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APPLICATION OF ABS METHODS TO THE PRIMAL-DUAL INTERIOR POINT METHOD FOR LINEAR PROGRAMMING

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Abstract

We consider the application of the ABS procedure to the linear system arising in the primal-dual interior point method where Newton method is used to compute the path to the solution. When approaching the solution the linear system becomes more and more ill conditioned. We show how the use of the Huang algorithm in the ABS class can reduce the ill conditioning. Preliminary numerical experiments show that the proposed approach can provide a residual in the computed solution up to sixteen orders lower.

Key words : ABS methods, Huang algorithm, primal-dual interior point method, Newton method.

1. Introduction

ABS algorithms have been introduced in 1984 by Abaffy, Broyden and Spedicato [1] to solve determined or underdetermined linear systems and have been later extended to linear least squares, nonlinear equations, optimization problems and integer (Diophantine) equations and LP problems. The ABS literature consists presently of over 400 papers, see Abaffy and Spedicato's monograph [2] for a presentation of such techniques as available at the beginning of 1989 and Spedicato et al [3] for a review dealing with later results. The class of ABS methods unifies most of existing methods for solving linear systems and provides a variety of alternative ways of implementing a specific algorithm. Extensive computational experience has shown that ABS methods are implementable in a stable way, being often more accurate than the corresponding traditional algorithms, and that on vector/parallel machines they often perform faster, see for example Bodon [4,5].

The linear programming problem is one of the most important problems in optimization, both in the continuous and discrete form. The classical algorithm is the simplex method, developed by Dantzig following a suggestion by Von Neumann, which approaches the solution by moving over the vertices of the polytope defining

the feasible region. This algorithm, while usually performing efficiently, may in the worst case require an exponentially high cost, since all vertices have to be reached. About a quarter of century ago a new class of methods, called the interior point methods, has been developed where the iterates are generated along a path which lies inside the boundary of the polytope, with possible computational advantages. The problem has been widely investigated in close to five thousand papers. The most efficient implementation appears to be the one based upon the primal-dual approach, leading to solving a sequence of structured nonlinear equations, containing partly linear and partly nonlinear equations in a diagonal form. The linear system to be solved at each iteration of Newton's method has the form

$$AA^T x = b$$

where $x \in R^n, b \in R^m, m < n$, and the matrix A has the form

$$A = BD^{-1}$$

where B is m by n and D is the diagonal matrix whose $i-th$ element in approching the solution tends either to zero or to infinity, resulting in a very ill conditioned matrix A and an even more ill conditioned matrix AA^T.

The system to be solved has the form of the normal equations of the second type, which arise when one is looking for the unique solution of least Euclidean norm of an underdetermined system. In the context of ABS methods such equation can be solved in a way similar to that proposed for the normal equations of the first kind, see [2], without the need of forming the product AA^T, thereby reducing the condition number essentially to the square root. Moreover the diagonal matrix responsible for the ill conditioning can be removed to some extent from the matrix to be applied to the right hand side. This will be discussed in the next section. The last section presents preliminary results of the proposed approach, which show that extremely better accuracy in the residual error can be obtained than by working on the system in the original form.

For convenience of the reader, we recall the basic steps of the (unscaled) ABS class for solving the linear system $Ax = b$, $x \in R^n$, $b \in R^m$, $m \leq n$, $A = (a_1, \cdots, a_m)^T$.

(A) Set $x_1 \in R^n$ arbitrary, $H_1 \in R^{n,n}$ arbitrary nonsingular, $i = 1$.
(B) Compute the vector $s_1 = H_i a_i$ and the scalar $\tau_i = a_i^T x_i - b_i$. If $s_i \neq 0$ go to (C), the $i-$th equation is linearly independent from the previous equations. Otherwise if $\tau_i \neq 0$ stop, the system has no solution, while if $\tau_i = 0$ the $i-$th equation is removed,

being a linear combination of the previous ones, so set $x_{i+1} = x_i, H_{i+1} = H_i$ and go to (F).
(C) Compute the search direction $p_i = H_i^T z_i,\ z_i \in R^n$ arbitrary save that $z_i^T H_i a_i \neq 0$.
(D) Update the estimate of the solution by $x_{i+1} = x_i - \alpha_i p_i$, with $\alpha_i = ({a_i}^T x_i - b_i)/a_i^T p_i$.
(E) Update the (Abaffian) matrix H_i by $H_{i+1} = H_i - H_i a_i w_i^T H_i / w_i^T H_i a_i$, where $w_i \in R^n$ is arbitrary save that $w_i^T H_i a_i \neq 0$.
(F) Stop if $i = m,\ x_{m+1}$ solves the system, otherwise increment i by one and go to (B).

