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Financial models with Lévy processes

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Abstract

This dissertation studies option pricing, portfolio selection, and risk management assuming exponential-Lévy models in financial markets. Option pricing of European, American, and path-dependent derivatives is dealt with the markovian approach. Markovian approach has been introduced by Duan and Simonato [26] to price American options under Wiener and GARCH processes, and then Duan *et al.* [27] has shown how to price barrier options. This dissertation proposes to extend the markovian approach to Lévy processes and shows numerical results where the price convergence is observed. European, American, and barrier options are priced using the same procedure of Duan *et al.*, while for compound and lookback options we propose a new pricing method. Specifically, we explain how to price compound and lookback options assuming a Markov chain evolutions of the asset price. Portfolio selection is studied assuming financial markets where asset log returns follow subordinated Lévy processes. Firstly, we propose a Mean-Value at Risk analysis under two financial markets, one without transaction costs, and the other one with proportional and constant transaction costs. Secondly, we study a multi-period model with unlimited short sales where investors look only at the mean and variance of the final wealth. Finally, we propose a Mean-Variance-Skewness analysis assuming a financial market with no short sales and without transaction costs. Our numerical results confirm the better performance of the studied subordinated Lévy processes with respect the Normal model. Risk management is studied proposing two conditional heteroscedastic models of portfolio returns. The first one is an extension of the EWMA RiskMetrics model and assumes Lévy distributed returns. The second one is a more sophisticated analysis and consists in a generalization of the GHICA model of Chen *et al.* [17].

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Chapter 1

Introduction

In this dissertation we discuss financial models of option pricing, portfolio selections, and risk management when asset log prices follow Lévy processes. Precisely, we assume the exponential-Lévy model

$$S_t = S_0 e^{X_t}, \quad t \geq 0,$$

where S_t is the price at time t of a financial asset and $\{X_t : t \geq 0\}$ is a stochastic process with some remarkable properties, such as independent and stationary increments. The first exponential-Lévy model proposed in literature to price contingent claims is the Black and Scholes one [11], where the dynamics of the asset price follows the geometric Brownian motion

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are called respectively drift term and volatility, and $\{W_t : t \geq 0\}$ is a standard Brownian motion. Then, under the Black and Scholes model, asset log returns on time intervals $[t, t+s]$ are normally distributed with mean $(\mu - \frac{1}{2}\sigma^2)s$ and variance $\sigma^2 s$. But, several empirical investigations, see Fama [30], Kraus and Litzenberg [46], and Mandelbrot [53], reject the Brownian evolution of asset log returns and stress the necessity to find better distributional assumptions. In particular, Mandelbrot and Fama observe empirical distributions with peaks higher and tails fatter than the Normal distribution and propose stable Paretian distributions as possible models for asset returns. Stable Paretian distributions constitute a subclass of Lévy processes characterized by the so-called stability property: if X has stable distribution and $X_1, \ldots, X_n$ are independent copies of X, then there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that for any n

$$a_n(X_1 + \cdots + X_n) + b_n \stackrel{d}{=} X_1.$$

After the proposal of Mandelbrot and Fama to look at stable Lévy processes as possible distributional assumptions, other financial researchers have confirmed the reliability of this hypothesis and designed models for option pricing, portfolio selections, and risk management (see, among others, Mittnik and Rachev [59],[60], and Chobanov *et al.* [19]). However, stable Lévy processes are not the unique ones to have fat-tails distributions, but others can be considered as possible choices to model asset log prices. A remarkable result due to Monroe [62] shows that every semimartingales can be written as a Brownian motion evaluated at a random time. Thus, we can always assume that a Lévy process is a subordinated process. In subordinated Lévy processes the concept of intrinsic or business time is invoked. The idea to consider a time scale different from the calendar one was due to Mandelbrot and Taylor [54]. Even Clark [20] has defined a model with an intrinsic time to describe arrival rates of new information. Specifically, he justifies the time change according to the principle that, when no information arrives, trading is slow and the price process evolves slowly, while, when new information arrives, trading is brisk and the prices process evolves more quickly. The existence of a business time is also studied by Geman, Madan, and Yor [34], which relate the time change to the information provided by demand and supply shocks in the market. Their analysis proposes infinite activity processes as possible models for asset log returns and shows the link between these stochastic processes and time-changed Brownian motions. In particular, Geman, Madan, and Yor observe that continuous stochastic processes can accurately describe market prices only in economies that instantaneously and continuously equilibrate to information flows driven by diffusion or Ito processes. Thus, they study the validity of diffusion processes as appropriate models for the underlying uncertainties and represent price processes as instantaneously and continuously adjustments to exogenous demand and supply shocks. In their economic model, the underlying uncertainties are increasing stochastic processes and consequently price processes result to be differences of two increasing random processes, representing respectively the up and down moves of the market. Therefore, in contrast with the Black and Scholes model, their economy implies price processes of finite variation and with jumps. The possibility of asset prices with discontinuities or jumps is also considered into the models of Merton [57] and more recently of Kou [45], where a diffusion component is added to a low or finite activity jump part. In particular, the diffusion components describes the high activity in price fluctuations while the jump component describes rare and extreme movements. However, the economy proposed by Geman, Madan, and Yor identifies pure jump processes as unique models able to represent asset prices. Thus, in these models "the high activity in price fluctuations is accounted for by a large (in fact infinite) number of small jumps and the activity at various jump sizes is analytically connected by the requirement that small jumps occur at higher rates than large jumps" (Geman, Madan, and Yor [34], page 3 lines 19-21). Moreover, Geman, Madan, and Yor remark the consistency of pure jump processes with the assumption of no arbitrage opportunities and show as the continuity of price processes can be recovered by measuring of time in units of business time. Specifically, since pure jump processes are of finite variation, then they are semimartingales consistent with the theory of no arbitrages. Indeed, the studies carry out on the consequence of no arbitrages (see Harrison and Kreps [40], Harrison and Pliska [41], and Delbaen and Schachermayer [25]) has concluded that price processes have to be semimartingales. However, time-changed Brownian motions can have infinite variation too, but the economic model proposed by Geman, Madan, and Yor views only pure jump processes of bounded variation as realist model to describe asset prices, since they can be decomposed into the difference of two increasing processes. Two important subordinated Lévy processes which can be assumed into the economic model of Geman, Madan, and Yor are the Variance Gamma and CGMY processes. The Variance Gamma (VG) process was introduced by Madan and Seneta [51] as a model for stock returns. They studied the symmetric case and, then, Madan *et al.* [52] defined the general case with skewness. The VG process can be used into the economy of Geman, Madan, and Yor because it can be built as the difference of two independent Gamma processes, and, moreover, it can also be obtained as a time-change Brownian motion with a Gamma process as subordinator. In order to have a more flexible process, Carr, Geman, Madan, and Yor [15] introduced the CGMY process. In particular, they modified with an additional parameter the Lévy measure of the VG process and obtained a process which could be, for different values of this parameter, finite or infinite active and, in the case of infinite activity, of finite or infinite variation. Obviously, the case of infinite variation does not adapt to the economic model of Geman, Madan, and Yor. Now, if we do not consider more the economy of Geman, Madan, and Yor as unique realistic representation of asset markets, other subordinated Lévy processes can be studied in order to model the random evolutions of asset prices. In particular, we can consider subordinated Lévy processes of infinite variation which, however, are supported by the result of Monroe to be consistent with the absence of arbitrages. Three important subordinated Lévy processes for financial applications are the Normal Inverse Gaussian (NIG), Hyperbolic, and Generalized Hyperbolic (GH) processes. Actually, the NIG and Hyperbolic processes are special cases of the GH one. The GH distributions were introduced by Barndorff-Nielsen [5] as a model for the grain-size distribution of wind-blown sand. Then, in the year 1995, two subclasses, the Hyperbolic and NIG distributions, was respectively studied by Eberlein and Keller [28] and Barndorff-Nielsen [6] as financial models for asset prices. Finally, Eberlein and Prause [29] and Prause [68] studied

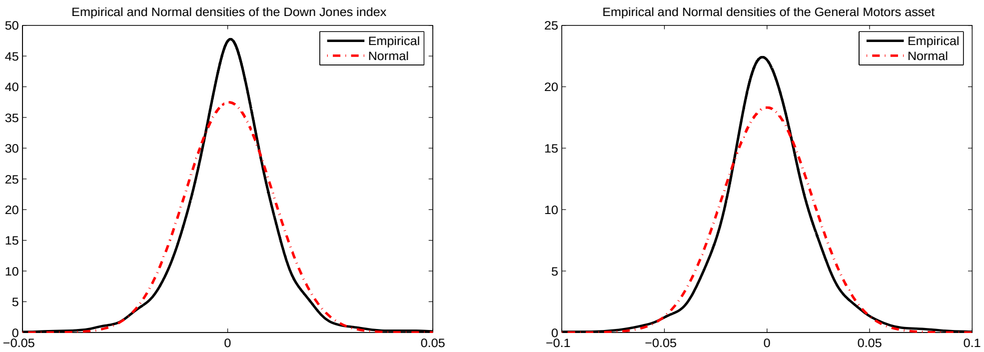

Figure 1.1: *Normal and empirical densities of the daily log returns of the Down Jones Index and General Motors stock.*

the whole family of GH distributions in order to describe random evolutions of asset prices.

In order to identify the distributional assumptions which better describe the random behaviour of asset log returns, let us carry out a little empirical investigation. In Figure 1.1 we display empirical densities of daily log returns from Down Jones index and General Motors stock observed from 13/09/1995 to 12/09/2007 and compare these densities with the Normal one. Given n independent observations $x_i, \ldots, x_n$, the empirical densities $f(x)$ can be estimated using the kernel density estimator

$$\widehat{f}_h(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x_i - x}{h}\right),$$

where $K(x) = \exp(-x^2/2)/\sqrt{2\pi}$ is the Gaussian kernel and h is the bandwidth. Under the Gaussian kernel, an optimal choice of h is $1.06\sigma n^{-1/5}$, where σ is an estimate of the standard deviation, see Silverman [79]. In Figure 1.1 we can see how the empirical densities exhibit a peak higher and tails fatter than the Normal one, that is the empirical distributions are leptokurtic. In order to have information about the tails of the empirical distributions, a better analysis is to plot the log densities. In Figure 1.2 we display the empirical log densities $\log \widehat{f}_h(x)$ and the corresponding log of the Normal density. The log Normal density has a

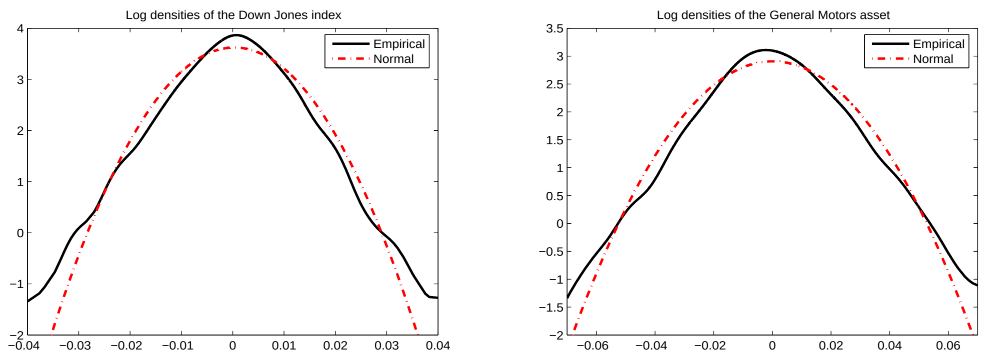

Figure 1.2: *Normal and empirical log densities of the daily log returns of the Down Jones Index and General Motors stock.*

quadratic bahaviour when it tends towards extreme values, while the empirical log densities seem to behave in linear way. This trend of the empirical distributions is generally called semi-heaviness of the tails, and we say that a density function $f(x)$ has semi-heavy tails if it satisfies

$$f(x) \sim \begin{cases} C_-|x|^{\rho_-} \exp(-\eta_-|x|) & \text{as} \quad x \to -\infty \\ C_+|x|^{\rho_+} \exp(-\eta_+|x|) & \text{as} \quad x \to +\infty, \end{cases}$$

for some $\rho_-, \rho_+ \in \mathbb{R}$ and $C_-, C_+, \eta_-, \eta_+ \geq 0$. In conclusion, this simple empirical analysis suggests to turn the interest towards those probability distributions which are leptokurtic and with semi-heavy tails. Lévy models provide a large class of distributions, including those with semi-heavy tails and skewness. Clearly, the complexity increases very much with respect to the Black and Scholes model. Not always there exists a closed formula for European contingent claims and pricing of exotic derivatives can be extremely hard by a computational point of view. In particular, Carr and Madan [14] derived the analytic expression of the Fourier transform of European call prices and proposed the Fast Fourier Transform as pricing procedure, and Boyarchenko and Levendorskiĭ [13] priced barrier options using the Wiener-Hopf decomposition and analytic techniques. But, the application of these methodologies is not straightforward and thus the option pricing under Lévy-exponential models is often based on simulation techniques and Monte Carlo methods. Another remarkable problem afflicting option pricing in an exponential-Lévy framework is the incompleteness of the market, i.e., there are more equivalent martingale measures. However, real asset markets are incomplete, since it is impossible to replicate any derivative instrument with a self-financing strategy. Thus, the incompleteness of the market does not represent a lack of the model but instead a better description of the real world. But, under an exponential-Lévy model, it remains the problem to find the equivalent martingale measure that better summarizes the investor's choices.

In this dissertation we discretize the Lévy process distributions using a multinomial model. This discretization process simplifies the computation of European, American, and Exotic options. In this contest, Amin [3] has been among the first researchers to propose a multinomial model. Differently by Amin, we adopt the analysis developed by Duan and Simonato [26] to describe the markovianity of contingent claims. Precisely, we propose to extent the Duan and Simonato methodology even when the underlying asset follows a Lévy process. This multinomial model simplifies greatly the option pricing under an exponential-Lévy model. Indeed now the price of any contingent claim (even path-dependent derivatives) can be studied under a Markov chain framework. Duan *et al.* [27] show how to price European, American, and barrier options through a Makov chain, and thus we can use the same technique in order to price these instruments under exponential-Lévy models. Furthermore, we are able to give a certain originality to our applications, explaining how to price compound and lookback options under a Markov chain framework and thus pricing these instruments in an exponential-Lévy context.

Another purpose of this dissertation is to select portfolios of financial assets assuming multi-dimensional subordinated Lévy models. According to the portfolio theory, investors allocate their wealth among available assets so that the expected value of their utility function is maximized. Markowitz [55], De Finetti [69], and Tobin [83], [84] were among the first researchers to face the problem of portfolio selections, and they established the basis of the mean-variance approach. According to this approach investors allocate their wealth looking only at two parameters of the distributions of portfolio returns, the mean and variance. Later, the mean-variance model was extended to the capital asset pricing model (CAPM) (see Sharpe [78], Lintner [49], and Mossin [63]). With the CAPM we have an equilibrium theory based just on the first two moments of portfolio distributions. In the successive years the CAPM was justified and extended by the arbitrage pricing theory (APT) and the stochastic dominance analysis. In particular, APT and fund separation theorems justify and extend CAPM to multi-parameter linear models (see Ross [71], [72]). While the stochastic dominance analysis justifies the partial consistency of the mean-variance model with the expected utility maximization when portfolios are elliptically distributed (see Bawa [7], Chamberlein [16], Owen and Rabinovitch [66]). Elliptical distributions are particular symmetric distributions that generalize the Gaussian one. However, sample data often display a certain level of skewness and tails fatter than the Gaussian one. Since subordinated Brownian motions have distributions with skewness different from zero and kurtosis greater than three, in this dissertation we propose a mean-risk measure analysis where asset log returns follow multi-dimensional Lévy processes. In particular, we present a model where each asset follows a subordinated Lévy process with the same subordinator and consider two possible distributional assumptions, the Normal Inverse Gaussian and Variance Gamma distributions. Moreover, we carry out empirical comparisons among these two distributional assumptions and the multi-Normal one.

A last analysis developed in this dissertation concerns risk management. In particular, we study two Lévy models. The first one extends the EWMA RiskMetrics model (see Longerstaey and Zangari [50]) in order to describe conditional heteroscedastic portfolio returns with Lévy distributions. As in the RiskMetrics model, the time-dependent portfolio volatility is described by the exponential weighted moving average (EWMA) model. This EWMA-Lévy model keeps the same computational complexity of the RiskMetrics model and the risk measures VaR and CVaR can be computed with very simple formulas. The second studied risk management model is a generalization of the GHICA model of Chen *et al.* [17]. Their model uses Independent Component Analysis (ICA) to define conditional heteroscedastic portfolio returns with independent innovations. Thus, they suggest to model these independent variables as Normal Inverse Gaussian distributions. Our generalization consists in to assume each innovation distributed as the Lévy law which better describes it. This ICA-Lévy model finds the characteristic function of portfolio return, and thus the Fast Fourier Transformation (FFT) algorithm can be implemented to compute the portfolio density function. Given the density function, the risk measures VaR and CVaR can be easily computed.

The other chapters of the dissertation are organized as follows. Chapter 2 is an introduction to Lévy processes where we report definitions and main results. In particular, we study subordinated Lévy processes and present the exponential-Lévy model as mathematical model to describe the random evolution of financial assets. Chapter 3 faces option pricing under Lévy processes. Specifically, we explain some methodologies to price European options and then introduce a markovian approach to price European, American, and Exotic options. Chapter 4 studies portfolio selection and risk management under Lévy processes. In the portfolio selection part we present a model to describe asset log returns and compare this model with the Normal one, in the risk management part we describe two Lévy models without numerical results. Finally, in chapter 5 we conclude the dissertation summarizing the results obtained in the other chapters and proposing new future researches.

Chapter 2

Lévy processes and exponential-Lévy models

Stochastic processes are mathematical models describing the time evolutions of random phenomena, for example the daily price of a risky asset or the payments made in one year by an insurance company. Lévy processes constitute a large class of stochastic processes with some remarkable properties, such as the independence and stationarity of the increments. Since Lévy processes are able to capture the skewness and kurtosis of the observed asset log returns, their use in finance is becoming very widespread. Indeed, there are several empirical investigations, for example Fama [30], Mandelbrot [53], and Kraus and Lintzenberg [46], which show how the behavior of the log returns is more skew and with tails fatter than the Normal distribution. The chapter is organized as follows. First, Section 2.1 presents Lévy processes, defining them and stating their characterization by infinitely divisible distributions. Moreover, we give the concept of Lévy measure and recall the Lévy-Ito decomposition, where any Lévy process is just a combination of a Gaussian term with drift and a possible infinite sum of compound Poisson processes. Second, Section 2.2 shows how any Lévy process satisfies Markov property, simplifying its tractability. Third, Section 2.3 focus on the construction of Lévy processes and, in particular, we explain how to define a Lévy process by a subordinator. Examples are the Variance Gamma (VG) and Normal Inverse Gaussian (NIG) processes. Fourth, section 2.4 generalizes the VG and NIG processes by the CGMY and Generalized Hyperbolic processes. Fifth, Section 2.5 describes briefly the Meixner process which is a good Lévy process to model asset log returns. Finally, Section 2.6 describes the market model generally assumed in order to price contingent claims, for example American options. We highlight the incompleteness of the market when the underlying asset follows a stochastic process with jumps and discuss several ways to define an equivalent martingale measure.

2.1 Definitions

In this section we present a class of stochastic processes, called Lévy processes, whose use in financial problems is becoming more and more widespread.

Definition 2.1. *A stochastic process* $\{X_t : t \geq 0\}$ *on* $\mathbb{R}^d$ *is a Lévy process if the following conditions are satisfied:*

(1) *For any choice of* $n \geq 1$ *and* $0 \leq t_0 < t_1 < \cdots < t_n$*, random variables* $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ *are independent.*

(2) $X_0 = 0$ *a.s.*

(3) *The distribution of* $X_{s+t} - X_s$ *does not depend on* s*.*

(4) *It is stochastically continuous.*

(5) *There is* $\Omega_0 \in \Im$ *with* $\mathrm{P}[\Omega_0] = 1$ *such that, for every* $\omega \in \Omega_0$*,* $X_t(\omega)$ *is right-continuous in* $t \geq 0$ *and has left limits in* $t > 0$*.*

Let us use the notation $\{X_t\}$. The properties **(1)**, **(2)**, and **(3)** are generally recalled saying that the stochastic process $\{X_t\}$ has independent and stationary increments, while the fourth property means

$$\lim_{s \to t} \mathrm{P}[|X_s - X_t| > \varepsilon] = 0, \tag{2.1}$$

for every $t \geq 0$ and $\varepsilon > 0$. Equation (2.1) does not imply that the stochastic process $\{X_t\}$ is continuous, but only that a discontinuity at a fixed time t has probability zero, in other words $\{X_t\}$ is discontinuous at random times. A stochastic process satisfying only the first four conditions is called Lévy processes in law. Finally, the fifth property is usually recalled saying that the stochastic process $\{X_t\}$ is cadlag.

An important role into the theory of Lévy processes is that one of the infinitely divisible distributions. A distribution μ is said infinitely divisible if, for any positive integer n, there exists a distribution μ_n such that μ is equal to the n-th convolution of μ_n with itself. Remembering that the convolution of two probability measure μ_1 and μ_2, denoted as $\mu_1 \star \mu_2$, is the distribution of the sum of two independent random variables with distributions μ_1 and μ_2, respectively, then the infinite divisibility of μ implies that, for each n, there are n independent and identically distributed random variables $Y_1, \dots, Y_n$ such that $Y_1 + \cdots + Y_n$ has distribution μ. We have the following proposition.

Proposition 2.1. *Let $\{X_t : t \geq 0\}$ be a Lévy process. Then, for every t, X_t has an infinitely divisible distribution. Conversely, if μ is an infinitely divisible distribution, then there exists a Lévy process $\{X_t\}$ such that the distribution of X_1 is μ.*

Proof. See Cont and Tankov [21], Proposition 3.1.

Given an infinitely divisible distribution μ on $\mathbb{R}^d$, it is possible to prove (see Sato [73], Lemma 7.6) the existence of an unique continuous function $\psi(z)$ from $\mathbb{R}^d$ into $\mathbb{C}$ such that $\psi(0) = 0$ and

$$\phi_\mu(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle} \mu(dx) = e^{\psi(z)}, \quad z \in \mathbb{R}^d,$$

where $\phi_\mu(z)$ is the characteristic function of μ. Moreover, (see Sato [73], Lemma 7.9) for every $t \in [0, \infty)$ there exists the infinitely divisible distribution μ^t with characteristic function $\phi_{\mu^t}(z) = \exp(t\psi(z))$. Now, if $\{X_t : t \geq 0\}$ is a Lévy process and $\mathrm{P}_{X_1} = \mu$, then it is possible to prove that $\mathrm{P}_{\mathrm{X_t}} = \mu^t$ (see Sato [73], Theorem 7.10), and thus

$$\phi_{X_t}(z) = \mathrm{E}[\exp(i\langle z, X_t\rangle)] = \exp(t\psi(z)), \quad z \in \mathbb{R}^d, \tag{2.2}$$

where $\phi_{X_t}(z)$ is the characteristic function of X_t and $\psi(z)$ is called the characteristic exponent of $\{X_t\}$. From equation (2.2) we obtain that the knowledge of the law of X_t is determined by the knowledge of the law of X_1.

Let us introduce the Lévy measure of Lévy processes starting by a compound Poisson process.

Definition 2.2. *A compound Poisson process with intensity $\lambda > 0$ and jump size distribution F is a stochastic process $\{X_t : t \geq 0\}$ defined as*

$$X_t = \sum_{j=1}^{N_t} Y_j,$$

where jump sizes Y_j are independent and identically distributed with distribution F and $\{N_t : t \geq 0\}$ is a Poisson process with intensity λ and independent from $\{Y_j : j = 1, 2, \ldots\}$.

Let $\phi_F(z)$ be the characteristic function of F:

$$\begin{aligned}
\phi_{X_1}(z) &= \mathrm{E}[e^{i\langle z, X_1\rangle}] = \mathrm{E}\left[\mathrm{E}\left[e^{i\langle z, X_1\rangle} | N_1\right]\right] \\
&= \mathrm{E}\left[\mathrm{E}\left[\exp\left(i\sum_{j=1}^{N_1}\langle z, Y_j\rangle\right)\middle| N_1\right]\right], \\
&= \mathrm{E}\left[\phi_F(z)^{N_1}\right] = \sum_{n=0}^{\infty}\phi_F(z)^n\frac{\lambda^n e^{-\lambda}}{n!} \\
&= \exp\left(\lambda(\phi_F(z) - 1)\right) = \exp\left(\lambda\int_{\mathbb{R}^d}(e^{i\langle z, x\rangle} - 1)F(dx)\right), \quad z \in \mathbb{R}^d.
\end{aligned}$$

Thus, $\phi_{X_1}(z)$ is the characteristic function of an infinitely divisible distribution, exactly, the compound Poisson distribution (see Sato [73], Definition 4.1), and then $\{X_t\}$ is a Lévy process with characteristic exponent

$$\psi(z) = \lambda \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)F(dx), \quad z \in \mathbb{R}^d.$$

The compound Poisson process is showed to be the unique Lévy process whose sample paths are piecewise constant functions (see Cont and Tankov [21], Proposition 3.3). By equation (2.2), the distribution of X_t has characteristic function

$$\phi_{X_t}(z) = \exp\left(t\lambda \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)F(dx)\right), \quad z \in \mathbb{R}^d,$$

and, introducing a new measure $\nu(A) = \lambda F(A)$, we have

$$\phi_{X_t}(z) = \exp\left\{t \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(dx)\right\}, \quad z \in \mathbb{R}^d.$$

The measure ν, called Lévy measure, is positive on $\mathbb{R}^d$ but it is not a probability measure, because $\int \nu(dx) = \lambda \neq 1$.

To every compound Poisson process $\{X_t\}$ on $\mathbb{R}^d$ we can assign a random measure on $[0,\infty) \times \mathbb{R}^d$ defined by

$$J_X(B) = \#\{(t, X_t - X_{t-}) \in B\},$$

where B is a measurable subset of $[0,\infty) \times \mathbb{R}^d$. Thus, for every measurable set $A \subset \mathbb{R}^d$, $J_X([t_1,t_2] \times A)$ counts the number of times between t_1 and t_2 such that the size of the jumps of $\{X_t\}$ belongs to A. The measure J_X is exactly a Poisson random measure on $\mathbb{R}^d \times [0,\infty)$ with intensity measure $\mu(dx \times dt) = \nu(dx)dt = \lambda F(dx)dt$ (see Cont and Tankov [21], Definition 2.18 and Proposition 3.5), that is for every measurable set $B \subset \mathbb{R}^d \times [0,\infty)$,

$$\mathrm{P}[J_X(B) = k] = e^{-\mu(B)} \frac{\mu(B)^k}{k!}, \quad \forall k \in \mathbb{N}. \tag{2.3}$$

Equation (2.3) suggests the interpretation of the Lévy measure of a compound Poisson process as the average number of jumps per unit of time:

$$\nu(A) = \mathrm{E}[\#\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}], \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where $\Delta X_t = X_t - X_{t-}$. Every compound Poisson process can be represented by

$$X_t = \sum_{s\in[o,t]} \Delta X_s = \int_{[o,t]\times\mathbb{R}^d} x J_X(ds \times dx),$$

where J_X is a Poisson random measure with intensity measure $\nu(dx)dt$.

Consider a Lévy process $\{X_t^0\}$ with piecewise constant paths, that is a compound Poisson process. Then, X_t^0 can be represented by

$$X_t^0 = \int_{[o,t]\times\mathbb{R}^d} x J_X(ds \times dx),$$

where J_X is a Poisson random measure with intensity measure $\nu(dx)dt$ and ν is a finite measure defined by

$$\nu(A) = \mathrm{E}[\#\{t \in [0,1] : \Delta X_t^0 \neq 0, \Delta X_t^0 \in A\}], \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Moreover, consider a Brownian motion with drift $\gamma t + W_t$ independent of X^0. Then, the sum $X_t = \gamma t + W_t + X_t^0$ is another Lévy process which can be expressed as

$$X_t = \gamma t + W_t + \sum_{s\in[0,t]} \Delta X_s^0 = \gamma t + W_t + \int_{[o,t]\times\mathbb{R}^d} x J_X(ds \times dx). \tag{2.4}$$

An expression as (2.4) can be proved for every Lévy process. Indeed, given a Lévy process $\{X_t\}$, we can define its Lévy measure as we have only just done for a compound Poisson process, that is

$$\nu(A) = \mathrm{E}[\#\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}], \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Then, the measure ν should satisfy $\nu(A) < \infty$ for any compact set $A \subset \mathbb{R}^d \setminus \{0\}$, otherwise the process, contradicting the cadlag property, would have an infinite number of jumps with size in A on $[0,T]$. However, the measure ν could not be finite (i.e. $\nu(\mathbb{R}^d \setminus 0) = \infty$), indeed the process X could have an infinite number of small jumps on $[0,T]$.

