k}_- \tilde{r}_-^\alpha}{c_-} - \alpha \end{aligned}$$

and

$$\begin{aligned} \mu - (r-d) \ &= \Gamma(-\alpha)c_-((\lambda_- + 1)^\alpha - \lambda_-^\alpha) + \Gamma(-\alpha)c_+((\lambda_+ - 1)^\alpha - \lambda_+^\alpha) \\ &- H_\alpha(-i; \tilde{k}_+, \tilde{r}_+, \tilde{p}_+) - H_\alpha(i; \tilde{k}_-, \tilde{r}_-, \tilde{p}_-) \\ &+ \alpha\Gamma(-\alpha)\left((\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}) - \left(\frac{k_+ r_+}{p_+ + 1} - \frac{k_- r_-}{p_- + 1} \right) \right). \end{aligned}$$

Anyway, we do not test this approach empirically.

In the CGMY and GTS case we have only one free parameter but in the KR case we have 3 free parameters to fit model prices to market prices; therefore, we can obtain a better solution to the optimization problem. The KR distribution is more flexible in order to find an equivalent change of measure and, at the same time, takes into account the historical estimates.

Contrary to the classical Black-Scholes case, in the exponential Lévy models there is no explicit formula for call option prices, since the probability density of a Lev́y process is typically not known in closed form. Due to the easy form of the characteristic functions of the CGMY, GTS and KR distributions, we follow the generally used pricing method for standard vanilla options, which can be applied in general when the characteristic function of the risk-neutral stock-price process is known [23, 111]. Let ρ be a positive constant such that the ρ-th moment of the price exists and ϕ the characteristic function of the random variable $\log S_T$. A value of ρ $= 0.75$ will typically do fine [111]. In [23, 111] is showed that

$$C(K,T) = \frac{\exp(-\rho \log K)}{\pi} \int_0^\infty \exp(-iv \log K)\varrho(v) dv,$$

where

$$\varrho(v) = \frac{\exp(-rT)\phi(v - (\rho+1)i)}{\rho^2 + \rho - v^2 + i(2\rho+1)v}.$$

Furthermore, we need to guarantee the analyticity of the integrand function in the horizontal strip of the complex plane, on which the line $L_\rho = \{x+i\rho \in \mathbb{C}| -\infty < x < \infty\}$ lies [77, 78]. If we consider the exponential KR model, we obtain the following additional inequality constraint,

$$r_+^{-1} \geq 1 + \rho,$$

by Proposition 2.23. Since α is less than 1 in the estimated market parameter for the given time-series data, we have to consider an additional condition

$$p_+, \ p_- \in (1/2 - \alpha, \infty),$$

by Equation 4.15.

The optimization procedure is run by considering five different maturities. Even if, due to the independence and stationarity of their increments, exponential Lévy models perform poorly when calibrating several maturities at the same time [29], the flexibility of the model allow one to obtain satisfactory results. In Table 5.5, we resume the error estimator of our option price fits. The classical Brownian motion case is considered for completeness, where the constant market volatility σ, estimated in the previous section, is used to price option. The risk-neutral parameters for different models are given in Table 5.3. To measure the performance of the option pricing model, we consider four statistics, by following the approach of [111]. Let $\bar{C}_i$ be the mean of options prices C_i, we evaluate the average absolute error as a percentage of the mean price

$$\text{APE} = \frac{1}{\bar{C}_i} \sum_{i=1}^{N} \frac{|C_i - C^\theta(\tau_i, K_i)|}{N},$$

the average absolute error

$$\text{AAE} = \sum_{i=1}^{N} \frac{|C_i - C^\theta(\tau_i, K_i)|}{N},$$

the root mean square error

$$\text{RMSE} = \sqrt{\sum_{i=1}^{N} \frac{(C_i - C^\theta(\tau_i, K_i))^2}{N}},$$

and the average relative percentage error

$$\text{ARPE} = \frac{1}{N} \sum_{i=1}^{N} \frac{|C_i - C^\theta(\tau_i, K_i)|}{C_i} \times 100.$$

If we consider the exponential TS models, we can estimate simultaneously market and risk-neutral parameters using historical prices and observed option prices. The flexibility of the KR distribution allows one to obtain a suitable solution to the calibration problem (see Table 5.5).

The CGMY model has four risk-neutral parameters to be estimated and three restrictions in their EMM conditions, the GTS model has six risk-neutral parameters to be estimated and five restrictions in their EMM conditions, while the exponential KR model has seven risk-neutral parameters and four restrictions. Hence the CGMY and the GTS models have only one free parameter, while the KR model has three free parameters for the estimation.

The error estimators of the KR distributed EMM are less than those of the CGMY and GTS parameter fits. The relatively flexible change of measure for the KR distribution seems to generate the better performance, at least for this data set taken into consideration.

Chapter 6

Simulation

In this chapter we analyze some methods to simulate Lévy processes and infinitely divisible (ID) distributions. In particular, algorithms to generate random numbers from an infinitely divisible random variable, can be easily modified to obtain the corresponding Lévy process, and the converse is true as well. First, we study a general framework to obtain random paths of Lévy processes, and then we apply it to some particular cases.

6.1 Simulation techniques for ID random variables and Lévy processes

Computer methods for construction of stochastic processes involve at least two kinds of discretization techniques. First, we have to consider the discretization of the time parameter and then an approximate representation of random variates with the aid of artificially produced finite time series data sets [59]. A Lévy process has stationary and independent increments, therefore the easiest idea, we have in mind to solve the problem of simulating them only for a discrete time set, is equivalent to the problem of generating random numbers from an infinitely divisible distribution.

Let us consider a Lévy process $(X_t)_{t\geq 0}$. If the density f_t has a simple form, then the random numbers generation can be implemented in a rather easy way and if the evaluation of the function f_t involves special functions, then the algorithm implementing the simulation becomes slow but yet easy to implement. In the literature we have many cases in which the density function f_t is not known in closed formula and we have to work with the characteristic function. In some bad cases, in which we are interested, we have a closed form for the characteristic function, but the Lévy measure ν has not a closed form or it is not simple. By inverting the distribution function, we can obtain random variates, but this method involve three numerical procedures, first we have to invert the characteristic function to obtain the density function, then we have to integrate the density function and finally we have to find the solution of a nonlinear equation: this seems not to be a fast way to proceed. In general also other methods, involving the inversion of the Lévy measure, do not seem to be easily implementable, thus we need to recall a general framework for simulating Lévy processes. Anyway, exact simulation of such processes is obviously impossible. A process that is close to the original one is generated instead [106]. We will recall some general results from [105].

The following methods will be considered

(a) random walk approximation,

(b) shot noise representation.

6.1.1 Taking care of small jumps

Series representation of Lévy processes involve at least one discretization error, due to the impossibility to deal with infinite summations. Furthermore, we have also to take into account the approximation of small jumps. In the infinite activity case, this truncation of small jumps involves an approximation error, since for a non compound Lévy process $(X_t)_{t\geq 0}$ with non zero Lévy measure the set of jumps is dense in $[0,\infty)$. Without loss of generality, we can consider a Lévy process without drift and gaussian part. Contents, we will be going to discuss, are based on [5, 3, 106]. Suppose we want to simulate a Lévy process with Lévy measure ν without gaussian part neither drift. Let us define the process $(X_t^\varepsilon)_{t\geq 0}$ as a compound Poisson process with a drift and the distribution of jumps proportional to $\nu^\varepsilon = \nu_{|\{|x|>\varepsilon\}}$ and the process $(R_t^\varepsilon)_{t\geq 0}$ with no gaussian part, zero mean and Lévy measure $\nu^\varepsilon = \nu_{|\{|x|\leq\varepsilon\}}$, then we have

$$X_t = X_t^\varepsilon + R_t^\varepsilon$$

In the sequel we will consider an approximation of X_t. First, in the compensated case,

$$\int_{|x|<1} |x|\nu(dx) = \infty,$$

we consider a compound Poisson process with a drift, the approximation is obtained by removing small jumps

$$X_t \approx \sum_{s\leq t} \Delta X_s I_{|\Delta X_s|\geq 1} + \left(\sum_{s\leq t} \Delta X_s I_{\varepsilon\leq|\Delta X_s|<1} - t\mu_\varepsilon\right)$$

where we set

$$\mu_\varepsilon := \int_{\varepsilon\leq|x|\leq 1} x\nu(dx).$$

If we do not consider small jumps and set only the compound Poisson process with drift we obtain a poor approximation. In the finite variation case,

$$\int_{|x|<1} |x|\nu(dx) < \infty,$$

one can use zero truncation function in the Lévy-Khinchin representation and it is possible to discard small jumps and replace them by their mean value, then, also in this case, the resulting process is a compound Poisson process with a drift. In both methods we have a Poisson approximation of a Lévy process and large jumps are precisely simulated. In this case we put

$$X_t \approx bt + \sum_{s\leq t} \Delta X_s I_{|\Delta X_s|\geq\varepsilon} + ta_\varepsilon$$

where we define

$$a_\varepsilon := \int_{|x|\le\varepsilon} x\nu(dx)$$

that is, rather then just removing small jumps, we replace them by their expected value. However, when the intensity of small jumps is high, discarding them may produce a substantial error. In such case, instead of discarding a small jump part of a Lévy process, one can often approximate it by a Brownian motion with small variance. This approximation is applicable when the series converges slowly and in this case

$$X_t \approx X_t^\varepsilon + A_t \tag{6.1}$$

where the process A_t is defined as

$$A_t^W = a_\varepsilon t + \sigma_\varepsilon W_t$$

where we define

$$\sigma_\varepsilon^2 := \int_{|x|<\varepsilon} x^2\nu(dx).$$

and

$$a_\varepsilon = \begin{cases} 0, & \int_{|x|<1} |x|\nu(dx) = \infty, \\ \int_{|x|\le\varepsilon} x\nu(dx), & \int_{|x|<1} |x|\nu(dx) < \infty, \end{cases}$$

and W_t is a standard Brownian motion. Also if a series representation is available, under some additional conditions, we can apply the method above. To be more precisely, we will show under which conditions the Brownian approximation of small jumps can be used. Let R_t^ε be a Lévy process with characteristic function

$$E[e^{iuR_t^\varepsilon}] = \exp\left\{ t\int_{|x|<\varepsilon} (e^{iux} - 1 - iux)\nu(dx) \right\},$$

then the following result is verified

Theorem 6.1. *We have $R^\varepsilon/\sigma_\varepsilon \xrightarrow{d} W$ as $\varepsilon \to 0$ if and only if for each $k > 0$*

$$\sigma_{k\sigma_\varepsilon \wedge \varepsilon} \sim \sigma_\varepsilon. \tag{6.2}$$

Proof. See [5, Theorem 2.1]. □

Error bound conditions can be found in [29, 6]. We want to recall an equivalent formulation to cover most of practical interest cases.

Proposition 6.2. *Assume that ν has a density of the form $L(x)/|x|^{\alpha+1}$ for all small x, where $L(x)$ is slowly varying as $x \to 0$ and $0 < \alpha < 2$. Then*

$$\frac{R_t^\varepsilon - a^\varepsilon t}{\sigma_\varepsilon} \xrightarrow{d} W_t$$

Sometimes it seems to be easy to check an alternative condition.

Proposition 6.3. *Condition (6.2) is implied by*

$$\lim_{\varepsilon\to 0} \frac{\sigma_\varepsilon}{\varepsilon} = +\infty. \tag{6.3}$$

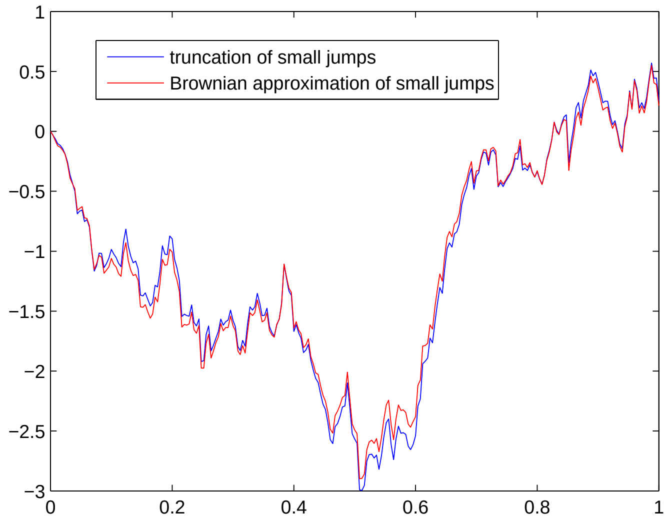

Figure 6.1: Simulation of a CGMY process with $C = 2$, $G = 3$, $M = 10$ and $Y = 1.5$ with and without Brownian approximation of small jumps, $\epsilon = 1e - 5$.

Proof. See [5, Proposition 2.1]. □

We will obtain an useful result for TS processes.

Proposition 6.4. *In the TS case we can always consider the Brownian approximation of small jumps.*

Proof. To prove this, we will show that in TS case, condition (6.3) is always fulfilled. By Theorem 2.4 and equation 2.8 we can write

$$\begin{aligned}\int_{|x|\le\varepsilon}|x|^2\nu(dx) &= \int_{\mathbb{R}}|x|^2\int_0^{\frac{\varepsilon}{|x|}}t^{1-\alpha}e^{-t}R(dx)\\ &\ge \int_{|x|\le1}|x|^2\int_0^{\varepsilon}t^{1-\alpha}e^{-t}R(dx)+\int_{|x|>1}|x|^2\int_0^{\frac{\varepsilon}{|x|}}t^{1-\alpha}e^{-\varepsilon}R(dx)\\ &\ge e^{-\varepsilon}\varepsilon^{2-\alpha}(2-\alpha)^{-1}\int_{|x|\le1}|x|^2R(dx)+e^{-\varepsilon}\varepsilon^{2-\alpha}(2-\alpha)^{-1}\int_{|x|>1}|x|^{\alpha}R(dx)\end{aligned}$$

By conditions (2.9), we can write

$$\lim_{\varepsilon\to0}\frac{\sigma_\varepsilon^2}{\varepsilon^2}\ge\lim_{\varepsilon\to0}\frac{Ke^{-\varepsilon}\varepsilon^{2-\alpha}}{\varepsilon^2}=+\infty,$$

with K finite constant and $0 < \alpha < 2$. Thus (6.3) holds. □

6.1.2 Series representation: a general framework

Series representation plays an important rule also in the construction of stochastic integral of a deterministic function respect to a random measure, i.e. the integral

$$I(f) = \int_E f(x)M(dx)$$

where $(E, \mathcal{E}, m)$ is a finite measure space and M is a random measure with finite control measure m, see [108, 59] and references therein.

Now, we recall a general result form [105]. As said before, a classical method to simulate infinite activity Lévy process is find a compound Poisson process which approximates the initial process. The basic idea is to consider a Lévy process with zero Lévy measure in a neighborhood $(-\varepsilon, \varepsilon)$, with $\varepsilon > 0$. If possible, jumps less than ε can be approximated by a Brownian approximation 6.1, otherwise by their mean. Let $\{X(t), t \geq 0\}$ a Lévy process, we can remove small jumps of magnitude less than ε_n and write

$$X^{\varepsilon_n}(t) = ta + \int_{\varepsilon_n \leq x \leq 1} x(N_*([0,t], dx)) - t\nu(dx)) + \int_{|x|>1} (N_*([0,t], dx)),$$

then by (6.4) we obtain

$$X_{\varepsilon_n}(t) = \sum_{\{i \geq 1 : |J_i(\omega)| \geq \varepsilon_n\}} J_i I_{\{U_i \leq t\}} - tb_n,$$

where

$$b_n = \int_{\varepsilon_n \leq x \leq 1} x\nu(dx) - a$$

The process of jumps N_*

$$N_* = \sum_{i=0}^{\infty} \delta_{(U_i, J_i)}, \tag{6.4}$$

where $\{J_i\}$ is a sequence of random variables in $\mathbb{R}$ independent of the sequence $\{U_i\}$ of i.i.d. uniform on $(0, t)$ random variables, can be represented in different ways, depending on the choice of $\{J_i\}$. By the Lévy-Itô decomposition, Theorem 1.38, we have the following convergence result

$$\sum_{\{i \geq 1 : |J_i(\omega)| \geq \varepsilon_n\}} J_i I_{\{U_i \leq t\}} - tb_n \to X(t) \quad a.s.$$

Therefore, we get a series representation of the following form

$$X(t) = \sum_{i=0}^{\infty} (J_i I_{\{U_i \leq t\}} - tc_i) \quad a.s.$$

where c_i depends on the choice on J_i. In order to simulate a Lévy processes, we need to find suitable representation on J_i and calculate the value of c_i. In computer simulation we cannot deal with infinite summation, therefore we have to truncate the series and calculate the summation only for a finite number of addends, some small jumps are inevitably truncated.

Let us now consider a measurable function

$$H : (0, \infty) \times S \to \mathbb{R}$$

where S is a measurable space and H is non increasing respect to the first variable. For most of the cases J_i is defined as

$$J_i = H(\Gamma_i, V_i)$$

for some i.i.d. sequence $\{V_i\}$, where $\{\Gamma_i\}$ are arrival times of a unit rate Poisson process on $[0, \infty)$, $\{U_i\}$ are uniform independent random variable in $\mathbb{R}$. Let $\{U_i\}$ be independent of $\{V_i\}$ and $\{\Gamma_i\}$. The Lévy measure can be written as

$$\nu(A) = \int_0^\infty P(H(r, V_i) \in A) dr \qquad A \in \mathcal{B}(\mathbb{R}).$$

By Corollary 1.36, indeed if we consider the Poisson point process

$$N = \sum_{i=1}^{\infty} \delta_{(U_i, \Gamma_i, V_i)},$$

then by taking a function h so defined

$$h(u, \gamma, v) = (u, H(\gamma, v))$$

we obtain

$$\tilde{N} = \sum_{i=1}^{\infty} \delta_{(U_i, H(\Gamma_i, V_i))}$$

where the sequences $\{U_i, \Gamma_i, V_i\}$ can be defined on the same probability space as $\tilde{N}$. The different choice of the function H gives us different algorithms to simulate infinitely divisible distributions and Lévy processes as well, see [105] for all details. We now recall the converge result of the above construction proved in [105]. First define measures on $\mathbb{R}$ by

$$\sigma(r; \cdot) = P(H(r, V_i) \in \cdot), \qquad r > 0$$

and

$$\nu(\cdot) = \int_0^\infty \sigma(r; \cdot) dr.$$

Put

$$A(s) = \int_0^s \int_{|x| \leq 1} x \sigma(r; dx) dr \qquad s \geq 0.$$

Theorem 6.5. *([105, Theorem 4.1])*

(A) *The series* $X = \sum_{i=1}^{\infty} H(\Gamma_i, V_i)$ *converges a.s. if and only if*

- *(i)* ν *is a Lévy measure on* $\mathbb{R}_0$.
- *(ii)* $a := \lim_{s \to \infty} A(s)$ *exists in* $\mathbb{R}$.

If (i) and (ii) are satisfied, then X is infinitely divisible with characteristic triplet $(a, 0, \nu)$.

(B) *If only (i) holds, then* $X = \sum_{i=1}^{\infty} |H(\Gamma_i, V_i) - c_i|$ *converges a.s. for* $c_i = A(i) - A(i-1)$. *In this case the characteristic triplet is* $(0, 0, \nu)$.

Furthermore if we want to simulate a Lévy process, by Theorem 5.1 of [105] the following convergence result follows

$$X(t) = at + \sum_{i=1}^{\infty} (H(\Gamma_i, V_i) I_{\{U_i \le t\}} - tc_i) \qquad a.s.$$

for each $t \in [0, 1]$. The speed of convergence is determined by the choice of the function H.

If the Lévy measure is written in spherical coordinates

$$\nu(A) = \int_{S^0} \int_0^{\infty} I_{\{rs \in B\}} \rho(dr, s) \sigma(ds)$$

where $\rho(\cdot, s)$ is a family of Lévy measures on $(0, \infty)$ and σ a measure on S^0, then a function

$$\rho^{\leftarrow}(u, v) := \inf\{x > 0 : \rho([x, \infty), v) < u\} \qquad r > 0$$

can be defined. By changing variable, the Lévy measure ν can be written as

$$\begin{aligned} \nu(A) &= \int_0^{\infty} P(H(r, V) \in A) dr \\ &= \int_0^{\infty} \int_{S^0} I_{\{H(r,V) \in A\}} \sigma(ds) dr \\ &= \int_{S^0} m(ds) \int_0^{\infty} I_{\{s\rho^{\leftarrow}(r,s) \in A\}} \sigma(ds) dr \end{aligned}$$

If we set H as

$$H(\gamma, v) = \rho^{\leftarrow}(\gamma, v) v,$$

then we can apply the construction above and obtain the so called LePage's series representation [75]. In the α-stable case this series converges awfully slowly, as noted in [59].

