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Modelling of Auction Type Markets

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Modelling of Auction Type Markets

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Abstract. We propose several new equilibrium models for auction based markets and show that they admit equivalent variational inequality formulations. This approach simplifies essentially both the derivation of existence and uniqueness results and construction of efficient solution methods. The new models may be considered as alternative to the known perfectly and imperfectly competitive economic equilibrium models.

Keywords. Equilibrium models, auction markets, variational inequalities, solution methods.

MS Classification. 49J40, 91B26, 65K10

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1 Introduction

In comparison with the classical perfectly (Walrasian) and imperfectly (Cournot - Bertrand) competitive models, auction market models are paid considerably less attention and usually restricted with several game-theoretic problems which evaluate strategies of players for capturing the desired lot for different kinds of auction; see e.g. Moulin (1981), Weber (1985) and references therein. This situation is due to the very popular opinion that just the above classical models give the most adequate description of behavior of the existing economic systems and that the decisions relying upon the classical principles provide both the stability and efficiency of an economy. However, the recent history of economic development shows clearly the necessity of certain control of economic processes. For instance, together with the privatization of great parts of the state property, say, in energy sector, the state usually keeps real tools for influence on these parts. Of course, these control mechanisms should be rather subtle and transparent, but they are behind the classical Walrasian or Bertrand-Cournot type models. The auction market principles may represent one of the possible ways in resolving this problem.

In this paper, following the approach from Konnov (2006b, 2007b, 2007c), we describe several auction based equilibrium models which admit equivalent variational inequality formulations. This property enables us to obtain rather easily existence and uniqueness properties and computational methods by utilizing directly the results from the theory of variational inequalities, which is now developed rather well. Therefore, the models can be applied for investigation and solution of problems in real economic systems.

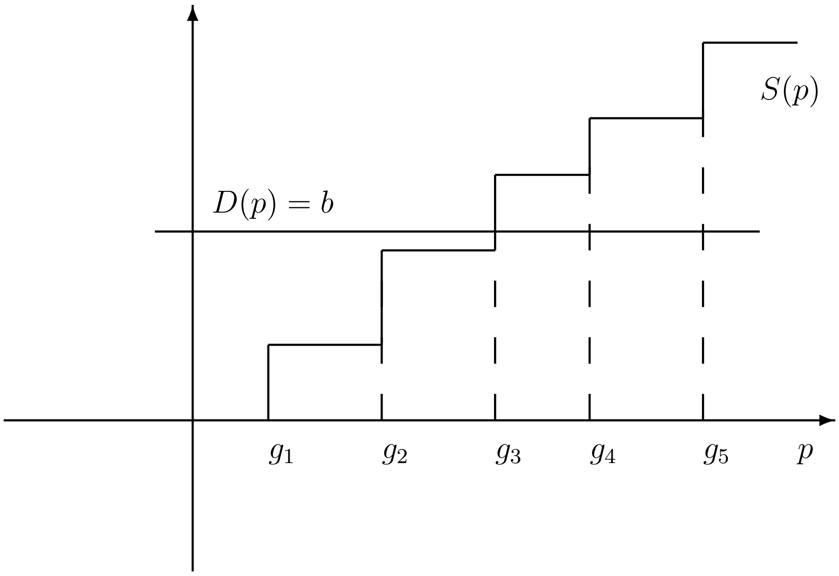
2 Single auctions of a homogeneous commodity with fixed prices

We start our considerations from the simplest auction market models where sellers and bidders announce their fixed prices and maximal offer/bid volumes.

A: Auction of sellers

Consider first the auction market where m sellers announce their fixed prices g_i and maximal offer volumes a_i for covering the prescribed bid volume b of a homogeneous commodity. Since the prices are fixed, the problem can

Figure 1: Auction of sellers



be solved very easily. Namely, without loss of generality, we suppose that $i < j$ implies $g_i \le g_j$ and find the index k such that

$$\sum_{i<k} a_i < b \quad \text{and} \quad \sum_{i\le k} a_i \ge b.$$

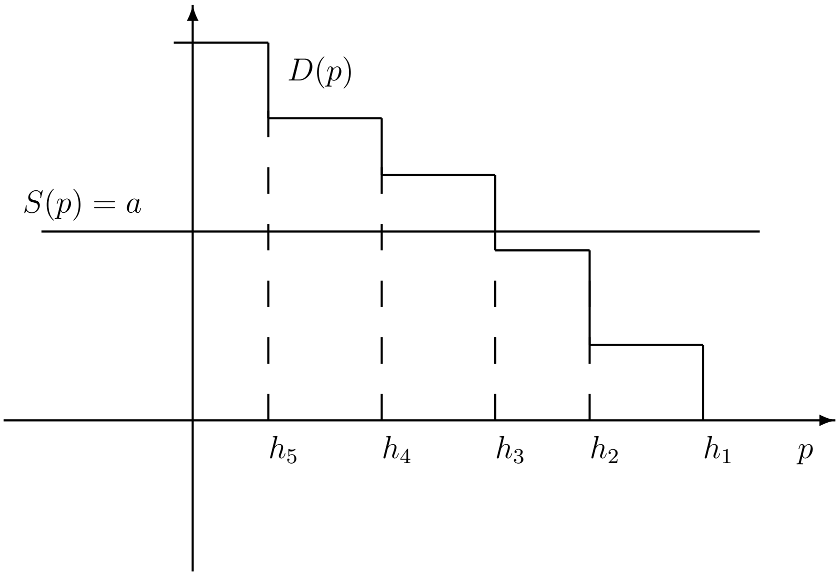
It follows that the optimal offers are the following: $x_i = a_i$ if $i < k$ and $x_k = \min\{a_k, b - \sum_{i<k} a_i\}$ and that the auction price is defined by $p^* = g_k$; see Figure 1.

B: Auction of buyers

Similarly, we can consider the auction market where l buyers announce their fixed prices h_j and maximal bid volumes b_j for covering the prescribed offer volume a of a homogeneous commodity. Then, without loss of generality, we suppose that $i < j$ implies $h_i \ge h_j$ and find the index k such that

$$\sum_{j<k} b_j < a \quad \text{and} \quad \sum_{j\le k} b_j \le a.$$

Figure 2: Auction of buyers



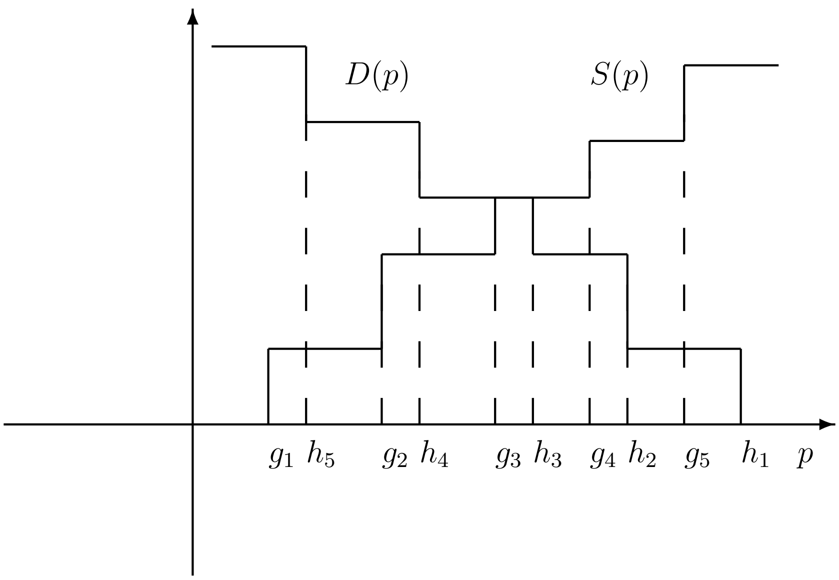
It follows that the optimal bids are the following: $y_j = b_j$ if $j < k$ and $y_k = \min\{b_k, a - \sum\limits_{j<k} b_j\}$ and that the auction price is defined by $p^* = h_k$; see Figure 2.