Proposition 2.2 (Lévy-Ito decomposition). *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ and ν its Lévy measure. Then*

- *ν is a measure on $\mathbb{R}^d \setminus \{0\}$ and verifies*

$$\int_{|x|\le 1} |x|^2 \nu(dx) < \infty \quad \int_{|x|\ge 1} \nu(dx) < \infty.$$

- *The jump measure of X, denoted by J_X, is a Poisson random measure on $[0,\infty) \times \mathbb{R}^d$ with intensity measure $\nu(dx)dt$.*

- *There exist a vector γ and a d-dimensional Brownian motion $\{W_t\}$ with covariance matrix* A *such that*

$$X_t = \gamma t + W_t + X_t^l + \lim_{\epsilon \downarrow 0} \widetilde{X}_t^\epsilon, \tag{2.5}$$

where

$$\begin{aligned} X_t^l &= \int_{|x|\ge 1, s\in[0,t]} x J_X(ds \times dx), \\ \widetilde{X}_t^\epsilon &= \int_{\epsilon \le |x| < 1, s\in[0,t]} x\{J_X(ds \times dx) - \nu(dx)ds\} \\ &\equiv \int_{\epsilon \le |x| < 1, s\in[0,t]} x \widetilde{J}_X(ds \times dx). \end{aligned}$$

The terms in (2.5) *are independent and the convergence in the last term is almost sure and uniform in t on $[0,T]$.*

Proof. See Cont and Tankov [21], Proposition 3.7.

The Lévy-Ito decomposition says that a Lévy process is uniquely determined by a vector γ, a positive definite matrix A and a positive measure ν. The triplet $(\mathrm{A}, \nu, \gamma)$ is said characteristic triplet or Lévy triplet of the process $\{X_t\}$. The two terms $\{X_t^l\}$ and $\{\widetilde{X}_t^\epsilon\}$ represent the jumps of $\{X_t\}$ and are described by the Lévy measure ν. $\{X_t^l\}$ is a compound Poisson process, while $\{\widetilde{X}_t^\epsilon\}$ a compensated compound Poisson process whose characteristic function at time t is

$$\phi_{\widetilde{X}_t^\epsilon}(z) = \exp\left\{ t \int_{\epsilon \le |x| < 1} (e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle)\nu(dx) \right\}, \quad z \in \mathbb{R}^d.$$

From Lévy-Ito decomposition, we can deduce several other properties. First, when a Lévy process is continuous then, by (2.5), we see that it must be a Brownian motion with drift. Second, the condition $\int_{|x|\geq 1}\nu(dx)<\infty$ says that a Lévy process must have a finite number of jumps with absolute value larger than 1. Third, every Lévy process is a combination of a Brownian motion with drift and a possibly infinite sum of independent compound Poisson processes. Finally, using the Lévy-Ito decomposition, we can express the characteristic function of a Lévy process in terms of its characteristic triplet (A,ν,γ).

Theorem 2.1 (Lévy-Khintchine representation)**.** *Let* $\{X_t\}$ *be a Lévy process on* $\mathbb{R}^d$ *with characteristic triplet* (A,ν,γ)*. Then*

$$\mathrm{E}\left[e^{i\langle z,X_t\rangle}\right]=e^{t\psi(z)},\quad z\in\mathbb{R}^d,$$

with

$$\psi(z)=-\frac{1}{2}\langle z,\mathrm{A}z\rangle+i\langle\gamma,z\rangle+\int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle}-1-i\langle z,x\rangle 1_{|x|\leq 1}\right)\nu(dx).$$

Proof. See Cont and Tankov [21], Theorem 3.1.

Since any infinitely divisible distribution is the distribution of a Lévy process at time $t=1$, then the Lévy-Khintchine representation characterizes the characteristic function of any infinitely divisible distribution as well.

Some path properties of a Lévy process can be deduced by the characteristic triplet. The next result gives the conditions which characterize Lévy processes of finite variation, that is whose trajectories are functions of finite variation with probability 1.

Proposition 2.3. *A Lévy process is of finite variation if and only if its characteristic triplet* (A,ν,γ) *satisfies:*

$$\mathrm{A}=0\quad\text{and}\quad\int_{|x|\leq 1}|x|\nu(dx)<\infty.$$

Proof. See Cont and Tankov [21], Proposition 3.9.

In the case of a Lévy process with finite variation, the Lévy-Ito decomposition and Lévy-Khintchine representation can be simplified.

Corollary 2.1. *Let $\{X_t : t \geq 0\}$ be a Lévy process of finite variation with Lévy triplet given $(0, \nu, \gamma)$. Then $\{X_t\}$ can be expressed as the sum of its jumps between 0 and t and a linear drift term:*

$$X_t = bt + \int_{[0,t]\times\mathbb{R}^d} x J_X(ds \times dx) = bt + \sum_{s\in[0,t]}^{\Delta X_s \neq 0} \Delta X_s,$$

and its characteristic function can be expressed as

$$\mathrm{E}[e^i \langle x, X_t \rangle] = \exp\left(t \left(i\langle z, b\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(dx) \right) \right), \quad z \in \mathbb{R}^d,$$

where $b = \gamma - \int_{|x|\leq 1} x\nu(dx)$.

Proof. See Cont and Tankov [21], Corollary 3.1.

Given a pure jump Lévy process, that is one with no Brownian component ($\sigma^2 = 0$), it is usual to distinguish between finite or infinite activity. When $\int_{-1}^{1} \nu(dx) < \infty$ the Lévy process is said of finite activity and thus there are finitely many jumps in any finite interval. Instead, when $\int_{-1}^{1} \nu(dx) = \infty$ the Lévy process is called of infinite activity and in this case there are infinitely many jumps in any finite interval.

Let us conclude the section reporting the notion of completely monotone Lévy density. Given a Lévy measure with density, i.e. $\nu(dx) = k(x)dx$, the Lévy density $k(x)$ is called completely monotone if it can be written in the form

$$k(x) = \int_0^\infty e^{-ax}\zeta(da),$$

for some positive measure ζ. Thus, a completely monotone Lévy density relates arrival rates of large jump sizes to smaller jump sizes in such a way that large jumps arrive less frequently than small jumps.

2.2 Markov property

Lévy processes are also Markov processes and thus satisfy the so-called Markov property. In this section we define temporally homogeneous Markov processes through temporally homogeneous transition functions, then recall a theorem which characterize Lévy processes as temporally homogeneous Markov processes with spatially homogeneous transition functions, finally introduce the concept of Markov property.

Definition 2.3. *A mapping* $\mathrm{P}_{s,t}(x,B)$ *of* $x \in \mathbb{R}^d$ *and* $B \in \mathcal{B}(\mathbb{R}^d)$ *with* $0 \le s \le t < \infty$ *is called a transition function on* $\mathbb{R}^d$ *if*

(1) *it is a probability measure as a mapping of* B *for any fixed* x*;*

(2) *it is measurable in* x *for any fixed* B*;*

(3) $\mathrm{P}_{s,s}(x,B) = \delta_x(B)$ *for* $s \ge 0$*;*

(4) *it satisfies*

$$\int_{\mathbb{R}^d} \mathrm{P}_{s,t}(x,dy)\mathrm{P}_{t,u}(y,B) = \mathrm{P}_{s,u}(x,B) \quad \textit{for} \quad 0 \le s \le t \le u.$$

If, in addition,

(5) $\mathrm{P}_{s+h,t+h}(x,B)$ *does not depend on* h*,*

then it is called a temporally homogeneous transition function and it is given by $\mathrm{P}_t(x,B)$ *such that*

$$\mathrm{P}_t(x,B) = \mathrm{P}_{s,s+t}(x,B) \quad s \ge 0.$$

The property **(4)** is called the Chapman-Kolmogorov identity, and, when the transition function is also temporally homogeneous, we have

$$\int_{\mathbb{R}^d} \mathrm{P}_s(x,dy)\mathrm{P}_t(y,B) = \mathrm{P}_{s+t}(x,B) \quad \text{for} \quad s \ge 0 \quad \text{and} \quad t \ge 0. \tag{2.6}$$

Before to define Markov processes, we recall a theorem which is fundamental for the theory of stochastic processes, the celebrated Kolmogorov's extension theorem. Let $\Omega = (\mathbb{R}^d)^{[0,\infty)}$, the collection of all functions $\omega = (\omega(t))_{t\in[0,\infty)}$ from $[0,\infty)$ into $\mathbb{R}^d$ and define $X_t(\omega) = \omega(t)$. A set

$$C = \{\omega : X_{t_1}(\omega) \in B_1, \ldots, X_{t_n}(\omega) \in B_n\}$$

for $0 \le t_1 < \cdots < t_n$ and $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}^d)$ is called a cylinder set. Let $\Im$ be the σ-algebra generated by the cylinder sets.

Theorem 2.2 (Kolmogorov's extension theorem)**.** *Suppose that, for any choice of* n *and* $0 \le t_1 < \cdots < t_n$*, a distribution* $\mu_{t_1,\ldots,t_n}$ *is given and that, if* $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}^d)$ *and* $B_k = \mathbb{R}^d$*, then*

$$\mu_{t_1,\ldots,t_n}(B_1 \times \cdots \times B_n) \\ = \mu_{t_1,\ldots,t_{k-1},t_{k+1},\ldots,t_n}(B_1 \times \cdots \times B_{k-1} \times B_{k+1} \times \cdots \times B_n). \tag{2.7}$$

Then, there exists an unique probability measure P *on* $\Im$ *such that*

$$\mathrm{P}[B_1 \times \cdots \times B_n] = \mu_{t_1,\ldots,t_n}(B_1 \times \cdots \times B_n),$$

for any choice of n*,* $0 \le t_0 < \cdots < t_n$*, and* $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}^d)$*.*

Proof. See Billingsley [10], Theorem 36.1.

Assume that a temporally homogeneous transition function $\mathrm{P}_t(x, B)$ on $\mathbb{R}^d$ is given. Then, for any $a \in \mathbb{R}^d$, we can construct a stochastic process $\{Y_t : t \ge 0\}$ as follows. Let $\Omega^0 = (\mathbb{R}^d)^{[0,\infty)}$, the collection of all functions ω from $[0,\infty)$ into $\mathbb{R}^d$, $Y_t(\omega) = \omega(t)$ for $t \ge 0$, and $\Im^0$ be the σ-algebra generated by Y_t, $t \ge 0$. Define, for any $0 \le t_0 < \cdots < t_n$ and $B_0, \ldots, B_n$,

$$\mu^a_{t_0,\ldots,t_n}(B_0, \ldots, B_n) \\ = \int \mathrm{P}_{t_0}(a, dx_0) 1_{B_0}(x_0) \int \mathrm{P}_{t_1-t_0}(x_0, dx_1) 1_{B_1}(x_1) \\ \int \mathrm{P}_{t_2-t_1}(x_1, dx_2) 1_{B_2}(x_2) \cdots \int \mathrm{P}_{t_n-t_{n-1}}(x_{n-1}, dx_n) 1_{B_n}(x_n)$$

The function $\mu^a_{t_0,\dots,t_n}$ can be uniquely extended to a probability measure on $(\mathbb{R}^d)^{n+1}$ and, moreover, the family $\{\mu^a_{t_0,\dots,t_n}\}$ satisfies the condition (2.7) by equation (2.6). Therefore, by Theorem 2.2, there exists an unique probability measure P^a extending this family. Thus we can define univocally a probability space $(\Omega^0, \Im^0, \mathrm{P}^a)$. The next definition is that one of temporally homogeneous Markov process and it is mentioned the notion of stochastic processes identical in law. Two stochastic processes $\{X_t\}$ and $\{Y_t\}$ are called identical in law if, for any choice of n, $0 \le t_1 < \cdots < t_n$, and $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathrm{P}[X_{t_1} \in B_1, \dots, X_{t_n} \in B_n] = \mathrm{P}[Y_{t_1} \in B_1, \dots, Y_{t_n} \in B_n].$$

Definition 2.4. *A stochastic process $\{X_t : t \ge 0\}$ defined on a probability space $(\Omega, \Im, \mathrm{P})$ is called a temporally homogeneous Markov process with temporally homogeneous transition function $\{\mathrm{P}_t(x, B)\}$ and starting point a, if it is identical in law with the process $\{Y_t : t \ge 0\}$ define above on $(\Omega^0, \Im^0, \mathrm{P}^a)$. The process $\{Y_t\}$ is the path space representation of the process $\{X_t\}$.*

Temporally homogeneous transition functions can satisfy a further property, the homogeneous spatiality.

Definition 2.5. *A transition function $\mathrm{P}_{s,t}$ on $\mathbb{R}^d$ is said to be spatially homogeneous if*

$$\mathrm{P}_{s,t}(x, B) = \mathrm{P}_{s,t}(0, B - x)$$

for any s, t ,x and B, where $B - x = \{y - x : y \in B\}$.

The next theorem gives the characterization of Lévy processes as temporally homogeneous Markov processes with spatially homogeneous transition functions.

Theorem 2.3. (i) *Let μ be an infinitely divisible distribution on $\mathbb{R}^d$ and let $\{X_t\}$ be the Lévy process corresponding to μ. Define $\mathrm{P}_t(x, B)$ by*

$$\mathrm{P}_t(x, B) = \mu^t(B - x).$$

Then $\mathrm{P}_t(x, B)$ *is a temporally and spatially homogeneous transition function and* $\{X_t\}$ *is a Markov process with this transition function and starting point* 0*.*

(ii) *Conversely, any stochastically continuous, temporally homogeneous Markov process on* $\mathbb{R}^d$ *with spatially homogeneous transition function and starting point* 0 *is a Lévy process.*

Proof. See Sato [73], Theorem 10.5.

Markov processes satisfy an important property, called just Markov property, which simplifies their tractability. We introduce Markov property with the next proposition.

Proposition 2.4. *Consider* $\{Y_t : t \geq 0\}$*, the path space representation of a temporally homogeneous Markov process with a transition function* $\mathrm{P}_t(x, B)$*. Let* $0 \leq t_0 < \cdots < t_n$ *and let* $f(x_0, \ldots, x_n)$ *be a bounded measurable function. Then* $\mathrm{E}^a[f(Y_{t_0}, \ldots, Y_{t_n})]$ *is measurable in* a *and*

$$\begin{aligned} \mathrm{E}^a[f(Y_{t_0}, \ldots, Y_{t_n})] &= \int \mathrm{P}_{t_0}(a, dx_0) \int \mathrm{P}_{t_1-t_0}(x_0, dx_1) \\ &\quad \int \mathrm{P}_{t_2-t_1}(x_1, dx_2) \cdots \int \mathrm{P}_{t_{n-1}-t_n}(x_{n-1}, dx_n) f(x_0, \ldots, x_n). \end{aligned}$$

Moreover, for any $0 \leq s_0 < \cdots < s_m \leq s$ *and for any bounded measurable function* $g(x_0, \ldots, x_m)$*, we have*

$$\begin{aligned} &\mathrm{E}^a[g(Y_{s_0}, \ldots, Y_{s_m}) f(Y_{s+t_0}, \ldots, Y_{s+t_n})] \\ &\qquad = \mathrm{E}^a[g(Y_{s_0}, \ldots, Y_{s_m}) \mathrm{E}^{Y_s}[f(Y_{s+t_0}, \ldots, Y_{s+t_n})]]. \end{aligned} \tag{2.8}$$

Proof. See Sato [73], Proposition 10.6.

Equation (2.8) is the Markov property and is generally expressed by the filtration of a stochastic process $\{Y_t\}$. A filtration is a set of σ-algebras $\{\Im_t\}$ such that

$\Im_s \subseteq \Im_t$, for $s \leq t$, and Y_t is measurable with respect to $\Im_t$. Let $\Im_t$ the σ-algebra generated by the random variables Y_s such that $s \leq t$, then the family $\{\Im_t\}$ is the smallest filtration associated to $\{Y_t\}$. Equation (2.8) says that if we consider the conditional expectation of $f(Y_{s+t_0}, \ldots, Y_{s+t_n})$ with respect to $\Im_s$, then it is equal to the conditional expectation of $f(Y_{s+t_0}, \ldots, Y_{s+t_n})$ with respect to Y_s:

$$\mathrm{E}[f(Y_{s+t_0}, \ldots, Y_{s+t_n})|\Im_s] = \mathrm{E}[f(Y_{s+t_0}, \ldots, Y_{s+t_n})|Y_s].$$

More in general, Lévy processes satisfy the strong Markov property, where it is present the notion of stopping time. A stopping time T is a mapping from Ω into $[0, \infty]$ such that $\{T \leq t\} \in \Im_t$ for every $t \in [0, \infty)$. From a stopping time T, we could define a σ-algebra $\Im_T$ and a random variable Y_T, and further prove

$$\mathrm{E}[f(Y_{T+t_0}, \ldots, Y_{T+t_n})|\Im_T] = \mathrm{E}[f(Y_{T+t_0}, \ldots, Y_{T+t_n})|Y_T]. \tag{2.9}$$

Equation (2.9) is called the strong Markov property, and we have the Markov property when T is equal to a constant time t.

2.3 Subordinators and subordinated Lévy processes: VG and NIG processes

A subordinator is a Lévy process whose paths are nondecreasing almost surely, thus it is a Lévy process of finite variation and satisfies Corollary 2.1. Subordinators are often applied in financial models, because they can be used as time changes for other Lévy processes.

Proposition 2.5. *Let $\{X_t : t \geq 0\}$ be a Lévy process on $\mathbb{R}$. The following conditions are equivalent:*

(i) $X_t \geq 0$ *almost surely for some* $t > 0$.

(ii) $X_t \geq 0$ *for every* $t > 0$.

(iii) *Sample paths of* $\{X_t\}$ *are almost surely nondecreasing:* $t \geq s \Rightarrow X_t \geq X_s$ *almost surely.*

(iv) *The characteristic triplet of* $\{X_t\}$ *satisfies* $A = 0$, $\nu((-\infty, 0]) = 0$, $\int_0^\infty (x \wedge 1)\nu(dx) < \infty$ *and* $b \geq 0$, *that is, has no diffusion components, only positive jumps of finite variation and positive drift.*

Proof. See Cont and Tankov [21], Proposition 3.10.

Let $\{S_t : t \geq 0\}$ be a subordinator on $\mathbb{R}$ with Lévy measure ρ and drift b. For any time t, S_t is a positive random variable and thus we can describe it by its Laplace transform:

$$L_{S_t}(u) = \mathrm{E}[e^{uS_t}] = e^{tl(u)}, \quad u \leq 0,$$

where

$$l(u) = bu + \int_0^\infty (e^{ux} - 1)\rho(dx). \tag{2.10}$$

The function $l(u)$ is called Laplace exponent of $\{S_t\}$. The next theorem justifies the use of a subordinator $\{S_t\}$ as time change of another Lévy process.

Theorem 2.4. *Fixed a probability space* $(\Omega, \Im, \mathrm{P})$. *Let* $\{S_t : t \geq 0\}$ *a subordinator with Lévy measure* ρ, *drift* b, *and Laplace exponent* $l(u)$, *and let* $\{X_t : t \geq 0\}$ *be a Lévy process on* $\mathbb{R}^d$ *with Lévy triplet* (A, ν, γ) *and characteristic exponent* $\psi(z)$. *Then the process* $\{Y_t : t \geq 0\}$ *defined for each* $\omega \in \Omega$ *by* $Y_t(\omega) = X_{S_t(\omega)}(\omega)$ *is a Lévy process and its characteristic function is*

$$\phi_{Y_t}(z) = \mathrm{E}[e^{i\langle z, Y_t\rangle}] = e^{tl(\psi(z))}, \quad z \in \mathbb{R}^d. \tag{2.11}$$

The Lévy triplet $(A^\#, \nu^\#, \gamma^\#)$ *of* $\{Y_t\}$ *is given by*

$$\begin{aligned} A^\# &= bA, \\ \nu^\#(B) &= b\nu(B) + \int_0^\infty \mathrm{P}_{X_s}(B)\rho(ds), \quad \forall B \in \mathcal{B}(\mathbb{R}^d), \\ \gamma^\# &= b\gamma + \int_0^\infty \rho(ds) \int_{|x|\leq 1} x\mathrm{P}_{X_s}(dx), \end{aligned}$$

where P_{X_t} *is the distribution of* X_t.

Proof. See Cont and Tankov [21], Theorem 4.2.

A way to find a subordinator is to define a Lévy triplet which satisfies the condition **(iv)** of Proposition 2.5. The tempered stable subordinator is defined assuming that its drift b is zero and its Lévy measure is

$$\rho(dx) = \frac{ce^{-\lambda x}}{x^{\alpha+1}} 1_{x>0} dx,$$

where c and λ are positive and $0 \leq \alpha < 1$. By equation (2.10) the Laplace exponent of the tempered stable subordinator is given by (see appendix B.1 for the case $\alpha \neq 0$)

$$\begin{cases} l(u) = c\Gamma(-\alpha)\{(\lambda - u)^{\alpha} - \lambda^{\alpha}\} & \text{if} \quad \alpha \neq 0 \\ l(u) = -c\log(1 - u/\lambda) & \text{if} \quad \alpha = 0. \end{cases} \tag{2.12}$$

For financial applications two important subordinators are the Gamma process, $\alpha = 0$, and the Inverse Gaussian process, $\alpha = 1/2$, which have probability density function in explicit form. If $\alpha = 0$ in (2.12), then we have the Gamma (G) process $\{X_t^{(G)} : t \geq 0\}$ with parameters $a > 0$ and $b > 0$, where $a = c$ and $b = \lambda$. The Laplace transform and probability density function of X_t^G are, respectively,

$$\begin{aligned} L_{X_t^{(G)}}(u) &= (1 - u/b)^{-at}, \quad u \leq 0. \\ f_{X_t^{(G)}}(x; a, b) &= \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-xb) 1_{x>0}. \end{aligned}$$

The Lévy triplet of the Gamma process $\{X_t^{(G)}\}$ is

$$\left[\frac{a(1 - \exp(-b))}{b}, \quad 0, \quad \frac{a\exp(-bx)}{x} 1_{x>0} dx \right].$$

Instead, if $\alpha = 1/2$ in (2.12), then we have the Inverse Gaussian (IG) process $\{X_t^{(IG)} : t \geq 0\}$ with parameters $a > 0$ and $b > 0$, where $a = c\sqrt{2\pi}$ and $b = \sqrt{2\lambda}$. The Laplace transform and probability density function of $X_t^{(IG)}$ are, respectively,

$$\begin{aligned} L_{X_t^{(IG)}}(u) &= \exp\left(-at\left(\sqrt{b^2 - 2u} - b\right)\right), \quad u \leq 0, \\ f_{X_t^{(IG)}}(x; a, b) &= \frac{ta}{x^{3/2}\sqrt{2\pi}} \exp(tab) \exp\left(-\frac{1}{2}\left((ta)^2 x^{-1} + b^2 x\right)\right) 1_{x>0}. \end{aligned}$$

The Lévy triplet of the Inverse Gamma process $\{X_t^{(IG)}\}$ is

$$\left[\frac{a}{b}(2N(b)-1), \quad 0, \quad \frac{a}{x^{3/2}\sqrt{2\pi}}\exp\left(-\frac{b^2}{2}x\right)1_{x>0}dx\right].$$

The celebrated Black & Scholes model [11] assumes that the underlying asset follows a geometric Brownian motion, that is

$$S_t = S_0 e^{X_t},$$

where $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$ is a Brownian motion with drift. Thus, the volatility σ, that is the standard deviation of the log return over a time unit, is constant. But, the volatility should depend on the number of transactions occurred during a time unit, that is it should be stochastic. A way to include this other element of randomness is just to model the asset log-return as a subordinated Brownian motion with drift:

$$X_t = \mu Z_t + \sigma W_{Z_t},$$

where $\{Z_t\}$ is a subordinator. In this way we obtain the stochastic volatility $\sigma\sqrt{Z_1}$. Two important subordinated Lévy processes in finance are the Variance Gamma and Normal Inverse Gaussian processes. The Variance Gamma (VG) process can be defined subordinating a Brownian motion with drift $\{X_t = \theta t + \sigma W_t\}$ by a gamma process $\{Z_t^{(G)}\}$ with parameters $a = 1/\nu > 0$ and $b = 1/\nu > 0$, where $\theta \in \mathbb{R}$ and $\sigma > 0$, and adding a drift term $\mu \in \mathbb{R}$:

$$X_t^{(VG)} = \mu t + \theta Z_t^{(G)} + \sigma W_{Z_t^{(G)}}.$$

By equation (2.11), the characteristic function of $X_t^{(VG)}$ is

$$\phi_{X_t^{(VG)}}(z; \sigma\sqrt{t}, \nu/t, \theta t, \mu t) = \left(1 - iz\theta\nu + \frac{1}{2}\sigma^2\nu z^2\right)^{-t/\nu} e^{iz\mu t}, \quad z \in \mathbb{R}.$$

The probability density function of $X_t^{(VG)}$ is known in explicit form and is given by

$$\begin{aligned} f_{X_t^{(VG)}}(x; \sigma\sqrt{t}, \nu/t, \theta t, \mu t) \quad &= \quad \frac{2e^{\frac{\theta(x-\mu t)}{\sigma^2}}\left(\frac{(x-\mu t)^2}{2\sigma^2/\nu+\theta^2}\right)^{\frac{t}{2\nu}-\frac{1}{4}}}{\nu^{t/\nu}\sqrt{2\pi}\sigma\Gamma(t/\nu)} \times \\ &\times \mathrm{K}_{\frac{t}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{(x-\mu t)^2(2\sigma^2/\nu+\theta^2)}\right), \end{aligned}$$

where $K_{\frac{t}{\nu}-\frac{1}{2}}(x)$ is the modified Bessel function of the third kind with index $\frac{t}{\nu}-\frac{1}{2}$ (see appendix B.2 for the definition of Bessel functions). It is possible to show that this density function is leptokurtic and with semi-heavy tails. In particular, at time $t = 1$ we have the following characteristics:

mean	$\theta + \mu$
variance	$\sigma^2 + \nu\theta^2$
skewness	$\theta\nu(3\sigma^2 + 2\nu\theta^2)/(\sigma^2 + \nu\theta^2)^{3/2}$
kurtosis	$3(1 + 2\nu - \nu\sigma^4(\sigma^2 + \nu\theta^2)^{-2})$.

In order to determine the Lévy triplet of the Variance Gamma process $\{X_t^{(VG)}\}$ is preferable to consider a different definition. Indeed, we can show (see Madan *et al.* [52]) that the Variance Gamma process is also equal to the difference of two independent Gamma process:

$$X_t^{(VG)} = \mu t + X_t^{(G^1)} - X_t^{(G^2)},$$

where $\{X_t^{(G^1)}\}$ is a Gamma process with parameters $a = C$ and $b = M$, and $\{X_t^{(G^2)}\}$ is an independent Gamma process with parameters $a = C$ and $b = G$. This new parametrization is related to that one above by

$$\begin{cases} C = 1/\nu > 0, \\ G = \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\theta\nu\right)^{-1} > 0, \\ M = \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} + \frac{1}{2}\theta\nu\right)^{-1} > 0. \end{cases}$$

With this second definition the Lévy triplet is immediately given by $[\gamma, 0, \nu_{VG}(dx)]$, where

$$\begin{aligned} \nu_{VG}(dx) &= \begin{cases} C\exp(Gx)|x|^{-1}dx, & x < 0, \\ C\exp(-Mx)x^{-1}dx, & x > 0, \end{cases} \qquad (2.13) \\ \gamma &= \mu + \frac{-C(G(\exp(-M)-1) - M(\exp(-G)-1))}{MG}. \end{aligned}$$

The Normal Inverse Gaussian (NIG) process is defined subordinating the Brownian motion with drift $\{X_t = \beta\delta^2 t + \delta W_t\}$ by an Inverse Gaussian process $\{Z_t^{(IG)}\}$

with parameters $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$, where $\alpha > 0$, $-\alpha < \beta < \alpha$ and $\delta > 0$, and adding a drift term $\mu \in \mathbb{R}$:

$$X_t^{(NIG)} = \mu t + \beta\delta^2 Z_t^{(IG)} + \delta W_{Z_t^{(IG)}}.$$

By equation (2.11), the characteristic function of $X_t^{(NIG)}$ is

$$\phi_{X_t^{(NIG)}}(z; \alpha, \beta, t\delta, \mu t) = \exp\left(-t\delta\left(\sqrt{\alpha^2 - (\beta + iz)^2} - \sqrt{\alpha^2 - \beta^2}\right)\right) e^{iz\mu t}.$$

The Lévy triplet $[\gamma, 0, \nu_{NIG}]$ is computed using Theorem 2.4, and we have

$$\begin{aligned} \gamma &= \mu + \frac{2\delta\alpha}{\pi}\int_0^1 \sinh(\beta x)\mathrm{K}_1(\alpha x)dx, \\ \nu_{NIG}(dx) &= \frac{\delta\alpha}{\pi}\frac{\exp(\beta x)\mathrm{K}_1(\alpha|x|)}{|x|}dx, \end{aligned}$$

where $\mathrm{K}_1(x)$ is the modified Bessel function of the third kind with index 1. Then, the probability density function of $X_t^{(NIG)}$ is known in explicit form and is given by

$$\begin{aligned} f_{X_t^{(NIG)}}(x; \alpha, \beta, t\delta, \mu t) &= \frac{\alpha t\delta}{\pi}\exp\left(t\delta\sqrt{\alpha^2 - \beta^2} + \beta(x - \mu t)\right) \times \\ &\quad \times \frac{\mathrm{K}_1\left(\alpha\sqrt{(t\delta)^2 + (x - \mu t)^2}\right)}{\sqrt{(t\delta)^2 + (x - \mu t)^2}}. \end{aligned}$$

Thus, We have a leptokurtic density function with semi-heavy tails, since

$$f_{X_t^{(NIG)}}(x; \alpha, \beta, t\delta, \mu t) \sim |x|^{-3/2}\exp((\mp\alpha + \beta)x) \quad \text{as} \quad x \to \pm\infty,$$

up to a multiplicative constant, and at time $t = 1$

mean	$\delta\beta/\sqrt{\alpha^2 - \beta^2} + \mu$
variance	$\alpha^2\delta(\alpha^2 - \beta^2)^{-3/2}$
skewness	$3\beta\alpha^{-1}\delta^{-1/2}(\alpha^2 - \beta^2)^{-1/4}$
kurtosis	$3\left(1 + \frac{\alpha^2 + 4\beta^2}{\delta\alpha^2\sqrt{\alpha^2 - \beta^2}}\right).$

The Variance Gamma process was introduced by Madan and Seneta [51] which considered the symmetric case $\theta = 0$, then Madan *et al.* [52] studied the general case with skewness. Instead, the Normal Inverse Gaussian process was introduced by Barndorff-Nielsen [6] as model of asset log returns.