6.1.3 Rosińsky rejection method

In this section we will introduce the rejection method of [105]. Let $(X_t)_{t \ge 0}$ be the Lévy process, we want to simulate, and $(X_t^0)_{t \ge 0}$ another Lévy process. If we can find an easy way to generate $(X_t^0)_{t \ge 0}$ and if the ratio

$$\frac{d\nu}{d\nu^0} \le 1,$$

then we can construct the following algorithm, where J_i^0 is an approximation of the i-th jump of X_i^0 which can be easily generate. Additionally, let $\{W_i\}$ be a i.i.d. sequence of uniform random variables on (0,1) independent of $\{U_i, J_i^0\}$.

Algorithm 10

1. Generate a uniform variable W_i on (0,1).

2. Generate a variable J_i^0 independent from W_i.

3. Define
$$J_i = \begin{cases} J_i^0, & \frac{d\nu}{d\nu^0}(J_i^0) \geq W_i, \\ 0, & otherwise. \end{cases}$$

4. Take J_i as an approximation of the i-th jump of X_i.

The key to this method is to find an easy way to generate the Lévy process X^0 from which only a small finite number of jumps must be removed to get the jumps of X. In practical applications of this method one only needs to consider nonzero jumps. The proof of this result is a direct consequence by Corollary 1.36, indeed if we consider the Poisson point process

$$N = \sum_{i=1}^{\infty} \delta_{(U_i, W_i, J_i^0)},$$

then by taking a function h so defined

$$h(u, w, j) = (u, jI_{\{\frac{\nu}{\nu_0}(j) \geq w\}})$$

we obtain
$$\tilde{N} = \sum_{i=1}^{\infty} \delta_{(U_i, J_i)}$$

where the sequences $\{U_i, W_i, J_i^0\}$ can be defined on the same probability space as $\tilde{N}$.

6.1.4 Time-changed Brownian motion

The construction will be using throughout is well known from the theory of stochastic processes under the name of *Skorokhod embedding problem*, for a review of this problem, readers are referred to [94] and references therein. A process can be embedded in a Brownian motion if and only if it is a local semimartingale [92]. In particular every semimartingale can be written as a time-changed Brownian motion, where the random time G_t is a subordinator. As consequence of Theorem 1.38, every Lévy process Y_t is a semimartingale, thus there exists a subordinator G_t such that Y_t and X_{G_t} coincide.

A large part of modern finance has been concerned with modelling the evolution of return process over time [56, 83, 45, 46, 24]. By subordination, it is possible to capture empirically observed anomalies that contradict the classical log-normality assumption for asset prices [56, 57, 31]. In periods of high volatility, time runs faster than in periods of low volatility. The subordinator models operational time and provides distribution tail effects often observed in the market [31]. The Skorokhod embedding problem is also related to the subordination of Lévy processes. In order to obtain the generating triplet of the subordinated process, we recall a general result of [109, Theorem 30.1].

Theorem 6.6. *Let $(G_t)_{t\geq 0}$ be a subordinator with Lévy triplet $(b_0, 0, \rho)$, characteristic exponent ψ and $P_{G_1} = \lambda$. Let $(X_t)_{t\geq 0}$ be a Lévy process on R^d with generating triplet (a, σ, ν) and let $\mu = P_{X_1}$. Suppose that $(G_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ are independent. Define*

$$Y_t(\omega) = X_{G_t(\omega)}(\omega), \qquad t \geq 0.$$

Then $(Y_t)_{t\geq 0}$ is a Lévy process on $\mathbb{R}^d$ which satisfies

$$P\{Y_t \in B\} = \int_0^\infty \mu^s(B)\lambda^t(ds)$$

with characteristic function

$$E[e^{i\langle z, Y_t\rangle}] = e^{t\psi(\log \hat{\mu}(z))}, \qquad z \in \mathbb{R}^d \tag{6.5}$$

and generating triplet $(a^\sharp, \sigma^\sharp, \nu^\sharp)$ of the form

$$\begin{aligned} \nu^\sharp(B) &= b_0\nu(B) + \int_0^\infty \mu^s(B)\rho(ds), \qquad B \in \mathcal{B}(\mathbb{R}^d \backslash \{0\}), \\ \sigma^\sharp &= b_0\sigma, \\ a^\sharp &= b_0 a + \int_0^\infty \rho(ds) \int_{|x|\geq 1} x\mu^s(dx). \end{aligned} \tag{6.6}$$

Remark 6.7. *If the subordinated Lévy process is of the form*

$$Y_t = \theta G_t + W(G_t) \tag{6.7}$$

then the Lévy measure of the process Y_t is given by

$$\nu^\sharp(dx) = \int_0^\infty \frac{e^{-\frac{(x-\theta y)^2}{2y}}}{\sqrt{2\pi y}} \rho(ds). \tag{6.8}$$

The knowledge of the *business time* is very convenient for the simulation. First we can can generate the subordinator and then the Brownian motion. Since normal random variables are the building blocks of many simulation algorithms, it is then clear that all the difficulty comes from the generation of increments of the new time scale, represented by the subordinator.

Another problem comes from the theory of stochastic processes. Although the representation via Brownian subordination is a nice properties, we do not know in a general constructive method to find the process $(G_t)_{t\geq 0}$ such that $Y_t = X_{G_t(\omega)}$, thus this algorithm can be applied only for some particular Lévy processes, see for example [31, 29, 79]. If one knows how to simulate the increments of the subordinator, the increments of $(G_t)_{t\geq 0}$ can be simulated using a random walk approximation with a fixed time grid and a Brownian motion with volatility σ and mean μ.

Algorithm 11

1. Fix a time grid $t_1, \ldots, t_n$ and $G_0 = 0$.
2. Simulate increments of the subordinator $\Delta G_i = G_{t_i} - G_{t_i - 1}$.
3. Simulate n independent standard normal random variables $N_1, \ldots, N_n$.
4. Calculate increments $\Delta Y_i = \sigma N_i \sqrt{\Delta G_i} - \mu \Delta G_i$
5. Set $Y_{t_i} = \sum_{k=1}^{i} \Delta Y_i$

6.1.5 Alpha stable processes

We can consider a random walk approximation of a stable process, by using the Algorithm 9. Another approach is a shot noise representation, by following the general result of section 6.1.2. The stochastic integral of a deterministic function, measurable in the sense of [101], respect to an α-stable random measure, i.e. the integral

$$I(f) = \int_E f(x)M(dx)$$

where $(E, \mathcal{E}, m)$ is a finite measure space and M is a random measure with finite control measure m, can be defined also by the series representation described in [108]. A α-stable Lévy motion $S_\alpha(1, \beta, 0)$ can be viewed as

$$X_t \stackrel{d}{=} \int_0^T I_{[0,t]} M(dx) \qquad 0 \le t \le T$$

where M is an α-stable random measure on $([0,T], \mathcal{B}([0,T]))$ with Lebesgue control measure m and skewness intensity $\beta(x) \equiv \beta$. Therefore by Theorem 3.10.1 in [108], we have that if $0 < \alpha < 1$, then

$$X_t = C_\alpha^{1/\alpha} T^{1/\alpha} \sum_{i=1}^{\infty} V_i \Gamma^{-1/\alpha} I_{\{U_i \le t\}} \qquad 0 \le t \le T, \tag{6.9}$$

for $\alpha = 1$

$$X_t = \frac{2}{\pi} T^{1/\alpha} \sum_{i=1}^{\infty} (V_i \Gamma^{-1/\alpha} I_{\{U_i \le t\}} - \beta \frac{t}{T} b_i^{(1)}) + \beta t \frac{2}{\pi} \log \frac{2}{\pi} \qquad 0 \le t \le T \tag{6.10}$$

and $1 < \alpha < 2$

$$X_t = C_\alpha^{1/\alpha} T^{1/\alpha} \sum_{i=1}^{\infty} (V_i \Gamma^{-1/\alpha} I_{\{U_i \le t\}} - \beta \frac{t}{T} b_i^{(\alpha)}) \qquad 0 \le t \le T, \tag{6.11}$$

where we define three independent sequences, $\{V_i\}$ a sequence of i.i.d. random variables satisfying

$$P(V_i = 1) = 1 - P(V_i = -1) = \frac{1+\beta}{2},$$

$\{\Gamma_i\}$ a sequence of a Poisson point process with unit arrival rate, that is arrival times of a standard Poisson process, and $\{U_i\}$ a sequence of i.i.d. random variables uniformly distributed in $[0,T]$. Furthermore $b_i^{(\alpha)}$ is given by

$$b_i^{(\alpha)} = \begin{cases} 0, & 0 < \alpha < 1 \\ \int_{1/i}^{1/(i-1)} x^{-2} \sin x dx, & \alpha = 1 \\ \frac{\alpha}{\alpha-1}(i^{\frac{\alpha-1}{\alpha}} - (i-1)^{\frac{\alpha-1}{\alpha}}), & 1 < \alpha < 2 \end{cases} \tag{6.12}$$

6.1.6 CGMY processes

A CGMY process can be described as time changed Brownian motion, the explicit time change is known [79] and a procedure bases on this result can be developed. The time changed Brownian motion framework and the Rosińsky rejection method can be combined to obtain CGMY random numbers. The subordinator $(G_t)_{t\geq 0}$ is absolutely continuous respect to one side stable $Y/2$ subordinator with Lévy measure

$$\nu^0(dx) = \frac{K}{x^{\frac{Y}{2}+1}}.$$

The Lévy measure of the subordinator $(G_t)_{t\geq 0}$ is

$$\nu(dx) = s(x)\nu^0(dx)$$

where the function s is defined as

$$s(x) = \frac{2^{\frac{Y}{2}}\Gamma\left(\frac{Y}{2}+\frac{1}{2}\right)e^{\frac{x}{2}A^2-\frac{y}{4}B^2}}{\sqrt{\pi}}D_{-Y}(B\sqrt{x})$$

where C, G, M and Y are parameters of the process and

$$A = \frac{G-M}{2} \qquad B = \frac{G+M}{2}$$

and $D_\alpha(x)$ is a parabolic cylinder function of parameter α, see [1, 116]. Since stable distributions and processes are relatively easy to simulate, in this case the Rosińsky rejection method 6.1.3 can be applied in order to obtain a feasible simulation of the subordinator process. Indeed, we can take

$$\frac{d\nu}{d\nu^0} = f(x),$$

which can be proved to be strictly less that one, see [95] for details.

Another way to approximate jumps of a CGMY process is by applying the feasible density transformation in Proposition 4.4, see also [95], where an absolutely continuity result respect to a stable process is given. By a suitable change of measure, a CGMY process becomes an α-stable process, which can be easily simulated. This approach seems to be particulary useful for Monte Carlo methods.

The theory of TS processes allow us to consider CGMY processes as an example of the more general class of proper TS processes. Furthermore, the particular structure of this process provides a easy implementable shot noise representation as already noted in [3, 65]. We will see such representation in the following section, from a more general point of view.

6.1.7 Proper TS processes

In this section we will see a method for simulating TS distributions, as well as TS processes. There are different methods to simulate Lévy processes, but most of these methods are not suitable for the simulation of TS processes, due to the complicated structure of their Lévy measure. As already underlined in [107], the

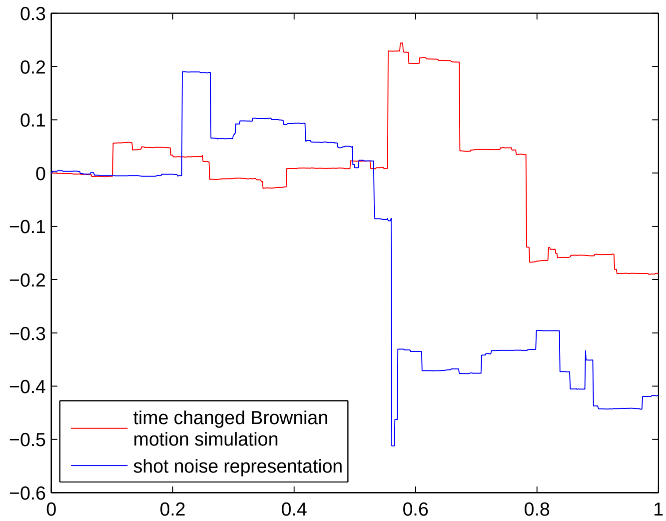

Figure 6.2: Simulation of a CGMY process with $C = 0.5$, $G = 2$, $M = 3$ and $Y = 0.5$ by time changing Brownian motion [95] and by series representation. Simulations are not comparable, due to differences of the generating algorithms.

usual method of the inverse of Lévy measure is hard to implement, even if the spectral measure R has a simple form. To overcome this problem, we will find a shot noise representation for proper TS distributions, and consequently also TS processes, without constructing any inverse. This representation holds for every TS process, therefore we obtain another procedure for simulating CGMY process, see [3, Example 4.5]. The representation, we will show, is based on results in [105] and [107]. Let ν be the Lévy measure of a proper TS distribution on $\mathbb{R}$, given by (2.3), and Q and R corresponding measures defined in (2.5) and (2.6). Let us define $\|\sigma\|$ as

$$\|\sigma\| := \sigma(S^0), \tag{6.13}$$

and by equality (2.29) and

$$Q(\mathbb{R}) = \int_{\mathbb{R}} |x|^\alpha R(dx),$$

we obtain

$$\|\sigma\| = Q(\mathbb{R}) = \int_{\mathbb{R}} |x|^\alpha R(dx) < \infty.$$

Let $\{V_j\}$ be an i.i.d. sequence of random variables in $\mathbb{R}$ with distribution $Q/\|\sigma\|$. Let $\{U_j\}$ and $\{T_j\}$ [1] be an i.i.d. sequences of uniform random variables on $(0, 1)$ and $(0, T)$ respectively, and let $\{E_j\}$ and $\{E'_j\}$ be i.i.d. sequences of exponential random variables with parameters 1. Furthermore, we assume that $\{V_j\}$, $\{U_j\}$, $\{T_j\}$, $\{E_j\}$ and $\{E'_j\}$ are independent. We consider $\Gamma_j = E'_1 + \ldots + E'_j$ and, by definition of $\{E'_j\}$, $\{\Gamma_j\}$ is a Poisson point process on $(0, \infty)$ with Lebesgue intensity measure,

[1] The random sequence $\{T_j\}$ is referred as $\{U_j\}$ in Section 6.1.2.

that is the distribution of Γ_j is $\Gamma(j,1)$, arrival times of a standard Poisson process. First, we consider a simple case.

Theorem 6.8. *Suppose that all the above assumption are fulfilled. If $\alpha \in (0,1)$, or if $\alpha \in [1,2)$ and Q is symmetric, the series*

$$X_t = \sum_{j=1}^{\infty} I_{\{T_j \le t\}} \left(\left(\frac{\alpha \Gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \wedge E_j U_j^{1/\alpha} |V_j|^{-1} \right) \frac{V_j}{|V_j|} \tag{6.14}$$

converges a.s. and uniformly in $t \in [0,T]$ to a Lévy process such that $X_t \sim TS^0_\alpha(tR,0)$ for $\alpha \in (0,1)$ and $X_t \sim TS_\alpha(tR,0)$ for $\alpha \in [1,2)$.

Proof. See [107, Theorem 5.1] □

Then we consider the general case.

Theorem 6.9. *Suppose that all the above assumption are fulfilled. If $\alpha \in [1,2)$ and Q is a non-symmetric, assume additionally that*

$$\int_{\mathbb{R}} |x| \log |x| R(dx) < \infty \tag{6.15}$$

when $\alpha = 1$ and that

$$\int_{\mathbb{R}} |x| R(dx) < \infty \tag{6.16}$$

when $\alpha \in (1,2)$. Put

$$X_t = \sum_{j=1}^{\infty} \left[I_{\{T_j \le t\}} \left(\left(\frac{\alpha \Gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \wedge E_j U_j^{1/\alpha} |V_j|^{-1} \right) \frac{V_j}{|V_j|} - \frac{t}{T} \left(\frac{\alpha j}{T\|\sigma\|} \right)^{-1/\alpha} x_0 \right] + t b_T, \tag{6.17}$$

where

$$b_T = \begin{cases} \alpha^{-1/\alpha} \zeta\left(\frac{1}{\alpha}\right) T^{-1} (T\|\sigma\|)^{1/\alpha} x_0 - \Gamma(1-\alpha) x_1, & 1 < \alpha < 2 \\ (2\gamma + \log(T\|\sigma\|)) x_1 - \int_{\mathbb{R}} x \log |x| R(dx), & \alpha = 1. \end{cases} \tag{6.18}$$

ζ denotes the Riemann zeta function [1, 23.2], γ is the Euler constant [1, 6.1.3], and

$$\begin{aligned} x_0 &= E\left[\frac{V_j}{\|V_j\|} \right] = \|\sigma\|^{-1} \int_{S^0} u \sigma(du), \\ x_1 &= \int_{\mathbb{R}} x R(dx). \end{aligned} \tag{6.19}$$

Then the series (6.17) converges a.s. uniformly in $t \in [0,T]$ to a Lévy process such that $X_t \sim TS^0_\alpha(tR,0)$

Proof. See [107, Theorem 5.1] □

This method allows one to simulate a stable process as well, by considering all jumps of the form

$$\left(\frac{\alpha \Gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \frac{V_j}{|V_j|}$$

without tempering big jumps throughout the minimum function. This method is equivalent to the procedure described in section 6.1.5.

6.1.8 Series representation for KR processes

Now, we will show a method based on the previous Theorem to simulate KR processes.

Proposition 6.10. *Let $\{X_t\}_{t\geq 0}$ be a KR process with parameters (α, k_+, k_-, r_+, r_-, p_+, p_-, m) as in Definition 2.36. Then the i.i.d. sequence of random variables $\{V_j\}$ in $\mathbb{R}$ has distribution $Q/\|\sigma\|$ with density*

$$f_V(r) = \frac{1}{\|\sigma\|}\left(k_+ r_+^{-p_+} I_{\{r>\frac{1}{r_+}\}} r^{-\alpha-p_+-1} + k_- r_-^{-p_+} I_{\{r<-\frac{1}{r_-}\}} |r|^{-\alpha-p_--1}\right)$$

where by equation (2.44)

$$\|\sigma\| = \frac{k_+ r_+^\alpha}{\alpha+p_+} + \frac{k_- r_-^\alpha}{\alpha+p_-}.$$

If $\alpha \in (0,1)$, or if $\alpha \in [1,2)$ with $k_+ = k_-$, $r_+ = r_-$ and $p_+ = p_-$, then the series

$$X_t = \sum_{j=1}^{\infty} I_{\{T_j\leq t\}} \left(\left(\frac{\alpha\Gamma_j}{T\|\sigma\|}\right)^{-1/\alpha} \wedge E_j U_j^{1/\alpha} |V_j|^{-1}\right) \frac{V_j}{|V_j|} + tb \tag{6.20}$$

converges a.s. and uniformly in $t \in [0,T]$ to a KR process with parameters (α, k_+, k_-, r_+, r_-, p_+, p_-, m) with

$$b = -\Gamma(1-\alpha)\left(\frac{k_+ r_+}{p_+ + 1} - \frac{k_- r_-}{p_- + 1}\right).$$

If $\alpha \in [1,2)$ and $k_+ \neq k_-$ (or $r_+ \neq r_-$ or alternatively $p_+ \neq p_-$), then

$$X_t = \sum_{j=1}^{\infty} \left[I_{\{T_j\leq t\}} \left(\left(\frac{\alpha\Gamma_j}{T\|\sigma\|}\right)^{-1/\alpha} \wedge E_j U_j^{1/\alpha} |V_j|^{-1}\right) \frac{V_j}{|V_j|} - \frac{t}{T}\left(\frac{\alpha j}{T\|\sigma\|}\right)^{-1/\alpha} x_0 \right] + t b_T, \tag{6.21}$$

converges a.s. and uniformly in $t \in [0,T]$ to a KR process with parameters (α, k_+, k_-, r_+, r_-, p_+, p_-, m), where we set

$$b_T = \begin{cases} \alpha^{-1/\alpha}\zeta\left(\frac{1}{\alpha}\right) T^{-1} (T\|\sigma\|)^{1/\alpha} x_0 - \Gamma(1-\alpha) x_1, & 1<\alpha<2 \\ (2\gamma + \log(T\|\sigma\|)) x_1 - \left(\frac{k_+ r_+}{p_+ + 1}\left(\log r_+ - \frac{1}{p_+ + 1}\right)\right. & \\ \quad \left. - \frac{k_- r_-}{p_- + 1}\left(\log r_- - \frac{1}{p_- + 1}\right)\right), & \alpha = 1. \end{cases}$$

with

$$\begin{aligned} x_0 &= \|\sigma\|^{-1}\left(\frac{k_+ r_+^\alpha}{\alpha+p_+} - \frac{k_- r_-^\alpha}{\alpha+p_-}\right), \\ x_1 &= \frac{k_+ r_+}{p_+ + 1} - \frac{k_- r_-}{p_- + 1}, \end{aligned}$$

ζ denotes the Riemann zeta function [1, 23.2], γ is the Euler constant [1, 6.1.3].