C: Auction of sellers and buyers

Moreover, we can consider the general auction market problem where m sellers announce their fixed prices g_i and maximal offer volumes a_i and l buyers announce their fixed prices h_j and maximal bid volumes b_j.

In order to find a solution, we should rearrange the sellers indices such that $i < j$ implies $g_i \leq g_j$ and rearrange the buyers indices such that $i < j$ implies $h_i \geq h_j$. Then we find any intersection point for $S(p)$ and $D(p)$, which gives the desired auction price; see Figure 3, where $p^* \in [g_3, h_3]$.

Figure 3: Auction of sellers and buyers



3 Single auctions of a homogeneous commodity with price functions

We are interested in investigation of more complicated behavior of participants, when the announced prices depend on offer/bid values. Let us now consider the auction market which involves m sellers and l buyers of a homogeneous commodity, the i-th seller announcing his minimal α'_i and maximal β'_i offer values and his price (inverse supply) function $g_i : \mathbb{R}^{m+l} \to \mathbb{R}$, the j-th buyer announcing his minimal α''_j and maximal β''_j bid values and his price (inverse demand) function $h_j : \mathbb{R}^{m+l} \to \mathbb{R}$, i.e. their prices depend on offer/bid values (x, y) where $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_l)$. The standard situation corresponds to the case when $\alpha'_i = 0$ and $\alpha''_j = 0$ for all i and j. Additionally, we can take into account the "passive" economic agents who do not participate explicitly in the auction process but agree beforehand with its price. We suppose that their total excess demand is fixed and equal to b, i.e. in case $b = 0$ we have the usual auction market. The value of b may be positive or negative and may in principle determine the prescribed dis-balance value. The usual choice $b = 0$ leads to the precise balance and forces the auction market to be a closed system. However, if b is an arbitrary parameter, we can place the model in more general settings and take into account the reaction of some other economic agents. The solution of the problem is constituted by a volumes vector (x^*, y^*) and a price p^* such that

$$g_i(x^*, y^*) \begin{cases} \geq p^* & \text{if } x_i^* = \alpha'_i, \\ = p^* & \text{if } x_i^* \in (\alpha'_i, \beta'_i), \\ \leq p^* & \text{if } x_i^* = \beta'_i, \end{cases} \quad \text{for} \quad i = 1, \dots, m; \tag{1}$$

and

$$h_j(x^*, y^*) \begin{cases} \leq p^* & \text{if } y_j^* = \alpha''_j, \\ = p^* & \text{if } y_j^* \in (\alpha''_j, \beta''_j), \\ \geq p^* & \text{if } y_j^* = \beta''_j, \end{cases} \quad \text{for} \quad j = 1, \dots, l; \tag{2}$$

and also

$$(x^*, y^*) \in Z, \tag{3}$$

where

$$Z = \left\{ (x, y) \in \mathbb{R}^{m+l} \;\middle|\; \begin{array}{c} \sum\limits_{i=1}^{m} x_i - \sum\limits_{j=1}^{l} y_j = b, \\ \alpha'_i \leq x_i \leq \beta'_i, i = 1, \dots, m, \\ \alpha''_j \leq y_j \leq \beta''_j, j = 1, \dots, l. \end{array} \right\}.$$

Thus, the choice of bid/offer volumes must be feasible in the sense of restrictions for volumes of economic agents and equilibrate the supply and demand, furthermore, each trader sells the minimal (respectively, maximal) value if its price is greater (less) than the (unknown) auction price p^*, and each buyer purchases the maximal (respectively, minimal) value if its price is greater (less) than the (unknown) auction price p^*, which conforms to the auction principle.

The main difficulty of the formulation (1)–(3) is in the fact that it involves the superfluous unknown auction price. We propose an equivalent variational inequality (VI) formulation of the problem for excluding the unknown price p^*.

Theorem 1 *(i) If (x^*, y^*, p^*) is a solution of problem (1)–(3), then (x^*, y^*) solves the problem:*

$$\begin{array}{c}\sum_{i=1}^{m} g_i(x^*, y^*)(x_i - x_i^*) - \sum_{j=1}^{l} h_j(x^*, y^*)(y_j - y_j^*) \geq 0 \\ \forall (x, y) \in Z.\end{array} \tag{4}$$

(ii) Conversely, if $(x^*, y^*) \in Z$ satisfies (4), then there exists a number p^* such that (x^*, y^*, p^*) is a solution of problem ((1)–(3).

Proof. (i) Let (1)–(2) hold and $(x^*, y^*) \in Z$. For brevity, set $c_i = g_i(x^*, y^*)$, $d_j = h_j(x^*, y^*)$. Then we can define the Lagrangian

$$L(x, y, p) = \sum_{i=1}^{m} c_i x_i - \sum_{j=1}^{l} d_j y_j - p \left(\sum_{i=1}^{m} x_i - \sum_{j=1}^{l} y_j - b \right) \tag{5}$$

and rewrite conditions (1)–(2) as follows:

$$\begin{array}{l} \dfrac{\partial L(x^*, y^*, p^*)}{\partial x_i}(x_i - x_i^*) \geq 0 \quad \forall x_i \in [\alpha_i', \beta_i'], \quad i = 1, \dots, m; \\ \dfrac{\partial L(x^*, y^*, p^*)}{\partial y_j}(y_j - y_j^*) \geq 0 \quad \forall y_j \in [\alpha_j'', \beta_j''], \quad j = 1, \dots, l. \end{array} \tag{6}$$

By using the suitable Karush-Kuhn-Tucker theorem (see e.g. Sukharev et al (1986), Chapter 4, Theorem 2.4 or Facchinei and Pang (2003), Proposition 1.3.4), we see that (x^*, y^*) must solve the problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} c_i x_i - \sum_{j=1}^{l} d_j y_j, \\ (x, y) \in Z & \end{array} \tag{7}$$

i.e. (x^*, y^*) solves problem (4).

(ii) If (x^*, y^*) solves problem (4), it solves (7). By using the other part of the same Karush-Kuhn-Tucker theorem, we obtain that there exists p^* such that (6) holds, i.e. the Lagrangian defined in (5) has the saddle point. But (6) implies (1)–(2) and the result follows. □

From the proof it follows that the auction price p^* coincides with the Lagrange multiplier for the balance constraint $\sum_{i=1}^{m} x_i - \sum_{j=1}^{l} y_j = b$. After solving VI (4) we can find the auction price easily from (1)–(2).

Observe that each participant, unlike the perfect competition conditions, may utilize additional information about the other agents, however, the auctioneer rule equilibrating consumers and producers differs from those in imperfect competition models; see e.g. Arrow and Hahn (1971), Okuguchi and Szidarovszky (1990). However, we have the clear principle for setting the price. Next, we can utilize the well-developed techniques from the theory and solution methods of VIs for investigation and solution of the initial problem. For instance, we apply the known result that any VI with continuous mapping and convex and compact feasible set is solvable; see e.g. Facchinei and Pang (2003), Corollary 2.2.5.