2.4 Generalization of the VG and NIG processes: CGMY and GH processes

In this section we present other two Lévy processes which generalize the VG and NIG processes. The first one is the CGMY process introduced by Carr, Geman, Madan, and Yor [15], and the second one is the Generalized Hyperbolic (GH) process introduced by Bandorff-Nielsen [5] and then studied by Eberlein and Prause [29] and Prause [68] as model to describe asset log returns.

The VG Lévy density in (2.13) can be generalize to the CGMY Lévy density as follows

$$k_{CGMY}(x) = \begin{cases} C\exp(Gx)(-x)^{-1-Y} & x < 0 \\ C\exp(-Mx)x^{-1-Y} & x > 0, \end{cases}$$

where $C, G, M > 0$ and $Y < 2$. The condition $Y < 2$ allows to integrate x^2 in the neighborhood of 0. Thus, the CGMY process reduces to a VG process when $Y = 0$. Denoting by $\{X_t^{(CGMY)} : t \geq 0\}$ the CGMY process, then the characteristic function at time t is given by

$$\phi_{X_t^{(CGMY)}}(z; tC, G, M, Y) = \exp\left\{tC\Gamma(-Y)[(M - iz)^Y - M^Y + (G + iz)^Y - G^Y]\right\},$$

and the first term of the Lévy triplet is

$$\gamma = C\left(\int_0^1 \exp(-Mx)x^{-Y}dx - \int_{-1}^0 \exp(Gx)|x|^{-Y}dx\right). \qquad (2.14)$$

Considering that for a general Lévy density $k(x)$, the random variable X representing the level of a Lévy process at time $t = 1$ satisfies

$$\begin{aligned} \mathrm{E}[X] &= \int_{-\infty}^{\infty} xk(x)dx \\ \mathrm{E}[(X - \mathrm{E}[X])^2] &= \int_{-\infty}^{\infty} x^2k(x)dx \\ \mathrm{E}[(X - \mathrm{E}[X])^3] &= \int_{-\infty}^{\infty} x^3k(x)dx \\ \mathrm{E}[(X - \mathrm{E}[X])^4] &= 3\left(\mathrm{E}[(X - \mathrm{E}[X])^2]\right)^2 + \int_{-\infty}^{\infty} x^4k(x)dx, \end{aligned}$$

then the following characteristic of the CGMY process at time $t = 1$ can be easily computed

mean	$C(M^{Y-1} - G^{Y-1})\Gamma(1-Y)$
variance	$C(M^{Y-2} + G^{Y-2})\Gamma(2-Y)$
skewness	$\frac{C(M^{Y-3}-G^{Y-3})\Gamma(3-Y)}{(C(M^{Y-2}+G^{Y-2})\Gamma(2-Y))^{3/2}}$
kurtosis	$3 + \frac{C(M^{Y-4}+G^{Y-4})\Gamma(4-Y)}{(C(M^{Y-2}+G^{Y-2})\Gamma(2-Y))^{2}}.$

The next result highlights the important role of the parameter Y which controls the path behaviour.

Theorem 2.5. *The CGMY process*

1. *has completely monotone Lévy density for* $Y > -1$*;*

2. *is a process of infinite activity for* $Y > 0$*;*

3. *is a process of infinite variation for* $Y > 1$*.*

Proof. See Carr *et al.* [15], Theorem 2.

The Inverse Gaussian process $\{X_t^{(IG)} : t \geq 0\}$ can be generalized to the Generalized Inverse Gaussian (GIG) process $\{X_t^{(GIG)} : t \geq 0\}$ adding a parameter $\lambda \in \mathbb{R}$ and defining the density function at time $t = 1$ as

$$f_{X_1^{(GIG)}}(x; \lambda, a, b) = \frac{(b/a)^\lambda}{2K_\lambda(ab)} x^{\lambda-1} \exp\left(-\frac{1}{2}(a^2x^{-1} + b^2x)\right) 1_{x>0},$$

where $a, b \geq 0$ and not simultaneously 0, $\lambda \in \mathbb{R}$, and $K_\lambda(x)$ is the modified Bessel function of the third kind with index λ. Considering that $K_{-1/2}(x) = \sqrt{\pi/2}x^{-1/2}\exp(-x)$, the GIG process reduces to the IG process for $\lambda = -1/2$ and $a, b > 0$. The characteristic function is

$$\phi_{X_1^{(GIG)}}(z; \lambda, a, b) = \frac{1}{K_\lambda(ab)}(1 - 2iz/b^2)^{\lambda/2} K_\lambda(ab\sqrt{1 - 2izb^{-2}}).$$

Subordinating the Brownian motion with drift $\{X_t = \beta t + W_t\}$ by a GIG process $\{Z_t^{(GIG)}\}$ with parameters $\lambda = \upsilon$, $a = \delta$, and $b = \sqrt{\alpha^2 - \beta^2}$ we have the

Generalized Hyperbolic (GH) process $\{X_t^{(GH)} : t \geq 0\}$:

$$X_t^{(GH)} = \beta Z_t^{(GIG)} + W_{Z_t^{(GIG)}}.$$

The characteristic function and density function at time $t = 1$ are, respectively,

$$\phi_{X_1^{GH}}(z; \alpha, \beta, \delta, \upsilon) = \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iz)^2} \right)^{\upsilon/2} \frac{K_\upsilon \delta \sqrt{\alpha^2 - (\beta + iz)^2}}{K_\upsilon(\delta \sqrt{\alpha^2 - \beta^2})},$$

$$\begin{aligned} f_{X_1^{(GH)}}(x; \alpha, \beta, \delta, \upsilon) &= a(\alpha, \beta, \delta, \upsilon)(\delta^2 + x^2)^{(\upsilon - 1/2)/2} K_{\upsilon - 1/2}(\alpha \sqrt{\delta^2 + x^2}) \exp(\beta x), \\ a(\alpha, \beta, \delta, \upsilon) &= \frac{(\alpha^2 - \beta^2)^{\upsilon/2}}{\sqrt{2\pi} \alpha^{\upsilon - 1/2} \delta^\upsilon K_\upsilon(\delta \sqrt{\alpha^2 - \beta^2})}, \end{aligned}$$

where

$$\begin{aligned} \delta \geq 0 \quad |\beta| < \alpha \quad &\text{if} \quad \upsilon > 0, \\ \delta > 0 \quad |\beta| < \alpha \quad &\text{if} \quad \upsilon = 0, \\ \delta > 0 \quad |\beta| \leq \alpha \quad &\text{if} \quad \upsilon < 0. \end{aligned}$$

The Normal Inverse Gaussian process is a special case of the GH process for $\upsilon = -1/2$.

2.5 Meixner process

In this section we briefly describe the Meixner process which will be used in our applications together with the NIG and VG processes.

The density function of the Meixner distribution, Meixner$(\mu, \alpha, \beta, \delta)$, is defined as

$$f_{\text{Meixner}}(x; \alpha, \beta, \delta, \mu) = \frac{(2\cos(\beta/2))^{2\delta}}{2\alpha\pi\Gamma(2\delta)} \exp\left(\frac{\beta(x - \mu)}{\alpha} \right) \left| \Gamma\left(\delta + \frac{i(x - \mu)}{\alpha} \right) \right|^2,$$

where $\mu \in \mathbb{R}$, $\alpha > 0$, $-\pi < \beta < \pi$, and $\delta > 0$. This is a leptokurtic density function with semi-heavy tails, given that

$$f_{\text{Meixner}}(x; \alpha, \beta, \delta, \mu) \sim \begin{cases} C_- |x|^{\rho_-} \exp(-\eta_- |x|) & \text{as} \quad x \to -\infty \\ C_+ |x|^{\rho_+} \exp(-\eta_+ |x|) & \text{as} \quad x \to +\infty. \end{cases}$$

where

$$\rho_- = \rho_+ = 2\delta - 1, \quad \eta_- = (pi - \beta)/\alpha \quad \eta_+ = (\pi + \beta)/\alpha,$$

and for some $C_-, C_+ \geq 0$, and that

mean	$\alpha\delta\tan(\beta/2) + \mu$
variance	$\frac{1}{2}\alpha^2\delta(\cos^{-2}(\beta/2))$
skewness	$\sin(\beta/2)\sqrt{2/\delta}$
kurtosis	$3 + (2 - \cos(\beta))/\delta.$

The Meixner$(\mu, \alpha, \beta, \delta)$ distribution has characteristic function given by

$$\phi_{\text{Meixner}}(z; \alpha, \beta, \delta, \mu) = \left(\frac{\cos(\beta/2)}{\cosh((\alpha z - i\beta)/2)}\right)^{2\delta} e^{iz\mu}. \tag{2.15}$$

The characteristic function (2.15) is infinitely divisible and thus we can define the Meixner process $\{X_t^{(\text{Meixner})} : t \geq 0\}$ as the stochastic process which starts at zero, that is $X_0^{(\text{Meixner})} = 0$ a.s., has independent and stationary increments, and so that $X_t^{(\text{Meixner})}$ has Meixner$(\alpha, \beta, \delta t, \mu t)$ distribution. The Lévy triplet is $[\gamma, 0, \nu_{\text{Meiner}}(dx)]$ where

$$\begin{aligned} \nu_{\text{Meixner}}(dx) &= \delta\frac{\exp(\beta x/\alpha)}{x\sinh(\pi x/\alpha)}dx, \\ \gamma &= \mu + \alpha\delta\tan(\beta/2) - 2\delta\int_1^\infty \frac{\sinh(\beta x/\alpha)}{\sinh(\pi x/\alpha)}dx. \end{aligned}$$

The Meixner process was introduced by Schoutens and Teugels [77], and then Schoutens [74], [75] applied this stochastic process to describe the random behaviour of asset prices.

2.6 Market model and equivalent martingale measure

Given a probability space $(\Omega, \Im, \mathrm{P})$, consider a market where a risky asset with price process $\{S_t : t \geq 0\}$ and a bank account $dB(t) = rB(t)dt$, with $r > 0$, are defined. Let $\{\Im_t : t \geq 0\}$ be the filtration generated by $\{S_t\}$, that is $\Im_t$ represents the information obtained observing the risky asset from 0 to t. Any contingent claim with maturity T defined on this market can be represented by its terminal payoff $H : \Omega \to \mathbb{R}$, which, depending on the price process $\{S_t\}$ up to T, has to be $\Im_T$-measurable. In order to determine the arbitrage-free price process $\{\Pi_t(H) : 0 \leq t \leq T\}$ of the contingent claim H, we recall the risk-neutral pricing (see Cont and Tankov [21], Proposition 9.1):

$$\Pi_t(H) = e^{-r(T-t)}\mathrm{E}^{\widetilde{\mathrm{P}}}[H|\Im_t], \tag{2.16}$$

where $\widetilde{\mathrm{P}}$ is an equivalent martingale measure. Therefore, the measure $\widetilde{\mathrm{P}}$ has to guarantee the equivalence with respect to P, that is $\widetilde{\mathrm{P}}(A) = 0$ if and only if $\mathrm{P}(A) = 0$ for each $A \in \Im$, and to imply

$$\widehat{S}_t = \mathrm{E}^{\widetilde{\mathrm{P}}}[\widehat{S}_T|\Im_t],$$

where $\{\widehat{S}_t = e^{-rt}S_t : t \geq 0\}$ is the discounted price process of the risky asset. If we are able to find a measure $\widetilde{\mathrm{P}}$ on $\Im$ with these properties, then we are sure that the formula (2.16) returns prices which do not generate arbitrage, that is there does not exist a self-financing strategy Θ with no intermediate losses and positive terminal gain:

$$\mathrm{P}[V_t(\Theta) \geq 0, \forall t \in [0,T]] = 1, \quad \mathrm{P}[V_T(\Theta) > V_0(\Theta)] \neq 0.$$

A strategy is a portfolio represented by a predictable (that is left continuous and with right limits) process $\Theta = (\Theta_t^S, \Theta_t^B)$, where Θ_t^S and Θ_t^B are the hold quantities of the risky and risk-less assets, respectively, at time t. Moreover, the strategy Θ is called self-financing if it does admit the possibility to add or withdraw money. The value at time T of the strategy Θ is given by the stochastic integral

$$V_t(\Theta) = V_0(\Theta) + \int_0^t \Theta_u^B dB(u) + \int_0^t \Theta_u^S dS_u,$$

where $V_0(\Theta) = \Theta_0^B + \Theta_0^S S_0$. The market that we have only just described can be either complete or incomplete. It is called complete if for any contingent claim H

with maturity T there exists a self-financing strategy Θ such that $V_T(\Theta) = H$ P-a.s., incomplete otherwise. The completeness would be an important property of the market because it would imply the existence of an unique equivalent martingale measure, however, there are only a few mathematical models satisfying this condition, for example the Black & Scholes model. In particular, the completeness of the market is related to the ability of the mathematical model to imply the predictable representation property (see Cont and Tankov [21], Remark 9.1, or Schoutens [76], Section 2.5), and it is showed that for all non-Gaussian Lévy processes, except the compensated Poisson process, this property fails.

In our applications the price process $\{S_t\}$ of the risk asset is an exponential-Lévy model, that is

$$S_t = S_0 e^{X_t},$$

where $\{X_t\}$ is a Lévy process with Lévy triplet $(\gamma_X, \sigma_X^2, \nu_X)$. Now, the Lévy process $\{X_t\}$ can be seen as a random variable on the scenario space $(\Omega, \Im)$ whose distribution defines a probability measure P^X on $(\Omega, \Im)$. So, the real world has the representation $(\Omega, \Im, \{\Im_t\}, \mathrm{P}^X)$, where $\{\Im_t\}$ is the history of the assets up to t. Given a contingent claim H, we know that a way to determine its price process $\{\Pi_t(H)\}$ is to define an equivalent martingale measure on $(\Omega, \Im)$. Under this measure, the asset $\{S_t\}$ will be driven by a Lévy process $\{Y_t\}$, different from $\{X_t\}$, with characteristic triplet $(\gamma_Y, \sigma_Y^2, \nu_Y)$. Therefore, we obtain an equivalent martingale measure if we are able to define a Lévy process Y_t on $(\Omega, \Im)$ so that its distribution P^Y is equivalent to P^X and the process $\{\widehat{S}_t = S_t/B_t\}$, where now $S_t = S_0 e^{Y_t}$, is a martingale under P^Y. The next proposition gives the conditions which assure the equivalence between P^X and P^Y.

Proposition 2.6. *Let $(\{X_t\}, \mathrm{P})$ and $(\{X_t\}, \mathrm{P}')$ be two Lévy processes on $\mathbb{R}$ with characteristic triplets (γ, σ^2, ν) and $(\gamma', \sigma'^2, \nu')$. Then $\mathrm{P}|_{\Im_t}$ and $\mathrm{P}'|_{\Im_t}$ are equivalent for all t if and only if the following conditions are satisfied:*

1. *$\sigma = \sigma'$.*

2. *The Lévy measures are equivalent with*

$$\int_{-\infty}^{\infty}(e^{\phi(x)/2}-1)^2\nu(dx)<\infty,$$

where $\phi(x)=\ln\left(\frac{d\nu'}{d\nu}\right)$.

3. *If* $\sigma=0$ *then we must in addition have*

$$\gamma-\gamma'=\int_{-1}^{1}x(\nu-\nu')(dx).$$

When P *and* P′ *are equivalent, the Radon-Nykodim derivative is*

$$\frac{dP'|_{\Im_t}}{dP|_{\Im_t}}=e^{U_t}$$

with

$$\begin{aligned} U_t &= \eta X_t^c-\frac{\eta^2\sigma^2 t}{2}-\eta\gamma t \\ &\quad +\lim_{\epsilon\downarrow 0}\left(\sum_{s\le t,|\Delta X_s|>\epsilon}\phi(\Delta X_s)-t\int_{|x|>\epsilon}(e^{\phi(x)}-1)\nu(dx)\right). \end{aligned} \tag{2.17}$$

Here $\{X_t^c\}$ *is the continuous part of* $\{X_t\}$ *and* η *is such that*

$$\gamma'-\gamma-\int_{-1}^{1}x(\nu'-\nu)(dx)=\sigma^2\eta$$

if $\sigma>0$ *and zero if* $\sigma=0$.

U_t *is a Lévy process with characteristic triplet* $(\gamma_U,\sigma_U^2,\nu_U)$ *given by*

$$\begin{aligned} &\sigma_U^2=\sigma^2\eta^2, \\ &\nu_U=\nu\phi^{-1}, \\ &\gamma_U=-\tfrac{1}{2}\sigma^2\eta^2-\textstyle\int_{-\infty}^{\infty}(e^y-1-y1_{|y|\le 1})(\nu\phi^{-1})(dy). \end{aligned}$$

Proof. See Sato [73], Theorem 33.1.

When the Lévy process $\{X_t\}$, specified in our model, has the diffusion component σ_X^2, then a way to find a Lévy process $\{Y_t\}$, whose distribution P^Y is equivalent to P^X and guarantees the martingale property of $\{\widehat{S}_t\}$, is to change the drift component to $\{X_t\}$. Indeed, if $\mathrm{E}[e^{-rt+X_t}] = 1\ \forall t \geq 0$ then the process $\{\widehat{S}_t\}$ is a martingale, because, from the independence and stationarity of the increments,

$$\begin{aligned}
\mathrm{E}[\widehat{S}_t|\Im_s] &= S_0\mathrm{E}[e^{-rt+X_t}|\Im_s] \\
&= S_0\mathrm{E}\left[e^{-rs-r(t-s)+X_s+(X_t-X_s)}|\Im_s\right] \\
&= S_0 e^{-rs+X_s}\mathrm{E}\left[e^{-r(t-s)+(X_t-X_s)}\right] \\
&= S_0 e^{-rs+X_s}.
\end{aligned}$$

Now, by the Lévy-Khintchine representation of the characteristic function, the condition $\mathrm{E}[e^{-rt+X_t}] = 1\ \forall t \geq 0$ is satisfied if $\psi(-i) = 0$, where $\psi(z)$ is given by

$$\psi(z) = -\frac{1}{2}\sigma_X^2 z^2 + i(\gamma_X - r)z + \int_{-\infty}^{\infty}(e^{izx} - 1 - izx1_{|x|\leq 1})\nu_X(dx).$$

Then we construct a Lévy process $\{Y_t\}$ which implies the martingale property of $\{\widehat{S}_t = S_0 e^{-rt+Y_t}\}$, if we define $\{Y_t\}$ by the Lévy triplet $(\gamma_Y, \sigma_Y^2, \nu_Y)$, where $\sigma_Y^2 = \sigma_X^2$, $\nu_Y = \nu_X$ and

$$\gamma_Y = r - \frac{1}{2}\sigma_X^2 - \int_{-\infty}^{\infty}(e^x - 1 - x1_{|x|\leq 1})\nu_X(dx).$$

When we use a stochastic process $\{X_t^{(I)}\}$, $I = VG, NIG, CGMY, GH,$ Meixner, to model asset log returns, then the process $\{Y_t\}$, obtained by drift change, is still a stochastic process of the same type of I and with the same parameters of $\{X_t^{(I)}\}$, but the drift parameter μ which becomes

$$\begin{cases}
\mu = r + \frac{1}{\nu}\log\left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right) & \text{if} \quad I = VG, \\
\mu = r + \delta\left(\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2}\right) & \text{if} \quad I = NIG, \\
\mu = r - C\Gamma(-Y)((M-1)^Y - M^Y + (G+1)^Y - G^Y) & \text{if} \quad I = CGMY, \\
\mu = r - \log\left(\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+1)^2}\right)^{\upsilon/2}\frac{K_\upsilon(\delta\sqrt{\alpha^2-(\beta+1)^2})}{K_\upsilon(\delta\sqrt{\alpha^2-\beta^2})}\right) & \text{if} \quad I = GH, \\
\mu = r - 2\delta(\log(\cos(\beta/2)) - \log(\cos((\alpha+\beta)/2))) & \text{if} \quad I = \text{Meixner}.
\end{cases} \tag{2.18}$$

Observe that when the Lévy measure ν_X is zero, thus our Lévy process is simply a Brownian motion with drift, we find the drift change usually applied in the

Black & Scholes model. The probability measure obtained by a drift change is generally said mean-correcting martingale measure.

Another way to find an equivalent martingale measure is to use the Esscher transform (see Gerber and Shiu [35], [36]). If $f_t(x)$ is the density function of X_t and $\int_{-\infty}^{\infty} e^{\theta y} f_t(y)dy < \infty$ for some real number θ we can define a new density as

$$f_t^{(\theta)}(x) = \frac{\exp(\theta x) f_t(x)}{\int_{-\infty}^{\infty} \exp(\theta y) f_t(y)dy}.$$

The Esscher transform consists into select θ such that the discounted price process $\{\widehat{S}_t = S_t e^{-rt}\}$ is a martingale, that is

$$S_0 = e^{-rt}\mathrm{E}^{(\theta)}[S_t], \quad \forall t \geq 0, \tag{2.19}$$

where the expectation is taken with respect to the law with density $f_t^{(\theta)}(x)$. Always from the Lévy-Khintchine representation, the condition (2.19) is satisfied if

$$r = \psi(-i(\theta+1)) - \psi(-i\theta), \tag{2.20}$$

where $\psi(z)$ is given by

$$\psi(z) = -\frac{1}{2}\sigma_X^2 z^2 + i\gamma_X z + \int_{-\infty}^{\infty} (e^{izx} - 1 - izx1_{|x|\leq 1})\nu_X(dx).$$

The solution θ to the equation (2.20) gives the Esscher transform martingale measure by the density function $f_t^{(\theta)}(x)$. If $\phi(z)$ is the characteristic function of X_1 then $\phi^{(\theta)}(z) = \phi(z - i\theta)/\phi(-i\theta)$ is the characteristic function of the Esscher transform and is infinitely divisible. Thus, there is a Lévy process Y_t under which the process $\{\widehat{S}_t = S_0 e^{-rt+Y_t}\}$ is a martingale and whose Lévy triplet $(\sigma_Y^2, \nu_Y, \gamma_Y)$ is given by

$$\begin{aligned}
\sigma_Y^2 &= \sigma_X^2, \\
\nu_Y(dx) &= e^{\theta x}\nu_X(dx), \\
\gamma_Y &= \gamma_X + \sigma_X^2\theta + \int_{-1}^{1} x(e^{\theta x} - 1)\nu_X(dx).
\end{aligned}$$

The Lévy process $\{Y_t\}$ and its distribution P^Y satisfy the conditions of Proposition 2.6, in particular we see that, when the diffusion component σ_X equals zero, the condition 3 is guaranteed. In the Black & Scholes model we have $\nu_X(dx) = 0$ and $\theta = (r - \frac{1}{2}\sigma_X^2 - \gamma_X)/\sigma_X^2$, so we have the new drift $\gamma_Y = r - \frac{1}{2}\sigma_X^2$ and the process $\{Y_t\}$ is the same given by the mean-correcting martingale measure. This equality was expected, because the completeness of the Black & Scholes model implies the existence of an unique equivalent martingale measure.

Another possible way to choose an equivalent martingale measure is to find that one which minimizes the relative entropy. Indeed, the relative entropy represents a measure of the distance between two equivalent probability measure, and thus a selection criteria could be to choose the equivalent martingale measure $\widetilde{\mathrm{P}}$ nearer to P. Important studies on this methodology are due to Csiszar [24], Stutzer [81], Miyahara [61], and Frittelli [32], in particular Frittelli gives sufficient conditions for the existence of a unique equivalent martingale measure minimizing the relative entropy and shows the equivalence between the maximization of expected exponential utility and the minimization of the relative entropy. Let $(\Omega, \Im)$ be the space of real-valued cadlag functions defined on $[0, T]$, $\Im_t$ the history of paths up to t, and P and $\widetilde{\mathrm{P}}$ two equivalent probability measures. The relative entropy between P and $\widetilde{\mathrm{P}}$ is defined as

$$I(\widetilde{\mathrm{P}}, \mathrm{P}) = \mathrm{E}^{\widetilde{\mathrm{P}}}\left[\ln \frac{d\widetilde{\mathrm{P}}}{d\mathrm{P}}\right] = \mathrm{E}^{\mathrm{P}}\left[\frac{d\widetilde{\mathrm{P}}}{d\mathrm{P}} \ln \frac{d\widetilde{\mathrm{P}}}{d\mathrm{P}}\right]$$

and, introducing the strictly convex function $f(x) = x\ln(x)$,

$$I(\widetilde{\mathrm{P}}, \mathrm{P}) = \mathrm{E}^{\mathrm{P}}\left[f\left(\frac{d\widetilde{\mathrm{P}}}{d\mathrm{P}}\right)\right].$$

It is possible to observe that the functional $\widetilde{\mathrm{P}} \to I(\widetilde{\mathrm{P}}, \mathrm{P})$ is strictly convex and that, for any $\widetilde{\mathrm{P}}$, $I(\widetilde{\mathrm{P}}, \mathrm{P}) \geq 0$ and $I(\widetilde{\mathrm{P}}, \mathrm{P}) = 0$ if and only if $\widetilde{\mathrm{P}} = \mathrm{P}$. Given a stochastic model $\{S_t : t \in [0, T]\}$, the minimal entropy martingale model is defined as the martingale $\{S_t^* : t \in [0, T]\}$ whose law P^* minimizes the relative entropy with respect to the law P of $\{S_t\}$. Under an exponential-Lévy model, there exists an analytic criterion for the existence of the minimal entropy martingale measure and it is possible to compute it explicitly. Furthermore, the minimal entropy martingale is still an exponential-Lévy model.

Proposition 2.7. *if $S_t = S_0 \exp(rt+X_t)$ where $\{X_t : t \in [0,T]\}$ is a Lévy process with Lévy triplet (σ^2, ν, b). If there exists a solution $\beta \in \mathbb{R}$ to the equation:*

$$b + (\beta + \tfrac{1}{2})\sigma^2 + \int_{-1}^{1} \nu(dx)[(e^x - 1)e^{\beta(e^x-1)} - x]$$
$$+ \int_{|x|>1} (e^x - 1)e^{\beta(e^x-1)}\nu(dx) = 0.$$

Then, the minimal entropy martingale S_t^* is also an exponential-Lévy process $S_t^* = S_0 \exp(rt + X_t^*)$ where $\{X_t^* :\in \mathbb{R}\}$ is a Lévy process with Lévy triplet (σ^2, ν^*, b^*) given by:

$$\begin{aligned} b^* &= b + \beta\sigma^2 + \int_{-1}^{1} \nu(dx)[xe^{\beta(e^x-1)} - x], \\ \nu^*(dx) &= \exp[\beta(e^x - 1)]\nu(dx). \end{aligned}$$

Proof. See Fujiwara and Miyahara [33], Theorem 3.1.

Chapter 3

Option pricing under Lévy processes

Option pricing under Lévy processes is a task very difficult, above all when the payoff of the option depends on the path of the price process of the underlying asset. Examples are lookback and barrier options which were studied (see Boyarchenko and Levendorskiĭ [13]) using the Wiener-Hopf decomposition and analytic techniques. In the case of European path dependent options we can easily determine the price simulating the exponential Lévy process. Appendix A gives a description of some techniques of simulation. Instead, when the option is American, then the simulation is not so straightforward because it is necessary to compute a conditional expectation and thus to use the least squares Monte Carlo method. In this chapter we propose a simple method to price American, compound, barrier, and lookback options assuming exponential-Lévy models for the underlying. The method is based on the idea that, given a Markov process, we can construct a sequence of Markov chains converging weakly to the Markov process. This idea was applied by Duan and Simonato [26] and Duan *et al.* [27] to approximate Wiener processes and GARCH processes with Gaussian residuals in order to price American and barrier options. While, Amin [3] used this idea to approximate jump-diffusion processes and price European and American options. Our method is equal to that one developed by Duan and Simonato [26] and Duan *et al.* [27], we just propose a method to compute the transition probabilities of the Markov chain. This discretization process presents the same advantages of the binomial model since permits us to price path dependent contingent claims. Thus, we first observe the convergence of compound option prices in the case analyzed by Geske [37] for the Brownian motion and then we extend the same analysis to the other three Lévy processes. While for American and barrier options we just apply Duan et al.'s method to Lévy processes, for lookback options we explain a method which allows to price these options in a Markov chain framework. Recall that in the Black and Scholes framework there is an analytical pricing formula for lookback options derived by Goldman *et al.* [38] and extended by Conze and Viswanathan [22]. However, in lookback contracts the maximum and/or the minimum of the underlying asset price are computed over some prespecified dates only, such as daily, weekly or monthly. In this sense the analytical continuous time models fail to predict the right price that in many cases is completely different (see Cheuk and Vorst [18]).

The chapter is organized as follows. Section 1 is an overview of the main methods used to price European options under Lévy processes. European options can be computed easily when the density function of the underlying asset is known in explicit form, but, for many Lévy processes, only the characteristic function is known and, thus, an alternative method could be one based on the Fourier transform. Moreover, when the Lévy triplet satisfies some conditions, we could apply the stochastic calculus to determine a partial integro-differential equation and thus to use a finite difference method. In Section 2 we explain how to build a sequence of Markov chains converging weakly to a Lévy process. In particular, we consider the Lévy process $\{X_t\}$ at discrete times $\{n\Delta t : n = 0, 1, \ldots\}$ and define a sequence $\{Y^{(m)}_{n\Delta t} : n = 0, 1, \ldots\}$ of Markov chains so that $Y^{(m)}_{n\Delta t}$ converges in distribution to $X_{n\Delta t}$ as m tends to ∞. In section 3 we introduce the markovian approach to price options when the underlying asset follows a Lévy process. In particular, we show how to use the markovian approach to price European and American options, and then to price three type of exotic options: compound, barrier, and lookback options.