Proof. If $\alpha \in (0,1)$, or if $\alpha \in [1,2)$ with $k_+ = k_-$, $r_+ = r_-$ and $p_+ = p_-$, we can apply Theorem 6.5 and the series 6.14 converges a.s. and uniformly in $t \in [0,T]$ to a KR process with parameters $(\alpha,\ k_+,\ k_-,\ r_+,\ r_-,\ p_+,\ p_-,\ m)$. By Remark 2.11, we have to consider the additional term b. If $\alpha \in [1,2)$ and $k_+ \neq k_-$ (or $r_+ \neq r_-$ or alternatively $p_+ \neq p_-$), by definition of the Rosiński measure for KR processes (2.45), then the integrals (6.15) and (6.16) are finite. Furthermore, integration by parts allows us to find the value of the integral

$$\int_{\mathbb{R}} x \log |x| R(dx)$$

and by (2.45), x_0 and x_1 (6.19), such as b_T (6.18), can be easily computed. □

Finally, we can write a procedure to simulate a KR process with parameters $(\alpha,\ k_+,\ k_-,\ r_+,\ r_-,\ p_+,\ p_-,\ m)$ for discrete values of time t_i, where $\{t_i\}_{0\leq i\leq K}$ is a partition of the interval $[0,T]$ with equal length subinterval and mesh

$$\Delta t = \frac{T}{K}$$

with $K \in \mathbb{N}$.

Algorithm 12

1. Fix a time T and consider a partition of the interval $[0,T]$ in K parts of same length.
2. Fix a number C ($\sim 10^5, 10^6$).
3. Simulate independent sequences $\{V_j\}$, $\{\Gamma_j\}$, $\{U_j\}$ and $\{E_j\}$ of length C.
4. Calculate the vector $\{X_{t_i}\}$ by equality (6.21) (or (6.20)).

By the algorithm above, we can simulate the entire trajectory of a TS process. This method seems to be particulary useful for path dependent options, such as barrier options or asian options [65].

6.1.9 Series representation for CGMY processes

By similar arguments, also a series representation of a CGMY process (Definition 2.35) can be obtained. By equalities (2.33), (2.34) and (2.35), we obtain a sequence $\{V_j\}$ of discrete random variable with distribution

$$P(V_j = -G) = P(V_j = M) = \frac{1}{2}$$

and

$$\|\sigma\| = 2C.$$

By recalling that $0 < Y < 2$, we have

$$X_t \stackrel{d}{=} \sum_{j=1}^{\infty} \left[\left(\frac{Y\Gamma_j}{2C} \right)^{-1/Y} \wedge E_j U_j^{1/Y} |V_j|^{-1} \right] \frac{V_j}{|V_j|} I_{\{T_j \leq t\}} + t b_T \qquad t \in [0,T], \quad (6.22)$$

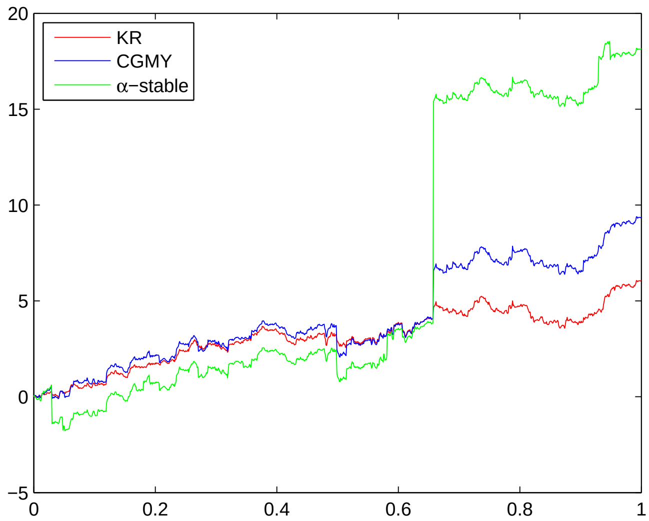

Figure 6.3: Series representation of a stable process and two different TS processes with truncation of small jumps.

where

$$b_T = \begin{cases} -\Gamma(1-Y)C(M^{Y-1} - G^{Y-1}), & Y \in (0,2)/\{1\} \\ (2\gamma + \log(2TC))C(M^{Y-1} - G^{Y-1}) & \\ \qquad -C(G^{Y-1}\log G - M^{Y-1}\log M), & Y = 1. \end{cases} \tag{6.23}$$

where γ is the Euler constant [1, 6.1.3].

6.1.10 Proper TID laws and processes

There are different methods to simulate Lévy processes, but most of these methods are not suitable for the simulation of TID processes due to the complicated structure of their Lévy measure. The usual method of the inverse of the Lévy measure is difficult to implement, even if the spectral measure R has a simple form, readers are referred to [107]. To overcome this problem, we will find a shot noise representation for proper TID distributions, and consequently also TID processes, without constructing any inverse. The representation, we will show, is based on results in [105] and [107].

Let ν be the Lévy measure of a proper TID distribution on $\mathbb{R}^d$, given by (3.3), and Q and R corresponding measures defined in (2.5) and (2.6). Let us define $\|\sigma\|$ as

$$\|\sigma\| := \sigma(S^{d-1}), \tag{6.24}$$

and by equalities (3.13) and (3.15), we obtain

$$\|\sigma\| = Q(\mathbb{R}^d) = \int_{\mathbb{R}^d} \|x\|^\alpha R(dx) < \infty.$$

Let $\{V_j\}$ be an i.i.d. sequence of random vector in $\mathbb{R}^d$ with distribution $Q/\|\sigma\|$. Let $\{U_j\}$ be an i.i.d. sequence of uniform random variables on $(0,1)$ and let $\{E_j\}$ and $\{E'_j\}$ be i.i.d. sequences of exponential random variables with parameters 1. Furthermore, we assume that $\{V_j\}$, $\{U_j\}$, $\{E_j\}$ and $\{E'_j\}$ are independent. We consider $\Gamma_j = E'_1 + \ldots + E'_j$ and, by definition of $\{E'_j\}$, $\{\Gamma_j\}$ is a Poisson point process on $(0,\infty)$ with Lebesgue intensity measure. Now, we will prove a useful lemma.

Lemma 6.11. *Under the above definitions, let us define the function*

$$H(\Gamma_j,(V_j,E_j,U_j)) := \left(\left(\frac{\alpha\Gamma_j}{\|\sigma\|}\right)^{-1/\alpha} \wedge \sqrt{2}E_j^{1/2}U_j^{1/\alpha}\|V_j\|^{-1}\right)\frac{V_j}{\|V_j\|}. \tag{6.25}$$

Then, for every non-empty set $A \in \mathcal{B}(\mathbb{R}^d)$, the equality

$$\int_0^\infty P(H(s,(V_1,E_1,U_1)) \in A)ds = \nu(A)$$

is verified.

Proof. Let A be a set of the form

$$A = \left\{x \in \mathbb{R}^d : \|x\| > a, \frac{x}{\|x\|} \in B\right\},$$

where $a > 0$ and $B \in \mathcal{B}(S^{d-1})$. Then, we can write

$$\begin{aligned}
&\int_0^\infty P(H(s,(V_1,E_1,U_1)) \in A)ds = \\
&= \int_0^\infty P\Bigg(\left(\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} \wedge \sqrt{2}E_1^{1/2}U_1^{1/\alpha}\|V_1\|^{-1}\right)\frac{V_1}{\|V_1\|} \in A\Bigg)ds, \\
&= E\int_0^\infty I\Bigg(\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} > a, \sqrt{2}E_1^{1/2}U_1^{1/\alpha} > a\|V_1\|, \frac{V_1}{\|V_1\|} \in B\Bigg)ds \\
&= \frac{\|\sigma\|a^{-\alpha}}{\alpha}EI\Big(\sqrt{2}E_1^{1/2}U_1^{1/\alpha} > a\|V_1\|, \frac{V_1}{\|V_1\|} \in B\Big) \\
&= \frac{a^{-\alpha}}{\alpha}\int_B\int_0^\infty P\Big(\sqrt{2}E_1^{1/2}U_1^{1/\alpha} > as\Big)Q(ds|u)\sigma(du).
\end{aligned}$$

By conditioning, the probability in the integral can be calculated

$$\begin{aligned}
P\Big(\sqrt{2}E_1^{1/2}U_1^{1/\alpha} > as\Big) &= \int_0^1\int_{\frac{a^2s^2}{2u^{2/\alpha}}}^\infty e^{-x}dxdu \\
&= \int_0^1 e^{-\frac{a^2s^2}{2u^{2/\alpha}}}du \\
&= a^\alpha\alpha\int_a^\infty e^{-r^2s^2/2}r^{-\alpha-1}dr,
\end{aligned}$$

therefore we obtain

$$\begin{aligned}
\int_0^\infty P(H(s,(V_1,E_1,U_1)) \in A)ds &= \int_B\int_0^\infty\int_a^\infty e^{-r^2s^2/2}r^{-\alpha-1}drQ(ds|u)\sigma(du) \\
&= \int_B\int_a^\infty q(r,u)r^{-\alpha-1}dr\sigma(du) = \nu(A).
\end{aligned}$$

□

First, we consider a simple case.

Theorem 6.12. *($\alpha \in (0,1)$ and symmetric case) Suppose that all the above assumption are fulfilled. If $\alpha \in (0,1)$, or if $\alpha \in [1,2)$ and Q is symmetric, the series*

$$S_0 = \sum_{j=1}^{\infty} \left(\left(\frac{\alpha \Gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} E_j^{1/2} U_j^{1/\alpha} \|V_j\|^{-1} \right) \frac{V_j}{\|V_j\|}. \tag{6.26}$$

converges a.s.. Furthemore, we have that $S_0 \sim TID_\alpha^0(R,0)$ for $\alpha \in (0,1)$ and $S_0 \sim TID_\alpha(R,0)$ for $\alpha \in [1,2)$.

Proof. To prove this theorem, we are going to use [105, Theorem 4.1] and [107, Theorem 5.1]. If H is defined as in (6.25), we can apply Lemma 6.11. Let us consider the case $\alpha \in (0,1)$, then by Proposition 3.13 we can write

$$\int_0^\infty E(\|H(s,(V_1,E_1,U_1))\| I(\|H(s,(V_1,E_1,U_1))\| \leq 1)) ds = \int_{\|x\|\leq 1} \|x\| \nu(dx) < \infty$$

and [105, Theorem 4.1(A)] proves the theorem in the case $\alpha \in (0,1)$.

If $\alpha \in [1,2)$, then by Proposition 3.11 we have

$$\int_0^\infty E(\|H(s,(V_1,E_1,U_1))\| I(\|H(s,(V_1,E_1,U_1))\| > 1)) ds = \int_{\|x\|>1} \|x\| \nu(dx) < \infty$$

and by [105, Theorem 4.1(B)], we can consider a series

$$\bar{S}_0 = \sum_{j=1}^{\infty} \left[\left(\left(\frac{\alpha \Gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} E_j^{1/2} U_j^{1/\alpha} \|V_j\|^{-1} \right) \frac{V_j}{\|V_j\|} - c_j \right]$$

which converges a.s. and $\bar{S}_0 \sim TID_\alpha(R,0)$, where

$$c_j = \int_{j-1}^{j} E\left[\left(\left(\frac{\alpha \Gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} E_j^{1/2} U_j^{1/\alpha} \|V_j\|^{-1} \right) \frac{V_j}{\|V_j\|} \right] ds. \tag{6.27}$$

If Q is symmetric, c_j is equal to zero. It follows that $S_0 = \bar{S}_0$. This completes the proof. □

Now we consider the non-symmetric case.

Theorem 6.13. *(Non-symmetric case) Under the above notation, suppose $\alpha \in [1,2)$, Q is non-symmetric and additionally that*

$$\int_{\mathbb{R}^d} \|x\| |\log \|x\|| R(dx) < \infty \tag{6.28}$$

when $\alpha = 1$ and that

$$\int_{\mathbb{R}^d} \|x\| R(dx) < \infty \tag{6.29}$$

when $\alpha \in (1,2)$. Then, the series

$$S_1 = \sum_{j=1}^{\infty} \left[\left(\left(\frac{\alpha \Gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} E_j^{1/2} U_j^{1/\alpha} \|V_j\|^{-1} \right) \frac{V_j}{\|V_j\|} - \left(\frac{\alpha j}{\|\sigma\|} \right)^{-1/\alpha} x_0 \right] + b \quad (6.30)$$

where

$$\begin{aligned} x_0 &= E \frac{V_j}{\|V_j\|} = \|\sigma\|^{-1} \int_{S^{d-1}} u \sigma(du), \\ x_1 &= \int_{\mathbb{R}^d} x R(dx), \end{aligned}$$

$$b = \begin{cases} \zeta\left(\frac{1}{\alpha}\right) \alpha^{-1/\alpha} \|\sigma\|^{1/\alpha} x_0 - 2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) x_1, & 1 < \alpha < 2, \\ (\frac{3}{2}\gamma - \frac{\log 2}{2} + \log \|\sigma\|) x_1 - \int_{\mathbb{R}^d} x \log \|x\| R(dx), & \alpha = 1, \end{cases} \quad (6.31)$$

ζ denotes the Riemann zeta function and γ is the Euler constant, converges a.s.. Furthermore, we have that $S_1 \sim TID_\alpha(R,0)$.

Proof. To prove this theorem, we are going to use [105, Theorem 4.1] and [107, Theorem 5.1]. If H is defined as in (6.25), we can apply Lemma 6.11. If $\alpha \in [1,2)$, then by Proposition 3.11 we have

$$\int_0^\infty E(\|H(s,(V_1,E_1,U_1))\| I(\|H(s,(V_1,E_1,U_1))\| > 1)) ds = \int_{\|x\|>1} \|x\| \nu(dx) < \infty$$

and [105, Theorem 4.1(B)], we can consider a series

$$S_1 = \sum_{j=1}^{\infty} \left[\left(\left(\frac{\alpha \Gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} E_j^{1/2} U_j^{1/\alpha} \|V_j\|^{-1} \right) \frac{V_j}{\|V_j\|} - c_j \right]$$

which converges a.s. and $S_1 \sim TID_\alpha(R,0)$, where

$$c_j = \int_{j-1}^{j} E \left[\left(\left(\frac{\alpha \Gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} E_j^{1/2} U_j^{1/\alpha} \|V_j\|^{-1} \right) \frac{V_j}{\|V_j\|} \right] ds.$$

We have to prove that the equality

$$\sum_{j=1}^{\infty} \left[\left(\frac{\alpha j}{\|\sigma\|} \right)^{-1/\alpha} x_0 - c_j \right] = b \quad (6.32)$$

holds, where b is given by (6.31).

First consider the case $\alpha \in (1,2)$. Define for $j \geq 1$ [107, equation (5.8)]

$$c_j' = \int_{j-1}^{j} E \left[\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} \frac{V_1}{\|V_1\|} \right] ds = \frac{\alpha^{1-1/\alpha} \|\sigma\|^{1/\alpha}}{\alpha - 1} [j^{1-1/\alpha} - (j-1)^{1-1/\alpha}] x_0. \quad (6.33)$$

We have

$$\|c_j' - c_j\| \leq \int_{j-1}^{j} E \left\{ \left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} - \left[\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} E_1^{1/2} U_1^{1/\alpha} \|V_1\|^{-1} \right] \right\} ds.$$

Furthermore by [107, equation 5.9], for every $\theta > 0$ the equality

$$\int_0^\infty \left\{ \left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} - \left[\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} \wedge \theta \right] \right\} ds = \frac{\|\sigma\|}{\alpha(\alpha-1)} \theta^{1-\alpha} \tag{6.34}$$

holds. Using this identity for $\theta = \sqrt{2} E_1^{1/2} U_1^{1/\alpha} \|V_1\|^{-1}$ pointwise, we obtain

$$\begin{aligned}
\sum_{j=1}^\infty \|c_j' - c_j\| &= E \int_0^\infty \left\{ \left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} - \left[\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} E_1^{1/2} U_1^{1/\alpha} \|V_1\|^{-1} \right] \right\} ds \\
&= \frac{\|\sigma\|}{\alpha(\alpha-1)} E\left[2^{\frac{1}{2}(1-\alpha)} E_1^{\frac{1}{2}(1-\alpha)} U_1^{-1+\frac{1}{\alpha}} \|V_1\|^{\alpha-1} \right] \\
&= 2^{\frac{1}{2}(1-\alpha)} \Gamma\left(\frac{3}{2} - \frac{\alpha}{2} \right) \frac{\|\sigma\|}{\alpha-1} E\|V_1\|^{\alpha-1} \\
&= -2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2} \right) \int_{\mathbb{R}^d} \|x\| R(dx) < \infty.
\end{aligned} \tag{6.35}$$

By using (6.34) we obtain

$$\begin{aligned}
\sum_{j=1}^\infty (c_j' - c_j) &= E\left\{ \int_0^\infty \left(\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} - \left[\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} E_1^{1/2} U_1^{1/\alpha} \|V_1\|^{-1} \right] \right) ds \frac{V_1}{\|V_1\|} \right\} \\
&= E\left[\frac{\|\sigma\|}{\alpha(\alpha-1)} 2^{\frac{1}{2}(1-\alpha)} E_1^{\frac{1}{2}(1-\alpha)} U_1^{-1+\frac{1}{\alpha}} \|V_1\|^{\alpha-1} \frac{V_1}{\|V_1\|} \right] \\
&= 2^{\frac{1}{2}(1-\alpha)} \Gamma\left(\frac{3}{2} - \frac{\alpha}{2} \right) \frac{1}{\alpha-1} \int_{\mathbb{R}^d} x \|x\|^{\alpha-2} Q(dx) \\
&= -2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2} \right) \int_{\mathbb{R}^d} x R(dx) \\
&= -2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2} \right) x_1
\end{aligned}$$

Then we have

$$\sum_{j=1}^n \left[\left(\frac{\alpha j}{\|\sigma\|} \right)^{-1/\alpha} x_0 - c_j' \right] = \left(\sum_{j=1}^n j^{-1/\alpha} - \frac{\alpha}{\alpha-1} n^{1-1/\alpha} \right) \alpha^{-1/\alpha} \|\sigma\|^{1/\alpha} x_0.$$

From a classical formula [1, 23.2.9],

$$\sum_{j=1}^n j^{-z} - \frac{n^{1-z}}{1-z} = \zeta(z) + z \int_n^\infty \frac{s - [s]}{s^{z+1}} ds, \qquad \mathrm{Re}(z) > 0,\ \mathrm{Re}(z) \neq 1, \tag{6.36}$$

we obtain

$$\sum_{j=1}^\infty \left[\left(\frac{\alpha j}{\|\sigma\|} \right)^{-1/\alpha} x_0 - c_j' \right] = \zeta\left(\frac{1}{\alpha} \right) \alpha^{-1/\alpha} \|\sigma\|^{1/\alpha} x_0$$

and we can write

$$\begin{aligned}
\sum_{j=1}^\infty \left[\left(\frac{\alpha j}{\|\sigma\|} \right)^{-1/\alpha} x_0 - c_j \right] &= \sum_{j=1}^\infty \left[\left(\frac{\alpha j}{\|\sigma\|} \right)^{-1/\alpha} x_0 - c_j' \right] + \sum_{j=1}^\infty (c_j' - c_j) \\
&= \zeta\left(\frac{1}{\alpha} \right) \alpha^{-1/\alpha} \|\sigma\|^{1/\alpha} x_0 - 2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2} \right) x_1 = b
\end{aligned}$$

which proves (6.32). Now, we consider the case $\alpha = 1$. By the same computation above, define for $j \leq 2$

$$c'_j = \int_{j-1}^{j} E\left[\left(\frac{s}{\|\sigma\|}\right)^{-1/\alpha} \frac{V_1}{\|V_1\|}\right] ds = (\log j - \log(j-1))\|\sigma\| x_0, \tag{6.37}$$

and put $c'_1 = 0$. For every $\theta > 0$ [107, equation (5.14)], we have

$$\begin{aligned}
&\int_1^\infty \left\{\left(\frac{s}{\|\sigma\|}\right)^{-1} - \left[\left(\frac{s}{\|\sigma\|}\right)^{-1} \wedge \theta\right]\right\} ds \\
&\quad = \{\theta - \|\sigma\| \log\theta + \|\sigma\| \log\|\sigma\| - \|\sigma\|\} I(\theta \leq \|\sigma\|) \\
&\quad \leq \|\sigma\| \log^+\left(\frac{\|\sigma\|}{\theta}\right).
\end{aligned} \tag{6.38}$$

By assumption (6.28), we can write

$$\begin{aligned}
\sum_{j=1}^{\infty} \|c'_j - c_j\| &= E\int_1^\infty \left\{\left(\frac{s}{\|\sigma\|}\right)^{-1} - \left[\left(\frac{s}{\|\sigma\|}\right)^{-1} \wedge \sqrt{2}E_1^{1/2} U_1 \|V_1\|^{-1}\right]\right\} ds \\
&\leq \|\sigma\| E \log^+\left(\frac{\|\sigma\|\|V_1\|}{\sqrt{2}E_1^{1/2} U_1}\right) \\
&\leq \|\sigma\|(|\log\|\sigma\|| + E|\log\|V_1\|| + E|\log\sqrt{2}E_1^{1/2} U_1|) \\
&= \|\sigma\||\log\|\sigma\|| + \int_{\mathbb{R}^d} |\log\|x\|| \|x\| R(dx) + K\|\sigma\| < \infty,
\end{aligned} \tag{6.39}$$

where $K = E|\log\sqrt{2}E_1^{1/2} U_1| < \infty$.