In the above class (that can be generalized to the scaled ABS class by introducing an extra parameter, obtaining a complete realization of any Krylov-type process computing the solution in no more than m steps) algorithms are defined by specific choices of the parameters I, z_i, w_i. For the present problem we use the so called Huang or implicit Gram-Schmidt algorithm, defined by the choices

$$H_1 = I, z_i = w_i = a_i$$

where we denote by a_i the $i - th$ row of A. This algorithm has the remarkable property that it determines the unique solution of least Euclidean norm of an underdetermined system, if it is started with an arbitrary vector x_1 proportional to a_1, usually the zero vector. The solution is moreover approached, if $x_1 = 0$, monotonically from below. Additionally, the search directions p_i are orthogonal among themselves. The algorithm's stability can be improved in several ways, usually one does a reprojection on the search direction, i.e.

$$p_i = H_i(H_i a_i)$$

and one defines the update e.g. as

$$H_{i+1} = H_i - p_i p_i^T / p_i^T p_i$$

The above modification of the Huang algorithm is called the *modified Huang algorithm.*

For the use of the ABS algorithm in the normal equation of the second type step (B) plays an important role, since it allows the elimination of the redundant equations and in practice often the equations that are almost linearly dependent.

Notice that the norms of s_i, τ_i in practice are not tested against zero, due to the effects of roundoff, but against some positive small ϵ. The best determination of ϵ is still an open problem, but in practice we have found in all our extensive testing of ABS algorithm that ϵ can be taken rather safely as the product of the zero machine by the norm of the tested quantity.

2. ABS solution of the normal equations of the second kind

It is immediate to see that the normal equations of the second kind are equivalent to the following extended system, where only A appears

$$A^T x = y$$

$$Ay = b$$

The first of the above equations is an overdetermined linear system that must be solvable. Hence y must lie in the column space of the coefficient matrix A^T, i.e. in the row space of the matrix of the second system, which is an underdetermined system. Now an underdetermined system has one and only one solution in the row space, which is the solution of smallest Euclidean norm. Such a solution is computed by the Huang algorithm started with the zero vector. Therefore we are led to the following procedure, apparently never considered before, for solving the given normal equations of the second kind.

AA - Solve by Huang algorithm, started with the zero vector, system $Ay = b$

BB - Solve by any ABS algorithm the overdetermined system $A^T x = y$, where the $n - m$ linearly dependent equations are identified and removed at step (B).

By the above procedure we deal with systems having only matrix A, not the "squared" matrix AA^T, which reduces the condition number number. We also notice that if A is not full rank, then the Huang algorithm will still compute the solution of $Ay = b$ of smallest Euclidean norm, while use of the Huang algorithm in the overdetermined system, which is in such a case compatible but with infinite solutions, would compute the solution of smallest Euclidean norm. With such a solution Newton method would be well defined and convergent under mild conditions. Not many years ago exactly this property of Huang algorithm allowed us to eliminate occasional nonconvergence of Newton method used to solve the equations arising in the simulation of a large plant system defined by algebraic and differential equations. Our contractor, ENEL, was able to solve the stabilized code to its French counterpart

for a substantial amount.

Now we pay more attention to the case arising in the primal-dual interior point method, where A has the form $A = BD^{-1}$. The extended system is written now as

$$D^{-1}B^T x = y$$
$$BD^{-1}y = b$$

In the primal-dual method the matrix B is fixed, D changes at every iteration. Since D^{-1} scales the columns of the underdetermined system, the solution will require different search vectors at every Newton iteration. Once y is computed, we notice that the overdetermined system can be written in the form

$$B^T x = Dy$$

Since the coefficient matrix B is unchanged in the course of Newton's iterations, hence the following can be done only once at the first iteration:

AAA - computation of the search vectors

BBB - identification of the redundant equations, which are exactly those equations associated with dependent rows in B^T, since A and B have exactly the same set of dependent/independent equations. Therefore only the diagonal components of D associated with the selected independent rows of B are used in the computation of the solution (all of them enter in the underdetermined system)

Hence it is enough to determine independent equations and the associated search vectors (which do not depend on D) only at the first Newton iteration. This approach in other term is equivalent to essentially deal with the linear part of the primal-dual equations only once at the beginning. The number of operations to solve the overdetermined system after the first step is order(m^2). The number of operations to solve the underdetermined system by Huang algorithm is order (mn^2). These estimates can be certainly reduced when A is sparse, albeit a full study of the best implementation of a sparse Huang algorithm (possibly via the vector-wise formulation that applies to all ABS algorithms) is yet to be presented.

3. Some numerical experiments

Here we present a few numerical experiments obtained on two types of matrices:

1 - the ill conditioned Micchelli matrix where $A_{i,j}$ is the difference $i-j$ in absolute value. Such a matrix is symmetric nonsingular with condition number growing fast with the dimension. It appears in radial basis function approximation.

2 - a well conditioned Householder matrix with truncated last rows and post-multiplied by diagonal matrix with elements tending alternatively to zero and infinity. This was done by multiplication by the diagonal matrix whose elements had the form alternatively $10^k, 10^{-k}$, for $k = 1, 2, 3, 4, 5,$. The values taken for n were $n = 10, 100, 200$, for m were $m = 2, n/4, n/2, 3n/4, n-2$. The right hand side of the original equation was computed assigning equal to one all components of x. Notice that this nominal solution is the one computed by the algorithm in exact arithmetic only for full rank problems; for rank deficient problems, or problems with deficient numerical rank as determined in step (B) of the ABS algorithm, the Huang algorithm would determine the solution of smallest Euclidean norm, generally different from the nominal solution. The proposed algorithm via the extended system and use of the modified Huang algorithm is compared in the following experiments with the results of applying the modified Huang algorithm directly on the original system $AA^Tx = b$.