3.1 Review of European option pricing methods

A European call option with maturity T, strike price K, and written on an asset with price process $\{S_t\}$ gives the holder the right to buy the asset at date T for the fixed price K. Since the holder at the date T can immediately sell the asset at its price S_T, the option can be seen as a contingent claim with payoff $H(S_T) = (S_T - K)^+$ at date T. Instead, a European option is called put when the holder can sell the asset at date T for the fixed price K, and in this case the terminal payoff is $H(S_T) = (K - S_T)^+$. The price at time t of a European call option with maturity T and strike price K is denoted by $C_t(T, K)$ and, using arbitrage arguments, we can show the so-called call-put parity:

$$C_t(T, K) - P_t(T, K) = S_t - Ke^{-r(T-t)},$$

where $P_t(T, K)$ is the price at time t of a European put on the same asset and with the same maturity and strike price.

In this section we explain three possible methods in order to price European options under Lévy processes. The first one is based on the density function at time T of the Lévy process and, thus, it can be applied only when we know in explicit form the density function. The second one is more general and uses the characteristic function of the Lévy process, which is known in explicit form for any Lévy process. Finally, the last one can be used only when the price process of the underlying asset admits second moment and consists in to solve numerically a partial integro-differential equation.

Density function method

Under an exponential-Lévy model, the risk-neutral dynamics of the asset satisfies

$$S_t = S_0 e^{X_t},$$

where $\{X_t\}$ is a Lévy process with triplet (γ, σ^2, ν). Suppose that, under the risk-neutral probability $\widetilde{\mathrm{P}}$, the density function of X_t is known in explicit form and given by $f_{\widetilde{\mathrm{P}}}(x,t)$, and consider a European call option with maturity T and strike price K. Then, its price $C_0(T,K)$ at time $t=0$ is given by

$$\begin{aligned} C_0(T,K) &= \mathrm{E}^{\widetilde{\mathrm{P}}}[e^{-rT}(S_T - K)^+] \\ &= e^{-rT}\int_{-\infty}^{\infty}(S_0e^x - K)^+ f_{\widetilde{\mathrm{P}}}(x,T)dx \\ &= e^{-rT}\int_{\log(S_0/K)}^{\infty}(S_0e^x - K) f_{\widetilde{\mathrm{P}}}(x,T)dx \\ &= e^{-rT}S_0\int_{\log(S_0/K)}^{\infty} e^x f_{\widetilde{\mathrm{P}}}(x,T)dx - e^{-rT}K\widetilde{\mathrm{P}}[S_T > K]. \end{aligned} \tag{3.1}$$

Formula (3.1) can be applied when we model the asset log-return by one of the NIG, VG, and Meixner processes.

Characteristic function method (Carr and Madan (1998))

This method was developed by Carr and Madan [14], which were able to determine an analytic expression of the Fourier transform of the option price and, thus, to propose the Fast Fourier Transform as procedure to establish the option price.

Consider a European call with maturity T and strike price $K = e^k$ and written on asset with price process $\{S_t\}$. Let the risk-neutral density of the log price $s_T = \log(S_T)$ be $q_T(s)$, then the characteristic function of s_T is given by

$$\phi_T(u) = \int_{\infty}^{\infty} e^{ius} q_T(u).$$

In this setting, the price $C_0(T,K)$, denoted by $C_T(k)$, of the European call at time $t=0$ is given by

$$C_T(k) = \int_k^{\infty} e^{-rT}(e^s - e^k)q_T(s)ds.$$

Since the function $C_T(k)$ tends to S_0 as k tends to $-\infty$, then it is not square integrable, and thus, in order to obtain a square integrable function, we define the modified call price

$$c_T(k) \equiv \exp(\alpha k) C_T(k),$$

for $\alpha > 0$. The Fourier transform of $c_T(k)$ is given by

$$\psi_T(\upsilon) = \int_{\infty}^{\infty} e^{i\upsilon k} c_T(k) dk.$$

If we find an analytic expression for $\psi_T(\upsilon)$ in terms of $\phi_T(u)$, then we can price the European option using the inverse transform

$$C_T(k) = \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{\infty} e^{-i\upsilon k} \psi_T(\upsilon) d\upsilon = \frac{\exp(-\alpha k)}{\pi} \int_{0}^{\infty} e^{-i\upsilon k} \psi_T(\upsilon) d\upsilon, \quad (3.2)$$

where the second equality holds because $C_T(k)$ is real and thus $\psi_T(\upsilon)$ is odd in its imaginary part and even in its real part. We have

$$\begin{aligned} \psi_T(\upsilon) &= \int_{-\infty}^{\infty} e^{i\upsilon k} \int_{k}^{\infty} e^{\alpha k} e^{-rT} (e^s - e^k) q_T(s) ds dk \\ &= \int_{-\infty}^{\infty} e^{-rT} q_T(s) \int_{-\infty}^{s} (e^{s+\alpha k} - e^{(1+\alpha)k}) e^{i\upsilon k} dk ds \\ &= \int_{-\infty}^{\infty} e^{-rT} q_T(s) \left[\frac{e^{(\alpha+1+i\upsilon)s}}{\alpha + i\upsilon} - \frac{e^{(\alpha+1+i\upsilon)s}}{\alpha+1+i\upsilon} \right] ds \\ &= \frac{e^{-rT} \phi_T(\upsilon - (\alpha+1)i)}{\alpha^2 + \alpha - \upsilon^2 + i(2\alpha+1)\upsilon}. \end{aligned} \quad (3.3)$$

Thus, substituting (3.3) into (3.2), the price at $t = 0$ of the European call is given by

$$C_T(k) = \frac{exp(-\alpha k)}{\pi} \int_0^{\infty} e^{-i\upsilon k} \frac{e^{-rT} \phi_T(\upsilon - (\alpha+1)i)}{\alpha^2 + \alpha - \upsilon^2 + i(2\alpha+1)\upsilon} d\upsilon. \quad (3.4)$$

Consider now the right choice of the coefficient α. Indeed, a positive α guarantees the integrability of $c_T(k)$ on the negative log strike axis, but worsens the same condition on the positive axis. A sufficient condition for the integrability of $c_T(k)$ on the positive log strike axis, and thus for its square integrability, is $\psi_T(0) < \infty$. From equation (3.3), we have $\psi_T(0) < \infty$ if $\phi_T(-(\alpha+1)i) < \infty$, and thus a sufficient condition is

$$\mathrm{E}^{\widetilde{\mathrm{P}}}[S_T^{\alpha+1}] < \infty,$$

where $\widetilde{\mathrm{P}}$ is the risk-neutral probability. Finally, we could use the Fast Fourier Transform in order to solve numerically the expression (3.4). In particular, Carr and Madan [14] show that the option price can be written as

$$C_T(k_u) \approx \frac{\exp(-\alpha k_u)}{\pi} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)} e^{ib\upsilon_j} \psi(\upsilon_j) \frac{\eta}{3}(3 + (-1)^j - \delta_{j-1}),$$

where $b = N\lambda/2$, $\lambda\eta = 2\pi/N$, $k_u = -b + \lambda(u-1)$, $u = 1, \ldots, N$, $\upsilon_j = (j-1)\eta$, and η small.

Partial integro-differential equation (PIDE) method

Consider a market where the risk-neutral dynamics of the asset satisfies the exponential-Lévy model

$$S_t = S_0 e^{X_t},$$

where $\{X_t\}$ is a Lévy process with characteristic triplet (γ, σ^2, ν). Assume that the price process $\{S_t\}$ has finite second moment, which is equivalent to assume

$$\int_{|y|\geq 1} e^{2y} \nu(dy) < \infty.$$

Then, we can write the risk-neutral dynamics of S_t as

$$S_t = S_0 + \int_0^t r S_{u-} du + \int_0^t S_{u-} \sigma dW_u + \int_0^t \int_{-\infty}^{\infty} (e^x - 1) S_{u-} \widetilde{J}_X(dudx), \quad (3.5)$$

where $\widetilde{J}_X$ is the compensated jump process of $\{X_t\}$ and $\{W_t\}$ its Brownian component (see Cont and Tankov [21], Proposition 8.20). Given a European call on this asset with maturity T and strike price K, we have that its price at time t is

$$C_t(T, K) = e^{-r(T-t)} \mathrm{E}^{\widetilde{\mathrm{P}}}[(S_T - K)^+ | \Im_t],$$

where $\widetilde{\mathrm{P}}$ is the risk-neutral probability. The Markov property of the Lévy process $\{X_t\}$ implies that we can consider the option price $C_t(T, K)$ as a function $C(t, S_t)$ of the time t and asset price S_t. Let us assume that $C(t, S_t) \in C^{1,2}$. Applying the

Ito formula for Lévy processes (see Cont and Tankov [21], Proposition 8.15) to the discounted option price $\widehat{C}_t = e^{-rt}C(t, S_t)$ and using equation (3.5), we have

$$\begin{aligned} d\widehat{C}_t &= e^{-rt}\left[-rC_t + \frac{\partial C}{\partial t}(t, S_{t-}) + \frac{\sigma^2 S_{t-}^2}{2}\frac{\partial^2 C}{\partial S^2}(t, S_{t-})\right]dt + e^{-rt}\frac{\partial C}{\partial S}(t, S_{t-})dS_t \\ &\quad + e^{-rt}\left[C(t, S_{t-}e^{\Delta X_t}) - C(t, S_{t-}) - S_{t-}(e^{\Delta X_t} - 1)\frac{\partial C}{\partial S}(t, S_{t-})\right] \\ &= a(t)dt + dM_t, \end{aligned}$$

where

$$\begin{aligned} a(t) &= e^{-rt}\left[-rC_t + \frac{\partial C}{\partial t} + \frac{\sigma^2 S_{t-}^2}{2}\frac{\partial^2 C}{\partial S^2} + rS_{t-}\frac{\partial C}{\partial S}\right](t, S_{t-}) \\ &\quad + \int_{-\infty}^{\infty} \nu(dx)e^{-rt}\left[C(t, S_{t-}e^x) - C(t, S_t) - S_{t-}(e^x - 1)\frac{\partial C}{\partial S}(t, S_{t-})\right], \end{aligned}$$

$$dM_t = e^{rt}\left\{\frac{\partial C}{\partial S}(t, S_{t-})\sigma S_{t-}dW_t + \int_{\mathbb{R}} \left[C(t, S_{t-}e^x) - C(t, S_{t-})\right]\widetilde{J}_X(dtdx)\right\}.$$

The next step is to show that the stochastic process $\{M_t\}$ is a martingale. We have

$$\begin{aligned} C(t,x) - C(t,y) &= e^{-r(T-t)}\left\{\mathrm{E}[(xe^{X_{T-t}} - K)^+] - \mathrm{E}[(ye^{X_{T-t}} - K)^+]\right\} \\ &\leq e^{-r(T-t)}|x-y|\mathrm{E}[e^{X_{T-t}}] \\ &\leq |x-y|, \end{aligned} \tag{3.6}$$

because $\{e^{-rt+X_t}\}$ is a martingale. Thus, the predictable random function $\psi(t,x) = C(t, S_{t-}e^x) - C(t, S_{t-})$ satisfies

$$\begin{aligned} \mathrm{E}\left[\int_0^T dt \int_{\mathbb{R}} \nu(dx)|\psi(t,x)|^2\right] &= \mathrm{E}\left[\int_0^T dt \int_{\mathbb{R}} \nu(dx)|C(t, S_{t-}e^x) - C(t, S_{t-})|^2\right] \\ &\leq \mathrm{E}\left[\int_0^T dt \int_{\mathbb{R}} (e^{2x}+1)S_{t-}^2\nu(dx)\right] \\ &\leq \int_{\mathbb{R}} (e^{2x}+1)\nu(dx)\mathrm{E}\left[S_{t-}^2 dt\right] < \infty, \end{aligned}$$

and, then, the compensated Poisson integral (see Cont and Tankov [21], Proposition 8.8)

$$\int_0^t \int_{-\infty}^{\infty} e^{-rt}[C(t, S_{t-}e^x) - C(t, S_{t-})]\widetilde{J}_X(dtdx)$$

is a square-integrable martingale. Moreover, by inequality (3.6), we have

$$\|\frac{\partial C}{\partial S}(t,.)\|_{L^\infty} \leq 1$$

and thus

$$\mathrm{E}\left[\int_0^T S_{t-}^2 \left|\frac{\partial C}{\partial S}(t,S_{t-})\right| dt\right] \leq \mathrm{E}\left[\int_0^T S_{t-}^2 dt\right] < \infty$$

and $\int_0^t \sigma S_{t-} \frac{\partial C}{\partial S}(t,S_{t-}) dW_t$ is a square-integrable martingale as well (see Cont and Tankov [21], Proposition 8.6). Therefore, $\{M_t\}$ is a square-integrable martingale, and, since $\{\widehat{C}_t\}$ is a martingale by construction, $\{\widehat{C}_t - M_t\}$ is a martingale as well. But $\widehat{C}_t - M_t = \int_0^t a(s)ds$ and thus $\{\widehat{C}_t - M_t\}$ is also a continuous process with finite variation. Then, the stochastic process $a(t)$ has to be equal to zero $\widetilde{\mathrm{P}}$-almost surely (see Cont and Tankov [21], Proposition 8.9), and so the option price $C(t,S)$ has to satisfy the partial integro-differential equation

$$\begin{aligned}\frac{\partial C}{\partial t}(t,S) + rS\frac{\partial C}{\partial S}(t,S) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}(t,S) - rC(t,S)\\ + \int \nu(dy)\left[C(t,Se^y) - C(t,S) - S(e^y-1)\frac{\partial C}{\partial S}(t,S)\right] = 0 \quad (3.7)\end{aligned}$$

on $[0,T) \times (0,\infty)$ with terminal condition:

$$C(T,S) = (S-K)^+, \quad \forall S > 0.$$

The PIDE (3.7) can be solved applying a finite difference method which takes into account the integral term. Section 12.4 of Cont and Tankov [21] is devoted to the study of this type of numerical algorithms.

3.2 Construction of a sequence of Markov chains converging weakly to a Lévy process

In this section we explain the Duan *et al.*'s procedure in order to construct a sequence of Markov chains converging weakly to a Lévy process on a discrete set of times. Considering the maturity T of the contingent claim, our task is to approximate, under the risk neutral probability $\widetilde{\mathrm{P}}$, the log price process $\{\ln(S_t)\}_{0\leq t\leq T}$ at times $\{0, \Delta t, 2\Delta t, \ldots, s\Delta t = T\}$ by a sequence of Markov chains $\{Y^{(m)}_{n\Delta t}, n = 0, 1, 2, \ldots, s\}_{m=2i+1, i\in\mathbb{N}}$ with state space $\{p_1, p_2, \ldots, p_m\}$ and transition probability matrix $Q_{(m)} = [q_{ij}]_{i,j=1,\ldots,m}$, where m is an odd integer and $p_{(m+1)/2} = \ln(S_0)$. In order to fix the ideas, we adopt the mean correcting martingale measure and observe that under this measure the asset price follows

$$S_t = S_0 e^{\mu t + X_t},$$

where μ is given by formula (2.18) and $\{X_t\}$ is one of the NIG, VG, and Meixner processes with drift term zero. Thus, we build a sequence of Markov chains $\{Y^{(m)}_{n\Delta t}, n = 0, 1, 2, \ldots, s\}_{m=2i+1, i\in N}$ with state space $\{p_1, p_2, \ldots, p_m\}$, converging weakly to the risk neutral Lévy process $\{\ln(S_0) + \mu t + X_t : t = 0, \Delta t, 2\Delta t, \ldots, T\}$ as the state number m tends to infinite. Given the current price S_0, we define an interval $[\ln(S_0) - I(m), \ln(S_0) + I(m)]$ such that

$$\widetilde{\mathrm{P}}\left[\ln(S_T) \in [\ln(S_0) - I(m), \ln(S_0) + I(m)]\right] \approx 1.$$

The m states of the Markov chain are defined as $p_i = \ln(S_0) + \frac{2i-m-1}{m-1} I(m)$, $i = 1, \ldots, m$. Note that $p_1 = \ln(S_0) - I(m)$, $p_m = \ln(S_0) + I(m)$ and $p_{(m+1)/2} = \ln(S_0)$. In order to get the convergence, we have to guarantee that $I(m) \to \infty$ and $I(m)/m \to 0$ as the number of the states converges to infinity ($m \to \infty$). For example, when the Markov process $\{\ln(S_t)\}_{0\leq t\leq T}$ admits finite mean (i.e., $\mathrm{E}^{\widetilde{\mathrm{P}}}\left(|\ln(S_{\Delta t})|\right) < \infty$), we can use $I(m) = \max(|z_{1/m}|, |z_{1-1/m}|)$, where z_k is the k% quantile (under the risk-neutral probability) of $\ln(S_T)$. Since $I(m) \to \infty$ and $I(m)/m \to 0$, we can guarantee the convergence of the Markov chain sequence. However, the speed of convergence is strictly linked to the choice of $I(m)$. Thus, we have to choose opportunely $I(m)$. Duan *et al.* suggest to use $I(m) = (2 + \ln(\ln(m)))\,\sigma\sqrt{T}$ for the Brownian Motion. When we assume the mean correcting risk neutral valuation for the NIG, VG, and Meixner processes, we observe an higher speed of convergence using $I(m) = z + \frac{\log(\log(m))}{2}$, where with log we mean logarithm with base 10, $z = \max(|z_{0.05}|, |z_{0.95}|)$, and $z_{0.05}$ and $z_{0.95}$ are respectively the 5% and 95% quantiles of the distribution $\mu T + X_T$. In order to construct the Markov chain, we define the cells $(c_j, c_{j+1}]$, $j = 1, \ldots, m$, where $c_1 = -\infty$, $c_j = (p_j + p_{j-1})/2$, $j = 2, \ldots, m$, and $c_{m+1} = \infty$, and observe that $c_2 \to -\infty$, $c_m \to \infty$, and

$$c_{j+1} - c_j = 2\left(\frac{I(m)}{m-1}\right) \to 0, \quad j = 2, \ldots, m-1,$$

as $m \to \infty$. The transition probability between the i-th state and the j-th state is given by

$$q_{ij} = \widetilde{\mathrm{P}}[p_i + \mu\Delta t + X_{\Delta t} \in (c_j, c_{j+1}]],$$

and by the convergence to zero of the cell width we can deduce the weak convergence of the sequence of Markov chains $\{Y^{(m)}_{n\Delta t}, n = 0, 1, 2, \ldots, s\}_{m=2i+1, i\in\mathbb{N}}$ to the Lévy process $\{\ln(S_0) + \mu t + X_t : t = 0, \Delta t, 2\Delta t, \ldots, T\}$ as $m \to \infty$. Indeed, we have

$$\begin{aligned} \widetilde{\mathrm{P}}[Y^{(m)}_{n\Delta} \leq y | Y^{(m)}_{(n-1)\Delta t} = p_i] &= \sum_{j : p_j \leq y} q_{ij} \\ &= \sum_{j \leq j^*} \widetilde{\mathrm{P}}[p_i + \mu\Delta t + X_{\Delta t} \in (c_j, c_{j+1}]], \end{aligned}$$

where $j^* = \max\{j : p_j \leq y\}$, and, by the convergence to zero of the cell width,

$$\widetilde{\mathrm{P}}[Y^{(m)}_{n\Delta t} \leq y | Y^{(m)}_{(n-1)\Delta t} = p_i] \quad \to \quad \widetilde{\mathrm{P}}[p_i + \mu\Delta t + X_{\Delta t} \leq y]$$

as $m \to \infty$. Therefore, as a consequence of Theorem 2.3, the sequence of Markov chains $\{Y^{(m)}_{n\Delta t}, n = 0, 1, 2, \ldots, s\}$ converges weakly to the Lévy process $\{\ln(S_0) + \mu t + X_t\}$ at times $\{0, \Delta t, 2\Delta t, \ldots, s\Delta t\}$.

Fixed the m values p_i, we can always determine other m values starting form any other state by $p^i_k = p_i + \frac{2k-m-1}{m-1} I(m)$. In particular, $p^i_k = p_j$ if and only if $k = j - i + \frac{m+1}{2}$, that is

$$p^i_k = p_i + \frac{2k - m - 1}{m - 1} I(m) = \ln(S_0) + \frac{2(i + k - \frac{m+1}{2}) - m - 1}{m - 1} I(m).$$

Then, we can determine for any state p_i the cells $(c_k^i, c_{k+1}^i]$, $k = 1, \ldots, m$, where $c_1^i = p_1^i - \frac{\log(\log(m))}{2}$, $c_k^i = (p_k^i + p_{k-1}^i)/2$, $k = 2, \ldots, m$, $c_{m+1}^i = p_m^i + \frac{\log(\log(m))}{2}$, and, defining $k(j) = j - i + \frac{m+1}{2}$, $j = 1, \ldots, m$, we can compute the entries of the transition matrix $Q_{(m)}$ by:

$i < \frac{m+1}{2}$

$$q_{ij} = \begin{cases} \sum_{k=1}^{1+\frac{m+1}{2}-i} \int_{c_k^i - p_i - \mu\Delta t}^{c_{k+1}^i - p_i - \mu\Delta t} f_{X_{\Delta t}}(x)dx & \text{if} \quad j = 1 \\ \int_{c_{k(j)}^i - p_i - \mu\Delta t}^{c_{k(j)+1}^i - p_i - \mu\Delta t} f_{X_{\Delta t}}(x)dx & \text{if} \quad j = 2, \ldots, i + \frac{m-1}{2} \\ 0 & \text{if} \quad j = i + \dfrac{m+1}{2}, \ldots, m, \end{cases}$$

if $i > \frac{m+1}{2}$

$$q_{ij} = \begin{cases} 0 & \text{if} \quad j = 1, \ldots, i - \frac{m+1}{2} \\ \int_{c_{k(j)}^i - p_i - \mu\Delta t}^{c_{k(j)+1}^i - p_i - \mu\Delta t} f_{X_{\Delta t}}(x)dx & \text{if} \quad j = i - \frac{m-1}{2}, \ldots, m-1 \\ \sum_{k=m-i+\frac{m+1}{2}}^{m} \int_{c_k^i - p_i - \mu\Delta t}^{c_{k+1}^i - p_i - \mu\Delta t} f_{X_{\Delta t}}(x)dx & \text{if} \quad j = m, \end{cases}$$

if $i = \frac{m+1}{2}$:

$$q_{ij} = \int_{c_j^i - p_i - \mu\Delta t}^{c_{j+1}^i - p_i - \mu\Delta t} f_{X_{\Delta t}}(x)dx, \quad j = 1, \ldots, m,$$

where $f_{X_{\Delta t}}(\cdot)$ is the density function of $X_{\Delta t}$. When m increases the intervals $(c_k^i, c_{k+1}^i]$ become so small that we can well approximate any integral with the area of only one rectangle, i.e.,

$$\int_{c_k^i - p_i - \mu\Delta t}^{c_{k+1}^i - p_i - \mu\Delta t} f_{X_{\Delta t}}(x)dx \approx f_{X_{\Delta t}}\left(\frac{c_k^i + c_{k+1}^i}{2} - p_i - \mu\Delta t\right)(c_{k+1}^i - c_k^i).$$

3.3 Option pricing under the markovian approach

In this section we show how to use a sequence of Markov chains $\{Y_{n\Delta t}^{(m)} : n = 0, 1, \ldots, s\}_{m=2i+1, i\in\mathbb{N}}$ converging weakly to $\{\ln(S_{n\Delta t}) : n = 0, 1, \ldots, s\}$, where $s\Delta t = T$, to price European, American, compound, barrier, and lookback options. This method possesses the same ductility of the binomial model and thus it can be

NIG	$\alpha = 153.866$	$\beta = 7.603$	$\delta = 1.562$	$\mu = -0.00029$	$D = 0.0653$
VG	$\theta = 0.0756$	$\sigma = 0.0984$	$\nu = 0.0024$	$\mu = 0.00055$	$D = 0.0667$
Meixner	$\alpha = 0.0146$	$\beta = 0.1116$	$\delta = 94.676$	$\mu = -0.00026$	$D = 0.0661$

Table 3.1: *MLE of parameters and Kolmogorov-Smirnoff test of daily S&P500 log-returns assuming or a Normal Inverse Gaussian process, or a Variance-Gamma process, or a Meixner process.*

used to determine the price of almost every path dependent contingent claim. In this section we also exhibit some numerical results, where it is possible to observe the price convergence. These results concern the market Index S&P500 observing its daily prices from January 2006 to March 2007. In Table 3.1 we display the maximum likelihood estimates of parameters on annual basis of the NIG, VG, and Meixner processes. In the last column we show Kolmogorv-Smirnoff distances

$$D = \sup_{x \in \mathbb{R}} |F(x) - F_E(x)|,$$

where F_E is the empirical cumulative distribution and F the assumed distribution. Considering that the Brownian Motion hypothesis gives a value of the distance $D = 0.0766$, then the other three distributional hypotheses present a better approximation. This empirical result is confirmed by the QQ-plot analysis of Figure 3.1. Thus we can see how the empirical and theoretical distributions are closer on the whole real line when we use the NIG or VG distributions to model the log-returns.

3.3.1 European options

When the maturity of a European option is T and we consider s steps (i.e., $s\Delta t = T$), then the price of the European option is given by the $((m+1)/2)$-th component of the price vector

$$V(p, 0) = e^{-rT} Q^s_{(m)} Z,$$

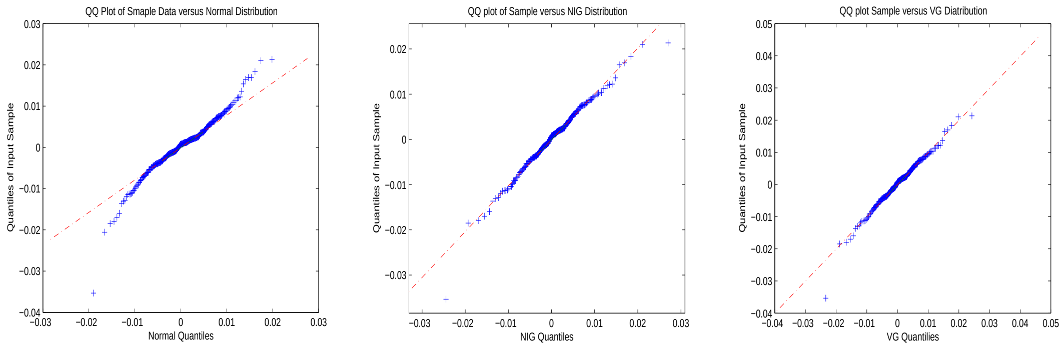

Figure 3.1: *QQ-plots among the sample and the Gaussian, NIG, and VG distributions.*

	NIG process		VG process		Meixner process	
States	weekly	daily	weekly	daily	weekly	daily
m=101	1.7428	1.7984	1.6795	1.7489	1.7343	1.8022
m=501	1.7442	1.7442	1.6809	1.6852	1.7357	1.7357
m=1001	1.7442	1.7442	1.6810	1.6840	1.7358	1.7358
m=1501	1.7442	1.7442	1.6810	1.6810	1.7358	1.7358
m=2001	1.7442	1.7442	1.6810	1.6810	1.7358	1.7358
df method	1.7442		1.6810		1.7358	

Table 3.2: *European put option prices under NIG, VG, and Meixner processes.*

where Z is the m-dimensional vector of payoff at the maturity correspondent to the vector of log prices $p = [p_1, p_2, \dots, p_m]$. So, we assume that the payoff vector is given by $Z = [g_{w,1}, \dots, g_{w,m}]'$, where $g_{w,i} = \max\{w[\exp(p_i) - K], 0\}$, w is equal to 1 for a call and -1 for a put, and K is the strike price.

Analogously to the example reported by Duan and Simonato [26] with the Black and Scholes model, in Table 3.2 we show the convergence of this methodology under the three different distributional assumptions. In order to determine some prices which refer to the same underlying stock process, for this table and all the following ones we use the mean correcting risk neutral measure applied to the parameters estimated in Table 3.1. Table 3.2 reports European put option prices at the money under NIG, VG, and Meixner processes on a stock price with current value $S_0 = 100$ euro, maturity $T = 0.5$ years, and short interest rate $r = 5\%$ a.r.. Moreover, we consider that the temporal horizon is shared either in 24 periods or in 126 periods (i.e. Δt is equal respectively either to one week or to one day). In both cases we observe the convergence of the option prices with respect to the number of the states m. The last row reports put option prices computed by the density function (df) method described in Section 3.1.