Before computing the series $\sum_{j=1}^{\infty}(c'_j - c_j)$, we recall some useful relations [107]. For every $\theta > 0$

$$\int_0^1 \left(\frac{s}{\|\theta\|}\right)^{-1} \wedge \theta ds = \theta I(\theta \leq \|\sigma\|) + \{\|\sigma\| - \|\sigma\| \log\|\sigma\| + \|\sigma\| \log\theta\} I(\theta > \|\sigma\|)$$

and by (6.38) we get

$$-\int_0^1 \left(\frac{s}{\|\theta\|}\right)^{-1} \wedge \theta\, ds + \int_1^\infty \left\{\left(\frac{s}{\|\theta\|}\right)^{-1} - \left[\left(\frac{s}{\|\theta\|}\right)^{-1} \wedge \theta\right]\right\} = \|\sigma\|(\log\|\sigma\| - \log\theta - 1).$$

By using this formula for $\theta = \sqrt{2}E_1^{1/2} U_1 \|V_1\|^{-1}$ we get

$$\begin{aligned}
\sum_{j=1}^{\infty}(c'_j - c_j) &= E\left\{\left[-\int_0^1 \left(\left(\frac{s}{\|\theta\|}\right)^{-1} \wedge \sqrt{2}E_1^{1/2} U_1 \|V_1\|^{-1}\right) ds\right.\right. \\
&\qquad \left.\left. + \int_1^\infty \left(\left(\frac{s}{\|\theta\|}\right)^{-1} - \left[\left(\frac{s}{\|\theta\|}\right)^{-1} \wedge \sqrt{2}E_1^{1/2} U_1 \|V_1\|^{-1}\right]\right) ds\right] \frac{V_1}{\|V_1\|}\right\} \\
&= \|\sigma\| E\left\{(\log\|\sigma\| + \log\|V_1\| - \log(\sqrt{2}E_1^{1/2} U_1) - 1)\frac{V_1}{\|V_1\|}\right\}.
\end{aligned}$$

The following expectation can be calculated

$$E\log(\sqrt{2}E_1^{1/2} U_1) = \frac{\log 2}{2} - 1 - \frac{1}{2}\gamma,$$

where $\gamma = -\int_0^\infty \log(x)e^{-x}dx$ is the Euler constant, see [1, 6.1.3]. By equation (3.7), the series above can be rewritten as

$$\begin{aligned}\sum_{j=1}^{\infty}(c_j' - c_j) &= \|\sigma\|E\Big\{\frac{V_1}{\|V_1\|}(\log\|\sigma\| + \log\|V_1\| + \frac{1}{2}\gamma - \frac{\log 2}{2}\Big\}\\ &= \int_{\mathbb{R}^d}\frac{x}{\|x\|}\Big(\log\|\sigma\| + \log\|x\| + \frac{1}{2}\gamma - \frac{\log 2}{2}\Big)Q(dx)\\ &= \int_{\mathbb{R}^d} x\Big(\log\|\sigma\| - \log\|x\| + \frac{1}{2}\gamma - \frac{\log 2}{2}\Big)R(dx)\\ &= \Big(\frac{1}{2}\gamma - \frac{\log 2}{2} + \log\|\sigma\|\Big)x_1 - \int_{\mathbb{R}^d} x\log\|x\|R(dx).\end{aligned}$$

By [107, Theorem 5.1], the equality

$$\sum_{j=1}^{\infty}\left[\left(\frac{j}{\|\sigma\|}\right)^{-1}x_0 - c_j'\right] = \gamma x_1$$

holds, where γ is the Euler constant, thus we obtain

$$\begin{aligned}\sum_{j=1}^{\infty}\left[\left(\frac{j}{\|\sigma\|}\right)^{-1}x_0 - c_j\right] &= \sum_{j=1}^{\infty}\left[\left(\frac{j}{\|\sigma\|}\right)^{-1}x_0 - c_j'\right] + \sum_{j=1}^{\infty}(c_j' - c_j)\\ &= \Big(\frac{3}{2}\gamma - \frac{\log 2}{2} + \log\|\sigma\|\Big)x_1 - \int_{\mathbb{R}^d} x\log\|x\|R(dx) = b,\end{aligned}$$

which completes the proof. □

A series representation for TID processes can be obtained.

Theorem 6.14. *Under the above notation and assumptions, given a fixed $T > 0$, let $\{T_j\}$ be a i.i.d. sequence of uniform random variables in $[0, T]$. Assume $\{T_j\}$ independent of the random sequences $\{V_j\}$, $\{U_j\}$, $\{E_j\}$ and $\{\Gamma_j\}$.*

(i) If $\alpha \in (0, 1)$, or if $\alpha \in [1, 2)$ and Q is symmetric, set for every $t \in [0, T]$

$$X_0(t) = \sum_{j=1}^{\infty} I_{\{T_j\le t\}}\left(\left(\frac{\alpha\Gamma_j}{T\|\sigma\|}\right)^{-1/\alpha} \wedge \sqrt{2}E_j^{1/2}U_j^{1/\alpha}\|V_j\|^{-1}\right)\frac{V_j}{\|V_j\|}, \qquad (6.40)$$

then the series converges a.s. uniformly in $t \in [0, T]$ to a Lévy process such that $X_0(t) \sim TID_\alpha^0(tR, 0)$ if $\alpha \in (0, 1)$ and $X_0(t) \sim TID_\alpha^0(tR, 0)$ if $\alpha \in [1, 2)$.

(ii) If $\alpha \in [1, 2)$, Q is non-symmetric and additionally

$$\int_{\mathbb{R}^d}\|x\||\log\|x\||R(dx) < \infty$$

when $\alpha = 1$ and that

$$\int_{\mathbb{R}^d}\|x\|R(dx) < \infty$$

when $\alpha \in (1,2)$, then, the series

$$X_1(t) = \sum_{j=1}^{\infty} \left[I_{\{T_j \le t\}} \left(\left(\frac{\alpha \Gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} E_j^{1/2} U_j^{1/\alpha} \|V_j\|^{-1} \right) \frac{V_j}{\|V_j\|} - \frac{t}{T} \left(\frac{\alpha j}{T\|\sigma\|} \right)^{-1/\alpha} x_0 \right] + t b_T. \tag{6.41}$$

where

$$b_T = \begin{cases} \zeta\left(\frac{1}{\alpha}\right) \alpha^{-1/\alpha} T^{-1} (T\|\sigma\|)^{1/\alpha} x_0 - 2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) x_1, & 1 < \alpha < 2, \\ (\frac{3}{2}\gamma - \frac{\log 2}{2} + \log T\|\sigma\|) x_1 - \int_{\mathbb{R}^d} x \log\|x\| R(dx), & \alpha = 1, \end{cases} \tag{6.42}$$

the series converges a.s. uniformly in $t \in [0,T]$ to a Lévy process such that $X_1(t) \sim TID_\alpha(tR, 0)$.

Proof. It is enough to show the convergence in distribution of series (6.40) and (6.41) for a fixed t, see [105, 107]. By the same arguments of Lemma 6.11, we obtain

$$\int_0^\infty P(H(s,(V_1,E_1,U_1,T_1)) \in A) ds = t\nu(A)$$

where we define

$$H(\Gamma_j,(V_j,E_j,U_j,T_j)) := I_{\{T_j \le t\}} \left(\left(\frac{\alpha \Gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} E_j^{1/2} U_j^{1/\alpha} \|V_j\|^{-1} \right) \frac{V_j}{\|V_j\|}. \tag{6.43}$$

By following the proof of Theorem 6.12, (i) is verified in the case $\alpha \in (0,1)$. By Proposition 3.11 if $\alpha \in [1,2)$, then $\int_{\|x\|>1} \|x\| \nu(dx) < \infty$. By [105, Theorem 4.1(B)] we can consider the series

$$\bar{X}_1(t) = \sum_{j=1}^{\infty} \left[I_{\{T_j \le t\}} \left(\left(\frac{\alpha \Gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} E_j^{1/2} U_j^{1/\alpha} \|V_j\|^{-1} \right) \frac{V_j}{\|V_j\|} - a_j^T \right]$$

which converges a.s. and $\bar{X}_1(t) \sim TID_\alpha(tR, 0)$, where

$$a_j^T = \int_{j-1}^{j} E\left[I_{(0,t]}(T_j) \left(\left(\frac{\alpha \Gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} E_j^{1/2} U_j^{1/\alpha} \|V_j\|^{-1} \right) \frac{V_j}{\|V_j\|} \right] ds.$$

If Q is symmetric then $a_j^T = 0$ and (i) is proved. To complete the proof, by following [107, Theorem 5.3] and equation (6.27), c_j can be viewed as a function of the measure Q, thus we have

$$a_j^T(t) = \frac{t}{T} c_j(TQ).$$

By Theorem 6.13, where TQ and TR have to be considered instead of Q and R, we have

$$\sum_{j=1}^{\infty} \left[\left(\frac{\alpha \Gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} x_0 - c_j(TQ) \right] = \begin{cases} \zeta\left(\frac{1}{\alpha}\right) \alpha^{-1/\alpha} (T\|\sigma\|)^{1/\alpha} x_0 - 2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) T x_1 \alpha - 1, & 1 < \alpha < 2, \\ (\frac{3}{2}\gamma - \frac{\log 2}{2} + \log T\|\sigma\|) T x_1 - T \int_{\mathbb{R}^d} x \log\|x\| R(dx), & \alpha = 1. \end{cases}$$

By definition of $a_j^T(t)$, we obtain

$$\sum_{j=1}^{\infty}\left[\frac{t}{T}\left(\frac{\alpha\Gamma_j}{T\|\sigma\|}\right)^{-1/\alpha} x_0 - a_j^T(t)\right] = tb_T,$$

which completes the proof. □

Remark 6.15. *By removing the tempering part* $\sqrt{2}E_j^{1/2}U_j^{1/\alpha}\|V_j\|^{-1}$ *in the shot noise representation, a well-known result for* α*-stable processes can be found, see [107, Theorem 5.4] or [108].*

6.1.11 Series representation for RDTS processes

By similar arguments, also a series representation of a RDTS process can be obtained. Let us suppose that $c_+ = c_- = C$. By equalities (3.41), (3.42) and (3.43), we obtain a sequence $\{V_j\}$ of discrete random variable with distribution

$$P(V_j = -\lambda_-) = P(V_j = \lambda_+) = \frac{1}{2}$$

and

$$\|\sigma\| = 2C.$$

By recalling that $\alpha \in (0,2)/\{1\}$, we have

$$X_t \stackrel{d}{=} \sum_{j=1}^{\infty}\left[\left(\frac{\alpha\Gamma_j}{2C}\right)^{-1/\alpha} \wedge \sqrt{2}E_j^{1/2}U_j^{1/\alpha}|V_j|^{-1}\right]\frac{V_j}{|V_j|}I_{\{T_j\le t\}} + tb_T \qquad t\in[0,T], \tag{6.44}$$

where

$$b_T = -2^{-\frac{1+\alpha}{2}}\Gamma\left(\frac{1-\alpha}{2}\right)C(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}). \tag{6.45}$$

6.1.12 A Monte Carlo example

In this section, we assess the goodness of fit of random number generators proposed in the previous section. A brief Monte Carlo study is performed and prices of European put options with different strikes are calculated. By taking into account equation 5.3, the stock price process under a risk neutral martingale measure Q is of the form

$$S_t = S_0 e^{(r+\omega)t+X_t}. \tag{6.46}$$

where ω is the *convexity correction* and the price $\Pi(t,C)$ ad time t of an European call can be calculated as the expected value of the discounted value of its payout, that is

$$\Pi(0,C) = E_Q[e^{-r(T-t)}(S_T - K)^+|\mathcal{F}_0]$$

We take into consideration a CGMY process with the same artificial parameters of the work [95] that is $C = 0.5$, $G = 2$, $M = 3.5$, $Y = 0.5$, interest rate $r = 0.04$, initial stock price $S_0 = 100$ and annualized maturity $T = 0.25$. Furthermore we consider also a GTS process defined by the characteristic exponent (2.36) and

Table 6.1: European put option prices computed using the Fourier transform method (Price) and by Monte Carlo simulation (Monte Carlo).

CGMY			GTS		
Strike	Price	Monte Carlo	Strike	Price	Monte Carlo
80	1.7444	1.7472	80	3.2170	3.2144
85	2.3926	2.3955	85	4.2132	4.2179
90	3.2835	3.2844	90	5.4653	5.4766
95	4.5366	4.5383	95	7.0318	7.0444
100	6.3711	6.3724	100	8.9827	8.9968
105	9.1430	9.1532	105	11.3984	11.4175
110	12.7632	12.7737	110	14.3580	14.3895
115	16.8430	16.8551	115	17.8952	17.9394
120	21.1856	21.2064	120	21.9109	21.9688

parameters $c_+ = 0.5$, $c_- = 1$, $\lambda_+ = 3.5$, $\lambda_- = 2$ and $\alpha = 0.5$, interest rate r, initial stock price S_0 and maturity T as in the CGMY case.

The expectation above is calculate via Monte Carlo simulation, where 50,000 sample paths are generated. The Esscher transform with $\theta = -1.5$ is considered to reduce the variance [65]. We want to emphasize that the Esscher transform is an exponential tilting [107], thus if applied to a CGMY or a GTS process, it modifies only parameters but not the form of the characteristic function.

In Table 6.1 simulated prices and prices obtained by using the Fourier transform method [23] are compared. Even if there is a competitive CGMY random number generator, where a time changed Brownian motion is considered [95], we prefer to use an algorithm based on series representation. Contrary to the CGMY case, in general there is not a constructive method to find the subordinator process that changes the time of the Brownian motion, that is we do not know the process T_t such that the TS process X_t can be rewritten as $W_{T(t)}$ [29]. The shot noise representation allows one to generate any TS_α process.

Chapter 7

Non Gaussian GARCH models

7.1 Introduction

Volatility clustering is the tendency for extreme returns to be followed by other extreme returns, although not necessarily with the same sign [88].

There is a general consensus that asset returns exhibit variances that change through time. In the financial literature GARCH models are a popular choice to model these changing variances [18]. However the success of GARCH in modeling volatility clustering only partially extends to option pricing [35, 104, 54]. Within the last 30 years a vast amount of literature on the option pricing problem has been published. Since the seminal work of Black and Scholes [17] and Merton [91] who derived the arbitrage free option price solely from the stochastic dynamic of the underlying stock and the risk free rate, the research has focussed on improving the fit of the theoretical stock price dynamics to market data. The homoskedasticity and the lognormality postulated in the Black and Scholes framework cannot deal effectively with the volatility clustering and the leptokurtosis observed in asset prices. Although asset return distributions are known to be conditionally leptokurtic, only few works consider non gaussian innovations in the recent GARCH model literature, [67, 89, 90, 9, 27, 26].

The importance of GARCH option pricing has recently expanded due to their linkage with stochastic volatility models [37]. Indeed, even if GARCH models are a bit mechanical, the methodology is useful since their diffusion limits contain many well known stochastic volatility models. From an estimation perspective, GARCH models may have distinct advantages over stochastic volatility models. Continuous time stochastic volatility models are difficult to implement, because, with discrete observations on the underlying asset price process, the volatility is not readily identifiable. If the volatility level cannot be established, option prices cannot be computed. Furthermore, time continuity models impose the possibility of continuous trading in order to construct the hedge portfolio which is not feasible in reality. To overcome this problem implied volatilities are often established from concurrent option prices. Indeed, a common technique for estimating stochastic volatility models, as adopted in [7] for example, is to use a cross section of option data to estimate all the parameters, including volatility, on a daily basis. If the parameters of the process are required to be constant through time, then a time series of daily option records are used in the analysis and a daily sequence of implied volatilities has to be estimated. Since the number of unknown volatilities increases linearly with the number of days, the computational effort involved in the optimization problem soon becomes severe. This approach has been used in [11] and in [51], for example.

In contrast, GARCH models have the advantage that the volatility is observable from the history of asset prices. Consequently, it is possible to price options, solely on the basis of observable history of the underlying asset process, without requiring information on derivative prices. As a result, option prices can be generated in illiquid markets where concurrent information on derivative prices may not exist. Most of the empirical analysis on option pricing with GARCH is based on the fact that S&P 500 index is one of the best markets for testing a European option valuation model and it is easy to hedge since there is a very active market for futures. By following the classical literature, we will consider this market as well, pointing out the our approach could be adopt also for over the counter markets, since the risk neutral dynamics is calculate with a pure mathematical argument, without assuming any economic reason.

While the continuous time approach is an elegant way to deal with the modelling of stocks markets, some practical problem could be difficult to solve. For example, when hedging option positions, rebalancing decisions must be made in discrete time. In the case of American and exotic options, early exercise decisions must be made in discrete time as well. Moreover, as only discrete observations are available for empirical study, discrete time models are often more econometrically tractable.

In last years a general idea has been that for the purpose of option valuation, parameters estimated from option prices are preferable to parameters estimated from the underlying returns (see for instance [25]). Alternatively, the most recent results are based on a different approach. Both historical asset prices and option prices are considered to assess the model performance. Parametric models [67, 26] and a nonparametric one [9] have been proposed by connecting the statistical with the risk neutral measure.

In general, the asset return model is specified under the historical measure P and cannot be directly used to price options. One possibility is to specify a change of measure between P and possible risk neutral measure Q. This approach is particularly attractive because the GARCH parameters can be easily estimated using historical asset returns and used for pricing purposes. Unfortunately, the failure to explain observed option prices only by considering time series information is well known. This approach leads to a rather poor pricing performance and it is largely dominated by option pricing models estimated only using option prices [25, 28, 9].

To overcome this drawback the dynamic of the logarithmic stock price will be driven by a GARCH(1,1) process where the standardized innovations are governed by a TS or a TID distribution with zero mean and unit variance and finite moment generating function. Since discrete time markets with continuous return distribution fail to be complete, the problem of the appropriate choice of the equivalent martingale respectively pricing measure for the discounted asset price process will be solve considering the TS and the TID innovation assumption. Instead of imposing unrealistic conditions on investor's preferences or the Esscher transform [26], a change of measure between the class of TS and TID distribution will allow us to choice a suitable equivalent martingale measure and to perform a joint estimation of objective and risk neutral measures.

The change of measures between the class of Lévy processes takes its origin in the work of Sato [109], and has been widely used in continuous time modelling. We will see how the structure of the problem in the discrete time case is different in comparison with the continuous one. As already figure out in [26] the risk neutral distribution is not always the same for the entire time window, but on each time step it is governed by different parameters. This comes from the discrete time nature of this setting.