Corollary 3.1 *If the set Z is nonempty and bounded, and the functions $g_i : \mathbb{R}^{m+l} \to \mathbb{R}$, $i = 1, \dots, m$ and $h_j : \mathbb{R}^{m+l} \to \mathbb{R}$, $j = 1, \dots, l$ are continuous, then problem (4) is solvable.*

Obviously, solvability of VI (4) implies the solvability of the auction equilibrium problem (1)–(3) with the corresponding feasible set. The uniqueness may be derived under the strict monotonicity of the mapping $(x, y) \mapsto (g, -h)$. There are many other existence and uniqueness theorems for VIs, including the unbounded case (see e.g. Facchinei and Pang (2003) and references therein), which can be also applied to the above problems.

4 Iterative solution methods for single auction market problems

The results of the previous section also enable us to find a solution of auction market problems. In particular, using numerous iterative algorithms for VIs (see Patriksson (1999), Konnov (2001a, 2007b), Facchinei and Pang

(2003)) we can model various dynamic auction market processes and investigate their stability (convergence). Of course, these algorithms can be used for computation of a solution of auction market problems.

The simplest of such methods is the well-known projection method, which consists in generating the iteration sequence $\{(x^k, y^k)\}$ in conformity with the formula: Find $(x^{k+1}, y^{k+1}) \in Z$ such that

$$\begin{array}{l} \sum\limits_{i=1}^{m}(g_i(x^k, y^k) + \theta_k^{-1}(x_i^{k+1} - x_i^k))(x_i - x_i^{k+1}) \\ - \sum\limits_{j=1}^{l}(h_j(x^k, y^k) - \theta_k^{-1}(y_j^{k+1} - y_j^k))(y_j - y_j^{k+1}) \geq 0 \\ \forall (x, y) \in Z, \end{array} \tag{8}$$

where $\theta_k > 0$ is the stepsize parameter. The preference of this method is that it always gives a convex quadratic programming subproblem, which has a unique solution if the set Z is nonempty, i.e. under very mild assumptions. The convergence of the projection algorithm may require certain additional strengthened monotonicity or integrability assumptions; see e.g. Patriksson (1999), Konnov (2007b), Facchinei and Pang (2003).

Another basic procedure is the Frank-Wolfe or conditional gradient method, which represents the sequential solution of auction problems for each commodity with the corresponding sequence of fixed prices for arbitrary volumes. More precisely, we can find components of the next iterate (x^{k+1}, y^{k+1}) as solutions of series of linear programming problems

$$\begin{array}{ll} \text{minimize} & \sum\limits_{i=1}^{m} g_i(x^k, y^k)x_i - \sum\limits_{j=1}^{l} h_j(x^k, y^k)y_j. \\ (x, y) \in Z & \end{array} \tag{9}$$

The implementation of this procedure is also very simple, but it requires additionally the boundedness of the set Z. Moreover, the convergence of the "pure" Frank-Wolfe algorithm require certainly integrability assumptions; see e.g. Dem'yanov and Rubinov (1968) and Patriksson (1999).

If we are interested in creation of iterative methods which do not require a priori information about the problem and ensure stability (convergence) under rather mild conditions, we can incorporate the above iterations within the combined relaxation process; see Konnov (2001a).

For the sake of convenience, we set

$$z = (x, y) \in \mathbb{R}^{m+l}, \ F(z) = (g(z), -h(z));$$

then we can rewrite problem (4) in the general VI format: Find $z^* \in Z$ such that

$$\langle F(z^*), z - z^* \rangle \geq 0 \quad \forall z \in Z. \tag{10}$$

We denote by Z^* its solution set and introduce the following basic assumptions:

(A1) *Problem (10) is solvable;*

(A2) $G : Z \to \mathbb{R}^{m+l}$ *is a continuous monotone mapping.*

The above assumptions seem rather natural. Following Konnov (1993, 2001a), we apply the projection-based combined relaxation method, which is convergent under the above assumptions and describe the dynamic process where the participants utilize the extrapolated offer/bid values for their decisions.

Method (CRM). Choose a point $z^0 \in Z$ and numbers $\alpha \in (0,1)$, $\beta \in (0,1)$, $\gamma \in (0,2)$, $\eta > 0$. At the k-th iteration, $k = 0, 1, \ldots$, we have a point $z^k \in Z$, find $u^k \in Z$ such that

$$\langle F(z^k) + \eta^{-1}(u^k - z^k), z - u^k \rangle \geq 0 \quad \forall y \in X, \tag{11}$$

and set $p^k = u^k - z^k$. If $p^k = 0$, stop. Otherwise, choose $\theta_k \geq 0$, set $v^k = z^k + \theta_k p^k$ and stop if $F(v^k) = 0$. Otherwise, set

$$\begin{gathered} f^k = F(v^k), \omega_k = \langle f^k, z^k - v^k \rangle, \\ z^{k+1} = \pi_Z \left[z^k - \gamma(\omega_k / \|f^k\|^2) f^k \right], \end{gathered}$$

and $k = k + 1$. Here $\pi_Z[\cdot]$ denotes the projection mapping onto Z. The k-th iteration is complete.

Since the termination of (CRM) yields a solution, we shall consider only the case when it generates an infinite iteration sequence. Note that

$$u^k = \pi_Z \left[z^k - \eta F(z^k) \right],$$

i.e., (11) represents the projection iteration and can be implemented with finite algorithms. Convergence of (CRM) with linesearch was investigated in Konnov (1993, 2001a) under the additional Lipschitz continuity assumption. Applying Theorem 1.3.1 from Konnov (2001a), we obtain the convergence property for the case of a fixed stepsize.

Proposition 1 *Suppose that* **(A1)** *and* **(A2)** *are fulfilled and* $G : Z \to \mathbb{R}^{m+l}$ *is Lipschitz continuous. Then there exists* $\theta' > 0$ *such that Method (CRM) with* $\theta_k = \theta \in (0, \theta')$ *generates a sequence* $\{z^k\}$ *such that*

$$\lim_{k\to\infty} z^k = z^* \in Z^*. \tag{12}$$

Furthermore, we are able to ensure convergence under only continuity of G. Let us consider the following linesearch procedure.
(Rule L) Find s as the smallest non-negative integer such that

$$\langle F(z^k + \beta^s p^k), p^k\rangle \leq \alpha\langle F(z^k), p^k\rangle,$$

and set $\theta_k = \beta^s$.

Convergence of the method with linesearch was established in Konnov (2007a).

Proposition 2 *Suppose that* **(A1)** *and* **(A2)** *are fulfilled. If a sequence* $\{z^k\}$ *is generated by Method (CRM) with Rule L, then (12) holds.*

The similar combined relaxation method with the Frank-Wolfe iteration was substantiated in Konnov (2007c).