3.3.2 American options

An option is called American if its holder can choose to exercise it before the maturity. Consider an American option with maturity T and strike price K, and assume that the contract may be exercised at times $\{0, \Delta t, 2\Delta t, \dots, s\Delta t\}$, where $T = s\Delta t$. Fixed the number of states m we build the vector of the state values $p = [p_1, p_2, \dots, p_m]$ of an approximating Markov chain $\{Y^{(m)}_{n\Delta t} : n = 0, 1, \dots, s\}_{m=2i+1, i\in\mathbb{N}}$, with risk-neutral transition matrix $Q_{(m)}$. Since the states remain the same for all the time steps, then at each time $\{0, \Delta t, 2\Delta t, \dots, s\Delta t\}$ there is an unique payoff vector

$$g_w(p, K) = [g_{w,1}, ..., g_{w,m}]',$$

where $g_{w,i} = \max\{w[\exp(p_i) - K], 0\}$, and w is equal to 1 for a call and -1 for a put. For every couple of vectors $a = [a_1, ..., a_m]'$, $b = [b_1, ..., b_m]'$ we assume the vectorial notation $\max[a, b] := [\max(a_1, b_1), \max(a_2, b_2), ..., \max(a_m, b_m)]'$. Therefore, the price of the American option can be computed using the recursive vectorial formula:

$$\begin{gathered} V_w(p, T) = g_w(p, K), \\ V_w(p, t_i) = \max[g_w(p, K), e^{-r\Delta t} Q_{(m)} V_w(p, t_{i+1})], \\ i = 0, \dots, s-1, \quad t_i = 0, \Delta t, 2\Delta t, ..., s\Delta t = T. \end{gathered}$$

The option price at time 0 is given by the $((m+1)/2)$-th element of $V(p, 0)$.

	NIG process		VG process		Meixner process	
	K=98	K=102	K=98	K=102	K=98	K=102
m=501	1.2419	3.0101	1.2067	2.9527	1.2349	3.0025
delta	-0.2919	-0.5686	-0.2914	-0.5739	-0.2914	-0.5692
gamma	0.0560	0.0816	0.0572	0.0829	0.0561	0.0820
m=1001	1.2419	3.0101	1.1882	2.9529	1.2349	3.0025
delta	-0.2919	-0.5686	-0.2881	-0.5732	-0.2914	-0.5692
gamma	0.0560	0.0816	0.0571	0.0847	0.0561	0.0820
m=1501	1.2419	3.0101	1.1869	2.9509	1.2349	3.0025
delta	-0.2919	-0.5686	-0.2879	-0.5732	-0.2914	-0.5692
gamma	0.0560	0.0816	0.0571	0.0848	0.0561	0.0820
m=2001	1.2419	3.0101	1.1868	2.9507	1.2349	3.0025
delta	-0.2919	-0.5686	-0.2879	-0.5732	-0.2914	-0.5692
gamma	0.0560	0.0816	0.0571	0.0848	0.0561	0.0820
m=2501	1.2419	3.0101	1.1868	2.9508	1.2349	3.0025
delta	-0.2919	-0.5686	-0.2879	-0.5732	-0.2914	-0.5692
gamma	0.0560	0.0816	0.0571	0.0848	0.0561	0.0820

Table 3.3: *Delta, Gamma, and American put option prices under NIG, VG, and Meixner processes.*

When we price a contingent claim with the markovian approach we get the vector $V_w(p, 0)$ whose elements are option prices corresponding to discrete values of the stock price. Thus we can compute the Greeks in a way very similar to the finite-difference approach using the option prices adjacent to the $(m + 1)/2$-th element of $V(p, 0)$. However, as suggested by Duan *et al.*, in order to obtain higher quality Greeks it is advisable to have adjacent prices very close to the initial stock price. This approximation problem can be easily solved considering the states $p_{\frac{m+1}{2}} + \varepsilon$, and $p_{\frac{m+1}{2}} - \varepsilon$ in the Markov chain with ε opportunely small.

In this way we can use the following approximation of delta and gamma values:

$$\Delta = \frac{\partial V}{\partial \ln S_0}\frac{1}{S_0} \approx \frac{V\left(p_{\frac{m+1}{2}}+\varepsilon,0\right)-V\left(p_{\frac{m+1}{2}}-\varepsilon,0\right)}{2\varepsilon}\frac{1}{S_0},$$

$$\begin{aligned}\Gamma &= \frac{\partial}{\partial S_0}\left(\frac{\partial V}{\partial \ln S_0}\frac{1}{S_0}\right) \approx \left(\frac{V\left(p_{\frac{m+1}{2}}-\varepsilon,0\right)-V\left(p_{\frac{m+1}{2}}+\varepsilon,0\right)}{2\varepsilon}+\right.\\ &\left.+\frac{V\left(p_{\frac{m+1}{2}}+\varepsilon,0\right)+V\left(p_{\frac{m+1}{2}}-\varepsilon,0\right)-2V\left(p_{\frac{m+1}{2}},0\right)}{\varepsilon^2}\right)\frac{1}{S_0^2}.\end{aligned}$$

Consider American put options with exercise prices K=98 euro or K=102 euro under the assumption the log returns follow either a NIG, or a VG, or a Meixner process. We use the mean correcting risk neutral measure applied to the parameters estimated in Table 3.1 for puts on a stock price with current value $S_0 = 100$ euro, maturity $T = 0.5$ years, short interest rate $r = 5\%$ a.r.. In Table 3.3 we report the option prices and the values of delta and gamma when we assume $\varepsilon = 10^{-6}$. Even in this case we observe the convergence of these values for a number of states m greater than 500.

3.3.3 Exotic options

A European or American option is called exotic if its payoff at time t depends on the path up to t of the underlying asset. In this section we study three particular exotic options, compound, barrier, and lookback options, and describe how to price these options with the markovian approach. Therefore, we assume that there is a sequence of Markov chains $\{Y_{n\Delta t}^{(m)} : n = 0, 1, 2, \ldots, s\}$ converging weakly under the risk-neutral measure to the log return process at times $\{0, \Delta t, 2\Delta t, \ldots, s\Delta t = T\}$. In particular, the proposed methodology is innovative for compound, and lookback options that have not been dealt by Duan and Simonato [26] and Duan *et al.* [27].

Compound options

Compound options are options written on options and may be of four types: a call on call, a put on call, a call on put, and a put on put. Consider a call on call. At the first maturity T_1 the compound option holder has the right to pay the first exercise price K_1 and get a call. Then, the call gives to the compound option holder the right to buy the underlying asset at the second maturity T_2 paying the second exercise price K_2.

The markovian approach allows to price easily compound options. Using the recursive system to price an option with maturity $T_2 - T_1$ and exercise price K_2, we find a vector which represents the possible prices at time T_1 of the American (or European) option on which the first option is written. Denote this vector as

$$\tilde{V}_{w_1}(p, T_1) = [\tilde{V}_{w_1,1}, ..., \tilde{V}_{w_1,m}]' \tag{3.8}$$

where w_1 is equal to 1 for a call and -1 for a put. The payoff at time T_1 of the compound option is given by the vector

$$V_{w_2}(p, T_1) = \max\{w_2[\tilde{V}_{w_1}(p, T_1) - K_1\overline{1}], \overline{0}\}, \tag{3.9}$$

where $\overline{1}$ and $\overline{0}$ are respectively vectors of ones and zeros, and w_2 is equal to 1 for a call and -1 for a put. Thus, using again the recursive system with s steps (i.e., $s\Delta t = T_1$), the price at time 0 of an European option on an American (or European) option is given by the $((m+1)/2)$-th element of the vector $V_{w_2}(p, 0) = e^{-rT_1}Q^s_{(m)}V_{w_2}(p, T_1)$.

Table 2.4 exhibits the prices of compound options obtained under Brownian motion, NIG, VG, and Meixner processes (considering different number of states m). In particular, we compare the results obtained under the Brownian Motion and those given by the Geske's closed formula (see Geske (1979)). These prices concern European calls on European calls, where the current asset price is $S_0 = 100$, the first call has strike price K_1 and maturity $T_1 = 0.25$ years, and the second call has strike price K_2 and maturity $T_2 = 0.25$ years. We consider

	Brownian motion				Brownian motion		
$K_1 = 2$	$K_2 = 98$	$K_2 = 100$	$K_2 = 102$	$K_1 = 1.5$	$K_2 = 98$	$K_2 = 100$	$K_2 = 102$
m=101	3.7530	2.5803	1.6764	m=101	4.1629	2.9332	1.9609
m=501	3.7540	2.5851	1.6747	m=501	4.1637	2.9381	1.9598
m=1001	3.7542	2.5851	1.6747	m=1001	4.1637	2.9385	1.9598
m=1501	3.7542	2.5852	1.6746	m=1501	4.1637	2.9386	1.9598
m=2001	3.7542	2.5852	1.6747	m=2001	4.1637	2.9386	1.9597
Geske	3.7542	2.5852	1.6747	Geske	4.1637	2.9386	1.9597
	NIG process				NIG process		
$K_1 = 2$	$K_2 = 98$	$K_2 = 100$	$K_2 = 102$	$K_1 = 1.5$	$K_2 = 98$	$K_2 = 100$	$K_2 = 102$
m=101	3.7380	2.5584	1.6607	m=101	4.1479	2.9127	1.9438
m=501	3.7360	2.5655	1.6577	m=501	4.1459	2.9189	1.9415
m=1001	3.7359	2.5660	1.6574	m=1001	4.1459	2.9190	1.9413
m=1501	3.7359	2.5660	1.6575	m=1501	4.1459	2.9191	1.9414
m=2001	3.7359	2.5660	1.6575	m=2001	4.1458	2.9191	1.9414
	Meixner process				Meixner process		
$K_1 = 2$	$K_2 = 98$	$K_2 = 100$	$K_2 = 102$	$K_1 = 1.5$	$K_2 = 98$	$K_2 = 100$	$K_2 = 102$
m=101	3.7304	2.5519	1.6552	m=101	4.1394	2.9065	1.9365
m=501	3.7289	2.5578	1.6494	m=501	4.1389	2.9107	1.9330
m=1001	3.7288	2.5580	1.6496	m=1001	4.1388	2.9108	1.9329
m=1501	3.7287	2.5580	1.6495	m=1501	4.1388	2.9110	1.9329
m=2001	3.7287	2.5580	1.6495	m=2001	4.1387	2.9110	1.9330
	VG process				VG process		
$K_1 = 2$	$K_2 = 98$	$K_2 = 100$	$K_2 = 102$	$K_1 = 1.5$	$K_2 = 98$	$K_2 = 100$	$K_2 = 102$
m=101	3.6634	2.4874	1.5795	m=101	4.0738	2.8397	1.8610
m=501	3.6800	2.5043	1.5965	m=501	4.0904	2.8564	1.8776
m=1001	3.6805	2.5048	1.5971	m=1001	4.0909	2.8570	1.8781
m=1501	3.6806	2.5049	1.5971	m=1501	4.0910	2.8571	1.8782
m=2001	3.6807	2.5050	1.5972	m=2001	4.0911	2.8571	1.8783

Table 3.4: *Compound option prices under Brownian motion, NIG, VG, and Meixner processes.*

two possible strike prices K_1 ($K_1 = 1.5, 2$) and three possible strike prices K_2 ($K_2 = 98, 100, 102$). Moreover, the short interest rate is $r = 5\%$, the annual volatility of the Brownian motion is $\sigma = 10.14\%$, and the parameters of the NIG, Meixner and VG processes are always those ones of Table 3.1.

Barrier options

Barrier options may be of two types, knock-out and knock-in. We proceed explaining how to use the markovian approach to price knock-out options and refer to Duan *et al.* for knock-in options. An option is said knock-out when it becomes worthless if the underlying asset touches or crosses a constant barrier H at any monitoring time. The barrier H may be lower or upper. A barrier option is double when there are two barriers and the underlying asset must remain between these two barriers at the monitoring days. Following Duan *et al.*, we introduce an auxiliary variable a_t which takes the value 1 if the barrier condition is triggered before or at time t and the value 0 otherwise. If we denote with $v(p_i, t; a_t)$ the option price at t when the asset log return equals p_i, then for a knock-out option we have:

1. for every time
$$v_w(p_i, t_k; a_{t_k} = 1) = 0,$$

2. for $t_s = s\Delta t = T$
$$v_w(p_i, T; a_T = 0) = \max\{w[\exp(p_i) - K], 0\},$$

3. $t_k = k\Delta t$, $k = 0, 1, \ldots, s-1$,
$$v_w(p_i, t_k; a_{t_k} = 0) = \max\{g_w(p_i, K, a_{t_k} = 0), e^{-r\Delta t} \times$$
$$\times \textstyle\sum_{j=1}^{m} \widetilde{\mathrm{P}}[Y_{t_{k+1}}^{(m)} = p_j, a_{t_{k+1}} = 0 | Y_{t_k}^{(m)} = p_i, a_{t_k} = 0] v(p_j, t_{k+1}; a_{t_{k+1}} = 0)\},$$

where w is equal to 1 for a call and -1 for a put, K is the strike price, $\{Y_{t_k}^{(m)} : t_k = 0, \Delta t, 2\Delta t, \ldots, s\Delta t\}$ is the Markov chain with states number m, and

$$g_w(p_i, K, a_{t_k} = 0) = \begin{cases} \max\{w[\exp(p_i) - K], 0\} & \text{if} \quad \text{American} \\ 0 & \text{if} \quad \text{European.} \end{cases}$$

To compute the transition probability, we define the set of the states for which the option is knocked out and becomes worthless:

$$\Lambda = \begin{cases} \{i \in \{1, \ldots, m\} : \exp(p_i) \leq H\} & \text{down}-\text{and}-\text{out} \quad \text{option} \\ \{i \in \{1, \ldots, m\} : \exp(p_i) \geq H^*\} & \text{up}-\text{and}-\text{out} \quad \text{option} \\ \{i \in \{1, \ldots, m\} : \exp(p_i) \leq H \text{ or} \exp(p_i) \geq H^*\} & \text{double} \quad \text{option} \end{cases}$$

When the states p_i and p_j do not belong to Λ, the conditional probabilities are the same of the matrix $\mathrm{Q}_{(m)} = [q_{ij}]$ as described in Section 3.2, otherwise they are equal to zero. Therefore, the probability to transit from state p_i to state p_j is given by:

$$\begin{aligned} \pi_{ij} &= \widetilde{\mathrm{P}}[Y_{t_{k+1}}^{(m)} = p_j, a_{t_{k+1}} = 0 | Y_{t_k}^{(m)} = p_i, a_{t_k} = 0] \\ &= \begin{cases} q_{ij} & \text{if} \quad i \in \Lambda^c \quad \text{and} \quad j \in \Lambda^c \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where Λ^c is the complement of Λ. Therefore the matrixes that define the conditional probabilities (that we call quasi-transition probabilities matrices) for the down-and-out, up-and-out, and double barrier-out options are respectively given by:

$$\begin{aligned} \Pi_{DO} &= \begin{bmatrix} \mathbf{0}_{k-1,k-1} & \mathbf{0}_{k-1,m-k+1} \\ \mathbf{0}_{m-k+1,k-1} & Q(k,m;k,m) \end{bmatrix} \\ \Pi_{UO} &= \begin{bmatrix} Q(1,l;1,l) & \mathbf{0}_{l,m-l} \\ \mathbf{0}_{m-l,l} & \mathbf{0}_{m-l,m-l} \end{bmatrix} \\ \Pi_{DBO} &= \begin{bmatrix} \mathbf{0}_{k-1,k-1} & \mathbf{0}_{k-1,l-k+1} & \mathbf{0}_{k-1,m-l} \\ \mathbf{0}_{l-k+1,k-1} & Q(k,l;k,l) & \mathbf{0}_{l-k+1,m-l} \\ \mathbf{0}_{m-l,k-1} & \mathbf{0}_{m-l,l-k+1} & \mathbf{0}_{m-l,m-l} \end{bmatrix} \end{aligned}$$

where k is the index number of the log price located immediately above the lower barrier H, l is the index number of the price located immediately below the upper barrier H^*, $\mathbf{0}_{i,j}$ is an $i \times j$ matrix of zeros, and $Q(i,j;k,l)$ is the sub-matrix of $Q_{(m)}$ taken from rows i to j and from columns k to l inclusively. Thus the knock-out option price with maturity T and strike price K can be computed using the recursive vectorial formula:

$$V_w(p,T;a_T=0) = [v_w(p_1,T;a_T=0),\ldots,v_w(p_m,T;a_T=0)]'$$

and for $t_k = k\Delta t$, $k = 0,\ldots,s-1$

$$\begin{aligned} V_w(p,t_k;a_{t_k}=0) &= [v_w(p_1,t_k;a_{t_k}=0),\ldots,v_w(p_m,t_k;a_{t_k}=0)]' \\ &= \max[g_w(p,K,a_{t_k}=0), e^{-r\Delta t}\Pi V_w(p,t_{k+1};a_{t_{k+1}}=0)], \end{aligned}$$

where

$$g_w(p,K,a_{t_k}=0) = [g_w(p_1,K,a_{t_k}=0),\ldots,g_w(p_m,K,a_{t_k}=0)]',$$

and Π is either Π_{DO}, or Π_{UO}, or Π_{DBO}, depending on the nature of the knock-out option. The knock-out option price at time 0 is given by the $((m+1)/2)$-th element of $V_w(p,0;a_0=0)$. Barrier option prices are very sensitive to the position between discrete asset prices and barrier value, thus to reduce this effect it is important to define the cells of the markovian approach so that the barrier value correspond exactly to the border of a cell $(c_k^i, c_{k+1}^i]$.

Table 3.5 exhibits European barrier option prices. We consider two possible strike prices K=100 and K=90 for different fixed barriers and different distributional assumptions (NIG, VG, and Meixner). Even for this table we assume that the temporal horizon is shared either in 24 periods or in 126 periods (i.e., Δt is equal respectively either to one week or to one day). These prices refer to European down-out and up-out call options on a stock price with current value $S_0 = 100$ euro, maturity $T = 0.5$ years, short interest rate $r = 5\%$ a.r.. Similarly, Table 3.6 displays American barrier option prices on a stock with the same current

	European down-out call options under NIG process					Eurpean up-out call options under NIG process			
Strike price	Weekly		Daily		Strike price	Weekly		Daily	
K=100	H=94	H=98	H=94	H=98	K=90	$H^* = 102$	$H^* = 106$	$H^* = 102$	$H^* = 106$
m=501	4.1358	3.1026	4.1059	2.8162	m=501	1.1594	4.1648	0.9289	3.8133
m=1001	4.1359	3.1033	4.1059	2.8183	m=1001	1.1563	4.1616	0.9203	3.8025
m=1501	4.1359	3.1031	4.1058	2.8177	m=1501	1.1568	4.1607	0.9217	3.7997
m=2001	4.1359	3.1029	4.1059	2.8171	m=2001	1.1565	4.1607	0.9206	3.7996
m=2501	4.1359	3.1028	4.1059	2.8168	m=2501	1.1564	4.1604	0.9204	3.7995
	European down-out call options under VG process					Eurpean up-out call options under VG process			
Strike price	Weekly		Daily		Strike price	Weekly		Daily	
K=100	H=94	H=98	H=94	H=98	K=90	$H^* = 102$	$H^* = 106$	$H^* = 102$	$H^* = 106$
m=501	4.0826	3.0813	4.0536	2.7955	m=501	1.1844	4.2817	0.9680	4.0230
m=1001	4.0825	3.0820	4.0625	2.8071	m=1001	1.1847	4.2820	0.9439	3.9200
m=1501	4.0825	3.0812	4.0553	2.7991	m=1501	1.1847	4.2818	0.9420	3.9126
m=2001	4.0825	3.0815	4.0544	2.7996	m=2001	1.1849	4.2820	0.9425	3.9123
m=2501	4.0825	3.0813	4.0546	2.7991	m=2501	1.1847	4.2818	0.9420	3.9119
	European down-out call options under Meixner process					Eurpean up-out call options under Meixner process			
Strike price	Weekly		Daily		Strike price	Weekly		Daily	
K=100	H=94	H=98	H=94	H=98	K=90	$H^* = 102$	$H^* = 106$	$H^* = 102$	$H^* = 106$
m=501	4.1288	3.0986	4.0893	2.8123	m=501	1.1610	4.1780	0.9301	3.8265
m=1001	4.1288	3.0993	4.0993	2.8145	m=1001	1.1579	4.1730	0.9210	3.8096
m=1501	4.1288	3.0991	4.0991	2.8139	m=1501	1.1579	4.1735	0.9210	3.8115
m=2001	4.1288	3.0989	4.0990	2.8132	m=2001	1.1580	4.1730	0.9214	3.8099
m=2501	4.1288	3.0988	4.0991	2.8129	m=2501	1.1579	4.1732	0.9211	3.8103

Table 3.5: *European barrier option prices under NIG, VG, and Meixner processes.*

asset price, short interest rate and maturity. In particular, we consider American down-out and up-out put option prices assuming a strike price $K = 101$ and that the early exercise and the monitoring are on daily basis. As for American and European vanilla options Tables 3.5 and 3.6 show a good tendency towards a specific price when we increase the number of states of the Markov chain.

Lookback options

An European floating strike lookback put option gives the right to sell the underlying asset at maturity for the maximum price monitored discretely during the

American down-out put with daily monitoring						
Strike price	NIG process		VG process		Meixner process	
K=101	H=96	H=99	H=96	H=99	H=96	H=99
m=501	2.2453	1.1477	2.2568	1.1579	2.2496	1.1452
m=1001	2.2453	1.1462	2.2394	1.1540	2.2496	1.1438
m=1501	2.2454	1.1459	2.2382	1.1535	2.2497	1.1436
m=2001	2.2454	1.1458	2.2380	1.1534	2.2497	1.1434
m=2501	2.2454	1.1455	2.2380	1.1533	2.2498	1.1432
American up-out put with daily monitoring						
Strike price	NIG process		VG process		Meixner process	
K=101	$H^* = 101$	$H^* = 104$	$H^* = 101$	$H^* = 104$	$H^* = 101$	$H^* = 104$
m=501	1.1425	2.0802	1.1174	2.0635	1.1302	2.0747
m=1001	1.1334	2.0800	1.1165	2.0417	1.1308	2.0744
m=1501	1.1335	2.0793	1.1164	2.0407	1.1309	2.0736
m=2001	1.1341	2.0793	1.1164	2.0405	1.1316	2.0736
m=2501	1.1337	2.0795	1.1165	2.0404	1.1312	2.0737

Table 3.6: *American down-out and up-out put option prices under NIG, VG, and Meixner processes; both early exercise and monitoring are on daily basis.*

time to maturity, while a call gives the right to buy the underlying asset for the minimum price. The option is American if the right is extended to the whole time to maturity. The pricing and hedging for a lookback option can be faced under the assumption that the asset follows a Markov chain. Consider an European floating strike lookback put option with maturity T and monitored at times $k = iT/n$, where $n+1$ is the number of dates of monitoring and $i = 0, 1, \ldots, n$. In this setting it is implicitly assumed that the asset is monitored at constant time intervals $\Delta t = T/n$. The payoff at maturity T is equal to

$$M_T - S_T,$$

where

$$M_k = \max\left\{S_{\frac{iT}{n}} : i = 0, 1, \dots, nk/T\right\}.$$

The evolution of the asset price $\{S_t\}$ at times $k = iT/n$, $i = 0, 1, \dots, n$, is described under the risk-neutral probability Q by the Markov chain $\{\widetilde{X}_i^{(m)} = \exp(Y_i^{(m)}) : i = 0, 1, \dots, n\}$ with state number m and transition matrix $Q_{(m)} = [q_{ij}]_{i,j=1,\dots,m}$. The random variables $\widetilde{X}_i$, $i = 1, \dots, n$, can assume the ordered values $\widetilde{x}(j)$, $j = 1, \dots, m$, (with $\widetilde{x}(j) < \widetilde{x}(j+1)$). Let us define the function

$$Z_k(h, w),$$

where $h, w = 1, \dots, m$, and $k = iT/n$, $i = 0, 1, \dots, n$. $Z_k(h, w)$ is the value at time k of a contingent claim with final payoff $M_T - S_T$ when the current asset price is equal to $\widetilde{x}(w)$ and the maximum asset price from time 0 to time $k - \Delta t$ has been $\widetilde{x}(h)$. Therefore, at time T we consider the final payoff matrix:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ Z_T(2,1) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Z_T(m,1) & Z_T(m,2) & \cdots & 0 \end{bmatrix}.$$

According to the risk-neutral pricing, at time $T - \Delta t$ we have

$$Z_{T-\Delta t}(h, w) = \sum_{j=1}^{m} q_{wj} Z_T(h, j) e^{-r\Delta t}, \quad \text{if} \quad h > w, \tag{3.10}$$

$$Z_{T-\Delta t}(h, w) = \sum_{j=1}^{m} q_{wj} Z_T(w, j) e^{-r\Delta t}, \quad \text{if} \quad h \leq w. \tag{3.11}$$

Formulas (3.10) and (3.11) have a quite immediate explanation: q_{wj} is just the probability to move from the state $\widetilde{x}(w)$ to the state $\widetilde{x}(j)$; on the right of (3.10) we have $Z_T(h, j)$ because $\widetilde{x}(h) > \widetilde{x}(w)$ and thus the maximum at time $T - \Delta t$ is $\widetilde{x}(h)$, while on the right of (3.11) we have $Z_T(w, j)$ because $\widetilde{x}(h) \leq \widetilde{x}(w)$ and the maximum is $\widetilde{x}(w)$; $e^{-r\Delta t}$ is the discount factor. Iterating the procedure, at time k we obtain

$$Z_k(h, w) = \sum_{j=1}^{m} q_{wj} Z_{k+\Delta t}(\max(h, w), j) e^{-r\Delta t}. \tag{3.12}$$

After n backward steps we obtain a matrix whose element $Z_0(h,w)$ is the value at time 0 of the contingent claim with payoff $M_T - S_T$ when the current asset price is $\widetilde{x}(w)$ and the maximum before time 0 has been $\widetilde{x}(h)$. Therefore, the price of the contingent claim is given by any value $Z_0(h, \frac{m+1}{2})$ with $h \leq \frac{m+1}{2}$. American style options can be priced using the formula for $k = iT/n$, $i = 0, 1, \ldots, n-1$,

$$Z_k(h,w) = \max\left\{\sum_{j=1}^{m} q_{wj} Z_{k+\Delta t}(\max(h,w),j)e^{-r\Delta t}, \widetilde{x}(h) - \widetilde{x}(w)\right\},$$

and then taking the element $Z_0\left(h, \frac{m+1}{2}\right)$ with $h \leq \frac{m+1}{2}$.

In order to show the methodology, we consider a simple numerical example. Let us assume that we have only three times of monitoring, $t = 0, 1, 2$. Thus, the maturity of the European lookback put is $T = 2$. The asset price is described by the Markov chain $\{\widetilde{X}_i^{(5)}, i = 0, 1, 2\}$ with state vector

$$\widetilde{x} = \begin{bmatrix} 97 \\ 98 \\ 100 \\ 102 \\ 103 \end{bmatrix}$$

and transition matrix

$$Q = \begin{bmatrix} 2/5 & 3/10 & 1/5 & 1/10 & 0 \\ 1/5 & 2/5 & 1/5 & 3/20 & 1/20 \\ 1/10 & 1/5 & 2/5 & 1/5 & 1/10 \\ 1/20 & 3/20 & 1/5 & 2/5 & 1/5 \\ 0 & 1/10 & 1/5 & 3/10 & 2/5 \end{bmatrix}.$$

Thus, the current asset price is $S = 100$ and the function $Z_2(h,w)$ is given by

$$Z_2(h,w) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 5 & 4 & 2 & 0 & 0 \\ 6 & 5 & 3 & 1 & 0 \end{bmatrix},$$

where $Z_2(h,w) = \max(\widetilde{x}(h) - \widetilde{x}(w), 0)$, $h, w = 1, \ldots, 5$, and $\widetilde{x}(h)$ and $\widetilde{x}(w)$ are the h-th and w-th element of the state vector $\widetilde{x}$. Now, assuming that the short interest rate is $r = 5\%$ and using formula (3.12), we obtain

$$Z_1(h,w) = \begin{bmatrix} 0 & 0.1902 & 0.6659 & 1.1890 & 1.3317 \\ 0.3805 & 0.1902 & 0.6659 & 1.1890 & 1.3317 \\ 1.7122 & 1.3317 & 0.6659 & 1.1890 & 1.3317 \\ 3.4244 & 2.8537 & 1.9976 & 1.1890 & 1.3317 \\ 4.3757 & 3.7574 & 2.8537 & 1.9500 & 1.3317 \end{bmatrix}$$

and

$$Z_0(h,w) = \begin{bmatrix} 0.2941 & 0.5044 & 1.0225 & 1.6559 & 1.9635 \\ 0.4388 & 0.5044 & 1.0225 & 1.6559 & 1.9635 \\ 1.2713 & 1.9121 & 1.0225 & 1.6559 & 1.9635 \\ 2.6105 & 2.3503 & 1.9816 & 1.6559 & 1.9635 \\ 3.4655 & 3.1466 & 2.7145 & 2.2825 & 1.9635 \end{bmatrix}.$$

Then, in this simple example, we have that the price of the lookback put is given by $Z_0(h, \frac{m+1}{2}) = 1.0255$ for $h \leq \frac{m+1}{2}$ (i.e., $Z_0(1,3) = Z_0(2,3) = Z_0(3,3) = 1.0225$).