Unfortunately, this approach does not provide analytical solutions to price European options and hence numerical procedures have to be considered. The use of non gaussian GARCH models combined with Monte Carlo simulation methods allows one to obtain very promising results. Technics for simulating some infinitely divisible distribution, as described in the previous chapter, will be used to obtain option prices.

7.2 GARCH models with infinitely divisible distributed innovations

In this section we will present a GARCH model with the infinitely divisible distributed innovation process. The GARCH stock price model is defined over a filtered probability space $(\Omega,\ \mathcal{F},\ (\mathcal{F}_t)_{t\in\mathbb{N}},\ \mathrm{P})$ which is constructed as follows. Consider a sequence $(\varepsilon_t)_{t\in\mathbb{N}}$ of iid real random variables on a sequence of probability spaces $(\Omega_t, \mathrm{P}_t)_{t\in\mathbb{N}}$, such that ε_t is an infinitely divisible distributed random variable with zero mean and unit variance on (Ω_t, P_t), and assume that $E[e^{x\varepsilon_t}] < \infty$ where $x \in (-a, b)$ for some $a, b > 0$. In order to construct this model, the distribution of the random variable ε_t, must have exponential moments of some or any order. A similar condition is necessary for the construction of exponential Lévy models, see [29]. Now we define

$$
\begin{aligned}
\Omega &:= \prod_{t\in\mathbb{N}} \Omega_t, \\
\mathcal{F}_t &:= \otimes_{k=1}^{t} \sigma(\varepsilon_k) \otimes \mathcal{F}_0 \otimes \mathcal{F}_0 \cdots, \\
\mathcal{F} &:= \sigma\left(\cup_{t\in\mathbb{N}} \mathcal{F}_t\right), \\
\mathrm{P} &:= \otimes_{t\in\mathbb{N}} \mathrm{P}_t,
\end{aligned}
$$

where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\sigma(\varepsilon_k)$ means the σ-algebra generated by ε_k on Ω_k.

We first propose the following stock price dynamics:

$$
\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t + \lambda_t \sigma_t - g_{\varepsilon_t}(\sigma_t) + \sigma_t \varepsilon_t, \quad t \in \mathbb{N}, \tag{7.1}
$$

where S_t is the stock price at time t, r_t and d_t denote the risk-free and dividend rate for the period $[t-1, t]$, respectively, and λ_t is a $\mathcal{F}_{t-1}$ measurable random variable. S_0 is the present observed price and $\hat{S}_t = S_t \exp(\sum_{k=1}^{t} d_k)$ is the stock price considering reinvestment of the dividends. The function $g_{\varepsilon_t}(x)$ is the *log-Laplace-transform* of ε_t, i.e, $g_{\varepsilon_t}(x) = \log(E[e^{x\varepsilon_t}])$, which is defined on the interval $(-a, b)$.

The one period ahead conditional variance σ_t^2 follows a GARCH(1,1) process with a restriction $0 < \sigma_t < b$, i.e,

$$\sigma_t^2 = (\alpha_0 + \alpha_1 \sigma_{t-1}^2 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2) \wedge \rho, \quad t \in \mathbb{N}, \ \varepsilon_0 = 0, \tag{7.2}$$

where α_0, α_1 and β_1 are non-negative, $\alpha_1 + \beta_1 < 1$, $\alpha_0 > 0$, and $0 < \rho < b^2$. Clearly the process $(\sigma_t)_{t\in\mathbb{N}}$ is predictable. The class of the TS (respectively TID) distributions having some exponential moments is a subclass of the infinitely divisible distribution, and suitable for constructing the new GARCH model having infinitely divisible distributed innovation. The stock price dynamics defined as (7.1) with the conditional variance defined as (7.2) over the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}}, \mathrm{P})$, where $(\varepsilon_t)_{t\in\mathbb{N}}$ is the sequence of TS (respectively TID) distributed iid real random variables, is called the TS-GARCH model (respectively TID-GARCH model). If ε_t equals the standard normal distributed random variable for all $t \in \mathbb{N}$ then $g_{\varepsilon_t}(x)$ is defined on the whole real line. This can be proved by considering the fact that the characteristic function of the normal distribution in an entire function, indeed it is analytic at all finite points of the complex plane $\mathbb{C}$. Consequently, we can ignore the restriction $\sigma_t < b$, since the normal distribution has exponential moments of any order, and hence the model becomes the normal-GARCH model introduced by Duan [35]. Furthermore by condition (d) of Proposition 3.11 and Equation (3.43), the same consideration holds also for the RDTS-GARCH model. Additionally, the finiteness of exponential moments of any order can be also proved by using the fact that the Kummer's or confluent hypergeometric function M in 3.45 is an entire function, indeed extendible to an analytical function on the complex plane $\mathbb{C}$.

7.2.1 Risk neutral dynamic

In order to price option we cannot use the physical measure P defined above. The objective in this section is to find a measure equivalent to the physical measure P that makes the price of the stock discounted by the riskless asset a martingale. A proper change of measure between zero mean and unit variance TS (respectively TID) distributions permits to derive the process dynamic of the log returns under an equivalent measure. By construction, the new measure Q makes the discounted stock price process a martingale. This result allows one to obtain the distribution of the stock return under a EMM and calculate option prices.

The model (7.1) defines an incomplete market, where the set of all possible EMM is infinite. Among the element of this set, we select those one such that the distribution on the innovation process remains the same in both market and risk neutral measure. Actually at least theoretically, the great flexibility given by the TS family, consent to use a given parametric zero mean and unit variance TS noise to model stock prices returns and a different parametric zero mean and unit variance TS noise to model option pricing, always by considering the equivalence relation between them. By following a similar idea of the previous chapter, in which we have considered a GTS distribution as statistical distribution, and a KR distribution, as risk neutral one, we may do the same for discrete time models. In the following, we will not consider a so general measure change, but we will focus on models considering the same distribution in the stock market and in the risk neutral ones.

In the following, the notation stdTS (respectively stdTID) will mean that we are considering a TS (respectively TID) distribution with zero mean and unit variance.

Proposition 7.1. *Let* P_t *be a measure under which* ε_t *is stdTS (respectively stdTID) distributed and* Q_t *be a measure under which* $\xi_t = \varepsilon_t + k_t$ *is stdTS distributed, satisfying the assumptions of the Proposition 4.6 for each* $1 \leq t \leq T$*, where* k_t *is defined as*

$$k_t := \lambda_t + \frac{1}{\sigma_t}(g_{\xi_t}(\sigma_t) - g_{\varepsilon_t}(\sigma_t))$$

and $T \in \mathbb{N}$ *be the time horizon. Define a measure* Q *on* $\mathcal{F}_T$ *equivalent to the measure* P*, with Radon-Nikodym derivative* $\frac{d\mathrm{Q}}{d\mathrm{P}} = Z_T$ *where the density process* $(Z_t)_{0 \leq t \leq T}$ *is defined according to*

$$\begin{aligned} Z_0 &\equiv 1, \\ Z_t &:= \frac{d(\mathrm{P}_1 \otimes \cdots \mathrm{P}_{t-1} \otimes \mathrm{Q}_t \otimes \mathrm{P}_{t+1} \otimes \cdots \mathrm{P}_T)}{d\mathrm{P}} Z_{t-1}, t = 1, \cdots, T, \end{aligned} \tag{7.3}$$

then the measure Q *satisfies the following properties:*

(i) *The stock price dynamics under* Q *can be written as*

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t - g_{\xi_t}(\sigma_t) + \sigma_t \xi_t, \quad 1 \leq t \leq T$$

and the variance process has the form

$$\sigma_t^2 = (\alpha_0 + \alpha_1 \sigma_{t-1}^2 (\xi_{t-1} - k_t)^2 + \beta_1 \sigma_{t-1}^2) \wedge \rho^1, \quad 1 \leq t \leq T, \quad \xi_0 = 0.$$

(ii) *The discount stock price process* $(e^{-r_t} \hat{S}_t)_{1 \leq t \leq T}$ *is a* Q*-martingale w.r.t. the filtration* $\mathcal{F}_{0 \leq t \leq T}$*, indeed the following equation holds for all*

$$E_{\mathrm{Q}}\left[\hat{S}_t | \mathcal{F}_{t-1}\right] = e^{r_t} \hat{S}_{t-1}$$

for all $1 \leq t \leq T$.

(iii) *We have*

$$Var_{\mathrm{Q}}\left(\log\left(\frac{S_t}{S_{t-1}}\right) \Big| \mathcal{F}_{t-1}\right) \overset{a.s.}{=} Var_{\mathrm{P}}\left(\log\left(\frac{S_t}{S_{t-1}}\right) \Big| \mathcal{F}_{t-1}\right), \quad 1 \leq t \leq T$$

Proof. To prove (i), we write the dynamic of the log returns of stock prices under the measure Q

$$\begin{aligned} \log\left(\frac{S_t}{S_{t-1}}\right) &= r_t - d_t + \lambda_t \sigma_t - g_{\varepsilon_t}(\sigma_t) + \sigma_t \varepsilon_t \\ &= r_t - d_t - g_{\xi_t}(\sigma_t) + \sigma_t(\varepsilon_t + k_t) \\ &= r_t - d_t - g_{\xi_t}(\sigma_t) + \sigma_t \xi_t. \end{aligned} \tag{7.4}$$

[1] The constant ρ disappear in the RDTS-GARCH model, since the RDTS distribution has exponential moment of any order

In the variance process, ε_t has to be replaced by $\xi_t - k_t$ in order to get the desired result. Since we have

$$Var_{\mathrm{Q}}(\varepsilon_t + k_t|\mathcal{F}_{t-1}) = 1 = Var_{\mathrm{P}}(\varepsilon_t|\mathcal{F}_{t-1}),$$

(ii) is verified. By the assumption (7.3), the definition of log Laplace transform and the measurability of σ_t respect to $\mathcal{F}_{t-1}$, we can write by the equation (7.4)

$$\begin{aligned}
E_{\mathrm{Q}}\big[\hat{S}_t|\mathcal{F}_{t-1}\big] &= E_{\mathrm{Q}}\big[\hat{S}_{t-1}\exp(r_t - g_{\xi_t}(\sigma_t) + \sigma_t\xi_t)|\mathcal{F}_{t-1}\big] \\
&= \hat{S}_{t-1}\exp(r_t - g_{\xi_t}(\sigma_t))E_{\mathrm{Q}}\big[\exp(\sigma_t\xi_t)|\mathcal{F}_{t-1}\big] \\
&= \hat{S}_{t-1}\exp(r_t - g_{\xi_t}(\sigma_t))E_{\mathrm{Q}}\big[E_{Q_t}[\exp(\sigma_t\xi_t)|\sigma_t]|\mathcal{F}_{t-1}\big] \\
&= \hat{S}_{t-1}\exp(r_t - g_{\xi_t}(\sigma_t))E_{\mathrm{Q}}\big[\exp(g_{\xi_t}(\sigma_t))|\sigma_t]|\mathcal{F}_{t-1}\big] \\
&= \hat{S}_{t-1}\exp(r_t).
\end{aligned}$$

□

Now, we are in the position to find the fair price of European call options. Under the risk neutral measure Q, we the arbitrage free price of a call option with strike price K and maturity T is given by

$$C_t = \exp(-\sum_{i=t+1}^{T} r_i)E_{\mathrm{Q}}[\max(S_T - K, 0)|\mathcal{F}_T]$$

where the stock price at maturity T can be calculate iteratively by the formula

$$S_T = S_t(\sum_{i=t+1}^{T}(r_i - d_i - g_{\xi_i}(\sigma_i)) + \sigma_i\xi_i)$$

and on each step also the conditional variance is evaluated through the equation

$$\sigma_i^2 = (\alpha_0 + \alpha_1\sigma_{i-1}^2(\xi_{i-1} - k_i)^2 + \beta_1\sigma_{i-1}^2) \wedge \rho, \quad t+1 \leq i \leq T.$$

We want to point out the dependence of k_t on the time t, which gives on each step a different set of parameters for the stdTS (respectively stdTID) distribution, we are considering. We will figure out this matter in the following parametric examples.

7.2.2 CGMY-GARCH model

Before considering the CGMY-GARCH model we are going to define the stdCGMY distribution, that is a CGMY distribution with zero mean and unit variance.

Definition 7.2. *Let X be a CGMY random variable with parameter (C, G, M, Y, m), where we define*

$$C = \frac{(M^{Y-2} + G^{Y-2})^{-1}}{\Gamma(2-Y)}$$

and

$$m = 0,$$

then we call this distribution stdCGMY with parameter (G, M, Y). In this case the function $g_X(u)$ is of the form

$$g_X(u) = \frac{(M-u)^Y - M^Y + uYM^{Y-1} + (G+u)^Y - G^Y - uYM^{Y-1}}{Y(Y-1)(M^{Y-2}+G^{Y-2})}$$

Consider the TS-GARCH model with the sequence $(\varepsilon_t)_{t\in\mathbb{N}}$ of iid random variables with $\varepsilon_t \sim \text{stdCGMY}(G, M, Y)$ for all $t \in \mathbb{N}$. We will call the TS-GARCH model the CGMY-GARCH model. Since $E[e^{x\varepsilon_t}] < \infty$ if $x \in (-G, M)$, ρ has to be in the interval $(0, M^2)$.

By Proposition 4.6, the following argument follows.

Proposition 7.3. *Consider the CGMY-GARCH model. Let $T \in \mathbb{N}$ be a time horizon, fix a natural number $t \leq T$. Suppose $\tilde{G}(t)$ and $\tilde{M}(t)$ satisfy the following conditions:*

$$\begin{cases} \tilde{M}^2 > \rho \\ \tilde{M}(t)^{Y-2} + \tilde{G}(t)^{Y-2} = M^{Y-2} + G^{Y-2} \\ \frac{M^{Y-1}-G^{Y-1}-\tilde{M}(t)^{Y-1}+\tilde{G}(t)^{Y-1}}{(1-Y)(M^{Y-2}+G^{Y-2})} \\ \qquad = \lambda_t + \frac{1}{\sigma_t}(g_{\xi(t)}(\sigma_t; \tilde{G}(t), \tilde{M}(t), Y) - g_{\varepsilon_t}(\sigma_t; G, M, Y)). \end{cases} \tag{7.5}$$

Then there is a measure Q_t equivalent to P_t such that

$$\varepsilon_t + k_t \sim \text{stdCGMY}(\tilde{G}(t), \tilde{M}(t), Y)$$

on the measure Q_t where

$$k_t = \lambda_t + \frac{1}{\sigma_t}(g_{\xi_t}(\sigma_t; \tilde{G}(t), \tilde{M}(t), Y) - g_{\varepsilon_t}(\sigma_t; G, M, Y)). \tag{7.6}$$

Proof. This result comes form the Proposition 4.6. In the CGMY case, the tempering function has the form

$$q(r, \pm 1) = e^{-\lambda_\pm r}, \qquad \lambda_\pm > 0,$$

therefore the condition (4.11) is easily verified. Some conditions on parameters are necessary to assure the equivalence, that is $Y = \tilde{Y}$, $C = \tilde{C}(t)$ and by equation (4.12) with $b_1 = 0$ and $b_2 = -k_t$ we have

$$k_t = \Gamma(1-Y)\Big(\int_{\mathbb{R}^d} xR(dx) - \int_{\mathbb{R}^d} x\tilde{R}(dx)\Big),$$

where R is the Rosiński measure of ε_t and $\tilde{R}$ is the Rosiński measure of $\varepsilon_t + k_t$. By Definition 7.2, we have

$$\frac{(M^{Y-2}+G^{Y-2})^{-1}}{\Gamma(2-Y)} = \frac{(\tilde{M}(t)^{Y-2}+\tilde{G}(t)^{Y-2})^{-1}}{\Gamma(2-Y)},$$

hence

$$M^{Y-2} + G^{Y-2} = \tilde{M}(t)^{Y-2} + \tilde{G}(t)^{Y-2}.$$

The evaluation of the integrals

$$\begin{aligned}\Gamma(1-Y)\int_{\mathbb{R}^d} xR(dx) &= C\Gamma(1-Y)(M^{Y-1}-G^{Y-1})\\ &= \frac{(M^{Y-1}-G^{Y-1})}{(1-Y)(M^{Y-2}+G^{Y-2})}\end{aligned}$$

and

$$\begin{aligned}\Gamma(1-Y)\int_{\mathbb{R}^d} x\tilde{R}(dx) &= \tilde{C}(t)\Gamma(1-Y)(\tilde{M}(t)^{Y-2}+\tilde{G}(t)^{Y-2})\\ &= \frac{(\tilde{M}(t)^{Y-1}-\tilde{G}(t)^{Y-1})}{(1-Y)(\tilde{M}(t)^{Y-2}+\tilde{G}(t)^{Y-2})},\end{aligned}$$

completes the proof. □

Suppose $\tilde{G}(t)$ and $\tilde{M}(t)$ satisfy the condition (7.5) in each time $t \in \mathbb{N}$. We have the stock price dynamic

$$\begin{aligned}\log\left(\frac{S_t}{S_{t-1}}\right) &= r_t - d_t + \lambda_t\sigma_t - g_{\varepsilon_t}(\sigma_t; G, M, Y) + \sigma_t\varepsilon_t\\ &= r_t - d_t - g_{\xi_t}(\sigma_t; \tilde{G}(t), \tilde{M}(t), Y) + \sigma_t(\varepsilon_t + k_t)\end{aligned} \tag{7.7}$$

where k_t is given by equation (7.6). By Proposition 7.3, there is a measure Q_t equivalent to P_t such that $\varepsilon_t + k_t \sim \text{stdCGMY}(\tilde{G}(t), \tilde{M}(t), Y)$ on the measure Q_t, and hence

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t - g_{\xi_t}(\sigma_t; \tilde{G}(t), \tilde{M}(t), Y) + \sigma_t(\xi_t) \tag{7.8}$$

with the following variance process

$$\sigma_t^2 = (\alpha_0 + \alpha_1\sigma_{t-1}^2(\xi_{t-1} - k_{t-1})^2 + \beta_1\sigma_{t-1}^2) \wedge \rho \tag{7.9}$$

The stock price dynamic is called the the CGMY-GARCH option pricing model, where $\tilde{G}(t)$ and $\tilde{M}(t)$ satisfy conditions (7.5), and k_t is equal to equation (7.6). Under the CGMY-GARCH option pricing model, a risk neutral stock price dynamic of the process S_t at time $t > 0$ is given by

$$S_t = S_0 \exp\left(\sum_{j=1}^{t}\left(r_j - d_j - g_{\xi_j}(\sigma_j; \tilde{G}(t), \tilde{M}(t), Y) + \sigma_j\xi_j\right)\right),$$

We recall that the martingale condition, indeed $E[S_t|\mathcal{F}_{t-1}] = S_{t-1}e^{r_t-d_t}$, follows by Proposition 7.1. Assume that the GARCH parameters (α_0, α_1, and β_1) the standard CGMY parameters (G, M, and Y) the constant market price of risk $\lambda_t = \lambda$, and the conditional variance $\sigma_{t_0}^2$ of the initial time t_0 are estimated from the historical data. Then we can generate the risk-neutral process for the CGMY-GARCH option pricing model by the following algorithm.

Algorithm:

1. Initialize $t := t_0$.
2. Find the parameters $\tilde{G}(t)$ and $\tilde{M}(t)$ satisfying condition (7.5).
3. Generate random number $\xi_t \sim \text{stdCGMY}(\tilde{G}(t), \tilde{M}(t), Y)$.
4. Let $\log\left(\frac{S_t}{S_{t-1}}\right)$ be equal to equation (7.8).
5. Let k_t be equal to equation (7.6).
6. Set $t = t + 1$ and then substitute
$$\sigma_t^2 = (\alpha_0 + \alpha_1\sigma_{t-1}^2(\xi_{t-1} - k_{t-1})^2 + \beta_1\sigma_{t-1}^2) \wedge \rho.$$
7. Repeat $2 \sim 6$ until $t > T$.

7.2.3 GTS-GARCH model

Before considering the GTS-GARCH model we are going to define the stdGTS distribution, that is a GTS distribution with zero mean and unit variance.