If we are interested in simultaneous finding offer/bid values and auction market prices, we can apply one of the multiplier methods, which also require only monotonicity of the mapping $(g, -h)$; see Konnov (2001b, 2002b). We describe such a method for problem (1)–(3). For brevity, set

$$X = [\alpha'_1, \beta'_1] \times \ldots \times [\alpha'_m, \beta'_m]$$

and

$$Y = [\alpha''_1, \beta''_1] \times \ldots \times [\alpha''_l, \beta''_l].$$

Method (ML). Choose a point $p_0 \in \mathbb{R}$ and a number $\eta > 0$. At the k-th iteration, $k = 0, 1, \ldots$, we have a point p_k and find the next point in conformity with the rule:

$$p_{k+1} = p_k + \eta\left(b - \sum_{i=1}^{m} x_i^k - \sum_{j=1}^{l} y_j^k\right),$$

where the pair $(x^k, y^k) \in X \times Y$ solves the problem

$$\begin{aligned}
&\sum_{i=1}^{m} g_i(x^k, y^k)(x_i - x_i^k) - \sum_{j=1}^{l} h_j(x^k, y^k)(y_j - y_j^k) \\
&- \left[p_k + \eta \left(b - \sum_{i=1}^{m} x_i^k + \sum_{j=1}^{l} y_j^k \right) \right] \\
&\times \left[\left(\sum_{i=1}^{m} x_i - \sum_{j=1}^{l} y_j \right) - \left(\sum_{i=1}^{m} x_i^k - \sum_{j=1}^{l} y_j^k \right) \right] \geq 0 \\
&\forall (x, y) \in X \times Y.
\end{aligned}$$

In this process, the participants also take into account extrapolated values of the auction price, but it is sufficient for convergence under the above assumptions.

Proposition 3 *Suppose that* **(A1)** *and* **(A2)** *are fulfilled and that the set Z is bounded. Then the sequence $\{p_k\}$ converges to the equilibrium auction price p^*, and the sequence $\{(x^k, y^k)\}$ has limit points such that each limit point (x^*, y^*), together with p^*, constitutes a solution of problem (1)–(3).*

The proof can be obtained by the modification of the corresponding proofs from Konnov (2001b, 2002b) and is omitted. Observe that the method with approximate solutions of subproblems possesses similar convergence properties.

5 A mixed multi-commodity equilibrium model

The auction market models described in the previous sections admit various modifications and extensions. We now present two extensions which reflect different roles of auction markets in economic systems. In this section, we describe a mixed type equilibrium model where auction markets subordinate other subsystems in the sense that the agents of these subsystems accept the price decisions of auction markets.

The model consists of n auction markets for n different commodities and of some other economic agents (consumers and producers) whose joint behavior is described by the excess demand mapping $p \mapsto E(p)$, where $p = (p_1, \dots, p_n)$ is a given price vector. We denote by I_k and J_k, respectively, the index sets of sellers and buyers of the k-th local market associated to the k-th commodity. It is supposed that the i-th seller chooses his offer value x_i within the segment $[\alpha'_i, \beta'_i]$ with $\alpha'_i \geq 0$ for $i \in I_k$ and the j-th buyer chooses his bid value y_j within the segment $[\alpha''_j, \beta''_j]$ with $\alpha''_j \geq 0$ for $j \in J_k$, however, their prices can also depend on the offer/bid volumes at this auction, i.e. given the volume vectors $x_{(k)} = (x_i)_{i \in I_k}$ and $y_{(k)} = (y_j)_{j \in J_k}$, the i-th seller (j-th buyer) determines his price $g_i = g_i(x_{(k)}, y_{(k)})$ (respectively, $h_j = h_j(x_{(k)}, y_{(k)})$). We define the sets of offer/bid bounds for the k-th auction

$$X_{(k)} = \prod_{i \in I_k} [\alpha'_i, \beta'_i], \qquad Y_{(k)} = \prod_{j \in J_k} [\alpha''_j, \beta''_j].$$

We say that vectors $(x^*_{(k)}, y^*_{(k)}) \in X_{(k)} \times Y_{(k)}$ for $k = 1, \dots, n$ and $p^* \in P$ constitute the equilibrium if

$$g_i(x^*_{(k)}, y^*_{(k)}) \begin{cases} \geq p^*_k & \text{if} \quad x^*_i = \alpha'_i, \\ = p^*_k & \text{if} \quad x^*_i \in (\alpha'_i, \beta'_i), \\ \leq p^*_k & \text{if} \quad x^*_i = \beta'_i, \end{cases} \quad \text{for } i \in I_k; \tag{13}$$

$$h_j(x^*_{(k)}, y^*_{(k)}) \begin{cases} \leq p^*_k & \text{if} \quad y^*_j = \alpha''_j, \\ = p^*_k & \text{if} \quad y^*_j \in (\alpha''_j, \beta''_j), \\ \geq p^*_k & \text{if} \quad y^*_j = \beta''_j, \end{cases} \quad \text{for } j \in J_k; \tag{14}$$

$k = 1, \dots, n$; and

$$\sum_{k=1}^{n} \left[\sum_{i \in I_k} x^*_i - \sum_{j \in J_k} y^*_j - E_k(p^*) \right] (p_k - p^*_k) \geq 0 \quad \forall p \in P, \tag{15}$$

where P denotes the set of feasible prices, which is supposed to be a nonempty and convex subset in $\mathbb{R}^n$. Obviously, (13) and (14) represent the auction price decisions whereas (15) is the usual market price equilibrium condition. In fact, if P is the non-negative orthant

$$\mathbb{R}^n_+ = \{ z \in \mathbb{R}^n \mid z_i \geq 0 \quad i = 1, \dots, n \},$$

it yields the complementarity conditions

$$p_k^* \geq 0, \sum_{i \in I_k} x_i^* - \sum_{j \in J_k} y_j^* - E_k(p^*) \geq 0, p_k^* \left[\sum_{i \in I_k} x_i^* - \sum_{j \in J_k} y_j^* - E_k(p^*) \right] = 0$$

for $k = 1, \ldots, n$, whereas $P = \mathbb{R}^n$ gives the balance condition (cf. (3)):

$$\sum_{i \in I_k} x_i^* - \sum_{j \in J_k} y_j^* - E_k(p^*) = 0 \quad \text{for } k = 1, \ldots, n.$$

However, conditions (13) and (14) are equivalent to the system of variational inequalities

$$\begin{array}{l} \sum_{i \in I_k} g_i(x_{(k)}^*, y_{(k)}^*)(x_i - x_i^*) - \sum_{j \in J_k} h_{jk}(x_{(k)}^*, y_{(k)}^*)(y_j - y_j^*) \\ -p_k^* \left[\left(\sum\limits_{i \in I_k} x_i - \sum\limits_{j \in J_k} y_j \right) - \left(\sum\limits_{i \in I_k} x_i^* - \sum\limits_{j \in J_k} y_j^* \right) \right] \geq 0 \\ \forall (x_{(k)}, y_{(k)}) \in X_{(k)} \times Y_{(k)} \end{array} \tag{16}$$

for $k = 1, \ldots, n$. Thus, the equilibrium problem is formulated as a primal-dual system of variational inequalities; see e.g. Konnov (2002a), (2003)–(2006a) and references therein. Observe that we can utilize the well-known approaches to model behavior of consumers and producers out of auction markets, which are accepted in the Walrasian equilibrium models; see Nikaido (1968), Arrow and Hahn (1971), Scarf and Hansen (1973). Therefore, in such a way we can obtain a mixed type economic system subordinated to auction markets. Again, we can deduce the existence and uniqueness results for this model by using the theory of variational inequalities; see Konnov (2001a), Facchinei and Pang (2003). For instance, we give the existence results for compact feasible sets.

Proposition 4 *Suppose that the sets $X_{(k)}$ and $Y_{(k)}, k = 1, \ldots, n$ are nonempty and bounded, the set P is nonempty, convex and compact, the functions $g_i, i \in I_k$ and $h_j, j \in J_k$ are continuous on $X_{(k)} \times Y_{(k)}$ for all i, j, k, and the mapping E is continuous on P. Then problem (15)–(16) has a solution.*

In the unbounded case, similar results are usually based upon a suitable coercivity condition.