Now we give some results obtained in double precision, where E1, E2 are the error in the solution respectively by using the original system and the extended system and R1, R2 are similarly the residual errors, both computed in the Euclidean norm. For more results see Bonomi, [6].

Table 1

Results with the Micchelli matrix

$m = 2$	$n = 200$	$E1 = 4.7E-12$	$E2 = 6.8E-11$	$R1 = 0$	$R2 = 5E-13$
$m = n/4$	$n = 200$	$E1 = 2.3E-7$	$E2 = 7.9E-7$	$R1 = 1.7E-7$	$R2$ =4E-11
$m = n/2$	$n = 200$	$E1 = 5.7E-7$	$E2 = 1.5E-6$	$R1 = 4.8E-7$	$R2$ =6.2E-11
$m = 3n/4$	$n = 200$	$E1 = 8.9E-7$	$E1.6 = E-7$	$R1 = 7.7E-7$	$R2$ =1.4E-10
$m = n-2$	$n = 200$	$E1 = 1.9E-6$	$E2 = 2.1E-6$	$R1 = 1.3E-6$	$R2$ =1E-10

Table 2

Results with the Householder matrix, scaled by diagonal matrix with $\epsilon = 10^{-1}$

$m = 2$	$n = 200$	$E1 = 0$	$E2 = 1E-15$	$R1 = 0$	$R2 = 3E-15$
$m = n/4$	$n = 200$	$E1 = 2.3E-13$	$E2 = 5.7E-13$	$R1 = 2.3E-13$	$R2 =$2E-14
$m = n/2$	$n = 200$	$E1 = 4.2E-12$	$E2 = 50E-12$	$R1 = 4.8E-13$	$R2 =$5.3E-14
$m = 3n/4$	$n = 200$	$E1 = 8.9E-12$	$E1.6E-11$	$R1 = 3.7E-13$	$R2 =$4.2E-14
$m = n-2$	$n = 200$	$E1 = 4.7E-11$	$E2 = 5.4E-11$	$R1 = 9.5E-13$	$R2 =$1E-13

Table 3

Results with the Householder matrix, scaled by diagonal matrix with $\epsilon = 10^{-5}$

$m = 2$	$n = 200$	$E1 = 2E-16$	$E2 = 2E-15$	$R1 = 1.8E-12$	$R2 = 1.4E-11$
$m = n/4$	$n = 200$	$E1 = 2.4$	$rankdef$	$R1 = 2.1E-5$	$R2 =$8E-9
$m = n/2$	$n = 200$	$E1 = 6.3$	$rankdef$	$R1 = 4.7E+10$	$R2 =$2E-6
$m = 3n/4$	$n = 200$	$E1 = 9$	$rankdef$	$R1 = 5.1E+10$	$R2 =$6.7E-4
$m = n-2$	$n = 200$	$E1 = 10$	$rankdef$	$R1 = 3.8E+10$	$R2 =$6E-5

From an inspection of the above Tables we can take the following preliminary considerations:

A - On the Micchelli matrix we see little difference in the solution error, while the algorithm using the extended system is up to four orders better in the residual error

B - There is a clear difference in performance between the two algorithms when we consider the ill conditioned matrix obtained by scaling the Householder matrix. The algorithm based on the extended system is up to 16 orders better in the residual error, while no comparison can be made in the solution error since the matrices turn out to be numerically rank deficient. There is little difference when the matrix, scaled by $\epsilon = 1/10$, is still well conditioned.

C - The accuracy obtained by solving the extended system tends to increase with m approaching n.

Further experiments are planned on several types of matrices, providing also estimates of the condition number and of the running time, that are not given in this case.

References

[1] Abaffy, J., Broyden C. G. and Spedicato, E., A class of direct methods for linear systems, *Numer. Math.* 45, 361-376, 1984.

[2] Abaffy, J. and Spedicato, E., ABS Projection Algorithms: Mathematical Techniques for Linear and Nonlinear Equations, Ellis Horwood, Chichester, 1989.

[3] Spedicato, E., Bodon E., DEl Popolo A. and Xia Z., ABS algorithms for linear systems and optimization: a review and a bibliography, Ricerca Operativa 89, 39-88, 2000

[4] Bodon E, Numerical experiments with ABS algorithms on banded systems of linear equations, Report TR/PA/93/13, CERFACS, Toulouse, 1993

[5] Bodon E, Numerical experiments with ABS algorithms on upper-banded systems of linear equations, Report TR/PA/93/14, CERFACS, Toulouse, 1993

[6] Bonomi M., Utilizzo dei metodi ABS nei metodi interior point, Tesi di Laurea, AA 2006/07, Bergamo University. 2007

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