Definition 7.4. *Let X be a GTS random variable with parameter (C_+, C_-, G, M, Y_+, Y_-, m), where we define*

$$\begin{aligned} C_+ &= \frac{pM^{2-Y_+}}{\Gamma(2-Y_+)}, \\ C_- &= \frac{(1-p)G^{2-Y_-}}{\Gamma(2-Y_-)}, \end{aligned}$$

where $p \in (0, 1)$ and

$$m = 0,$$

then we call this distribution stdGTS with parameter (G, M, Y_+, Y_-, p). In this case the function $g_X(u)$ is of the form

$$g_X(u) = p\frac{(M-u)_+^Y - M_+^Y + uYM^{Y_+-1}}{Y_+(Y_+-1)M^{Y_+-2}} + (1-p)\frac{(G+u)_-^Y - G_-^Y - uYM^{Y_--1}}{Y_-(Y_--1)G^{Y_--2}}$$

Consider the TS-GARCH model with the sequence $(\varepsilon_t)_{t\in\mathbb{N}}$ of iid random variables with $\varepsilon_t \sim \text{stdGTS}(G, M, Y_+, Y_-)$ for all $t \in \mathbb{N}$. We will call the TS-GARCH model the GTS-GARCH model. Since $E[e^{x\varepsilon_t}] < \infty$ if $x \in (-G, M)$, ρ has to be in the interval $(0, M^2)$.

By Proposition 4.10, the following argument follows.

Proposition 7.5. *Consider the GTS-GARCH model. Let $T \in \mathbb{N}$ be a time horizon, fix a natural number $t \le T$. Suppose $\tilde{G}(t)$, $\tilde{M}(t)$, and $\tilde{p}(t)$ satisfy the following conditions:*

$$\begin{cases} \tilde{M}(t)^2 > \rho \\ \tilde{p}(t)\tilde{M}(t)^{2-Y_+} = pM^{2-Y_+} \\ (1-\tilde{p}(t))\tilde{G}(t)^{Y_--2} = (1-p)G^{Y_--2} \\ p\frac{M^{Y_+-1} - \tilde{M}(t)^{Y_+-1}}{(1-Y_+)M^{Y_+-2}} + (1-p)\frac{\tilde{G}(t)^{Y_--1} - G^{Y_--1}}{(1-Y)G^{Y_--2}} \\ \qquad = \lambda_t + \frac{1}{\sigma_t}(g_{\xi(t)}(\sigma_t; \tilde{G}(t), \tilde{M}(t), Y_+, Y_-, \tilde{p}(t)) - g_{\varepsilon_t}(\sigma_t; G, M, Y_+, Y_-, p)). \end{cases} \tag{7.10}$$

Then there is a measure Q_t *equivalent to* P_t *such that*

$$\varepsilon_t + k_t \sim \mathrm{stdGTS}(\tilde{G}(t), \tilde{M}(t), Y_+, Y_-, \tilde{p}(t))$$

on the measure Q_t *where*

$$k_t = \lambda_t + \frac{1}{\sigma_t}(g_{\xi_t}(\sigma_t; \tilde{G}(t), \tilde{M}(t), Y_+, Y_-, \tilde{p}(t)) - g_{\varepsilon_t}(\sigma_t; G, M, Y_+, Y_-, p)). \quad (7.11)$$

Proof. This result comes form the Proposition 4.10 and an argument similar to the proof of Proposition 7.3, allows one to obtain (7.10). □

Suppose $\tilde{G}(t)$, $\tilde{M}(t)$ and $\tilde{p}(t)$ satisfy the condition (7.10) in each time $t \in \mathbb{N}$. We have the stock price dynamic

$$\begin{aligned} \log\left(\frac{S_t}{S_{t-1}}\right) &= r_t - d_t + \lambda_t \sigma_t - g_{\varepsilon_t}(\sigma_t; G, M, Y_+, Y_-, p) + \sigma_t \varepsilon_t \\ &= r_t - d_t - g_{\xi_t}(\sigma_t; \tilde{G}(t), \tilde{M}(t), Y_+, Y_-, \tilde{p}(t)) + \sigma_t(\varepsilon_t + k_t) \end{aligned} \quad (7.12)$$

where k_t is given by equation (7.11). By Proposition 7.5, there is a measure Q_t equivalent to P_t such that $\varepsilon_t + k_t \sim \mathrm{stdGTS}(\tilde{G}(t), \tilde{M}(t), Y_+, Y_-, \tilde{p}(t))$ on the measure Q_t, and hence

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t - g_{\xi_t}(\sigma_t; \tilde{G}(t), \tilde{M}(t), Y_+, Y_-, \tilde{p}(t)) + \sigma_t(\xi_t) \quad (7.13)$$

with the following variance process

$$\sigma_t^2 = (\alpha_0 + \alpha_1 \sigma_{t-1}^2 (\xi_{t-1} - k_{t-1})^2 + \beta_1 \sigma_{t-1}^2) \wedge \rho \quad (7.14)$$

The stock price dynamic is called the *the GTS-GARCH option pricing model*, where $\tilde{G}(t)$, $\tilde{M}(t)$ and $\tilde{p}(t)$ satisfy condition (7.10), and k_t is equal to equation (7.11). Under the GTS-GARCH option pricing model, a risk neutral stock price dynamic of the process S_t at time $t > 0$ is given by

$$S_t = S_0 \exp\left(\sum_{j=1}^{t} \left(r_j - d_j - g_{\xi_j}(\sigma_j; \tilde{G}(t), \tilde{M}(t), Y_+, Y_-, \tilde{p}(t)) + \sigma_j \xi_j\right)\right),$$

We recall that the martingale condition, indeed $E[S_t|\mathcal{F}_{t-1}] = S_{t-1}e^{r_t - d_t}$, follows by Proposition 7.1. Assume that the GARCH parameters (α_0, α_1, and β_1) the standard GTS parameters (G, M, Y_+, Y_-, and p), the constant market price of risk $\lambda_t = \lambda$, and the conditional variance $\sigma_{t_0}^2$ of the initial time t_0 are estimated from the historical data. Then we can generate the risk-neutral process for the GTS-GARCH option pricing model by the following algorithm.

Algorithm:

1. Initialize $t := t_0$.

2. Find the parameters $\tilde{G}(t)$, $\tilde{M}(t)$ and $\tilde{p}(t)$ satisfying condition (7.10).

3. Generate random number $\xi_t \sim \text{stdGTS}(\tilde{G}(t), \tilde{M}(t), Y_+, Y_-, \tilde{p}(t))$.

4. Let $\log\left(\frac{S_t}{S_{t-1}}\right)$ be equal to equation (7.13).

5. Let k_t be equal to equation (7.11).

6. Set $t = t + 1$ and then substitute

$$\sigma_t^2 = (\alpha_0 + \alpha_1 \sigma_{t-1}^2 (\xi_{t-1} - k_{t-1})^2 + \beta_1 \sigma_{t-1}^2) \wedge \rho.$$

7. Repeat $2 \sim 6$ until $t > T$.

7.2.4 KR-GARCH model

This further example shows the flexibility of the KR-distribution. In the previous examples, the parameters under the measure P together with conditions to find risk neutral parameters under the measure Q, do not leave any degree of freedom, in such a way the change of measure determines univocally risk neutral parameters, if a solution of the system 7.5, in the CGMY-GARCH case, or of the system 7.10, in the GTS-GARCH case, exists. Even though also in the KR-GARCH we have a similar system to solve, still we have a parameter free. Theoretically, we could find the parameter, which better fit the cross sectional option data, in order to reduce the distance between observed prices and model ones. To figure out this point, we will be going to go into this model.

Definition 7.6. *Let X be a KR random variable with parameter (r_+, r_-, k_+, k_-, p_+, p_-, α, m), where we define*

$$\begin{aligned} c &= \frac{1}{\Gamma(2-\alpha)}\left(\frac{\alpha + p_+}{2 + p_+} r_+^{2-\alpha} + \frac{\alpha + p_-}{2 + p_-} r_-^{2-\alpha}\right)^{-1} \\ k_+ &= c\frac{\alpha + p_+}{r_+^{\alpha}}, \\ k_- &= c\frac{\alpha + p_-}{r_-^{\alpha}}, \\ m &= 0 \end{aligned}$$

then we call this distribution stdKR with parameter (r_+, r_-, p_+, p_-, α). In this case the function $g_X(u)$ is defined in $u \in (-1/r_-, 1/r_+)$ and it is of the form

$$\begin{aligned} g_X(u) &= c\Gamma(-\alpha)\frac{\alpha + p_+}{p_+ r_+^{\alpha}}({}_2F_1(p_+, \alpha; 1 + p_+; r_+ u) - 1) \\ &\quad + c\Gamma(-\alpha)\frac{\alpha + p_-}{p_- r_-^{\alpha}}({}_2F_1(p_-, \alpha; 1 + p_-; -r_- u) - 1) \\ &\quad - u\Gamma(1-\alpha)\left(c\frac{\alpha + p_+}{p_+ + 1} r_+^{1-\alpha} - c\frac{\alpha + p_-}{p_- + 1} r_-^{1-\alpha}\right). \end{aligned}$$

Consider the TS-GARCH model with the sequence $(\varepsilon_t)_{t \in \mathbb{N}}$ of iid random variables with $\varepsilon_t \sim \text{stdKR}(r_+, r_-, p_+, p_-, \alpha)$ for all $t \in \mathbb{N}$. We will call the TS-GARCH model the KR-GARCH model. Since $E[e^{x\varepsilon_t}] < \infty$ if $x \in (-1/r_-, 1/r_+)$, ρ has to be in the interval $(0, 1/r_+^2)$. By Proposition 4.8, the following argument follows.

Proposition 7.7. *Consider the KR-GARCH model. Let $T \in \mathbb{N}$ be a time horizon, fix a natural number $t \leq T$. Suppose $\tilde{r}_+(t)$, $\tilde{r}_-(t)$, $\tilde{p}_+(t)$ and $\tilde{p}_-(t)$ satisfy the following conditions:*

$$\begin{cases} \tilde{r}_+(t)^{-2} > \rho \\ \frac{\alpha+p_+}{2+p_+} r_+^{2-\alpha} + \frac{\alpha+p_-}{2+p_-} r_-^{2-\alpha} = \frac{\alpha+\tilde{p}_+}{2+\tilde{p}_+} \tilde{r}_+^{2-\alpha} + \frac{\alpha+\tilde{p}_-}{2+\tilde{p}_-} \tilde{r}_-^{2-\alpha}, \\ \Gamma(1-\alpha)\Big(c\Big(\frac{\alpha+p_+}{p_++1} r_+^{1-\alpha} - \frac{\alpha+p_-}{p_-+1} r_-^{1-\alpha}\Big) - \tilde{c}\Big(\frac{\alpha+\tilde{p}_+}{\tilde{p}_++1} \tilde{r}_+^{1-\alpha} - \frac{\alpha+\tilde{p}_-}{\tilde{p}_-+1} r_-^{1-\alpha}\Big)\Big) \\ \qquad = \lambda_t + \frac{1}{\sigma_t}(g_{\xi(t)}(\sigma_t; \tilde{r}_+(t), \tilde{r}_-(t), \tilde{p}_+(t), \tilde{p}_-(t), \alpha) - g_{\varepsilon_t}(\sigma_t; r_+(t), r_-(t), p_+(t), p_-(t), \alpha)). \end{cases} \tag{7.15}$$

Then there is a measure Q_t equivalent to P_t such that

$$\varepsilon_t + k_t \sim \mathrm{stdKR}(\tilde{r}_+(t), \tilde{r}_-(t), \tilde{p}_+(t), \tilde{p}_-(t), \alpha)$$

on the measure Q_t where

$$k_t = \lambda_t + \frac{1}{\sigma_t}(g_{\xi_t}(\sigma_t; \tilde{r}_+(t), \tilde{r}_-(t), \tilde{p}_+(t), \tilde{p}_-(t), \alpha) - g_{\varepsilon_t}(\sigma_t; r_+, r_-, p_+, p_-, \alpha)). \tag{7.16}$$

Proof. The property of KR distribution

$$\sigma(A) = \frac{k_+ r_+^\alpha}{\alpha + p_+} I_A(1) + \frac{k_- r_-^\alpha}{\alpha + p_-} I_A(-1), \quad A \subset S^0,$$

and Proposition 4.8 by an argument similar to the proof of Proposition 7.3, allows one to obtain (7.15). □

By arguments similar to those in the previous sections, the stock price dynamic deduced from Proposition 7.7 is

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t - g_{\xi_t}(\sigma_t; \tilde{r}_+(t), \tilde{r}_-(t), \tilde{p}_+(t), \tilde{p}_-(t), \alpha) + \sigma_t \xi_t \tag{7.17}$$

for each $t \in \mathbb{N}$ and with

$$\xi_t \sim stdKR(\tilde{r}_+(t), \tilde{r}_-(t), \tilde{p}_+(t), \tilde{p}_-(t), \alpha)$$

possessing the following variance process

$$\sigma_t^2 = (\alpha_0 + \alpha_1 \sigma_{t-1}^2 (\xi_{t-1} - k_{t-1})^2 + \beta_1 \sigma_{t-1}^2) \wedge \rho$$

is called the KR-GARCH option pricing model, where $\tilde{r}_+(t)$, $\tilde{r}_-(t)$, $\tilde{p}_+(t)$ and $\tilde{p}_-(t)$ satisfy the condition (7.15), and k_t is equal to (7.16). Under the KR-GARCH option pricing model, the stock price S_t at time $t > 0$ is given by

$$S_t = S_0 \exp\left(\sum_{j=1}^{t} (r_j - d_j - g_{\xi_t}(\sigma_t; \tilde{r}_+(t), \tilde{r}_-(t), \tilde{p}_+(t), \tilde{p}_-(t), \alpha) + \sigma_j \xi_j)\right),$$

and the martingale condition $E[S_t|\mathcal{F}_{t-1}] = S_{t-1} e^{r_t - d_t}$ holds as well.

Assume that the GARCH parameters (α_0, α_1, and β_1), the standard KR parameters (r_+, r_-, p_+, p_- and α), the constant market price of risk $\lambda_t = \lambda$, and the conditional variance $\sigma_{t_0}^2$ of the initial time t_0 are estimated from historical data. Then we can generate the TS-GARCH option pricing model based on the standard KR distribution by the following algorithm.

Algorithm:

1. Initialize $t := t_0$.
2. Find the parameters $\tilde{r}_+(t)$, $\tilde{r}_-(t)$, $\tilde{p}_+(t)$ and $\tilde{p}_-(t)$ satisfying the conditions in (7.15).
3. Generate random number $\xi_t \sim \text{stdKR}(\alpha, \tilde{r}_+(t), \tilde{r}_-(t), \tilde{p}_+(t), \tilde{p}_-(t))$.
4. Put $\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t - g_{\xi_t}(\alpha, \tilde{r}_+(t), \tilde{r}_-(t), \tilde{p}_+(t), \tilde{p}_-(t)) + \sigma_t \xi_t$
5. Let $k_t = \lambda + \frac{1}{\sigma_t}(g_{\xi_t}(\alpha, \tilde{r}_+(t), \tilde{r}_-(t), \tilde{p}_+(t), \tilde{p}_-(t)) - g_{\varepsilon_t}(\alpha, r_+, \tilde{r}_-, \tilde{p}_+, \tilde{p}_-))$
6. Set $t = t+1$ and then substitute $\sigma_t^2 = (\alpha_0 + \alpha_1 \sigma_{t-1}^2(\xi_{t-1} - k_{t-1})^2 + \beta_1 \sigma_{t-1}^2) \wedge \rho$.
7. Repeat $2 \sim 6$ until $t > T$.

In Step 2 in this algorithm, we have to find the solution with four parameters satisfying the condition in (7.15). The solution is not unique. There are many way to select one of them. One way is to select one solution which minimizes the square root error between the market option prices and the simulated option prices. Another way is by fixing the parameter $\tilde{r}_+(t) = r_+$. Then, since $\rho < r_+$, the first condition in (7.15) is naturally satisfied.

7.2.5 IG-GARCH model

For sake of completeness, we will be going to recall some GARCH models with non gaussian innovations, proposed in [27, 26]. Both these models can be viewed as examples of TS-GARCH models, even though a more strict assumption on the change of measure has been made. An innovation IG distributed is considered in [27], together with a conditional variance of the Heston-Nandi type [51]. Thanks to this special GARCH specification, the characteristic function is calculated by means of a recursive procedure. Morover, options prices are found through the characteristic function [8] without having recourse to Monte Carlo simulation.

Proposition 7.8. *Let y_t be a IG random variable with parameter $(\sigma_t^2/\eta^2,\ 1,\ 0)$, then the random variable $X \sim \eta y_t$ has zero mean and variance σ_t^2. In this case the function $g_X(u)$ is of the form*

$$g_X(u) = \left(-u + \frac{1 - \sqrt{1 - 2u\eta}}{\eta} \right) \frac{\sigma_t^2}{\eta}.$$

Proof. The result comes from the properties of IG distribution, see section 5.3.4 in [111]. □

Let us define the innovation ε_t as

$$\varepsilon_t = \eta y_t + \frac{\sigma_t^2}{\eta}, \tag{7.18}$$

then return dynamic is of the form

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t + \zeta \sigma_t^2 + \varepsilon_t \tag{7.19}$$

where the conditional variance σ_t^2, recalling the work of Christoffersen et al. [26], is of this form

$$\sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + \alpha_1 y_{t-1} + \gamma \sigma_{t-1}^4 / y_{t-1}. \tag{7.20}$$

We call the model above the IG-GARCH model. In order to obtain the return dynamic under the risk neutral measure, the Esscher transform is considered.

Proposition 7.9. *Fix $t \in \mathbb{N}$ and let*

$$\varepsilon_t = \eta y_t + \frac{\sigma_t^2}{\eta}$$

define as above and let

$$\xi_t = \eta y_t + \frac{\sigma_t^2}{\eta\sqrt{1-2\theta\eta}} + k$$

be the equivalent random variable obtained by the Esscher transform of parameter θ. Then the following equality

$$k = 0 \tag{7.21}$$

holds.

Proof. By a slight modified version of Theorem 4.1, coming from the truncation function $h(x) = x$ of the Lévy-Khinchin formula, the following equation has to be fulfilled, that is

$$\tilde{a} - a = \int_{\mathbb{R}} x(e^{\theta x} - 1)\nu(dx).$$

In this particular case we have

$$\begin{aligned} \tilde{a} &= \frac{\sigma_t^2}{\eta\sqrt{1-2\theta\eta}} + k \\ a &= \frac{\sigma_t^2}{\eta} \end{aligned}$$

and the last integral, evaluated in (4.31), is

$$\int_{\mathbb{R}} x(e^{\theta x} - 1)dx = \frac{\sigma_t^2}{\eta\sqrt{1-2\theta\eta}} - \frac{\sigma_t^2}{\eta}.$$

Hence (7.21) is satisfied. □

Proposition 7.10. *Consider the IG-GARCH model. Let $T \in \mathbb{N}$ be a time horizon, fix a natural number $t \leq T$. Suppose $\theta(t)$ satisfies the following conditions:*

$$\begin{cases} \frac{1}{\eta} - 2\theta(t) > 2\rho \\ \theta(t) < \frac{1}{2\eta} \\ \lambda_t \sigma_t^2 + g_{\xi(t)}(1; \sigma_t^2/\eta^{3/2}, \sqrt{1/\eta}, \sigma_t^2/\eta\sqrt{1-2\theta(t)\eta}) = 0. \end{cases} \tag{7.22}$$

Then there is a measure Q_t equivalent to P_t such that

$$\xi_t = \eta y_t + \frac{\sigma_t^2}{\eta\sqrt{1-2\theta(t)\eta}} + k_t \sim \mathrm{IG}(\sigma_t^2/\eta^{3/2}, \sqrt{1/\eta - 2\theta(t)}, \sigma_t^2/\eta\sqrt{1-2\theta(t)\eta})$$

on the measure $\mathbb{Q}_t$ *and*

$$k_t = \lambda_t \sigma_t^2 + g_{\xi(t)}(1; \sigma_t^2/\eta^{3/2}, \sqrt{1/\eta}, \sigma_t^2/\eta\sqrt{1-2\theta(t)\eta}). \quad (7.23)$$

Furthermore the conditional variance under the measure $\mathbb{Q}_t$ *satisfied the following property*

$$\tilde{\sigma}_t^2 = \sigma_t^2/(1-2\theta\nu)^{3/2}. \quad (7.24)$$

Proof. By Theorem 4.1, Proposition 4.12 and equation (4.31) the result holds. The equation (7.24) follows by the evaluation of innovation process variance under the risk neutral measure $\mathbb{Q}_t$. □

By the value of $\theta(t)$ can be explicitly calculated, indeed it is enough to find the value of theta which satisfy the equation

$$\lambda\sigma_t^2 - \frac{\sigma_t^2}{\eta^{3/2}}\left(\sqrt{1-2\eta(\theta(t)+1)} - \sqrt{1-2\eta\theta(t)}\right) = 0,$$

thus, $\theta(t)$ does not depend on t and has the form

$$\theta(t) = \theta = \frac{1}{2\eta}\left(1 - \eta - \frac{1}{\eta^2\lambda^2} - \frac{1}{4}\eta^4\lambda^2\right). \quad (7.25)$$

By arguments similar to those in the previous sections, the stock price dynamic deduced from Proposition 7.10 is

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t - g_{\xi(t)}(1; \sigma_t^2/\eta^{3/2}, \sqrt{1/\eta - 2\theta(t)}, \sigma_t^2/\eta\sqrt{1-2\theta\eta}) + \xi_t \quad (7.26)$$

for each $t \in \mathbb{N}$ and with

$$\xi_t \sim IG(\sigma_t^2/\eta^{3/2}, \sqrt{1/\eta - 2\theta(t)}, \sigma_t^2/\eta\sqrt{1-2\theta\eta})$$

possessing the following variance process

$$\tilde{\sigma}_t^2 = \frac{\sigma_t^2}{(1-2\theta\nu)^{3/2}}.$$

is called the IG-GARCH option pricing model, where θ, satisfies the condition (7.22), and k_t is equal to (7.23). Under the IG-GARCH option pricing model, the stock price S_t at time $t > 0$ is given by

$$S_t = S_0 \exp\left(\sum_{j=1}^{t}\left(r_j - d_j - g_{\xi_t}(1; \sigma_t^2/\eta^{3/2}, \sqrt{1/\eta - 2\theta}, \sigma_t^2/\eta\sqrt{1-2\theta\eta}) + \xi_j\right)\right),$$

and the martingale condition $E[S_t|\mathcal{F}_{t-1}] = S_{t-1}e^{r_t - d_t}$ holds as well.