Various algorithms for the primal-dual systems of variational inequalities from Konnov (2002a), (2003)–(2006a) can be adjusted both for computation of equilibrium points and for modelling the dynamic processes in this system. We now give an illustration of one of dual Uzawa type methods (see Konnov (2002a)) applied to system (15)–(16).

Dual algorithm. Choose an initial price vector $p^0 \in P$. At the s-th iteration, $s = 0, 1, \ldots$, we have a price vector $p^s \in P$. For each $k = 1, \ldots, n$, we find $x^{s+1}_{(k)}$ and $y^{s+1}_{(k)}$ by solving problem (16) with setting $p^*_k = p^s_k$. Afterwards we find the next price vector by the formula

$$p^{s+1} = \pi_P[p^s - \lambda_s F(p^s)], \lambda_s > 0, \tag{17}$$

where $F_k(p^s) = \sum\limits_{i \in I_k} x^{s+1}_i - \sum\limits_{j \in J_k} y^{s+1}_j - E_k(p^*), k = 1, \ldots, n$.

Note that problem (16) reflects the decisions of sellers and buyers within the k-th auction market and can be solved easily by using the equivalent formulations (13) and (14), whereas (17) may be treated as a "tâtonnement" process governed by auctioneers with taking into account the total excess demand other economic agents. Convergence of this process requires strengthened monotonicity properties of the mappings g, $-h$, and $-E$. At the same time, there exist many other algorithms converging to a solution under rather general conditions.

6 Constrained spatial auction models

There are many kinds of spatial economic equilibrium models which take into account locations of economic agents involved in the system; see e.g. Nagurney (1999). These models are very popular in investigation both perfectly and imperfectly competitive distributed economic systems; various examples of applications are described e.g. in Facchinei and Pang (2003), Konnov (2007b). We now describe a spatial equilibrium model of distributed auction markets of a homogeneous commodity, which is based on the model presented in Section 3. In this model, unlike the previous section, the auction markets do not manage the whole system and their reactions are used for obtaining the general equilibrium.

The model involves n markets of a homogeneous commodity, which are joined by links (roads) in a network. We denote by I_k and J_k, respectively, the index sets of sellers and buyers of the k-th local market associated with the k-th node. It is supposed that the i-th seller chooses his offer value x_i within the segment $[\alpha'_i, \beta'_i]$ with $\alpha'_i \geq 0$ for $i \in I_k$ and the j-th buyer chooses his bid value y_j within the segment $[\alpha''_j, \beta''_j]$ with $\alpha''_j \geq 0$ for $j \in J_k$, moreover, their prices can depend on the offer/bid volumes at this auction, i.e. given the volume vectors $x_{(k)} = (x_i)_{i \in I_k}$ and $y_{(k)} = (y_j)_{j \in J_k}$, the i-th seller (j-th buyer) determines his price $g_i = g_i(x_{(k)}, y_{(k)})$ (respectively, $h_j = h_j(x_{(k)}, y_{(k)})$). We define the sets of offer/bid bounds for the k-th auction

$$X_{(k)} = \prod_{i \in I_k} [\alpha'_i, \beta'_i], \qquad Y_{(k)} = \prod_{j \in J_k} [\alpha''_j, \beta''_j].$$

Due to the auction principle, the solutions $(x^*_{(k)}, y^*_{(k)}) \in X_{(k)} \times Y_{(k)}$ must satisfy the auction market conditions:

$$g_i(x^*_{(k)}, y^*_{(k)}) \begin{cases} \geq p^*_k & \text{if} \quad x^*_i = \alpha'_i, \\ = p^*_k & \text{if} \quad x^*_i \in (\alpha'_i, \beta'_i), \quad i \in I_k; \\ \leq p^*_k & \text{if} \quad x^*_i = \beta'_i, \end{cases} \tag{18}$$

and

$$h_j(x^*_{(k)}, y^*_{(k)}) \begin{cases} \leq p^*_k & \text{if} \quad y^*_j = \alpha''_j, \\ = p^*_k & \text{if} \quad y^*_j \in (\alpha''_j, \beta''_j), \quad j \in J_k; \\ \geq p^*_k & \text{if} \quad y^*_j = \beta''_j, \end{cases} \tag{19}$$

where p^*_k is the (unknown) auction clearing price of the k-th market; i.e they coincide with (13) and (14). Next, the solutions must satisfy the market balance equation:

$$\sum_{i \in I_k} x^*_i - \sum_{j \in J_k} y^*_j - u^*_k = 0, \tag{20}$$

where u^*_k is the (unknown) value of external demand and these values give the total balance equation for the system:

$$\sum_{k=1}^{n} u^*_k = 0. \tag{21}$$

However, we have also to take into account the conditions on the graph associated with the system of distributed markets. We denote by $\mathcal{A}$ the set of all the arcs joining the nodes attributed to markets. Let f_a denote the commodity flow for arc $a = (k, l)$ and let $[b'_a, b''_a]$ be the segment of feasible upper capacity bounds for this arc. The formulation admits negative values both for the flow and for some bounds, which correspond to the reverse direction of the flow. Observe that bounds can be non-symmetric, i.e. $b'_a \neq -b''_a$ in general. Given the flow vector $f = (f_a)_{a \in \mathcal{A}}$ we can define the cost $c_a = c_a(f)$ of shipment of one unit of the commodity along arc $a \in \mathcal{A}$. Next, for a given node k, we denote by $\mathcal{A}^+_k$ and $\mathcal{A}^-_k$ the sets of incoming and outgoing arcs at k. Note that I_k and J_k can be empty for some k and this case corresponds to an intermediate node.

If f^* is the optimal flow distribution corresponding to $x^*_{(k)}, y^*_{(k)}, u^*_k$ in (18)–(21), then we have the node balance equation

$$\sum_{a \in \mathcal{A}^-_k} f^*_a - \sum_{a \in \mathcal{A}^+_k} f^*_a - u^*_k = 0, \quad k = 1, \dots, n \tag{22}$$

and the flow capacity constraints

$$f^*_a \in [b'_a, b''_a], \quad a \in \mathcal{A}. \tag{23}$$

Thus, the constrained spatial market equilibrium problem consists in finding (x^*, y^*, u^*, f^*) satisfying (18)–(20) for $k = 1, \dots, n$ and (21)–(23), where $x^* = (x^*_{(k)})_{k=1,\dots,n}$, $y^* = (y^*_{(k)})_{k=1,\dots,n}$, $u^* = (u^*_1, \dots, u^*_n)$.