In Table 2.7 we show the prices of European and American lookback put options, based on daily and weekly monitoring under the Brownian Motion and NIG, VG, and Meixner processes. The current asset price, the short interest rate and the maturity are $S_0 = 100$, $r = 5\%$, and $T = 0.25$, respectively. We compare the results for the European put with NIG and the VG processes with the prices obtained with 1000000 Montecarlo simulations and we obtain that the prices are respectively 2.6653 and 2.6025 with weekly monitoring and 3.0564 and 3.0011 with daily monitoring. Moreover we could observe that the results obtained with Montecarlo simulations are not very stable even when we simulate ten millions of values. While the prices obtained with the markovian approach are much more stable even with one thousand of states. As a matter of fact, for the European put with NIG and the VG processes we get 2.6691 and 2.6009 with weekly monitoring

	European lookback put							
	Brownian motion		NIG process		VG process		Meixner process	
	weekly	daily	weekly	daily	weekly	daily	weekly	daily
m=501	2.7121	3.1344	2.6680	3.0511	2.5998	3.0058	2.6605	3.0439
m=801	2.7125	3.1355	2.6683	3.0524	2.6000	2.9866	2.6609	3.0452
m=1001	2.7126	3.1358	2.6684	3.0528	2.6001	2.9843	2.6610	3.0456
m=1501	2.7127	3.1361	2.6685	3.0531	2.6002	2.9832	2.6611	3.0459
	American lookback put							
	Brownian motion		NIG process		VG process		Meixner process	
	weekly	daily	weekly	daily	weekly	daily	weekly	daily
m=501	2.8587	3.2919	2.8176	3.2253	2.7528	3.1780	2.8113	3.2195
m=801	2.8695	3.3216	2.8180	3.2266	2.7532	3.1646	2.8117	3.2209
m=1001	2.8696	3.3218	2.8181	3.2269	2.7533	3.1630	2.8118	3.2212
m=1501	2.8697	3.3221	2.8182	3.2273	2.7534	3.1624	2.8119	3.2215

Table 3.7: *European and American lookback put option prices with weekly and daily monitoring under Brownian motion and NIG, VG, and Meixner processes.*

and 3.0534 and 2.9838 with daily monitoring. Thus even if these prices are much more near to those obtained with the Markovian approach they require much more computational time and present an higher level of instability.

Chapter 4

Portfolio selection and risk management models

This chapter studies portfolio selection and risk management models under the assumption of asset returns distributed as multidimensional subordinated Lévy processes. In particular, we suggest two distributional hypotheses, the Normal Inverse Gaussian and Variance Gamma distribution. As explained in the previous chapters, the sample data often display a certain level of skewness and kurtosis greater than the Gaussian one. Thus, in order to describe better the random behaviour of asset returns, several alternative distributional assumptions have been proposed in literature (see, among the others, Ortobelli *et al.* [65] and Rachev and Mittnik [70]). Subordinated Brownian motions are stochastic processes whose distributions at any fixed time can have skewness different from zero and kurtosis greater than three. Besides, we often observe that Lévy processes present better Kolmogorov-Smirnov and Anderson-Darling tests. Therefore, subordinated Brownian motions can be good substitutes to the normality assumptions.

In our analysis asset log returns are modeled as multidimensional time-changed Brownian motion where the subordinator follows either an Inverse Gaussian process or a Gamma process. In this framework any two assets are characterized by the same subordinator, thus correlation coefficients differ from zero even when the components of the multidimensional Brownian motion are independent (see Cont and Tankov [21], chapter 5) and joint extremely events can be more possible. Moreover, subordinated Brownian motions take into account the jumps often observed in the stock prices that could imply large losses for the investors. Under these different distributional assumptions we discuss static and dynamic portfolio selection models in a mean-risk framework. In particular, we compare these models with the assumption that the log returns follow a Brownian motion and evaluate the distributional hypotheses by the point of view of several typologies of investors that recalibrate periodically their portfolios:

a) investors with exponential utility functions,

b) investors that maximize the mean-Value at Risk ratio.

Moreover, we present a model where portfolio choices are taken under a mean-variance-skewness framework.

Risk management is studied describing two possible modeling under Lévy distributions. The first one extends the EWMA RiskMetrics model and describes conditional portfolio returns as either NIG distributions or VG distributions. The conditional heteroscedastic volatility follows an exponential weighted moving average model. The second modeling generalizes the GHICA model of Chen *et al.* [17], and, after an Independent Component Analysis, describes each stochastic innovation through the Lévy distribution which better describes it. In this way we could have stochastic innovations with semi-heavy tails and others with heavy tails.

The chapter is organized as follows. Section 4.1 shows a first empirical comparison, where we assume several distributions for the market portfolio. Section 4.2 presents a multi-dimensional model where portfolio log returns follow subordinated Lévy processes. Section 4.3 shows empirical comparisons between the Lévy models of Section 4.2 and Normal one. In particular, there are two ex-post comparisons, one without transaction costs and the other one with constant and proportional transaction costs. Section 4.4 is an analysis of multi-period portfolio selection, where we compare subordinated Lévy models with the assumption of Normal distributed log returns. Section 4.5 shows an analysis where the skewness of portfolio is taken into account, and further it displays ex-post and ex-ante comparisons among several distributional hypotheses. Section 4.6 studies risk management and presents a simple extension of the RiskMetrics model where VaR and CVaR can be easily computed. Even Section 4.7 studies risk management and presents a model where the Independent Component Analysis (ICA) is applied and the conditional portfolio distribution is numerically computed by the Fast Fourier Transform (FFT) algorithm. The analyses of Sections 4.6 and 4.7 are only exhibited by a theoretical point of view.

4.1 A first empirical comparison among portfolio selection models based on different Lévy processes

In this section we compare the optimal portfolio composition under different distributional hypotheses, and, in particular, we consider Lévy processes with semi heavy tails. Thus, this analysis differs from other studies that assume Lévy processes with very heavy tails (see Rachev and Mittnik [70], Ortobelli *et al.* [65]). In order to facilitate the reading, we recall some notations and results discussed in chapter 2. Lévy processes are all processes with stationary and independent increments and stochastically continuous paths. Typical examples are the Normal Inverse Gaussian (NIG) and Variance Gamma (VG) processes. Many Lévy processes are often seen as subordinated Brownian motions where the subordinator is a Lévy process whose paths are almost surely non-decreasing. The NIG and VG processes can be seen as subordinated Lévy processes where the subordinators are respectively the Inverse Gaussian and Gamma process.

Inverse Gaussian: An Inverse Gaussian process $\{X_t^{(IG)} : t \geq 0\}$, denoted as $IG(a,b)$, assumes that the density function of $X_t^{(IG)}$ is

$$f_{IG}(x; ta, b) = \frac{ta}{x^{3/2}\sqrt{2\pi}} \exp(tab) \exp\left(-\frac{1}{2}\left((ta)^2 x^{-1} + b^2 x\right)\right) 1_{x>0},$$

where a and b are positive.

Gamma: A Gamma process $\{X_t^{(G)} : t \geq 0\}$, denoted as $G(a,b)$, assumes that the density function of $X_t^{(G)}$ is

$$f_G(x; ta, b) = \frac{b^{ta}}{\Gamma(ta)} x^{ta-1} \exp(-xb) 1_{x>0},$$

where a and b are positive.

Normal Inverse Gaussian: Subordinating the Brownian motion with an Inverse Gaussian process we obtain a Normal Inverse Gaussian process $NIG(\mu, \alpha, \beta, \delta)$ with parameters $\mu \in \mathbb{R}$, $\alpha > 0$, $\beta \in (-\alpha, \alpha)$, and $\delta > 0$, that is

$$X_t^{(NIG)} = \mu t + \beta \delta^2 I_t + \delta W_{I_t},$$

where $\{I_t\}$ is an Inverse Gaussian process with parameters $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$, and $\{W_t\}$ is a standard Brownian motion. The density function of $X_t^{(NIG)}$ is given by

$$\begin{aligned} f_{NIG}(x; t\mu, \alpha, \beta, t\delta) &= \frac{\alpha t \delta}{\pi} \exp\left(t\delta\sqrt{\alpha^2 - \beta^2} + \beta(x - \mu t)\right) \times \\ &\quad \times \frac{\mathrm{K}_1\left(\alpha\sqrt{(t\delta)^2 + (x - \mu t)^2}\right)}{\sqrt{(t\delta)^2 + (x - \mu t)^2}}, \end{aligned}$$

where $K_1(x)$ denotes the modified Bessel function of the third kind with index 1.

Variance Gamma: Subordinating the Brownian motion with a Gamma process we obtain a Variance Gamma process $VG(\mu, \theta, \sigma, \nu)$ with parameters $\mu \in \mathbb{R}$, $\theta \in \mathbb{R}$, $\sigma > 0$, and $\nu > 0$, that is

$$X_t^{(VG)} = \mu t + \theta G_t + \sigma W_{G_t},$$

where $\{G_t\}$ is a Gamma process with parameters $a = 1/\nu$ and $b = 1/\nu$. The Variance Gamma process can be also defined as the difference between two independent Gamma processes. The density function of $X_t^{(VG)}$ is given by

$$f_{VG}(x; t\mu, t\theta, \sqrt{t}\sigma, \nu/t) = \frac{2e^{\frac{\theta(x-\mu t)}{\sigma^2}}\left(\frac{(x-\mu t)^2}{2\sigma^2/\nu+\theta^2}\right)^{\frac{t}{2\nu}-\frac{1}{4}}}{\nu^{t/\nu}\sqrt{2\pi}\sigma\Gamma(t/\nu)} \times$$
$$\times \mathrm{K}_{\frac{t}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{(x-\mu t)^2(2\sigma^2/\nu+\theta^2)}\right),$$

where $\mathrm{K}_{\frac{t}{\nu}-\frac{1}{2}}(x)$ is the modified Bessel function of the third kind with index $\frac{t}{\nu} - \frac{1}{2}$.

In portfolio theory it has been widely used a standard Brownian motion to model the log return distribution, that is the asset log return follows the process $\{X_t^{(BM)} : t \geq 0\}$ where $X_t^{(BM)}$ is Normal distributed with mean $t\mu$ and standard deviation $\sqrt{t}\sigma$. In the next subsection we compare optimal portfolio strategies under NIG and VG processes.

4.1.1 A first empirical comparison

Consider the problem to select an optimal portfolio composed by d risky assets with log returns $\widetilde{X} = [X^{(1)}, \ldots, X^d]'$ and one risk-free asset with log return r_f. Let $w = [w_1, \ldots, w_d]'$ be the vector of the weights invested in the risky assets ad assume that no short sales are allowed (i.e., $w_i \geq 0$). In the classical mean-variance analysis, investors choose a portfolio that is the convex combination between the market portfolio and risk-free asset. The weights of the market portfolio w_M are given by the solution of the following optimization problem:

$$\max_w \frac{\mathrm{E}[w'\mu] - r_f}{w'Qw} \tag{4.1}$$
$$\text{s.t.}$$
$$\sum_{i=1}^{d} w_i = 1, \quad \text{and} \quad w_i \geq 0, i = 1, \ldots, d,$$

where μ and Q are respectively the mean and variance-covariance matrix of the log return vector $\widetilde{X}$. Now, let us suppose the investors' exponential utility function is:

$$u(x) = a\left(1 - \exp\left(-\frac{1}{a}x\right)\right), \tag{4.2}$$

where a is their risk tolerance parameter. In order to value the impact of different distributional hypotheses in the portfolio composition, we compute the optimal portfolio which maximizes investor's expected utility when the market portfolio follows a particular subordinated Lévy process. That is, we compute the riskless weight λ that maximizes

$$\mathrm{E}[u(\lambda r_f + (1-\lambda)w'_M\widetilde{X})], \tag{4.3}$$

when the market portfolio $w'_M\widetilde{X}$ follows or a Brownian motion (BM), or a Variance Gamma process (VG), or a Normal Inverse Gaussian process (NIG). Observe that the analytical value of the expression (4.3) for the exponential utility function can be easily found using the Laplace transform of the respective distributions (see, among others, Cont and Tankov [21]).

In this first empirical comparison, we consider daily log returns from 04/10/1992 to 01/01/2002 on 10 US market indexes: DJTM United States Automobiles, DJTM United States Oil & Gas, DJTM United States Basic Resource, Down Jones Industrials, Down Jones Utilities, Nasdaq Industrials, NYSE Composite, S&P100, S&P500, S&P900. We assume as risk-free asset the Treasury Bill 3-month $r_f = 1.61\%$ a.r. on 01/01/2002. Thus, first we determine the market portfolio solving the optimization problem (4.1), using the empirical mean and variance-covariance matrix as estimates of μ and Q, and then, assuming $\mu = 0$ under NIG and VG processes, we estimate the parameters of the market portfolio maximizing the log likelihood function (MLE) under the three distributional hypotheses. In Table 4.1 we report the maximum likelihood estimates (MLE) of the market portfolio parameters supposing that it follows or a Variance Gamma process, or a Normal Inverse Gaussian process, or a Brownian motion. Since we assume the investor's temporal horizon is three months, the distributional parameters are on 3 months basis. Secondly, we maximize the expected utility of

VG	$\theta = 0.0196$	$\sigma = 0.0637$	$\nu = 0.0142$
NIG	$\alpha = 107.2398$	$\beta = 4.7419$	$\delta = 0.4426$
BM	$\mu = 0.0196$	$\sigma = 0.0646$	

Table 4.1: *MLE parameter estimates of 3-months log returns of the market portfolio under VG, NIG, and BM distribution.*

a = 0.10	**NIG**	**BM**	**VG**
Riskless weight	0.5650	0.5241	0.5579
Expected utility	0.0997	0.0980	0.0997
Final wealth	0.9263	0.9178	0.9248
a = 0.15	**NIG**	**BM**	**VG**
Riskless weight	0.3475	0.2862	0.3369
Expected utility	0.1494	0.1468	0.1494
Final wealth	0.8814	0.8687	0.8792
a = 0.20	**NIG**	**BM**	**VG**
Riskless weight	0.1301	0.0483	0.1158
Expected utility	0.1982	0.1948	0.1982
Final wealth	0.8365	0.8196	0.8335

Table 4.2: *Quotes invested in the risk-free asset, maximum expected utility, and ex-post final wealth.*

the final wealth assuming in the utility function (4.2) three possible risk tolerance parameters: 0.10, 0.15, 0.20. Table 4.2 shows the quote invested in the risk-free asset, the maximum expected utility, and the ex-post final wealth after one year on date 01/01/2003 under the three different distributional hypotheses. From this table we observe that the NIG and VG processes take much more into account the possible losses. As a matter of fact, the quoted invested in the riskless is always higher than that one computed for the Brownian motion. Moreover, even the computed maximum expected utility is higher for the NIG and VG processes that implicitly underscores the better performance. These processes are more conservative with respect to the Brownian motion as confirmed by the ex-post final wealth of Table 4.2. As a matter of fact, during the 2002, year with very big losses on the US market, we observe a higher final wealth under the NIG and VG processes.

4.2 Multivariate subordinated Lévy processes and parameter estimates

The multivariate Lévy processes distributions are obtained as a logical extension of univariate ones. So, for example, the d-dimensional Multivariate Normal Inverse Gaussian (MNIG) process with parameters $\delta, \alpha > 0$, $\mu, \beta \in \mathbb{R}^d$ and $Q \in \mathbb{R}^{d\times d}$ valued at time t can be constructed from:

$$\widetilde{X}_t = \mu t + Z_t Q\beta + \sqrt{Z_t}Q^{1/2}\widetilde{Y},$$

where the intrinsic time Inverse Gaussian process Z_t is distributed as $IG(\delta t, \sqrt{\alpha^2 - \beta' Q\beta})$, $\widetilde{Y}$ is a standard d-dimensional Gaussian independent of Z_t and then the conditional distribution of vector $\widetilde{X}_t|Z_t$ is $\mathrm{N}_d(\mu t + Z_t Q\beta)$ (see Barndorff-Nielsen [5]). Thus the d-dimensional vector admits density probability function:

$$\begin{aligned} f_{X_t}(x) &= \int f_{X_t|Z_t}(x|z) f_{Z_t}(z)dz \\ &= \frac{\delta t}{2^{\frac{d-1}{2}}}\left(\frac{\alpha}{\pi q(x)}\right)^{\frac{d+1}{2}} K_{\frac{d+1}{2}}(\alpha q(x)\exp(p(x))), \end{aligned}$$

where

$$\begin{aligned} q(x) &= \sqrt{(\delta t)^2 + ((x-\mu t)' Q^{-1}(x-\mu t))} \\ p(x) &= \left(\beta'(x-\mu t) + \delta t\sqrt{\alpha^2 - \beta' Q\beta}\right), \end{aligned}$$

and $K_{\frac{d+1}{2}}$ denotes the modified Bessel function of the third kind with index $\frac{d+1}{2}$. Similarly, we can define the multivariate Variance-Gamma process. However, generally there exist many problems in the maximum likelihood estimation of multivariate Lévy process parameters, in particular when we assume a large number of assets (see, among others, Hanssen and Øigård [39], Bølviken and Benth [12]). For this reason we estimate the parameters of marginal distributions separately by the correlation matrix. Doing so we assume that every couple of subordinated components follow a joint bivariate subordinated process.

Suppose that in the market the vector of risky assets has log returns $\widetilde{X}_t = [X_t^{(1)}, \ldots, X_t^{(d)}]'$ distributed as

$$\widetilde{X}_t = \mu t + \gamma Z_t + Q^{1/2}\widetilde{W}_{Z_t}, \tag{4.4}$$

where $\{Z_t\}$ is the positive Lévy process, $\mu = [\mu_1, \ldots, \mu_d]'$, $\gamma = [\gamma_1, \ldots, \gamma_d]'$, $Q = [\sigma_{ij}^2]_{ij}$ is a fixed definite positive variance-covariance matrix (i.e., $\sigma_{ij}^2 = \sigma_{ii}\sigma_{jj}\rho_{ij}$ where ρ_{ij} is the correlation coefficients i-th component of $\widetilde{X}_t|Z_t$ and its j-th component), and $\{\widetilde{W}_t\}$ is a d-dimensional standard Brownian motion (i.e., $Q^{1/2}\widetilde{W}_{Z_t} = \sqrt{Z_t}Q^{1/2}\widetilde{Y}$ where $\widetilde{Y}$ is a standard d-dimensional Gaussian independent of Z_t). Under the above distributional hypotheses we approximate the log return of the portfolio $w = [w_1, \ldots, w_d]'$, where $w'e = 1$ and $e = [1, \ldots, 1]'$, through the portfolio of log returns, that is the convex combination of the log-returns:

$$X_t^{(w)} = w'\widetilde{X}_t = (w'\mu)t + (w'\gamma)Z_t + \sqrt{w'Qw}W_{Z_t}, \tag{4.5}$$

where $\{W_t\}$ is a 1-dimensional standard Brownian motion. At this point we will assume that the subordinator $\{Z_t\}$ is modeled either as an Inverse Gaussian process, $Z_1 \sim IG(1, b)$, or a Gamma process, $Z_1 \sim G(\frac{1}{\nu}, \frac{1}{\nu})$.

Inverse Gaussian subordinator: the NIG model

When $\{Z_t\}$ follows an Inverse Gaussian process $IG(1, b)$, then the i-th log return follows a NIG process $NIG(\mu_i, \alpha_i, \beta_i, \delta_i)$, where $\delta_i = \sigma_{ii}$, $\beta_i = \gamma_i/\delta_i^2$, and $\alpha_i = \sqrt{(b/\delta_i)^2 + \beta_i^2}$. Thus, the portfolio (4.5) follows a $NIG(\mu_w, \alpha_w, \beta_w, \delta_w)$ process whose parameters are:

$$\mu_w = w'\mu, \quad \alpha_w = \sqrt{\left(\frac{b}{\delta_w}\right)^2 + \beta_w^2}, \quad \beta_w = \frac{w'\gamma}{\delta_w^2}, \quad \delta_w = \sqrt{w'Qw}.$$

Mean, variance, Fisher-Pearson skewness, and kurtosis parameters of the portfolio $X_t^{(w)}$ are, respectively,

$$\mathrm{E}[X_t^{(w)}] = t\mu_w + \frac{t\delta_w\beta_w}{\sqrt{\alpha_w^2 - \beta_w^2}},$$

$$\mathrm{Var}[X_t^{(w)}] = t\alpha_w^2\delta_w\left(\alpha_w^2 - \beta_w^2\right)^{-3/2},$$

$$\mathrm{Sk}[X_t^{(w)}] = \frac{\mathrm{E}\left[\left(X_t^{(w)} - \mathrm{E}[X_t^{(w)}]\right)^3\right]}{\mathrm{E}\left[\left(X_t^{(w)} - \mathrm{E}[X_t^{(w)}]\right)^2\right]^{3/2}} = 3\beta_w\alpha_w^{-1}(t\delta_w)^{-1/2}(\alpha_w^2 - \beta_w^2)^{-1/4},$$

$$\mathrm{Ku}[X_t^{(w)}] = \frac{\mathrm{E}\left[\left(X_t^{(w)} - \mathrm{E}[X_t^{(w)}]\right)^4\right]}{\mathrm{E}\left[\left(X_t^{(w)} - \mathrm{E}[X_t^{(w)}]\right)^2\right]^{2}} = 3\left(1 + \frac{\alpha_w^2 + 4\beta_w^2}{t\delta_w\alpha_w^2\sqrt{\alpha_w^2 - \beta_w^2}}\right).$$

In order to estimate all these parameters, we estimate the parameters $(\mu_i, \alpha_i, \beta_i, \delta_i)$ for each asset maximizing the log likelihood function

$$L(\mu_i, \alpha_i, \beta_i, \delta_i) = \sum_{k=1}^{n} \log\left(f_{NIG}(y_k; \mu_i, \alpha_i, \beta_i, \delta_i)\right), \quad i = 1, \dots, d,$$

where f_{NIG} is the density of NIG process, y_k is the k-th observation of the i-th asset, and n is the sample size. Given the set of estimates $\{(\hat{\mu}_i, \hat{\alpha}_i, \hat{\beta}_i, \hat{\delta}_i)\}_{i=1}^d$, we compute the values $\hat{b}_i = \hat{\delta}_i\sqrt{\hat{\alpha}_i^2 - \hat{\beta}_i^2}$ and take its mean $\hat{b} = \frac{1}{d}\sum_{i=1}^d \hat{b}_i$ as estimate of the parameter b. Given $\hat{b}$, we again estimate $(\mu_i, \alpha_i, \beta_i, \delta_i)$ for each asset maximizing the log likelihood function $L(\mu_i, \alpha_i, \beta_i, \delta_i)$ subject to $\delta_i\sqrt{\alpha_i^2 - \beta_i^2} = \hat{b}$. Thus, we consider a multivariate NIG process where we have not a unique value α for all components of the vector (in this sense we get a generalization of the classic MNIG process). Since $\delta_i = \sigma_{ii}$, then we have to estimate the correlation matrix of the conditional Gaussian vector $X_t|Z_t$. Observe that the joint density function of the i-th and j-th assets is given by

$$f_{ij}(y^i, y^j; \mu_i, \beta_i, \delta_i, \mu_j, \beta_j, \delta_j, b, \rho_{ij}) = \int_0^\infty f_{\mathrm{N}_2}(y^i, y^j; \eta, \Sigma) f_{IG}(u; 1, b)du,$$

	α	β	δ	b	D_n
DJ-C65	86.4431	3.6188	0.0077	0.6650	0.0337
DJ-I	81.8854	3.3581	0.0081	0.6650	0.0325
DJ-U	80.5076	1.8582	0.0082	0.6650	0.0424
S&P500	81.2785	3.1116	0.0081	0.6650	0.0320
S&P100	77.6941	2.7709	0.0085	0.6650	0.0388
	DJ-C65	DJ-I	DJ-U	S&P500	S&P100
DJ-C65	1	0.9416	0.6061	0.9077	0.8907
DJ-I	0.9416	1	0.4858	0.9366	0.9373
DJ-U	0.6061	0.4858	1	0.5136	0.4804
S&P500	0.9077	0.9366	0.5136	1	0.9864
S&P100	0.8907	0.9373	0.4804	0.9864	1

Table 4.3: *Maximum likelihood estimates on daily basis under the NIG model.*

where f_{IG} is the density function of the Inverse Gaussian distribution with parameters $a = 1$ and $b > 0$, and f_{N_2} is the joint density function of the 2-dimensional Gaussian distribution with mean $\eta = (\mu_i + \beta_i \delta_i^2 u, \mu_j + \beta_j \delta_j^2 u)$ and covariance matrix

$$\Sigma = \begin{bmatrix} \delta_i^2 u & \delta_i \delta_j \rho_{ij} u \\ \delta_i \delta_j \rho_{ij} & \delta_j^2 u \end{bmatrix}.$$

Therefore, for each couple (i, j) of assets we estimate ρ_{ij} maximizing the log likelihood function

$$L(\rho_{ij}) = \sum_{k=1}^{n} \log \left(f_{ij}(y_k^i, y_k^j; \hat{\mu}_i, \hat{\beta}_i, \hat{\delta}_i, \hat{\mu}_j, \hat{\beta}_j, \hat{\delta}_j, \hat{b}, \rho_{ij}) \right).$$

In order to implement the estimate procedure above, we consider daily log returns of five market indexes (Down Jones Composite 65, Down Jones Industrials, Down Jones Utilities, S&P 500 Composite and S&P 100), observed during the period 04/10/1992-12/31/2005. In this empirical analysis we assume $\mu_i = 0$, $i = 1, \ldots, 5$. Table 4.1 reports, in the lower part, the estimate of the correla-

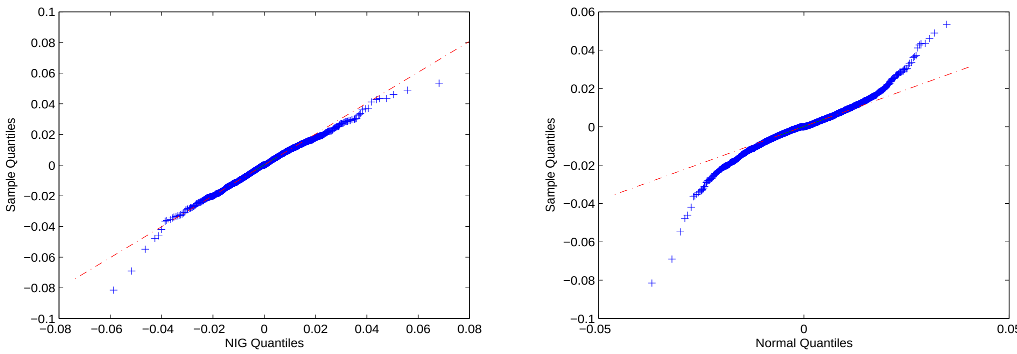

Figure 4.1: *QQ-plots of Down Jones Composite 65 sample data versus NIG and Normal distributions.*

tion matrix of $X_t|Z_t$, while, in the upper part, the estimates of the parameters $(\alpha_i, \beta_i, \delta_i)$ and the common parameter b. Further, In the last column we show the Kolmogorov-Smirnov distances $D_n = \sup_{-\infty<x<\infty} |F_{NIG}(x) - F_n(x)|$, where $F_{NIG}(x)$ is the theoretical distribution function and $F_n(x)$ the empirical one. Since under the Normal distribution the Kolmogorov-Smirnov distances are approximately 0.06 for all the indexes, we have an improvement by the NIG distribution. This result is confirmed by Figure 4.1 that displays QQplots of Down Jones Composite 65 sample data versus NIG and Normal distributions. From this analysis, it is worth noting that the NIG distribution describes the sample data better than the Normal distribution, in particular on the tails.

Gamma subordinator: the VG model

When $\{Z_t\}$ follows a Gamma process $G(\frac{1}{\nu}, \frac{1}{\nu})$, then the i-th log return follows a Variance Gamma process $VG(\mu_i, \theta_i, \sigma_i, \nu)$, where $\theta_i = \gamma_i$ and $\sigma_i = \sigma_{ii}$. Analogously, the portfolio (4.5) follows a $VG(\mu_w, \theta_w, \sigma_w, \nu)$ process whose parameters are:

$$\mu_w = w'\mu, \quad \theta_w = w'\gamma, \quad \sigma_w = \sqrt{w'Qw}.$$

Thus, mean, variance, skewness, and kurtosis of the portfolio $X_t^{(w)}$ are given by

$$\mathrm{E}[X_t^{(w)}] = t\mu_w + t\theta_w,$$

$$\mathrm{Var}[X_t^{(w)}] = t\sigma_w^2 + t\nu\theta_w^2,$$

$$\mathrm{Sk}[X_t^{(w)}] = \theta_w\nu(3\sigma_w^2 + 2\nu\theta_w^2)/(\sqrt{t}(\sigma_w^2 + \nu\theta_w^2)^{3/2}),$$

$$\mathrm{Ku}[X_t^{(w)}] = 3\left(1 + 2\nu/t - \nu\sigma_w^4/\left(t(\sigma_w^2 + \nu\theta_w^2)^2\right)\right).$$

As for the NIG process, in order to estimate all these parameters, we estimate the parameters $(\mu_i, \theta_i, \sigma_i, \nu_i)$ for each asset maximizing the log likelihood function

$$L(\mu_i, \theta_i, \sigma_i, \nu_i) = \sum_{k=1}^{n} \log\left(f_{VG}(y_k; \mu_i, \theta_i, \sigma_i, \nu_i)\right), \quad i = 1, \ldots, d,$$

where f_{VG} is the density function of the Variance Gamma process, y_k is the k-th observation of the i-th asset, and n is the sample size. Given the set of estimates $\{\hat{\mu}_i, \hat{\theta}_i, \hat{\sigma}_i, \hat{\nu}_i\}$, we take as estimate of ν the mean $\hat{\nu} = \frac{1}{d}\sum_{i=1}^{d} \hat{\nu}_i$. Then, for each asset we estimate again the parameters μ_i, θ_i, and σ_i, maximizing the log likelihood function

$$L(\mu_i, \theta_i, \sigma_i) = \sum_{k=1}^{n} \log\left(f_{VG}(y_k; \mu_i, \theta_i, \sigma_i, \hat{\nu})\right).$$

Finally, for each couple (i, j) of assets, we estimate the correlation coefficient ρ_{ij} maximizing the log likelihood function

$$L(\rho_{ij}) = \sum_{k=1}^{n} \log\left(f_{ij}(y_k^i, y_k^j; \hat{\mu}_i, \hat{\theta}_i, \hat{\sigma}_i, \hat{\mu}_j, \hat{\theta}_j, \hat{\sigma}_j, \hat{\nu}, \rho_{ij})\right),$$

where

$$f_{ij}(y^i, y^j; \mu_i, \theta_i, \sigma_i, \mu_j, \theta_j, \sigma_j, \nu, \rho_{ij}) = \int_0^\infty f_{\mathrm{N}_2}(y^i, y^j; \eta, \Sigma) f_G(u; 1/\nu, 1/\nu)du,$$

f_G is the density function of the Gamma distribution with parameters $a = b = 1/\nu$, and f_{N_2} is the density function of the 2-dimensional Normal distribution with mean $\eta = (\mu_i + \theta_i u, \mu_j + \theta_j u)$ and covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_i^2 u & \sigma_i\sigma_j\rho_{ij}u \\ \sigma_i\sigma_j\rho_{ij}u & \sigma_j^2 u \end{bmatrix}.$$

	θ	σ	ν	D_n	
DJ-C65	0.000321	0.000926	0.9661	0.0318	
DJ-I	0.000334	0.00982	0.9661	0.0308	
DJ-U	0.000192	0.01005	0.9661	0.0401	
S&P500	0.000314	0.00994	0.9661	0.0331	
S&P100	0.000310	0.01040	0.9661	0.0320	
	DJ-C65	DJ-I	DJ-U	S&P500	S&P100
DJ-C65	1	0.9434	0.6060	0.9079	0.8911
DJ-I	0.9434	1	0.4824	0.9371	0.9374
DJ-U	0.6060	0.4824	1	0.5068	0.4725
S&P500	0.9079	0.9371	0.5068	1	0.9868
S&P100	0.8911	0.9374	0.4725	0.9868	1

Table 4.4: *Maximum likelihood estimates on daily basis under the VG model.*

Consider again the daily log returns of the five market indexes introduced previously. Even under the VG model we assume $\mu_i = 0$, $i = 1, \ldots, 5$. The upper part of Table 4.2 exhibits the estimates (on daily basis) of the parameters $(\theta_i, \sigma_i, \nu)$, and the Kolmogorov-Smirnov distances which are quite similar to those of the NIG model. The lower part displays the estimate of the correlation matrix of $X_t|Z_t$. In Figure 4.2 we report the qq-plot of the Down Jones Composite 65 sample data versus the Variance Gamma distribution which can be compared with the analogous Normal qq-plot of Figure 4.1. Thus, even the Variance Gamma process provides a better distributional approximation with respect to the Brownian motion, since it takes into account heavier tails.