Assume that the GARCH parameters (α_0, α_1, and β_1), the IG parameters (η), and the conditional variance $\sigma_{t_0}^2$ of the initial time t_0 are estimated from historical data. Then we can generate the TS-GARCH option pricing model based on the IG distribution by the following algorithm.

Algorithm:

1. Initialize $t := t_0$.
2. Calculate the parameter θ by equation (7.25).
3. Generate random number $\xi_t \sim \mathrm{IG}(\sigma_t^2/\eta^{3/2}, \sqrt{1/\eta - 2\theta}, 0)$.
4. Put
$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_j - d_j - g_{\xi_t}(1; \sigma_t^2/\eta^{3/2}, \sqrt{1/\eta - 2\theta}, \sigma_t^2/\eta\sqrt{1-2\theta\eta}) + \xi_j$$
5. Let $k_t = \lambda_t \sigma_t^2 + g_{\xi(t)}(1; \sigma_t^2/\eta^{3/2}, \sqrt{1/\eta}, \sigma_t^2/\eta\sqrt{1-2\theta\eta})$.
6. Set $t = t+1$ and then substitute σ_t^2 by equation 7.20
7. Repeat $2 \sim 6$ until $t > T$.

7.2.6 SVG-GARCH model

In a further work, Christoffersen et al. [26] have developed a non gaussian GARCH framework. The risk neutral probability is found by means of the Esscher transform. This approach is close to that one proposed by Kim et al. [67], even if the change of measure problem of the last is solved in a more general way. The distribution considered to drive the stock return dynamic is the skewed Variance Gamma (SVG) distribution, a particular case of the BΓ distribution with parameter (c_+, c_-, λ_+, λ_-, m). Parameters of the SVG are chosen as follows. Let z_1 and z_2 be independent Gamma distributions
$$z_{\pm,t} \sim \Gamma(4/\tau_\pm^2, 1),$$
where τ_+ and τ_- are defined as
$$\tau_+ = \sqrt{2}\Big(s - \sqrt{\frac{2}{3}k - s^2}\Big) \quad \text{and} \quad \tau_- = -\sqrt{2}\Big(s + \sqrt{\frac{2}{3}k - s^2}\Big).$$
Let us now construct the Bilateral Gamma random variable [73] from the two Gamma variables as
$$z_t = \frac{1}{2\sqrt{2}}\Big(\tau_+ z_{+,t} - \tau_- z_{-,t}\Big) - \sqrt{2}\Big(\frac{1}{\tau_+} - \frac{1}{\tau_-}\Big)$$
In the notation of equation 2.37, we have that z_t has parameters (c_+, c_-, λ_+, λ_-, b) so defined
$$\begin{aligned} c_\pm &= \frac{4}{\tau_\pm^2} \\ \lambda_\pm &= \frac{2\sqrt{2}}{\tau_\pm} \\ b &= 0. \end{aligned}$$
By Equation (2.9)-(2.12) in [73], z_t has zero mean, unit variance, skewness and kurtosis equal to s and k respectively. The log Laplace transform g_{z_t} is
$$g_{z_t}(u) = -\sqrt{2}\Big(\frac{1}{\tau_+} - \frac{1}{\tau_-}\Big)u - 4\tau_+^{-2}\log\left(1 - \frac{1}{2\sqrt{2}}\tau_+ u\right) - 4\tau_-^{-2}\log\left(1 + \frac{1}{2\sqrt{2}}\tau_- u\right). \tag{7.27}$$

First let us consider a general result regarding BΓ distribution.

Proposition 7.11. *Let X a random variable $X \sim \mathrm{B\Gamma}(c_+, c_-, \lambda_+, \lambda_-, 0)$ under a measure* P *and $\tilde{X}$ the random variable obtained by the Esscher transform of X with parameter θ. If $\theta < \lambda_+$ and the equality*

$$k = \frac{c_+}{\lambda_+} - \frac{c_-}{\lambda_-} - \frac{c_+}{\lambda_+ - \theta} + \frac{c_-}{\lambda_- + \theta}, \tag{7.28}$$

then $\tilde{X} = X + k$ is a random variable $\tilde{X} \sim \mathrm{B\Gamma}(c_+, c_-, \lambda_+ - \theta, \lambda_- + \theta, 0)$ under the measure Q *obtained by the Esscher transform.*

Proposition 7.12. *Consider the SVG-GARCH model. Let $T \in \mathbb{N}$ be a time horizon, fix a natural number $t \leq T$. Suppose $\theta(t)$ satisfies the following conditions:*

$$\begin{cases} \lambda_+ - \theta(t) > \rho \\ \theta(t) < \lambda_+ \\ \frac{c_+}{\lambda_+} - \frac{c_-}{\lambda_-} - \frac{c_+}{\lambda_+ - \theta(t)} + \frac{c_-}{\lambda_- + \theta(t)} \\ \qquad = \lambda_t + \frac{1}{\sigma_t} g_{\xi(t)}(\sigma_t; c_+, c_-, \lambda_+ - \theta(t), \lambda_- + \theta(t), 0) - g_{\varepsilon_t}(\sigma_t; c_+, c_-, \lambda_+, \lambda_-, 0). \end{cases} \tag{7.29}$$

Then there is a measure Q_t equivalent to P_t such that

$$\xi_t = \varepsilon_t + k_t \sim \mathrm{SVG}(\sigma_t; c_+, c_-, \lambda_+ - \theta(t), \lambda_- + \theta(t), 0)$$

on the measure Q_t where

$$k_t = \lambda_t + \frac{1}{\sigma_t} g_{\xi(t)}(\sigma_t; c_+, c_-, \lambda_+ - \theta(t), \lambda_- + \theta(t), 0) - g_{\varepsilon_t}(\sigma_t; c_+, c_-, \lambda_+, \lambda_-, 0). \tag{7.30}$$

Proof. The result is a consequence of the Proposition 7.11. □

By arguments similar to those in the previous sections, the stock price dynamic deduced from Proposition 7.12 is

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t - g_{\xi(t)}(\sigma_t; c_+, c_-, \lambda_+ - \theta(t), \lambda_- + \theta(t), 0) + \sigma_t \xi_t \tag{7.31}$$

for each $t \in \mathbb{N}$ and with

$$\xi_t \sim SVG(c_+, c_-, \lambda_+ - \theta(t), \lambda_- + \theta(t), 0)$$

possessing the following variance process opportunely modified, is called the SVG-GARCH option pricing model, where $\theta(t)$, satisfies the condition (7.29), and k_t is equal to (7.30). Under the SVG-GARCH option pricing model, the stock price S_t at time $t > 0$ is given by

$$S_t = S_0 \exp\left(\sum_{j=1}^{t} (r_j - d_j - g_{\xi_t}(1; c_+, c_-, \lambda_+ - \theta(t), \lambda_- + \theta(t), 0) + \sigma_j \xi_j)\right),$$

and the martingale condition $E[S_t|\mathcal{F}_{t-1}] = S_{t-1} e^{r_t - d_t}$ holds as well.

Assume that the GARCH parameters (α_0, α_1, and β_1), the SVG parameters (c_+, c_-, λ_+, λ_-), and the conditional variance $\sigma_{t_0}^2$ of the initial time t_0 are estimated from historical data. Then we can generate the TS-GARCH option pricing model based on the standard SVG distribution by the following algorithm.

Algorithm:

1. Initialize $t := t_0$.
2. Find the parameters $\theta(t)$ satisfying the conditions in (7.29).
3. Generate random number $\xi_t \sim \text{SVG}(c_+, c_-, \lambda_+ - \theta(t), \lambda_- + \theta(t), 0)$.
4. Put $\log\left(\frac{S_t}{S_{t-1}}\right) = r_j - d_j - g_{\xi_t}(\sigma_t; c_+, c_-, \lambda_+ - \theta(t), \lambda_- + \theta(t), 0) + \sigma_t \xi_j$
5. Let $k_t = g_{\xi_t}(\sigma_t; c_+, c_-, \lambda_+ - \theta(t), \lambda_- + \theta(t), 0) - g_{\varepsilon_t}(\sigma_t; c_+, c_-, \lambda_+, \lambda_-, 0)$.
6. Set $t = t + 1$ and then substitute σ_t^2 depending on the chosen GARCH specification
7. Repeat $2 \sim 6$ until $t > T$.

7.2.7 RDTS-GARCH model

Now, it easy to understand how to construct a discrete time model with TID distributed innovation. Before considering the RDTS-GARCH model we are going to define the stdRDTS distribution, that is a RDTS distribution with zero mean and unit variance. We recall that, at least theoretically, the RDTS distribution has similar statistical properties to the CGMY distribution, even if the former one has exponential moment of any order, while the latter has not.

Definition 7.13. *Let X be a RDTS random variable with parameter $(C, \lambda_-, \lambda_+, \alpha, m)$, where we define*

$$C = \frac{2^{\alpha/2}(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})^{-1}}{\Gamma(1 - \frac{\alpha}{2})}$$

and

$$m = 0,$$

then we call this distribution stdRDTS with parameter $(\lambda_-, \lambda_+, \alpha)$. In this case the function $g_X(u)$ is of the form

$$g_X(u) = CG(u; \alpha, C, \lambda_+) + CG(-u; \alpha, C, \lambda_-)$$

where the function G is defined by Equation (3.45)

Consider the TID-GARCH model with the sequence $(\varepsilon_t)_{t\in\mathbb{N}}$ of iid random variables with $\varepsilon_t \sim \text{stdRDTS}(\lambda_-, \lambda_+, \alpha)$ for all $t \in \mathbb{N}$. We will call the TID-GARCH model the RDTS-GARCH model. The inequality $E[e^{x\varepsilon_t}] < \infty$ is satisfied for each $x \in \mathbb{R}$. By Proposition 4.15, the following argument follows.

Proposition 7.14. *Consider the RDTS-GARCH model. Let $T \in \mathbb{N}$ be a time horizon, fix a natural number $t \leq T$. Suppose $\tilde{\lambda}_-(t)$ and $\tilde{\lambda}_+(t)$ satisfy the following conditions:*

$$\left\{\begin{array}{l} \tilde{\lambda}_+(t)^{\alpha-2} + \tilde{\lambda}_-(t)^{\alpha-2} = \lambda_+^{\alpha-2} + \lambda_-^{\alpha-2} \\ \Gamma(\frac{1-\alpha}{2})\frac{\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1} - \tilde{\lambda}_+(t)^{\alpha-1} + \tilde{\lambda}_-(t)^{\alpha-1}}{\sqrt{2}\Gamma(1-\frac{\alpha}{2})(\lambda_+^{\alpha-2}+\lambda_-^{\alpha-2})} \\ \qquad = \lambda_t + \frac{1}{\sigma_t}(g_{\xi(t)}(\sigma_t; \tilde{\lambda}_-(t), \tilde{\lambda}_+(t), \alpha) - g_{\varepsilon_t}(\sigma_t; \lambda_-, \lambda_+, \alpha)). \end{array}\right. \tag{7.32}$$

Then there is a measure Q_t *equivalent to* P_t *such that*

$$\varepsilon_t + k_t \sim \text{stdRDTS}(\tilde{\lambda}_-(t), \tilde{\lambda}_+(t), \alpha)$$

on the measure Q_t *where*

$$k_t = \lambda_t + \frac{1}{\sigma_t}(g_{\xi_t}(\sigma_t; \tilde{\lambda}_-(t), \tilde{\lambda}_+(t), \alpha) - g_{\varepsilon_t}(\sigma_t; \lambda_-, \lambda_+, \alpha)). \tag{7.33}$$

Proof. This result comes form the Proposition 4.15 and Definition 7.13 □

Suppose $\tilde{\lambda}_-(t)$ and $\tilde{\lambda}_+(t)$ satisfy the condition (7.32) in each time $t \in \mathbb{N}$. We have the stock price dynamic

$$\begin{aligned} \log\left(\frac{S_t}{S_{t-1}}\right) &= r_t - d_t + \lambda_t\sigma_t - g_{\varepsilon_t}(\sigma_t; \lambda_-, \lambda_+, \alpha) + \sigma_t\varepsilon_t \\ &= r_t - d_t - g_{\xi_t}(\sigma_t; \tilde{\lambda}_-(t), \tilde{\lambda}_+(t), \alpha) + \sigma_t(\varepsilon_t + k_t) \end{aligned} \tag{7.34}$$

where k_t is given by equation (7.33). By Proposition 7.14, there is a measure Q_t equivalent to P_t such that $\varepsilon_t + k_t \sim \text{stdRDTS}(\lambda_-, \lambda_+, \alpha)$ on the measure Q_t, and hence

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t - g_{\xi_t}(\sigma_t; \tilde{\lambda}_-(t), \tilde{\lambda}_+(t), \alpha) + \sigma_t(\xi_t) \tag{7.35}$$

with the following variance process

$$\sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2(\xi_{t-1} - k_{t-1})^2 + \beta_1\sigma_{t-1}^2 \tag{7.36}$$

The stock price dynamic is called the the RDTS-GARCH option pricing model, where $\tilde{\lambda}_-(t)$ and $\tilde{\lambda}_+(t)$ satisfy condition (7.32), and k_t is equal to equation (7.33). Under the RDTS-GARCH option pricing model, a risk neutral stock price dynamic of the process S_t at time $t > 0$ is given by

$$S_t = S_0 \exp\left(\sum_{j=1}^{t}\left(r_j - d_j - g_{\xi_j}(\sigma_j; \tilde{\lambda}_-(t), \tilde{\lambda}_+(t), \alpha) + \sigma_j\xi_j\right)\right),$$

We recall that the martingale condition, indeed $E[S_t|\mathcal{F}_{t-1}] = S_{t-1}e^{r_t - d_t}$, follows by Proposition 7.1. Assume that the GARCH parameters (α_0, α_1, and β_1) the standard RDTS parameters (λ_-, λ_+, and α) the constant market price of risk $\lambda_t = \lambda$, and the conditional variance $\sigma_{t_0}^2$ of the initial time t_0 are estimated from the historical data. Then we can generate the risk-neutral process for the RDTS-GARCH option pricing model by the following algorithm.

Algorithm:

1. Initialize $t := t_0$.
2. Find the parameters $\tilde{\lambda}_-(t)$ and $\tilde{\lambda}_+(t)$ satisfying condition (7.32).
3. Generate random number $\xi_t \sim \text{stdRDTS}(\tilde{\lambda}_-(t), \tilde{\lambda}_+(t), \alpha)$.

4. Let $\log\left(\frac{S_t}{S_{t-1}}\right)$ be equal to equation (7.35).

5. Let k_t be equal to equation (7.33).

6. Set $t = t + 1$ and then substitute

$$\sigma_t^2 = (\alpha_0 + \alpha_1\sigma_{t-1}^2(\xi_{t-1} - k_{t-1})^2 + \beta_1\sigma_{t-1}^2) \wedge \rho.$$

7. Repeat $2 \sim 6$ until $t > T$.

7.3 Benchmark models and alternative GARCH pricing models

In order to asses the performance of our ID-GARCH models, we will be going to consider competing approaches. According with the literature on option pricing, we will consider the ad hoc BS model proposed by Dumas et al [39]. Although the BS model does not perform as well as other more complicated competing models in term of in-sample fitting and out-of-sample forecasting, its hedging performance in comparable to them, especially for in-the-money (ITM) calls [119] and furthermore ad hoc BS model is widely used in the financial industry. The implied volatility relation is smoothed across exercise prices and maturities, it is expressed by a function

$$\sigma_{ahBS} = a_0 + a_1K + a_2K^2 + a_3T + a_5KT \tag{7.37}$$

where K is the strike price and T the time to maturity. The standard BS pricing formula is used to find the implied volatilities of the observed option prices. The parameters $(a_1, \ldots, a_5)$ are then found by fitting the implied volatility function. To show the benefit of non gaussian innovation process, we will consider also the classical Duan model [35] and the quite recent nonparametric FHS-GARCH model. In a recent paper of Barone-Adesi et al. [9] has been shown that FHS-GARCH model outperform both Heston and Nandi model [51] and IG-GARCH model [27]. This last two models are not enough flexible to explain option prices with the only use of historical data, this is the reason why we do not test these models.

7.3.1 FHS-GARCH model

All GARCH models we have seen until now, may be evaluated my means of Monte Carlo simulation. A possible alternative to classical Monte Carlo methods, where a distributional assumption is always considered, can be the *filtering historical probability* (FHS) approach propose in [10] to compute portfolio risk measures and recently applied to the study of option pricing in the GARCH framework [9]. Permutations of the historical series are considered as the source of the randomness, without any distributional assumption. The idea comes by the observation that Monte Carlo simulations assume a particular distributional form, imposing the structure of the risk that they were suppose to investigate. In particular with the normal hypothesis we cannot incorporate excess skewness and kurtosis as well as cannot capture extreme events. Empirical studies show that residuals are not normal distributed, therefore one possibility to overcome this drawback is do not impose any theoretical distributional assumption. Historical simulations are usually sampled from past data assuming that returns are i.i.d., thus one needs to remove any serial correlation and volatility clusters present in the historical series. Of course a standard way to remove volatility clusters is by modeling returns with a GARCH(1,1) specification. A modified version of the GARCH model can be considered for model returns, that is under the historical measure P we have

$$\begin{aligned} r_t &= \mu + \varepsilon_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \gamma I_{t-1} \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \end{aligned} \quad (7.38)$$

where ε_t is the residual and $I_{t-1} = 1$ for positive residual, otherwise is zero. This is the asymmetric GJR-GARCH specification. The empirical innovation density captures potential non-normalities in the true innovation density. In order to use estimated residual for historical simulation, one needs to scale them respect to the volatility, that is

$$z_t = \frac{\varepsilon_t}{\sigma_t}.$$

It is clear that the first step is the estimation of parameters involved in the model then the extraction of estimated residuals. The historical simulation is provided by a random choice within the set of estimated residuals, after an opportune scale as above. On each step i the value of the innovation z_i is chosen and the conditional variance is updated, until the entire path is generated. Repeating this procedure 10^4 times we obtain the convergence to the option price. To ensure the convergence of the calibration algorithm, the FHS innovations used to simulate the GARCH sample paths are kept fix across all the iterations of the algorithm [9].

We have also to point out that innovations are the same under the market measure even in the risk neutral one but the conditional variance processes are not the same under the change of measure. The risk neutral dynamic of the FHS-GARCH model is

$$\begin{aligned} r_t &= \mu^* + \varepsilon_t \\ \sigma_t^2 &= \alpha_0^* + \alpha_1^* \varepsilon_{t-1}^2 + \gamma^* I_{t-1} \varepsilon_{t-1}^2 + \beta^* \sigma_{t-1}^2 \end{aligned} \quad (7.39)$$

Parameters of the volatility dynamics under the risk neutral measure are estimated by matching market option prices to model prices. The risk neutral drift μ^* ensures that the expected asset return equals the risk free rate. Furthermore, the variance of the historical simulation can be reduce by the empirical martingale simulation method. We will analyze the pricing performance of this approach in the following. The loss of a parametric model comes at a potential faster and easier to implement method. Due to its structure, this model cannot be used to price OTC options.