We now present a variational inequality problem, whose solutions satisfy the above conditions. Set

$$X = \prod_{k=1}^{n} X_{(k)}, Y = \prod_{k=1}^{n} X_{(k)}, F = \prod_{a \in \mathcal{A}} [b'_a, b''_a]$$

and define the set

$$W = \left\{ \begin{array}{l} (x, y, f) \\ \in X \times Y \times F \end{array} \middle| \begin{array}{l} \left(\sum\limits_{a \in \mathcal{A}^-_k} f_a - \sum\limits_{a \in \mathcal{A}^+_k} f_a \right) \\ - \left(\sum\limits_{i \in I_k} x_i - \sum\limits_{j \in J_k} y_j \right) = 0 \quad k = 1, \dots, n \end{array} \right\}. \tag{24}$$

The problem is to find $(x^*, y^*, f^*) \in W$ such that

$$\begin{aligned} & \sum_{k=1}^{n} \left[\sum_{i \in I_k} g_i(x^*_{(k)}, y^*_{(k)})(x_i - x^*_i) \right. \\ & \left. - \sum_{j \in J_k} h_j(x^*_{(k)}, y^*_{(k)})(y_j - y^*_j) \right] \\ & + \sum_{a \in \mathcal{A}} c_a(f^*)(f_a - f^*_a) \geq 0 \quad \forall (x, y, f) \in W. \end{aligned} \tag{25}$$

Theorem 2 *If (x^*, y^*, f^*) is a solution to VI (24)–(25), then there exist numbers p_k^* and $u_k^*, k = 1, \dots, n$ such that (18)–(20) for $k = 1, \dots, n$ and (21)–(23) hold true.*

Proof. Let (x^*, y^*, f^*) be a solution to (24)–(25), then $(x^*, y^*, f^*) \in X \times Y \times F$ and (23) holds. Next, from the Karush-Kuhn-Tucker theorem for problem (24)–(25); see e.g. Konnov (2007b), Proposition 11.7, it follows that there exist numbers p_k^*, $k = 1, \dots, n$ such that

$$\begin{aligned} &\sum_{k=1}^{n} \left[\sum_{i \in I_k} g_i(x^*_{(k)}, y^*_{(k)})(x_i - x_i^*) - \sum_{j \in J_k} h_j(x^*_{(k)}, y^*_{(k)})(y_j - y_j^*) \right] \\ &+ \sum_{a \in \mathcal{A}} c_a(f^*)(f_a - f_a^*) - \sum_{k=1}^{n} p_k^* \left[\sum_{i \in I_k} (x_i - x_i^*) - \sum_{j \in J_k} (y_j - y_j^*) \right. \\ &\left. - \sum_{a \in \mathcal{A}_k^-} (f_a - f_a^*) + \sum_{a \in \mathcal{A}_k^+} (f_a - f_a^*) \right] \geq 0 \quad \forall (x, y, f) \in X \times Y \times F \end{aligned} \tag{26}$$

and

$$\left(\sum_{a \in \mathcal{A}_k^-} f_a^* - \sum_{a \in \mathcal{A}_k^+} f_a^* \right) - \left(\sum_{i \in I_k} x_i^* - \sum_{j \in J_k} y_j^* \right) = 0 \quad k = 1, \dots, n. \tag{27}$$

i.e., $p^* = (p_1^*, \dots, p_n^*)$ is the vector of Lagrange multipliers for the node balance equation (27). If we determine the numbers $u_k^*, k = 1, \dots, n$ from (22), then (27) gives (20). Moreover, summing (22) over $k = 1, \dots, n$ gives

$$\sum_{k=1}^{n} u_k^* = \sum_{k=1}^{n} \left(\sum_{a \in \mathcal{A}_k^-} f_a^* - \sum_{a \in \mathcal{A}_k^+} f_a^* \right) = 0$$

since the right-hand side expression involves twice the flow value for each arc a with opposite signs. Hence, (21) also holds. Next, we see that (26) is equivalent to the following set of partial variational inequalities:

$$\begin{array}{c} (g_i(x^*_{(k)}, y^*_{(k)}) - p_k^*)(x_i - x_i^*) \geq 0 \quad \forall x_i \in [\alpha_i', \beta_i'], \\ i \in I_k, k = 1, \dots, n; \\ (p_k^* - h_j(x^*_{(k)}, y^*_{(k)}))(y_j - y_j^*) \geq 0 \quad \forall y_j \in [\alpha_j'', \beta_j''], \\ j \in J_k, k = 1, \dots, n; \\ (c_a(f^*) + p_k^* - p_l^*)(f_a - f_a^*) \geq 0 \quad \forall f_a \in [b_a', b_a''], \\ \forall a = (k, l) \in \mathcal{A}. \end{array} \tag{28}$$

The first two series of inequalities in (28) are equivalent to (18) and (19), respectively. The proof is complete. □

Observe that for any solution of problem (18)–(23) obtained from VI (24)–(25) the auction clearing prices $p_k^*, k = 1, \dots, n$ are Lagrange multipliers for the node balance constraints. At the same time, the reverse assertion is not true in general, i.e. for any solution (18)–(23) the auction clearing prices p_k^* need not be the Lagrange multipliers.

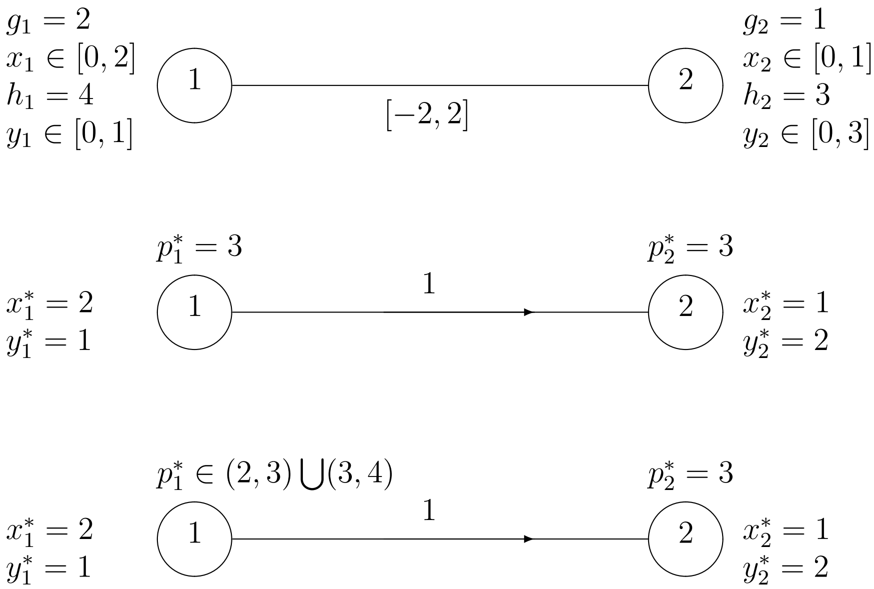
Example 6.1 Consider the simplest model of two auction markets joined by one costless two-directional link, i.e. $c_{12} = 0$. Each market involves one seller and one buyer whose prices are fixed, i.e. $I_1 = \{1\}, J_1 = \{1\}, I_2 = \{2\}, J_2 = \{2\}$; $\alpha_1' = \alpha_1'' = \alpha_2' = \alpha_2'' = 0$;
market 1: $g_1 = 2, h_1 = 4, \beta_1' = 2, \beta_1'' = 1$;
market 2: $g_2 = 1, h_2 = 3, \beta_2' = 1, \beta_2'' = 3$;
also, $b_{12}' = -2$, $b_{12}'' = 2$.
Then problem (11) has the following solution:
market 1: $x_1^* = 2, y_1^* = 1$;
market 2: $x_2^* = 1, y_2^* = 2$;
flow: $f_{12}^* = 1$; prices: $p_1^* = p_2^* = 3$; see Figure 4.

In fact, $g_1 < p_1^*$, $h_1 > p_1^*$, $g_2 < p_2^*$, $h_2 \geq p_2^*$, and $p_1^* = p_2^*$; i.e. all the optimality conditions hold true. At the same time we can choose the following auction prices: $p_1^* \in (2, 3) \bigcup (3, 4)$ and $p_2^* = 3$, which yield the same optimal volumes, but these prices are not the Lagrange multipliers.