Remark on the estimate of the correlation matrix of $X_t|Z_t$

Since the estimate procedure above requires very high computational times, an alternative method is necessary in order to select portfolios with a large number of assets. The estimate procedure can be simplified using the sample correlation

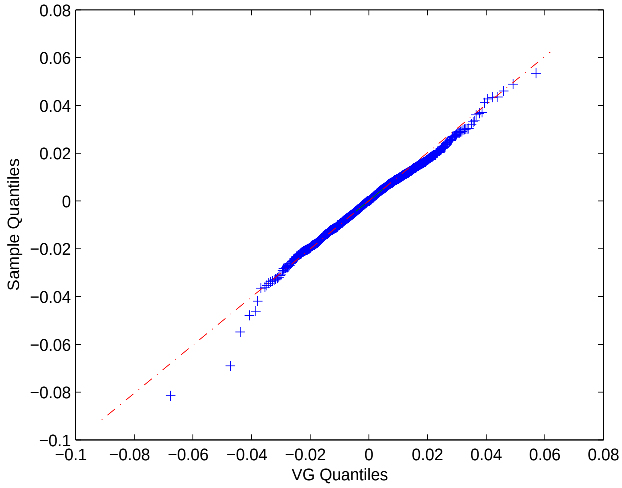

Figure 4.2: *QQ-plots of Down Jones Composite 65 sample data versus VG distribution.*

as estimator of the correlation matrix of $X_t|Z_t$. As a matter of fact, under the model (4.4), the correlation coefficient between the i-th and j-th log return is given by

$$\rho(X_t^{(i)}, X_t^{(j)}) = \frac{\sigma_{ii}\sigma_{jj}\rho_{ij}\mathrm{E}[Z_t] + \gamma_i\gamma_j\mathrm{Var}[Z_t]}{\left(\sigma_{ii}^2\mathrm{E}[Z_t] + \gamma_i^2\mathrm{Var}[Z_t]\right)^{1/2}\left(\sigma_{jj}^2\mathrm{E}[Z_t] + \gamma_j^2\mathrm{Var}[Z_t]\right)^{1/2}},$$

and, assuming $\gamma_i = \gamma_j = 0$,

$$\rho(X_t^{(i)}, X_t^{(j)}) = \rho_{ij}.$$

Then, since on daily basis the parameters γ_i, $i = 1, \ldots, d$, are very near to zero, we can assume the sample correlation as an approximation of the moment estimate of the correlation matrix of $X_t|Z_t$.

4.3 Ex-post comparison among optimal portfolios obtained under different Lévy processes

Consider the problem to select a portfolio among the previous five market indexes (Down Jones Composite 65, Down Jones Industrial, Down Jones Utilities, S&P

500 Composite, and S&P 100) assuming that the investor has a temporal horizon equal to one month. Then, suppose the investor decides to invest his money (1000 USD - his initial wealth) in the portfolio that maximizes the mean-Value at Risk ratio (see Favre and Galeano [31], Biglova *et al.* [9]):

$$\frac{\mathrm{E}[\mathrm{X}_{21}^{(\mathrm{w})} - \mathrm{r_f}]}{\mathrm{VaR}_{1\%}(X_{21}^{(w)} - r_f)},$$

where $r_f = 0.3884\%$ is the 1-month log return of LIBOR on 12/31/2005, $X_{21}^{(w)}$ is the portfolio of monthly log returns (i.e., $X_t^{(w)}$ valued at time $t = 21$ days), and the Value at Risk $\mathrm{VaR}_{1\%}$ of the continuous random variable $X_{21}^{(w)} - r_f$ is the opposite of the 1% quantile. Then, we assume no short sales are allowed, that is $w_i \geq 0$, $i = 1, \ldots, 5$, and $\sum_{i=1}^{5} w_i = 1$. Thus, we remark that the problem is well posed, since $\mathrm{VaR}_{1\%}$ of every portfolio is positive. In order to take into account skewness (generally different from zero) and kurtosis we approximate $\mathrm{VaR}_{1\%}(X_{21}^{(w)} - r_f)$ with the Iaquinta *et al.*'s approximation (see Iaquinta *et al.* [44]). Therefore,

$$\mathrm{VaR}_{1\%}(X_{21}^{(w)} - r_f) = -\left(\mathrm{E}[X_{21}^{(w)}] + h_w\sqrt{\mathrm{Var}[\mathrm{X}_{21}^{(\mathrm{w})}]} - r_f\right), \tag{4.6}$$

where

$$\begin{aligned} h_w &= \left(\frac{\mathrm{Ku}[X_{21}^{(w)}] - 1}{2\mathrm{Sk}[X_{21}^{(w)}]}\right) - f(X_{21}^{(w)})\frac{1}{2}\left(\left(\frac{\mathrm{Ku}[X_{21}^{(w)}] - 1}{\mathrm{Sk}[X_{21}^{(w)}]}\right)^2 + 4\right. \\ &\left. + f(X_{21}^{(w)})\frac{4p_{99\%}\sqrt{\left(\mathrm{Ku}[X_{21}^{(w)}] - 1 - (\mathrm{Sk}[X_{21}^{(w)}])^2\right)(\mathrm{Ku}[X_{21}^{(w)}] - 1)}}{|\mathrm{Sk}[X_{21}^{(w)}]|}\right)^{1/2}, \end{aligned}$$

$$f(X_{21}^{(w)}) = \begin{cases} -1 & \text{if} \quad \mathrm{Sk}[X_{21}^{(w)}] < 0 \quad \text{and} \quad d(X_{21}^{(w)}) \geq 0 \\ 1 & \text{otherwise,} \end{cases}$$

$$\begin{aligned} d(X_{21}^{(w)}) &= \left(\frac{\mathrm{Ku}[X_{21}^{(w)}] - 1}{\mathrm{Sk}[X_{21}^{(w)}]}\right)^2 + 4 \\ &- \frac{4p_{99\%}\sqrt{\left(\mathrm{Ku}[X_{21}^{(w)}] - 1 - (\mathrm{Sk}[X_{21}^{(w)}])^2\right)(\mathrm{Ku}[X_{21}^{(w)}] - 1)}}{|\mathrm{Sk}[X_{21}^{(w)}]|}, \end{aligned}$$

and $p_{99\%}$ is the 99% quantile of the standard Normal distribution. We add in passing that one could estimate $\text{VaR}_{1\%}$ using the Cornish-Fisher expansion (as suggested by Favre and Galeano [31]), in this case h_w is given by:

$$h_w = p_{1\%} + \frac{1}{6}\left(p_{1\%}^2 - 1\right)\text{Sk}[X_{21}^{(w)}] + \frac{1}{24}\left(p_{1\%}^3 - 3p_{1\%}\right)(\text{Ku}[X_{21}^{(w)}] - 3),$$

and $p_{1\%}$ is the 1% quantile of the standard Normal distribution. Taking this into account, we solve the optimization problem

$$\begin{cases} \max\limits_{w} \frac{\text{E}\left[X_{21}^{(w)} - r_f\right]}{\text{VaR}_{1\%}\left(X_{21}^{(w)} - r_f\right)} \\ \text{s.t.} \\ \sum_{i=1}^{5} w_i = 1, \quad w_i \geq 0, \quad i = 1, \ldots, 5, \end{cases} \tag{4.7}$$

under the three possible distributional assumptions:

1. Normal Inverse Gaussian, $X_{21}^{(w)} \sim NIG(\alpha_w, \beta_w, 21\delta_w)$;
2. Variance Gamma, $X_{21}^{(w)} \sim VG(21\theta_w, \sqrt{21}\sigma_w, \nu/21)$;
3. Brownian motion, $X_{21}^{(w)} \sim \text{N}(21\mu_w, 21\sigma_w)$.

Under the NIG and VG model the parameter estimates are those of Tables 4.3 and 4.4, respectively, and under the BM model $\mu_w = w'\mu$ and $\sigma_w = \sqrt{w'Qw}$ where μ and Q are the empirical mean and variance-covariance matrix. As solution of the problem (4.7) we obtain the optimal portfolio weights of Table 4.5. The three

	DJ-C65	DJ-I	DJ-U	S&P500	S&P100
$w_{(NIG)}$	0.3265	0.6735	0	0	0
$w_{(VG)}$	0.2307	0.7693	0	0	0
$w_{(BM)}$	0.3189	0.6811	0	0	0

Table 4.5: *Market portfolio W_M under the three distributional assumptions.*

optimal portfolios are composed by the same assets. In particular, under the BM and NIG distributional assumptions the portfolio composition is almost the same.

While, the VG model presents a significant difference in the portfolio compositions with respect to the other two processes. In order to value the impact of these choices we consider an investor who recalibrates the portfolio every month during the year 2006 such that the percentages in the portfolio composition remain the same under each distributional assumptions. Table 4.6 reports the ex-post

	NIG	**VG**	**BM**
01/01/06	1000	1000	1000
01/02/06	**1022.98**	1022.71	1022.95
01/03/06	**1035.14**	1034.03	1035.06
01/04/06	**1041.41**	1040.97	1041.38
01/05/06	**1058.52**	1058.49	1058.52
01/06/06	**1056.93**	1055.09	1056.79
01/07/06	**1061.37**	1057.33	1061.05
01/08/06	**1040.21**	1039.61	1040.16
01/09/06	1067.30	**1068.02**	1067.36
01/10/06	1084.39	**1085.74**	1084.50
01/11/06	1121.98	**1122.18**	1122.00
01/12/06	1136.25	**1136.73**	1136.29
01/01/07	1153.31	**1156.15**	1153.54

Table 4.6: *Monthly evolutions of* $w_{(NIG)}$, $w_{(VG)}$, *and* $w_{(BM)}$.

monthly evolutions of $w_{(NIG)}$, $w_{(VG)}$, and $w_{(BM)}$, supposing the investor's initial wealth is 1000 USD. Observe that there are not significant differences among the final wealths obtained under the NIG, VG processes and Brownian motion. However, both alternative processes (NIG, VG) present a better performance in different periods of the year, even if during the 2006 the market was growing and the asset prices did not show big jumps.

4.3.1 Ex-post comparison among optimal portfolio strategies with transaction costs and no short sales: dynamic selection

Let us compare dynamic strategies with constant and proportional transaction costs of $K = 0.05\%$ when short sales are not permitted. Assume an investor which has an initial wealth of 1000 USD and decides to invest this money in the portfolio that maximizes the mean-VaR ratio recalibrating it every month. As for the previous empirical analysis we consider five indexes (Down Jones Composite 65, Down Jones Industrials, Down Jones Utilities, S&P 500 composite, and S&P100) and a monthly riskless asset with log return $r_f = 0.3884\%$. Since we want to compare the ex-post sample paths of the investor's wealth under different distributional assumptions, then we follow the same algorithm proposed by Biglova *et al.* [9], Ortobelli *et al.* [65], and Leccadito *et al.* [47]. That is, we first consider an initial wealth W_0 and in the ex-post analysis we calibrate the portfolio 12 times. Once we have chosen a distributional assumption, after k periods, the main steps to compute the ex-post final wealth in the mean-VaR context are the following:

Step 1. At the k-th period, $k = 0, 1, \dots, 11$, we determine the market portfolio $w^{(k)}$ that maximizes the mean-VaR ratio, i.e., we solve the optimization problem:

$$\begin{cases} \max\limits_{w^{(k)}} \frac{\mathrm{E}\left[X_{21}^{(w^{(k)})} - r_f - t.c.(k)\right]}{\mathrm{VaR}_{1\%}\left(X_{21}^{(w^{(k)})} - r_f\right) + t.c.(k)}, \\ \text{s.t.} \\ \sum_{i=1}^{5} w_i^{(k)} = 1, \quad w_i^{(k)} \geq 0, \quad i = 1, \dots, 5, \end{cases}$$

where $\mathrm{VaR}_{1\%}\left(X_{21}^{(w^{(k)})} - r_f\right)$ is given by equation (4.6), the transaction costs are

$$t.c.(k) = \begin{cases} K \sum_{i=1}^{5} \left| w_i^{(k)} - \frac{w_i^{(k-1)}(1+r^{(k-1)})}{\sum_{i=1}^{5} w_i^{(k-1)}(1+r_i^{(k-1)})} \right| & \text{if} \quad k > 1 \\ K = 0.05\% & \text{if} \quad k = 0, \end{cases}$$

and $r_i^{(k-1)}$ is the observed i-th monthly return valued on the period $[t_{k-1}, t_k]$.

	DJ-C65	DJ-I	DJ-U	S&P500	S&P100
NIG	0.1967	0.8033	0	0	0
VG	0.0673	0.9327	0	0	0
BM	0.1386	0.8614	0	0	0

Table 4.7: *Portfolio* $w^{(0)}$ under the three distributional hypotheses.

Step 2. We value the ex-post final wealth after k periods by

$$W_k = W_{k-1}\left(\sum_{i=1}^{5} w_i^{(k-1)}(1+r_i^{(k-1)}) - t.c.(k)\right).$$

Step 3. We repeat steps 1 and 2 for each distributional hypotheses.

Since during the 2006 the indexes used in this analysis do not present very big jumps with respect to their expected value, then we do not observe very big differences among the optimal portfolios. Thus, the use of the transaction costs has implied that the optimal portfolio weights do not change at the times of calibration. That is, the investor chooses his first portfolio $w^{(0)}$ and all the other optimal portfolio $w^{(k)}$ are given by the evolution of $w^{(0)}$ up to the k-th period.

In Table 4.7 we report the weights of the portfolio $w^{(0)}$ under the three different distributional assumptions (NIG, VG, BM). The assets that appear in the optimal portfolio are the same for each distributional hypothesis and with small differences. Table 4.8 exhibits the ex-post final wealth sample paths under the three distributional assumptions. As for the previous comparison of Table 4.6 we observe a better performance of the VG and NIG processes in different periods of the year.

4.4 Multi-period portfolio selection with unlimited short sales

In this section we always model the asset log returns as a multidimensional time-changed Brownian motion where the subordinator follows or a Inverse Gaussian

	NIG	**VG**	**BM**
01/01/06	1000	1000	1000
01/02/06	**1022.12**	1021.75	1021.95
01/03/06	**1033.13**	1031.63	1032.46
01/04/06	**1040.29**	1039.69	1040.02
01/05/06	**1057.94**	1057.89	1057.92
01/06/06	**1053.91**	1051.42	1052.79
01/07/06	**1055.44**	1049.97	1052.98
01/08/06	**1038.72**	1037.97	1038.38
01/09/06	1067.57	**1068.59**	1068.03
01/10/06	1085.51	**1087.39**	1086.35
01/11/06	1121.48	**1121.83**	1121.64
01/12/06	1136.12	**1136.84**	1136.45
01/01/07	1155.82	**1159.71**	1157.57

Table 4.8: *Evolutions of market portfolios with transaction costs under NIG, VG, BM models.*

process or a Gamma process. Under these different distributional hypotheses, we compare the portfolio strategies with the assumption that the log returns follow a Brownian motion. The literature in the multi-period portfolio selection has been dominated by the results of maximizing expected utility function of terminal wealth and/or multi-period consumption. Differently from classic multi-period portfolio selection approaches, we consider mean-variance analysis alternative to that proposed by Li and Ng's [48] by giving a mean-dispersion formulation of the optimal dynamic strategies. Moreover, we also discuss a mean, variance, skewness, and kurtosis extension of the original multi-period portfolio selection problem. In order to compare the dynamic strategies under the different distributional assumptions, we analyze two investment allocation problems. The primary contribution of this empirical comparison is the analysis of the impact of distributional assumptions and different term structures on the multi-period asset allocation decisions. Thus, we propose a performance comparison among different Lévy processes and taking into consideration three different implicit term structures. For this purpose we discuss the optimal allocation obtained by different risk averse investors with different risk aversion coefficients. We determine the multi-period optimal choices given by the minimization of the variance for different levels of final wealth average. Each investor, characterized by his/her utility function, will prefer the mean-variance model which maximizes his/her expected utility on the efficient frontier. Thus, the portfolio policies obtained with this methodology represent the optimal investors' choices of the different approaches.

According to the multivariate and subordinated Lévy model introduced in section 4.2, given a market with d risky assets, the log return of the portfolio with weights $w = [w_1, \ldots, w_d]$, where $\sum_{i=1}^{d} w_i = 1$, is distributed at time t as

$$X_t^{(w)} = (w'\mu)t + (w'\gamma)Z_t + \sqrt{w'Qw}W_{Z_t},$$

where $\mu = [\mu_1, \ldots, \mu_d]'$, $\gamma = [\gamma_1, \ldots, \gamma_d]'$, $Q = [\sigma_{ij}^2]_{ij}$ is e fixed definitive positive variance-covariance matrix, $\{Z_t\}$ is a positive Lévy subordinator, and $\{W_t\}$ is a 1-dimensional Brownian motion. When $\{Z_t\}$ is an Inverse Gaussian process with parameters $a = 1$ and $b > 0$, $IG(1, b)$, then the log return process $\{X_t^{(w)}\}$ follows a Normal Inverse Gaussian process, $NIG(\mu_w, \alpha_w, \beta_w, \delta_w)$, with parameters

$$\mu_w = w'\mu, \quad \alpha_w = \sqrt{\left(\tfrac{b}{\delta_w}\right)^2 + \beta_w^2}, \quad \beta_w = \tfrac{w'\gamma}{\delta_w^2}, \quad \delta_w = \sqrt{w'Qw}.$$

Instead, when $\{Z_t\}$ is a Gamma process with parameters $a = b = 1/\nu$, $G(\frac{1}{\nu}, \frac{1}{\nu})$, then the log return process $\{X_t^{(w)}\}$ follows a Variance Gamma process, $VG(\mu_w, \theta_w, \sigma_w, \nu_w)$, with parameters

$$\mu_w = w'\mu, \quad \theta_w = w'\gamma, \quad \sigma_w = \sqrt{w'Qw} \quad \nu_w = \nu.$$

Generally in portfolio theory the vector of the asset log returns is modeled as a multivariate Brownian motion, and, under this hypothesis, the portfolio log return $\{X_t^{(w)}\}$ is distributed at time t as a normal distribution with mean $(w'\mu)t$ and standard deviation $\sqrt{t(w'Qw)}$.

Suppose an investor has temporal horizon t_T and calibrates its portfolio T times at some intermediate dates, say $t = t_0, \ldots, t_{T-1}$ (where $t_0 = 0$). Since Lévy processes have independent and stationary increments the distribution of the random vector of log returns on the period $(t_j, t_{j+1}]$ (i.e., $\widetilde{X}_{t_{j+1}} - \widetilde{X}_{t_j}$) is the same of $\widetilde{X}_{t_{j+1}-t_j} = [X^{(1)}_{t_{j+1}-t_j}, \ldots, X^{(d)}_{t_{j+1}-t_j}]'$. Considering that log returns represent a good approximation of returns when $t_{j+1} - t_j$ is little enough, we assume that $\max\limits_{j=0,\ldots,T-1}(t_{j+1} - t_j)$ is less or equal than one month and use $\widetilde{Y}_{t_j} \equiv \widetilde{X}_{t_{j+1}} - \widetilde{X}_{t_j} = [Y_{1,t_j}, \ldots, Y_{d,t_j}]'$ to estimate the vector of returns on the period $(t_j, t_{j+1}]$. Suppose the deterministic variable r_{0,t_j} represents the return on the period $(t_j, t_{j+1}]$ of the risk-free asset, x_{i,t_j} the amount invested at time t_j in the i-th risky asset, and x_{0,t_j} the amount invested at time t_j in the risk-free asset. Then, the investor's wealth at time t_{j+1} is given by

$$\mathbf{W}_{t_{j+1}} = \sum_{i=0}^{d} x_{i,t_j}(1 + Y_{i,t_j}) = \mathbf{W}_{t_j}(1 + r_{0,t_j}) + x'_{t_j}\widetilde{P}_{t_j}, \tag{4.8}$$

where $x_{t_j} = [x_{1,t_j}, \ldots, x_{d,t_j}]'$, $\widetilde{P}_{t_j} = [P_{1,t_j}, \ldots, P_{d,t_j}]'$ is the vector of excess returns $P_{i,t_j} = Y_{i,t_j} - r_{0,t_j}$. Thus, the final wealth is given by

$$\mathbf{W}_{t_T} = \mathbf{W}_0 \prod_{j=0}^{T-1}(1 + r_{0,t_j}) + \sum_{j=0}^{T-2} x_{t_j}\widetilde{P}_{t_j} \prod_{k=j+1}^{T-1}(1 + r_{0,t_k}) + x_{t_{T-1}}\widetilde{P}_{t_{T-1}}, \tag{4.9}$$

where the initial wealth $\mathbf{W}_0 = \sum_{i=0}^{d} x_{i,0}$ is known. Assume that the amounts $x_{t_j} = [x_{1,t_j}, \ldots, x_{d,t_j}]'$ are deterministic variables, while the amount invested in the risk-free asset is the random variable $x_{0,t_j} = \mathbf{W}_{t_j} - x'_{t_j}e$, where $e = [1, \ldots, 1]'$. Under these assumptions the mean, variance, skewness, and kurtosis of the final wealth are respectively

$$\mathrm{E}[\mathbf{W}_{t_T}] = \mathbf{W}_0 B_0 + \sum_{j=0}^{T-1} \mathrm{E}[x'_{t_j}\widetilde{P}_{t_j}]B_{j+1},$$

$$\mathrm{Var}[\mathbf{W}_{t_T}] = \sigma(\mathbf{W}_{t_T})^2 = \sum_{j=0}^{T-1}\left(x'_{t_j}Q_{t_j}x_{t_j}\right)B^2_{j+1},$$

$$\text{Sk}[\mathbf{W}_{t_T}] = \frac{\sum_{j=0}^{T-1} \text{E}\left[\left(x'_{t_j}\widetilde{P}_{t_j} - \text{E}[x'_{t_j}\widetilde{P}_{t_j}]\right)^3\right] B_{j+1}^3}{\sigma(\mathbf{W}_{t_T})^3},$$

$$\begin{aligned}\text{Ku}[\mathbf{W}_{t_T}] &= \frac{1}{\sigma(\mathbf{W}_{t_T})^4}\left(6\sum_{j=0}^{T-1}\sum_{k=j+1}^{T-1} B_{j+1}^2 B_{k+1}^2 x'_{t_j} Q_{t_j} x_{t_j}(x'_{t_k} Q_{t_k} x_{t_k})\right. \\ &\left. + \sum_{j=0}^{T-1} \text{E}\left[\left(x'_{t_j}\widetilde{P}_{t_j} - \text{E}[x'_{t_j}\widetilde{P}_{t_j}]\right)^4\right] B_{j+1}^4\right),\end{aligned}$$

where $B_T = 1$, $B_j = \prod_{k=j}^{T-1}(1 + r_{0,t_k})$, and the elements of the matrix $Q_{t_j} = [q_{ik,t_j}]_{ik}$, $j = 0, 1, \ldots, T-1$, are $q_{ik,t_j} = \text{Cov}[P_{i,t_j}, P_{k,t_j}] = \text{Cov}[X^{(i)}_{t_{j+1}-t_j}, X^{(k)}_{t_{j+1}-t_j}]$. Therefore, if we want to select the optimal portfolio strategies that solve the mean-variance problem

$$\begin{cases} \min\limits_{x_{t_0},\ldots,x_{t_{T-1}}} \text{Var}[\mathbf{W}_{t_T}] \\ \text{s.t.} \quad \text{E}[\mathbf{W}_{t_T}] = m, \end{cases}$$

we can use the closed form solutions determined by Ortobelli *et al.* [65]. These solutions for Lévy subordinated processes are given by

$$x_{t_j} = \frac{m - \mathbf{W}_0 B_0}{B_{j+1}\sum_{k=0}^{T-1} \text{E}[\widetilde{P}_{t_k}]' Q_{t_j}^{-1} \text{E}[\widetilde{P}_{t_k}]} Q_{t_j}^{-1}\text{E}[\widetilde{P}_{t_j}], \quad j = 0, 1, \ldots, T-1. \tag{4.10}$$

The optimal wealth invested in the riskless asset at time $t_0 = 0$ is the deterministic quantity $x_{0,t_0} = \mathbf{W}_0 - x'_{t_0}e$, while at time t_j it is given by the random variable $x_{0,t_j} = \mathbf{W}_{t_j} - x'_{t_j}e$, where $\mathbf{W}_{t_j}$ is formulated in equation (4.8). Observe that the covariance q_{ik,t_j} among components of the d-dimensional vector $\widetilde{X}_{t_{j+1}-t_j} = \mu(t_{j+1} - t_j) + \gamma Z_{t_{j+1}-t_j} + Q^{1/2}\widetilde{W}_{Z_{t_{j+1}-t_j}}$ is given by

$$q_{ik,t_j} = \sigma_{ik}^2 \text{E}[Z_{t_{j+1}-t_j}] + \mu_i\mu_k \text{Var}[Z_{t_{j+1}-t_j}],$$

where σ_{ik}^2 are the elements of the matrix $Q = [\sigma_{ik}^2]$ (see, among others, Cont and Tankov [21]). So, for example, in the case the vector of log returns $\{\widetilde{X}_t\}$ follows a NIG process we can rewrite the formulas of mean, variance, skewness, and kurtosis of final wealth by using the following equations:

$$\text{E}[x'_{t_j}\widetilde{P}_{t_j}] = (t_{j+1} - t_j)(b^{-1}x'_{t_j}\gamma + x'_{t_j}\gamma) - r_{0,t_j}x'_{t_j}e,$$

$$q_{ik,t_j} = \frac{\delta_i\delta_k\rho_{ik}}{b}(t_{j+1}-t_j) + \frac{\beta_i\beta_k\delta_i^2\delta_k^2}{b^3}(t_{j+1}-t_j),$$

$$\mathrm{Sk}[\mathbf{W}_{t_T}] = \frac{3\sum_{j=0}^{T-1} B_{j+1}^3(t_{j+1}-t_j)x'_{t_j}\gamma(b^2x'_{t_j}Q_{t_j}x_{t_j}+(x'_{t_j}\gamma)^2)}{\sigma(\mathbf{W}_{t_T})^3b^5},$$

$$\begin{aligned}\mathrm{Ku}[\mathbf{W}_{t_T}] \quad = \quad & \frac{6\sum_{j=0}^{T-1}\sum_{k=j+1}^{T-1} B_{j+1}^2B_{k+1}^2x'_{t_j}Q_{t_j}x_{t_j}(x'_{t_k}Q_{t_k}x_{t_k})}{\sigma(\mathbf{W}_{t_T})^4} \\ & +\frac{3\sum_{j=0}^{T-1} B_{j+1}^4(t_{j+1}-t_j)^2(b^2x'_{t_j}Q_{t_j}x_{t_j}+(x'_{t_j}\gamma)^2)^2}{\sigma(\mathbf{W}_{t_T})^4b^6} \\ & +\frac{3\sum_{j=0}^{T-1} B_{j+1}^4(t_{j+1}-t_j)(b^2x'_{t_j}Q_{t_j}x_{t_j}+5(x'_{t_j}\gamma)^2)(b^2x'_{t_j}Q_{t_j}x_{t_j}+(x'_{t_j}\gamma)^2)}{\sigma(\mathbf{W}_{t_T})^4b^7}.\end{aligned}$$

Instead, if $\{\widetilde{X}_t\}$ follows a Variance Gamma process these formulas become:

$$\mathrm{E}[x'_{t_j}\widetilde{P}_{t_j}] = (t_{j+1}-t_j)(x'_{t_j}\gamma + x'_{t_j}\mu) - r_{0,t_j}x'_{t_j}e,$$

$$q_{ik,t_j} = \sigma_{ii}\sigma_{kk}\rho_{ik}(t_{j+1}-t_j) + \nu\mu_i\mu_k(t_{j+1}-t_j),$$

$$\mathrm{Sk}[\mathbf{W}_{t_T}] = \frac{\sum_{j=0}^{T-1} B_{j+1}^3(t_{j+1}-t_j)\nu x'_{t_j}\gamma(3x'_{t_j}Q_{t_j}x_{t_j}+2\nu(x'_{t_j}\gamma)^2)}{\sigma(\mathbf{W}_{t_T})^3},$$

$$\begin{aligned}\mathrm{Ku}[\mathbf{W}_{t_T}] = & \frac{6\sum_{j=0}^{T-1}\sum_{k=j+1}^{T-1} B_{j+1}^2B_{k+1}^2x'_{t_j}Q_{t_j}x_{t_j}(x'_{t_k}Q_{t_k}x_{t_k})}{\sigma(\mathbf{W}_{t_T})^4} \\ & -\frac{\sum_{j=0}^{T-1} 3B_{j+1}^4(\nu(x'_{t_j}Q_{t_j}x_{t_j})^2(t_{j+1}-t_j))}{\sigma(\mathbf{W}_{t_T})^4} \\ & +\frac{\sum_{j=0}^{T-1} 3B_{j+1}^4((1+2\nu/(t_{j+1}-t_j))(x'_{t_j}Q_{t_j}x_{t_j}(t_{j+1}-t_j)+\nu(x'_{t_j}\gamma)^2(t_{j+1}-t_j))^2)}{\sigma(\mathbf{W}_{t_T})^4}.\end{aligned}$$

Clearly, a more realistic portfolio selection problem should consider the investor's preference for skewness (see, among others, Ortobelli [64]). Thus, under the above distributional assumptions and under institutional restrictions of the market, such as no short sales and limited liability, all risk-averse investors optimize their portfolios choosing the solution of the following constrained optimization problem:

$$\begin{cases} \min\limits_{x_{t_0},\ldots,x_{t_{T-1}}} \mathrm{Var}[\mathbf{W}_{t_T}] \\ s.t. \\ \mathrm{E}[\mathbf{W}_{t_T}], \quad \mathrm{Sk}[\mathbf{W}_{t_T}] \geq q_1, \quad \mathrm{Ku}[\mathbf{W}_{t_T}] \leq q_2, \\ x_{i,t_j} \geq 0, \quad i=1,\ldots,d, \quad j=0,1,\ldots,T-1, \end{cases}$$

	$\mathbf{t}_0$	$\mathbf{t}_1$	$\mathbf{t}_2$	$\mathbf{t}_3$	$\mathbf{t}_4$
term1	0.3884%	0.3984%	0.4084%	0.4184%	0.4284%
term2	0.3884%	0.3884%	0.3884%	0.3884%	0.3884%
term3	0.3884%	0.3784%	0.3684%	0.3584%	0.3484%

Table 4.9: *Term structures.*

for some mean m, skewness q_1, and kurtosis q_2. This problem has not generally closed form solution. However, using arguments similar to those proposed by Athayde and Flôres [2] based on a tensorial notation for the higher moments we can give an implicit analytical solution when unlimited short sales are allowed.