7.4 Empirical analysis

7.4.1 Data

On of the most used market to assess option pricing models is the S&P 500 index, [7, 39, 51]. Even though our method is constructed to price also options OTC, we will consider European call written on the S&P 500 index. For the purpose of option valuation there is a general consensus to prefer parameters estimated from option prices respect to parameters estimated from stock returns of the underlying asset. The heavy tailed innovations, together with the volatility clustering effect, we are going to consider in empirical tests, will be enough to obtain good results in both historical estimation and option valuation. It is well know that a large sample of observations is needed to estimate a GARCH time series model. We will consider adjusted closing prices of the S&P 500 index from Monday 12 April 1996 to Wednesday 12 April 2006 provided by Datastream for a total of 2501 observations. The size of this data set, 2501 observations, is large enough for GARCH model fitting, as remarked in [9]. The dividend yield will be not used, since adjusted closing pricing are taken into account, that is $d_t = 0$, for each t in our sample. For the daily interest rate process we take the time series of the above time window of the 3-months Treasury rate and the 1-year zero rate is calculated by using the bootstrap method [55].

European call data on Wednesday 12 April 2006 (with maturities 9, 37, 65, 156 and 247 days) and on Wednesday 19 April 2006 (with maturities 2, 30, 58, 149 and 240 days) will be considered for a total of 285 observations. European call prices are calculated by using the implied volatility provide by Ivy DB, via the BS formula. Option with time to maturity more than 100 days, implied volatility more than 0.7, price less than \$0.05 and such that $|S_0/K - 1| > 0.10$, where S_0 is the initial underlying price and K is the strike price, are discarded. The riskless interest rate for each given maturity is calculated by interpolating the U.S. Treasury yield curve.

7.4.2 In-sample model comparison

Market estimation

The time window between Monday 12 April 1996 and Wednesday 12 April 2006 is considered for the in-sample test. First we estimate the set of parameters θ of the normal-GARCH model, by using the MLE approach,

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t + \lambda_t\sigma_t - \frac{\sigma_t^2}{2} + \sigma_t\varepsilon_t, \quad 1 \leq t \leq T$$

where the conditional variance process has the form

$$\sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2\varepsilon^2 + \beta_1\sigma_{t-1}^2, \quad 1 \leq t \leq T, \quad \varepsilon_0 = 0.$$

Thus, the estimated innovations $\hat{\varepsilon}_t$ as well as the GARCH parameter $\hat{\theta}$ are taken into account in the QMLE estimation for the non gaussian models. This is a classical procedure to estimate parameters when the innovation distribution is not normal. For properties of this algorithm we refer to [100] and references therein. We want to point out the the estimated $\hat{\sigma}_t$ and $\hat{\varepsilon}_t$ are not equal for all GARCH model we consider, even if $\hat{\theta}$ is fixed. This is due to the fact that the log Laplace transform depends on the distribution of the innovation. Numerical procedure are needed for tempered stable and tempered infinitely divisible MLE estimation. Since the density function is not given in close form, but only the characteristic function is known, a discrete evaluation of the density together with an interpolation algorithm is used.

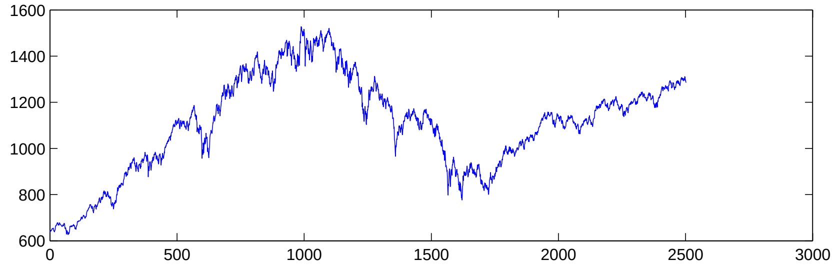

Figure 7.1: S&P 500 prices from April 12, 1996 to April 12, 2006.

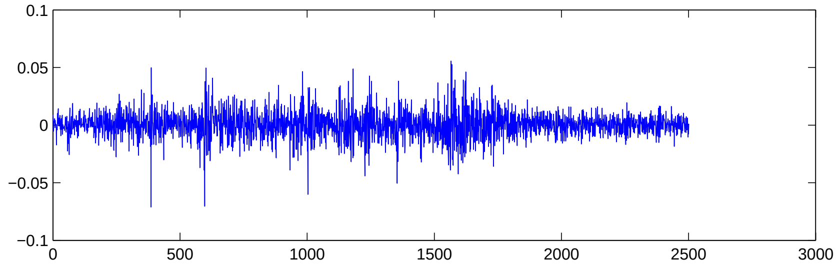

Figure 7.2: S&P 500 index log returns April 12, 1996 to April 12, 2006.

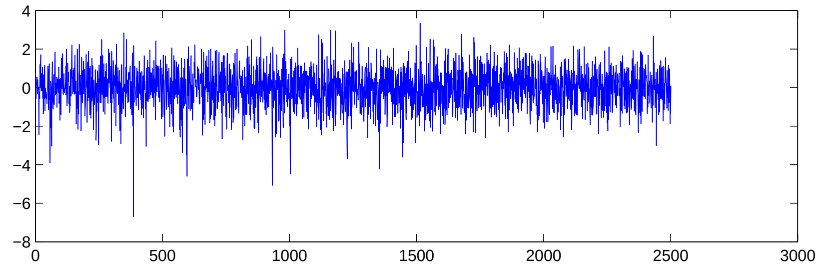

Figure 7.3: Estimated normal-GARCH innovations.

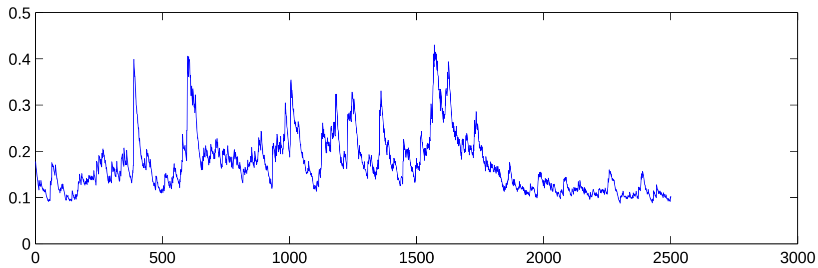

Figure 7.4: Annual estimated volatility.

Table 7.1: S&P 500 market parameters estimated by MLE approach and relative goodness of fit statistics on the time series from 12 April 1996 to 12 April 2006. The parameter m is always equal to zero.

	β	α_1	α_0	λ			
Normal-GARCH	0.9141	0.0756	1.2842e-6	0.0566			
	C	G	M	Y			
stdCGMY	0.1234	0.3689	1.2978	1.7517			
	C_+	C_-	G	M	Y_+	Y_-	
stdGTS	0.1565	0.1026	0.3128	1.7125	1.7517	1.7517	
	c_+	c_-	λ_+	λ_-			
stdSVG	3.3058	2.6199	2.5713	2.2891			
	k_+	k_-	r_+	r_-	p_+	p_-	α
stdKR	4.1487	0.0361	0.7839	3.6622	20.0000	1.0691	1.7517
	C	λ_+	λ_-	α			
stdRDTS	0.0745	1.1581	0.2863	1.8330			

	KS	AD	AD^2	AD^2_{up}	χ^2(p-value)
Normal-GARCH	0.0317	124.1306	3.4407	16.6561	135.1242(0.0765)
CGMY-GARCH	0.0304	0.0671	3.1712	8.3000	117.8472(0.2648)
GTS-GARCH	0.0301	0.0690	3.1269	8.0803	114.8468(0.2846)
SVG-GARCH	0.0263	0.2434	2.2150	21.4987	127.3840(0.1519)
KR-GARCH	0.0287	0.0648	2.6748	7.2551	117.3087(0.2130)
RDTS-GARCH	0.0307	0.0708	3.3837	14.0331	118.2469(0.2354)

By means of the classical FFT procedure, the characteristic function is inverted to give the density function. The classical MLE procedure involving both GARCH parameters and innovation parameters in one run is to time spending. The QMLE method gives one the possibility to skip this cumbersome optimization problem. In the first optimization step the normal- GARCH parameters (α_0, α_1 and β_1) are found by MLE. In the second step the innovation process and parameters of an infinitely divisible distribution are estimated.

The KS, the AD, the AD^2, the AD^2_{up}, and the χ^2 statistics are given, as described in Section 5.2.3. Furthermore, the qq-plots for the innovation fitting are also given.

The results in Table 7.1 show that all non normal distributions present a better fit performance. This fact is emphasized by AD statistics, which have a much less value in non normal cases. By looking to the plot of innovations Figure 7.2, this fact is not surprising, since we have tail events not explainable with a standard normal random variable.

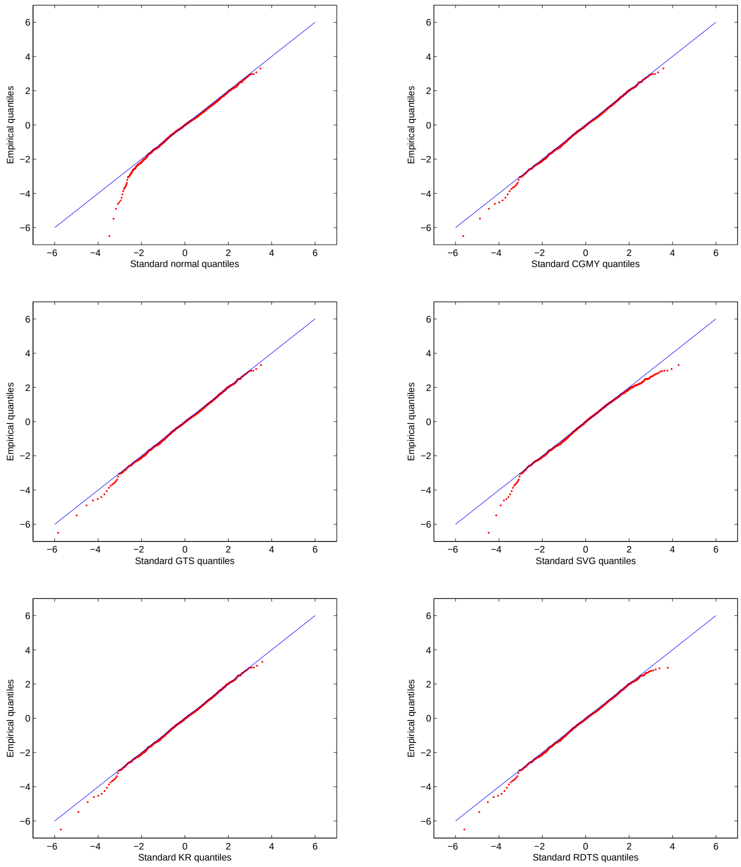

Figure 7.5: QQplots for different distributional assumption for the innovation on 12 April 2006.

Option pricing performances

In the second part of the in-sample analysis we evaluate option prices with different strike prices and maturities, considering the European call prices described in the previous section for Wednesday 12 April 2006. The Monte Carlo procedure is based on algorithms above with empirical martingale simulation. This last simulation technique, introduced in a work of Duan and Simonato [38], is a simple way to reduce the variance of the simulated sample and to preserve the martingale property of the simulated risk neutral process as well, which is in general lost with a crude Monte Carlo method. Let us consider a given market model and observed prices C_i of call options with maturities τ_i and strikes K_i, $i \in \{1, \dots, N\}$, where N is the number of options on Wednesday 19 April 2006. The FHS-GARCH model is fitted by matching model prices to market prices using nonlinear least squares. Hence, to obtain a practical solution to the calibration problem, our purpose is to find a

Table 7.2: Option pricing results on 12 April 2006. In the ad hoc BS model the paper of Dumas et al.[39] is considered, the normal-GARCH model is based on Duan et al. [35], the FHS-GARCH is a nonparametric model introduced by Barone-Adesi et al. [9].

	APE	AAE	RMSE	ARPE
ad hoc BS	0.0811	3.2302	5.6297	0.3819
FHS-GARCH	0.0182	0.7246	1.0528	0.1332
Normal-GARCH	0.0937	3.7292	6.4963	0.3944
CGMY-GARCH	0.0221	0.8788	1.2979	0.1545
GTS-GARCH	0.0581	2.3133	4.0034	0.3447
RDTS-GARCH	0.0363	1.4440	1.8835	0.2333

parameter set θ, such that the optimization problem

$$\min_{\theta} \sum_{i=1}^{N} (C_i - C^{\theta}(\tau_i, K_i))^2$$

is solved, where by C_i we denote the price of an option as observed in the market and by C_i^{θ} the price computed according to a pricing formula in the FHS-GARCH model with a parameter set θ. To measure the performance of the option pricing model, we consider four statistics(APE, AAE, RMSE and ARPE) as in Chapter 5.

Normal innovations are simulated with the `normrnd` command of Matlab, based on the Marsaglia and Tsang Ziggurat method [85] and TS innovations are simulated by series representation, as described in Chapter 6 with the exception of SVG innovation, which can be faster simulated by the `gamrnd` function of Matlab. Furthermore, in the FHS-GARCH model, the random choice is performed by the `randint` function of Matlab together with the `garchsim` function. Due to the structure of algorithms for non normal innovations, the risk neutral simulation is much more faster in the normal and the FHS-GARCH. We point out that for each time step and for each simulated path, we have to solve an optimization problem to find risk neutral parameters, that is each random number may have different parameters, which does not occur in the normal as well as the FHS case. The running time ranges from 3 to 6 hours to simulate 20.000 paths, by using Matlab R2007b on a Xeon Precision at 3.0 GHz with 3GB RAM.

Thus, Table 7.2 shows the performance of different option pricing models: the normal-GARCH perform worst respect to all other competitor models and the FHS-GARCH outperform all others. Also this result is not surprising, since this last model uses both historical and options information, while all ID-GARCH models take into account only historical information. To assess the benefit of our model, we show also the implied volatility surface. Even if we take in consideration only historical data, the model implied volatility is close to the real one.

7.4.3 Out-of-sample model comparison

Table 7.3: Option pricing results on 12 April 2006. The ad hoc BS [39], the normal-GARCH [35], and the FHS-GARCH model [9] are considered.

	APE	AAE	RMSE	ARPE
ad hoc BS	0.1201	4.3133	6.8064	0.9835
FHS-GARCH	0.0528	1.8965	2.3240	0.3499
Normal-GARCH	0.1613	5.7951	8.4702	0.7461
CGMY-GARCH	0.0609	2.1885	2.7170	0.4058
GTS-GARCH	0.1810	6.5035	10.0912	0.9611
RDTS-GARCH	0.0386	1.3858	2.1842	0.2162

In this section we analyze the out-of-sample performance of our models. Market parameters are estimated from the same data set previously considered, that is from Monday 12 April 1996 to Wednesday 12 April 2006 for a total of 2501 observations. We adopt the approach of Dumas et al. [39], Heston and Nandi [51] and Barone Adesi et al. [9] and estimated parameters are used to estimate European call options prices one week ahead, by using asset prices, time to maturities and interest rate on Wednesday 19 April 2006.

Performances are measured by the three statistics above, APE, AAE, RMSE and ARPE. The results given in Table 7.3 show the performance of different option pricing models: the normal-GARCH perform worst respect to all other competitor models and the RDTS-GARCH and FHS-GARCH outperform all others. At least for this data set the the CGMY-GARCH and RDTS-GARCH seem to be satisfactory in both in-sample and out-of-sample analysis, in comparison with the FHS-GARCH which is a non-parametric model and use market and cross sectional information. Consequently, the CGMY-GARCH and RDTS-GARCH models explain both the asset price behavior and European option prices better than the normal-GARCH model. Thus, we can say that the skewness and fat-tail properties of the innovation are also important for pricing of European options.

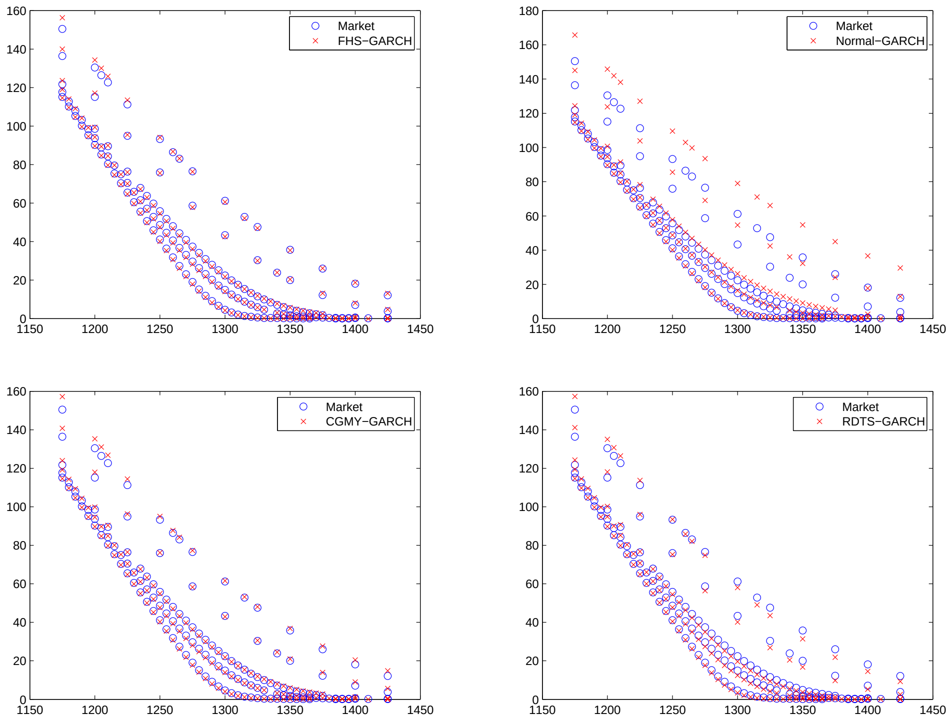

Figure 7.6: In-sample option pricing results on 12 April 2006. The the nonparametric FHS-GARCH [9], the normal-GARCH model [35], CGMY-GARCH, and the RDTS-GARCH model

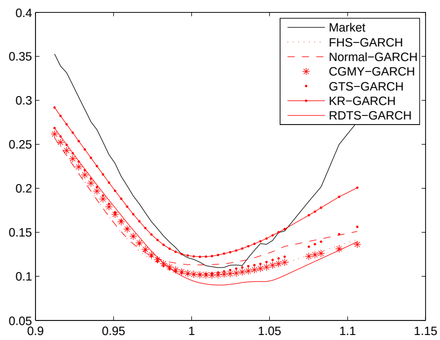

Figure 7.7: Implied volatility of European call options on Wednesday 12 April 2006 written on the S&P 500 index with maturity 21 April 2006.

Conclusions

In the first part of this thesis we have reviewed the construction of the tempered stable class, a subclass of infinitely divisible distributions, and presented some well known one dimensional parametric examples and well as some new one. Then, by using a similar argument we have developed the tempered infinitely divisible class and given some examples in one dimension.

In the second part, the change of measure problem has been discussed. In particular, by using the approach of Sato [109] concerning density transformations between infinitely divisible random variables, we have found relations between a given initial density and the transformed one. We have looked at changes of measure such that the initial distribution has the same parametric form of the transformed one, even if the great flexibility of infinitely divisible random variables allow a multitude of possibilities.

Then, we have empirically studied continuous and discrete models for financial stock returns and in the meanwhile also some algorithms to simulate random variates from tempered stable and tempered infinitely divisible distributions and processes. Efficient simulation algorithms are fundamental to price options in a discrete time setting, and they may be useful also in a continuous setting.

In the empirical study we have focused on methods to evaluate European call options by using information given by the underling asset. The effect on option pricing of fat tailed and skewed distributions, in continuous models, together with the volatility clustering, in discrete models, has been analyzed.

These models can be applied also to evaluate more complicated derivatives, in particular by using the more recent literature on these topics both continuous and discrete time frameworks can be easily extended to American and path dependent options. As example, we have reported market estimations and pricing errors on April 12, 2006. Anyway, the flexibility of our models allows one to obtain very promising results also by considering an more ample empirical study.

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