The sense of problem (24)–(25) is also clear: Find the feasible triplet $(x^*, y^*, f^*) \in W$ such that it minimizes the total diseconomies in the system for the corresponding offer/bid prices $g = g(x^*, y^*)$ and $h = h(x^*, y^*)$ and for the corresponding shipment costs $c = c(f^*)$. Observe that we do not impose any conditions on the functions g, h, and c, but it would be reasonable to suppose that they are continuous and have non-negative values and that the function c is symmetric, i.e. $c_a(f) = c_a(-f)$. For instance if $c \equiv 0$, then problem (24)–(25) reflects the maximization of pure auction markets profit. We can derive existence and uniqueness results for the auction market problem from the theory of VIs.

Proposition 5 *Suppose that the set W is nonempty and bounded, and that the mapping $(x, y, f) \mapsto (g(x, y), h(x, y), c(f))$ is continuous. Then problem (24)–(25) has a solution.*

Figure 4: Two auction markets



In fact, (24)–(25) is a VI with continuous cost mapping and nonempty, convex, and compact feasible set. Hence, the result then follows e.g. from Theorem 11.3 in Konnov (2007b).

Recall that a mapping T is said to be

(i) *monotone* if for each u', u'' it holds that

$$\langle T(u') - T(u''), u' - u'' \rangle \geq 0;$$

(ii) *strictly monotone* if for each u', u'', $u' \neq u''$, it holds that

$$\langle T(u') - T(u''), u' - u'' \rangle > 0.$$

Combining Proposition 5 with Proposition 1.14 in Konnov (2007b), we obtain also the uniqueness result.

Proposition 6 *Suppose that the set W is nonempty and bounded and that the mapping $(x, y, f) \mapsto (g(x, y), -h(x, y), c(f))$ is continuous and strictly monotone. Then problem (24)–(25) has a unique solution.*

7 Iterative solution methods for spatial auction market problems

Being based on the above results, we can propose various iterative solution methods for problem (24)–(25), which are similar to those in Section 4. For instance, the Frank-Wolfe or conditional gradient method, which represents the sequential solution of auction problems for each commodity with the corresponding sequence of fixed prices for arbitrary volumes. More precisely, we can find components of the next iterate $(x^{s+1}, y^{s+1}, f^{s+1})$ as solutions of series of linear programming problems

$$\begin{array}{ll} \text{minimize} & \sum\limits_{k=1}^{n} \left[\sum\limits_{i \in I_k} g_i(x^s_{(k)}, y^s_{(k)}) x_i - \sum\limits_{j \in J_k} h_j(x^s_{(k)}, y^s_{(k)}) y_j \right] \\ (x, y, f) & + \sum\limits_{a \in \mathcal{A}} c_a(f^s) f_a \\ \in W & \end{array}$$

(cf. (9)). The implementation of this procedure is very simple, but it again requires additionally the boundedness of the set W. Moreover, the convergence of the "pure" Frank-Wolfe algorithm requires certainly integrability assumptions.

Next, the well-known projection method now consists in generating the iteration sequence $\{(x^s, y^s, f^s)\}$ in conformity with the formula:
Find $(x^{s+1}, y^{s+1}, f^{s+1}) \in W$ such that

$$\begin{aligned}
&\sum_{k=1}^{n} \left[\sum_{i \in I_k} (g_i(x^s_{(k)}, y^s_{(k)}) + \theta_s^{-1}(x_i^{s+1} - x_i^s))(x_i - x_i^{s+1}) \right. \\
&\left. - \sum_{j \in J_k} (h_j(x^s_{(k)}, y^s_{(k)}) - \theta_s^{-1}(y_j^{s+1} - y_j^s))(y_j - y_j^{s+1}) \right] \\
&+ \sum_{a \in \mathcal{A}} (c_a(f^s) + \theta_s^{-1}(f_a^{s+1} - f_a^s))(f_a - f_a^{s+1}) \geq 0 \\
&\forall (x, y, f) \in W,
\end{aligned} \tag{29}$$

where $\theta_s > 0$ is the stepsize parameter (cf. (8)). Again we see that (29) is a convex quadratic programming subproblem, which has a unique solution if the set W is nonempty, i.e. under very mild assumptions. The convergence of the projection algorithm may also require strengthened monotonicity or integrability assumptions. In order to obtain a convergent process under more general assumptions, we can construct combined relaxation methods including iterations of the above methods as in Section 4.

Note that one can find a solution to (29) by finite algorithms. At the same time, the dual Uzawa type methods seem also very attractive for solving this problem. In fact, we can solve (29) via solving the dual problem

$$\begin{aligned} &\text{maximize } \varphi_s(\text{p}), \\ &p \in \mathbb{R}^n \end{aligned} \tag{30}$$

where

$$\varphi_s(p) = \min_{(x,y,f) \in X \times Y \times F} L_s(x, y, f, p), \tag{31}$$

$$\begin{aligned}
L_s(x, y, f, p) = &\sum_{k=1}^{n} \left[\sum_{i \in I_k} \left(g_i(x^s_{(k)}, y^s_{(k)}) + 0.5\theta_s^{-1} x_i \right) x_i \right. \\
&\left. - \sum_{j \in J_k} \left(h_j(x^s_{(k)}, y^s_{(k)}) - 0.5\theta_s^{-1} y_j \right) y_j \right] + \sum_{a \in \mathcal{A}} \left(c_a(f^s) + 0.5\theta_s^{-1} f_a \right) f_a \\
&+ \sum_{k=1}^{n} p_k \left[\left(\sum_{a \in \mathcal{A}_k^-} f_a - \sum_{a \in \mathcal{A}_k^+} f_a \right) - \left(\sum_{i \in I_k} x_i - \sum_{j \in J_k} y_j \right) \right].
\end{aligned}$$

Clearly, the computation of the values of φ_s and its gradient can be made componentwise, i.e. (31) is decomposed into a set of separable one-dimensional problems each of them has the explicit solution formula. In order to solve (30) we can apply a suitable conjugate gradient method.

The projection method (29) becomes very efficient if all the prices g_i and h_j are fixed and $c_a \equiv 0$, since it then coincides with the proximal point method and possesses the finite termination property; see e.g. Rockafellar (1976).

This combined proximal point and dual conjugate gradient method for subproblems was implemented for solving several spatial auction problems arising in electricity market systems and showed rather fast convergence to a solution. The norm of violations of conditions (26), (27) was taken as error evaluation and its accuracy 0.1 appeared sufficient for computation.

Example 7.1 Consider the model of five auction markets with fixed prices joined by two-directional links in a tree. Each market involves one buyer. The numbers of sellers are the following: market 1 – 108, market 2 – 63, market 3 – 48, market 4 – 22, market 5 – 44; see Figure 5. The problem was solved via the combined projection and dual conjugate gradient method in 72 iterations. In the picture, Δ denotes the excess supply at the market, p denotes the auction price at the market.

Example 7.2 Consider the model of nineteen auction markets with fixed prices joined by two-directional links in a tree. These markets involve 7 buyers and 106 sellers. The problem is depicted at Figure 6. It was solved via the combined projection and dual conjugate gradient method in 280 iterations. The solution is presented at Figure 7.

Figure 5: Five auction markets

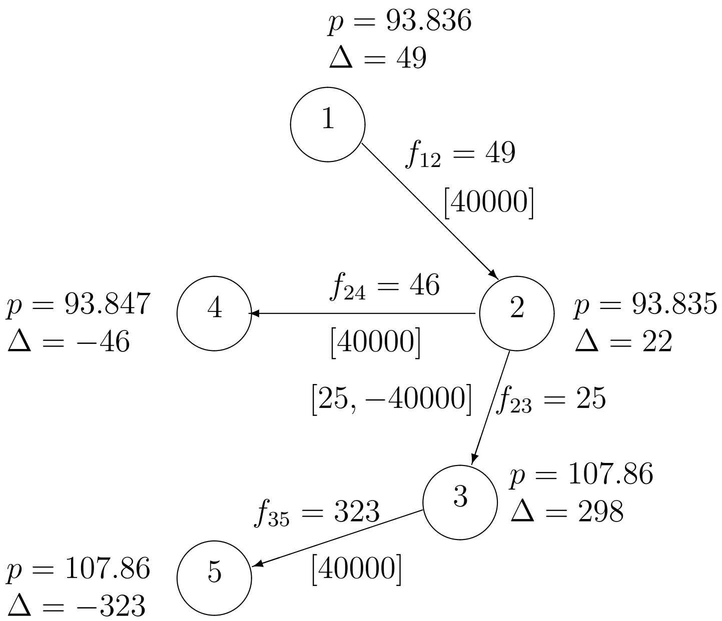


Figure 6: Nineteen auction markets: data

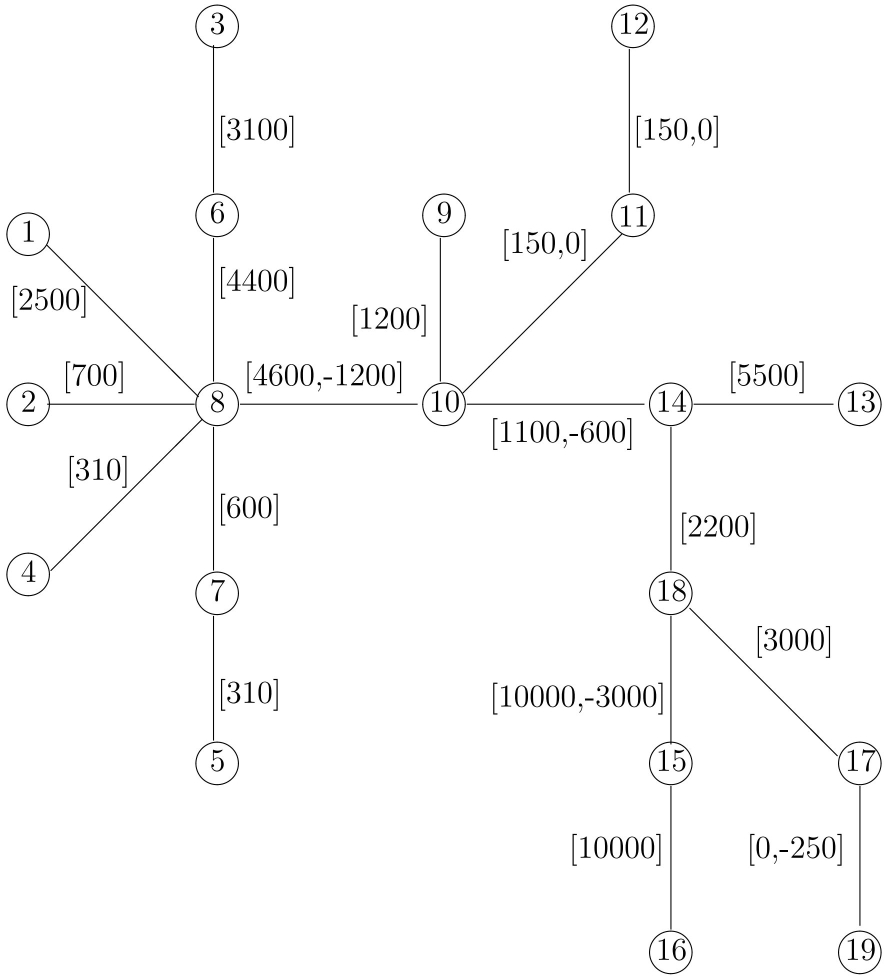
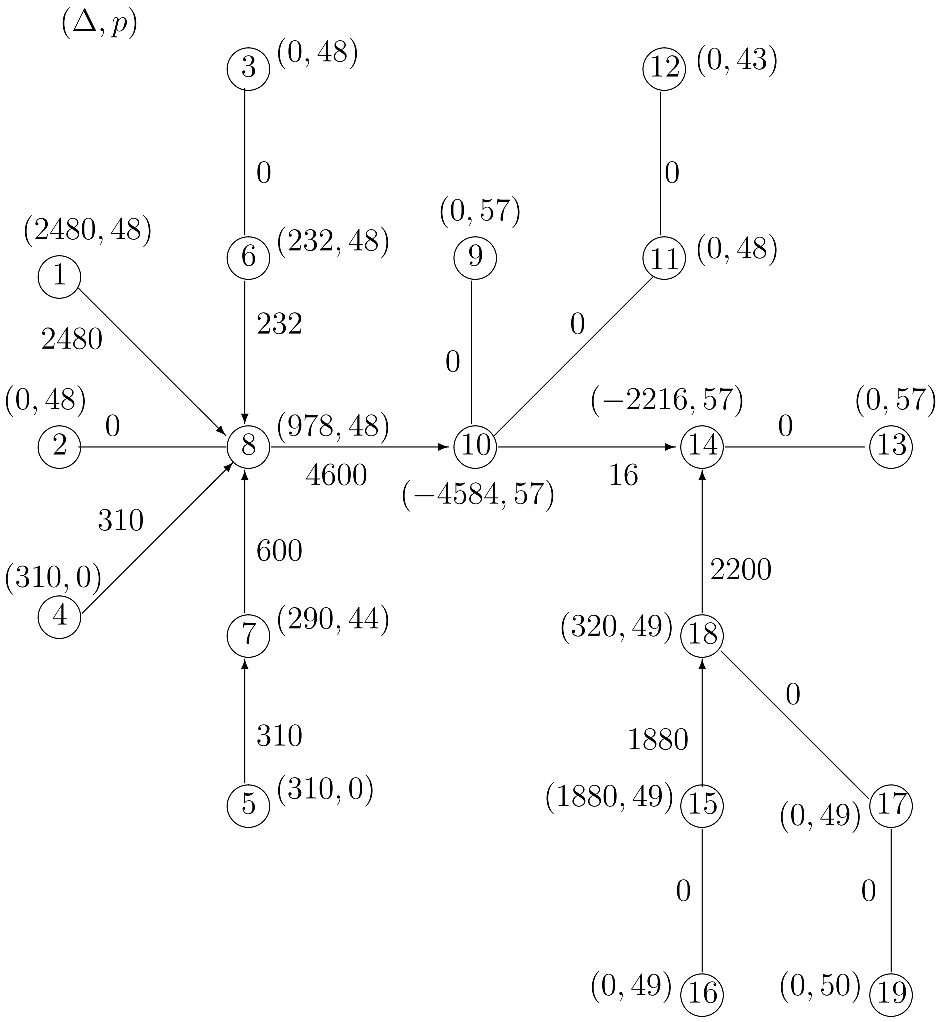


Figure 7: Nineteen auction markets: solution



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