Let us examine the performances of Lévy approaches and compare Gaussian and Lévy non-Gaussian dynamic portfolio choice strategies when short sales are allowed. Since we work in a mean-variance framework, we do not value the effects of skewness and kurtosis. First, we analyze the optimal dynamic strategies during a period of five months among the riskless return and 5 monthly index returns from 04/10/1992 to 12/31/2005. The market indexes used in this analysis are always those of section 4.3, and, under NIG and VG models, the parameter estimates are given by Tables 4.3 and 4.4, respectively, while under the Brownian motion (BM) model the estimates of μ and Q are given by the sample mean and covariances. We start with a monthly riskless return of 0.3884% and examine the different allocations considering three different implicit term structures. Table 4.9 reports the implicit term structures that we will use in this comparison. In particular, we approximate optimal solutions to the utility functional:

$$\max_{\{x_{t_j}\}_{j=0,1,\ldots,T-1}} \mathrm{E}\left[1 - \exp\left(-\frac{1}{a}\mathbf{W}_{t_T}\right)\right], \tag{4.11}$$

where a (we use $a = 0.5, 1, 1.5, 2$) is an indicator of the risk tolerance and $\mathbf{W}_{t_T}$ is defined by formula (4.9). Secondly, we consider the utility functional:

$$\max_{\{x_{t_j}\}_{j=0,1,\ldots,T-1}} \mathrm{E}[u(\mathbf{W}_{t_T})] \tag{4.12}$$

where

$$u(x) = \begin{cases} cx - \frac{1}{c}x^2 & \text{if} \quad x < \frac{c^2}{2} \\ \ln(x) + \left(\frac{c^3}{4} - \ln\left(\frac{c^2}{2}\right)\right) & \text{if} \quad x \geq \frac{c^2}{2}, \end{cases}$$

and thus for $x < c^2/2$ we have a quadratic utility function and for $x \geq c^2/2$ a logarithm utility function (we use $c = 1, 2, 3, 4, 5$). Clearly, we could obtain close form solutions to optimization problems (4.11) and (4.12) using arguments on the moments and on the Laplace transform. However, since we want to value the impact of different distributional assumptions in a mean-variance framework we will approximate formulas (4.11) and (4.12) using the historical observations of the final wealth valued for the optimal mean-variance portfolios. In particular, we use the same algorithm proposed by Ortobelli *et al.* [65] in order to compare the different models. Observe that transaction costs are not modeled in formula (4.10), but, as we have seen in section 4.3 there are not very big differences among portfolio choices with and without constant proportional transaction costs. Therefore, we do not consider transaction costs in this analysis. Thus, we select the optimal portfolio strategies on the efficient frontiers which are solutions of problems (4.11) and (4.12) for different coefficients a and c. Therefore, starting by an initial wealth $\mathbf{W}_0 = 1$ we compute for every multi-period efficient frontier:

$$\begin{cases} \max\limits_{\{x_{t_j}\}_{j=0,1,\ldots,4}} \frac{1}{N}\sum_{i=1}^{N} u(\mathbf{W}_5^{(i)}) \\ \text{s.t} \\ \{x_{t_j}\}_{j=0,1,\ldots,4} \quad \text{are optimal portfolio strategies (4.10)}, \end{cases}$$

where $\mathbf{W}_5^{(i)} = B_0 + \sum_{j=0}^{4} x'_{t_j} p_{t_j}^{(i)} B_{j+1}$ is the i-th observation of the final wealth and $p_{t_j}^{(i)} = [p_{1,t_j}^{(i)}, \ldots, p_{d,t_j}^{(i)}]'$ is the i-th observation of the vector of excess returns $p_{k,t_j}^{i} = r_{k,t_j}^{(i)} - r_{0,t_j}$ relative to the j-th period. Finally, we obtain Table 4.10 with the approximated maximum expected utility considering the three implicit term structures. In fact, we implicitly assume the approximation $\frac{1}{N}\sum_{i=1}^{N} u(\mathbf{W}_5^{(i)}) \approx \mathrm{E}[u(\mathbf{W}_5^{(i)})]$. Table 4.10 shows a superior performance of Lévy non Gaussian models with respect to the Gaussian one by the point of view of investors that max-

	Utility function (4.11)								
	Term1			Term2			Term3		
	BM	VG	NIG	BM	VG	NIG	BM	VG	NIG
a=0.5	0.8727	**0.8731**	0.8728	0.8727	**0.8731**	0.8728	0.8726	**0.8732**	0.8730
a=1	0.6468	**0.6479**	0.6473	0.6471	**0.6485**	0.6479	0.6477	**0.6491**	0.6484
a=1.5	0.5037	**0.5053**	0.5045	0.5044	**0.5062**	0.5053	0.5052	**0.5073**	0.5063
a=2	0.4117	**0.4136**	0.4126	0.4127	**0.4148**	0.4137	0.4138	**0.4161**	0.4150
	Utility function (4.12)								
	Term1			Term2			Term3		
	BM	VG	NIG	BM	VG	NIG	BM	VG	NIG
c=1	0.9942	**0.9973**	0.9964	0.9983	**1.0025**	1.0012	1.0031	**1.0083**	1.0065
c=2	1.5396	**1.5422**	1.5407	1.5404	**1.5436**	1.5422	1.5419	**1.5453**	1.5436
c=3	2.8763	**2.8994**	2.8880	2.8910	**2.9168**	2.9043	2.9073	**2.9361**	2.9225
c=4	4.3106	**4.3799**	4.3454	4.3578	**4.4359**	4.3980	4.4106	**4.4974**	4.4565
c=5	5.9522	**6.1025**	6.0276	6.0572	**6.2264**	6.1443	6.1745	**6.3623**	6.2738

Table 4.10: *Maximum expected utility under two utility functions, three distributional hypotheses, and three term structures.*

imize expected utility (4.11) and (4.12). In particular, the Variance Gamma model presents the best performance for different utility functions and term structures. Thus, from an ex-ante comparison among Variance Gamma, Normal Inverse Gaussian, and Brownian motion models, investors characterized by the utility functions (4.11) and (4.12) should select portfolios assuming a Variance Gamma distribution. The term structure determines the biggest differences in the portfolio weights of the same strategy and different periods. When the interest rates of the implicit term structure are growing (decreasing) we obtain that the investors are more (less) attracted to invest in the riskless in the sequent period. Generally it does not exist a common factor between portfolio weights of different periods and the same strategy. However, when we consider the flat term structure (2-nd term structure), the portfolio weights change over the time with the same capitalization factor. Table 4.11 shows the ex-post final wealth under

	Utility function (4.11)								
	Term1			Term2			Term3		
	BM	VG	NIG	BM	VG	NIG	BM	VG	NIG
a=0.5	1.0762	**1.0805**	1.0711	1.0716	**1.0755**	1.0670	1.0672	**1.0966**	1.0851
a=1	1.1319	**1.1404**	1.1470	1.1498	**1.1594**	1.1381	1.1401	**1.1487**	1.1295
a=1.5	1.1876	**1.2304**	1.1976	1.2019	**1.2154**	1.1856	1.1888	**1.2269**	1.1961
a=2	1.2433	**1.2903**	1.2482	1.2540	**1.2993**	1.2568	1.2617	**1.3050**	1.2627
	Utility function (4.12)								
	Term1			Term2			Term3		
	BM	VG	NIG	BM	VG	NIG	BM	VG	NIG
c=1	1.3269	**1.3503**	1.3241	1.3322	**1.3553**	1.3279	1.3347	**1.3571**	1.3294
c=2	1.1319	**1.1404**	1.1217	1.1237	**1.1594**	1.1381	1.1401	**1.1487**	1.1295
c=3	1.4383	**1.4702**	1.4252	1.4364	**1.4951**	1.4228	1.4320	**1.4873**	1.4404
c=4	1.8281	**1.9198**	1.8299	1.8532	**1.9428**	1.8261	1.8698	**1.9561**	1.8622
c=5	2.3573	**2.5194**	2.3358	2.3743	**2.5583**	2.3717	2.4049	**2.5812**	2.3951

Table 4.11: *Ex-post final wealths on date 06/01/06, investing 1 USD on 01/01/06 and using the optimal strategies solutions of the problems* (4.11) *and* (4.12) *under three distributional hypotheses and three term structures.*

the three term structures for the three distributional assumptions and two utility functions. These results confirm the better performance of the Variance Gamma approach with respect to the Normal Inverse Gaussian and Brownian motion ones. Moreover, in this ex-post comparison we observe a better performance of the Brownian motion with respect to the NIG model.

4.5 Ex-ante and ex-post Comparisons with more large portfolios and a proposal to take into account the skewness

In this section we show an analysis with 20 assets of the market index Down Jones. The previous analyses consider only a little number of assets, because the used values of the parameters are maximum likelihood estimates and the computational times are quite high. As observed at the end of section 4.2, we could use an approximation of the moment estimate in order to compute the correlation matrix of the d-dimensional Brownian motion specified into the model. The 20 studied assets are: 1) 3M Company; 2) Alcoa Inc; 3) Altria Group Inc; 4) Boeing Co; 5) Caterpillar Inc; 6) Coca Cola Co The; 7) Du Pont E I De Nem; 8) Exxon Mobil Cp; 9) Gen Electric Co; 10) Gen Motors; 11) Hewlett Packard Co; 12) Honeywell Intl Inc; 13) Intl Business Mach; 14) Johnson and Johns Dc; 15) McDonalds Cp; 16) Merck Co Inc; 17) Procter Gamble Co; 18) United Tech; 19) Wal Mart Stores; 20) Walt Disney-Disney C. We use daily log returns from 01/01/1985 to 30/12/2005 to estimate all the parameters, and then we consider the year 2006 for an ex-post analysis. Furthermore, our procedure to select portfolios allows an ex-ante comparison based on the expected utility. Let us suppose that the asset log returns satisfy the equation (4.4). Then, assuming that the log return of the portfolio $w = [w_1, \dots, w_d]'$ is well approximated by the convex combination of log returns, we have that the portfolio log return $\{X_t^{(w)}\}$ follows the subordinated Lévy process

$$X_t^{(w)} = (w'\mu)t + (w'\gamma)Z_t + \sqrt{w'Qw}W_{Z_t},$$

where $\{Z_t\}$ is the Lévy subordinator, and $\{W_t\}$ a 1-dimensional standard Brownian motion. As explained in section 4.2, considering the assets one at a time, we can compute the maximum likelihood estimates so that they are characterized by the same subordinator, that is we have the maximum likelihood estimates of μ,

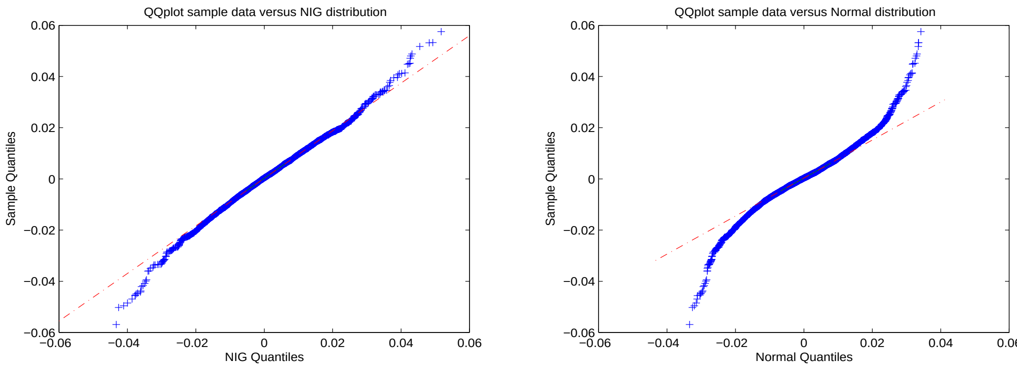

Figure 4.3: *QQplots of the portfolio with equal weights versus NIG distribution on the left and BM distribution on the right.*

γ, and parameters of $\{Z_t\}$. What remains not estimated is the definite positive matrix Q whose generic element is $\sigma_{ij}^2 = \sigma_i \sigma_j \rho_{ij}$, where σ_i and σ_j are already estimated and ρ_{ij} is the correlation coefficient between $X_t^{(i)}|Z_t$ and $X_t^{(j)}|Z_t$ ($X_t^{(i)}$ means the log return at time t of the portfolio with investment only in the i-th asset). But, as we have observed at the end of section 4.2, when the vector γ is near to zero then an approximation of the moment estimate of ρ_{ij} is the sample correlation between $X_t^{(i)}$ and $X_t^{(j)}$. The possibility to estimate quickly the covariance matrix Q allows to apply the NIG and VG models even though the portfolio is composed by a large number of assets. Figure 4.3 displays qq-plots of the empirical distribution of the portfolio composed by all assets with equal weights (i.e., $w_i = 1/20$) versus the NIG distribution on the left and Normal distribution on the right. According to this graphic comparison NIG distribution describes better than Normal distribution the sample data, and this result is confirmed by the Kolmogorov-Smirnov distances which are 0.0307 under NIG distribution and 0.0697 under Normal distribution. Thus, it is possible to estimate all the parameters of the NIG model so that computational times are not so high, and the description of the observed data is improved with respect to the Normal model. Similar results can be obtained under the VG model. Now, our analysis consists in to carry out an ex-post comparison among the NIG, VG and Normal models, thus we consider a risk-averse investor I that every day in the year 2006 selects a portfolio on these 20 assets with maturity 1 day. Suppose that I chooses the Normal model to describe the random behaviour of the market, then the vector of log returns $\widetilde{X}_t = [X_t^{(1)}, \dots, X_t^{(20)}]'$ follows

$$\widetilde{X}_t = \mu_t t + Q^{1/2}\widetilde{W}_t,$$

where $\mu_t = [\mu_{t,1}, \dots, \mu_{t,20}]'$, $Q = [\sigma_{ij}^2]$ is the variance-covariance matrix of $\widetilde{X}_t$ at time $t = 1$, and $\widetilde{W}_t$ is a standard 20-dimensional Brownian motion. Starting from 01/01/2006, the j-th day of the year, the investor I estimates μ_{t_j} by the mean of the sample data attained during the last 4 years, and Q by the sample variance-covariance matrix on the period 01/01/1985 - 30/12/2005. Then, the investor I solves for each $m \in [\mathrm{a1}, \mathrm{a2}]$, where a1 is the mean of the portfolio of minimum variance and a2 the highest value of the vector μ_{t_j}, the optimization problem

$$\begin{cases} \min\limits_{w} w'Qw \\ w'\mu_{t_j} = m \\ \sum_{i=1}^{20} w_i = 1, \quad w_i \geq 0, \quad i = 1, \dots, 20. \end{cases}$$

Thus, the investor I calculates the portfolio $w(m)$ which minimizes the variance for the level of mean m, under the assumption of no short sales. Finally, assuming that the investor's wealth at the j-th day is $\mathbf{W}_{t_j}$, for the next day the investor I chooses the portfolio w^* among the efficient portfolios $w(m)$ which maximizes the empirical expected utility of the last 4 years, that is, he solves the problem

$$\max_{w(m)} \frac{1}{N} \sum_{k=1}^{N} u\left(\mathbf{W}_{t_j}(1 + w(m)'x_{t_k})\right),$$

where N is the number of trading days during 4 years, and x_{t_k} is the return vector $N - k$ days before the j-th days of the year 2006. Suppose now that the investor I chooses a subordinated model to describe the random behaviour of the market, then the the log return vector $\widetilde{X}_t = [X_t^{(1)}, \dots, X_t^{(20)}]'$ follows

$$\widetilde{X}_t = \mu_t t + \gamma Z_t + Q^{1/2}\widetilde{W}_{Z_t}, \tag{4.13}$$

where $\mu_t = [\mu_{t,1}, \ldots, \mu_{t,20}]'$, $\gamma = [\gamma_1, \ldots, \gamma_{20}]'$, $Q = [\sigma^2_{ij}]$ is a definite positive matrix, $\{Z_t\}$ is a subordinator, and $\{\widetilde{W}_t\}$ is a standard 20-dimensional Brownian motion. If $\{Z_t\}$ is an Inverse Gaussian process with parameters $a = 1$ and $b > 0$ then we have the NIG model, if $\{Z_t\}$ is a Gamma process with parameters $a = b = 1/\nu > 0$ then we have the VG model. Every day, during the year 2006, the investor I selects a portfolio on the basis of the model (4.13). The estimates of γ and Q are obtained by the procedure explained at the beginning of the section, while μ_{t_j} at the j-th day is determined so that the mean of $\widetilde{X}_{t_j}$ is equal to the sample mean of the last 4 years. Let us remark an aspect of the subordinated model. Under this model, the log return of the portfolio w follows a subordinated process whose parameter $w'\gamma$ controls the skewness, that is, $w'\gamma > 0$ implies positive skewness and $w'\gamma < 0$ negative skewness. On the basis of this remark, we propose to select portfolio minimizing the objective function

$$f(w) = w'Qw - \frac{1}{c}w'\gamma,$$

where $c \in [1, 10]$. In this way we are able to select portfolios which maximize the skewness. As a matter of fact, Figure 4.4 plots under the NIG model the skewness of the portfolio for different values of $c \in [1, 10]$, and the maximum skewness is obtained for a value of c near to 4. When $c = 10$, we tend to select the portfolio which minimizes the risk measure $w'Qw$. Assuming that the investor I chooses our objective function, then at the j-th day of the year, for each $m \in [\mathrm{a1}, \mathrm{a2}]$, where a1 is the mean of the portfolio of minimum variance and a2 the highest value of $\mathrm{E}[\widetilde{X}_{t_j}]$, and for each $c \in [1, 10]$, the investor I solves the optimization problem

$$\begin{cases} \min\limits_{w} w'Qw - \frac{1}{k}w'\gamma \\ w'\mathrm{E}[\widetilde{X}_{t_j}] = m, \quad k = c, \\ \sum_{i=1}^{20} w_i = 1, \quad w_i \geq 0, \quad i = 1, \ldots, 20. \end{cases}$$

Thus, the investor I calculates the portfolio $w(m, c)$ which minimizes the objective function $w'Qw - \frac{1}{c}w'\gamma$ for the level of mean m, under the assumption of no short

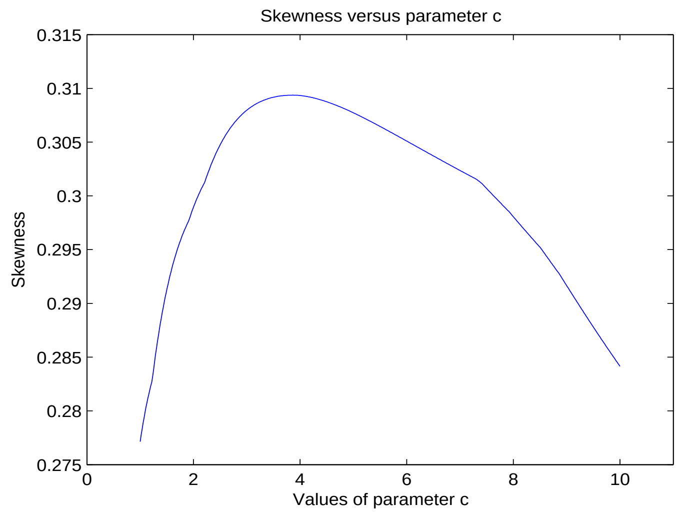

Figure 4.4: *Skewness of the portfolio as function of parameter c.*

sales. Finally, assuming that the investor's wealth at the j-th day is $\mathbf{W}_{t_j}$, for the next day the investor I chooses the portfolio w^* among the efficient portfolios $w(m,c)$ which maximizes the empirical expected utility of the last 4 years, that is he solves the problem

$$\max_{w(m,c)} \frac{1}{N} \sum_{k=1}^{N} u\left(\mathbf{W}_{t_j}(1 + w(m,c)' x_{t_k})\right),$$

where N is the number of trading days during 4 years, and x_{t_k} is the return vector $N-k$ days before the j-th days of the year 2006. Figure 4.5 reports ex-post and ex-ante comparisons between NIG and Normal models, assuming an investor with utility function

$$u(x) = 1 - e^{-\frac{1}{a}x}, \tag{4.14}$$

where a=0.5. In particular, on the left, day by day, we have the difference of the wealth between the two models, that is at day t_j the figure plots the value $\mathbf{W}_{t_j}^{(NIG)} - \mathbf{W}_{t_j}^{(N)}$, where $\mathbf{W}_{t_j}^{(NIG)}$ is the investor's wealth at day t_j under NIG model assuming at the beginning of the year an initial investment of 1 USD, while $\mathbf{W}_{t_j}^{(N)}$ is the wealth at same day under Normal model and with initial investment of 1 USD. Though these differences are small, we can see that NIG model is able to

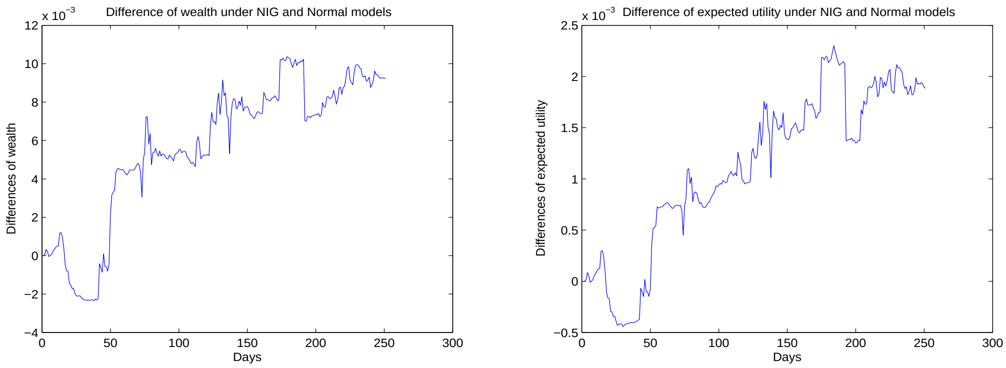

Figure 4.5: *Case: a=0.5 and comparison NIG and Normal models. On the left we have an ex-post comparison based on the difference of wealth between the NIG and Normal models, on the right an ex-ante comparison based on the difference of expected utility between the two models.*

guarantee a wealth greater than that one under Normal model, except a short period at the beginning of the year. On the right of Figure 4.5 we have an ex-ante comparison. In particular, day by day, it is plotted the difference of empirical expected utility of the last 4 years between the two models, that is at day t_j the figure shows the difference $U_{t_j}^{(NIG)} - U_{t_j}^{(N)}$, where

$$U_{t_j}^{(J)} = \frac{1}{N}\sum_{k=1}^{N} u\left(\mathbf{W}_{t_j}(1 + w^{*\prime} x_{t_k})\right), \quad J = NIG, N,$$

and where w^* is the portfolio that maximizes this empirical expected utility under the two models. Figure 4.5 shows that NIG models guarantees during the year an empirical expected utility greater than that one under Normal model, except a short period of the year. It is interesting to observe that the difference of the empirical expected utility has a behaviour similar to that one of the wealth, thus, during the year 2006, the maximization of the expected utility has really guaranteed higher future gains. Figure 4.6 shows the same type of analysis, but this time between the VG and Normal models and for an investor with utility function (4.14) and $a = 0.3$. On the left, day by day, we have an ex-post comparison based on the difference of wealth, that is the figure plots the

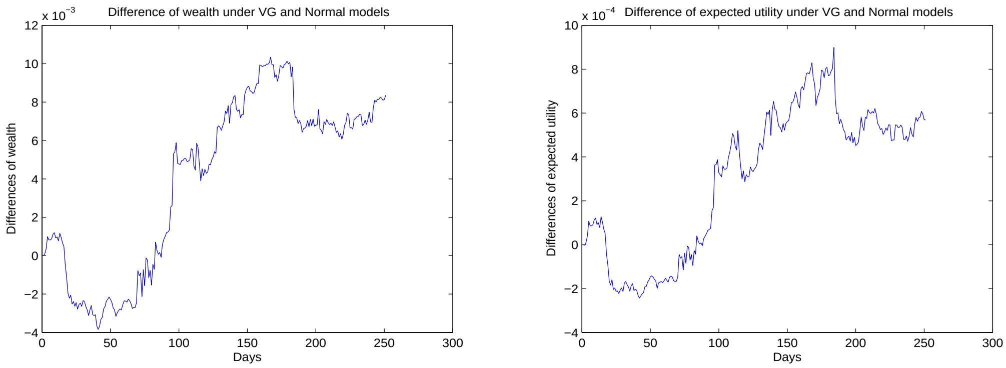

Figure 4.6: *Case: a=0.3 and comparison VG and Normal models. On the left we have an ex-post comparison based on the difference of wealth between the VG and Normal models, on the right an ex-ante comparison based on the difference of expected utility between the two models.*

value $\mathbf{W}_{t_j}^{(VG)} - \mathbf{W}_{t_j}^{(N)}$, where $\mathbf{W}_{t_j}^{(VG)}$ is the investor's wealth at day t_j under VG model assuming at the beginning of the year an initial investment of 1 USD, while $\mathbf{W}_{t_j}^{(N)}$ is the wealth at same day under Normal model and with initial investment of 1 USD. We can see that the VG model gives a better performance during the year, except a period which starts around the 10-th day of the year and finishes around the 100-th day. On the right of Figure 4.6 we have an ex-ante comparison. In particular, day by day, it is plotted the difference of empirical expected utility of the last 4 years between the two models, that is at day t_j the figure shows the difference $U_{t_j}^{(VG)} - U_{t_j}^{(N)}$, where

$$U_{t_j}^{(J)} = \frac{1}{N}\sum_{k=1}^{N} u\left(\mathbf{W}_{t_j}(1 + w^{*'} x_{t_k})\right), \quad J = VG, N,$$

and where w^* is the portfolio that maximizes this empirical expected utility under the two models. By the figure we have that the VG model gives empirical expected utilities greater than the Normal one, except a period which starts around the 10-th day of the year and finishes around the 100-th day. As in the case of the comparison between the NIG and Normal models, we can observe that the two graphs in Figure 4.6 are very similar, that is when the difference of wealth is positive then the difference of expected utility is positive too, thus, in the year 2006, maximizing expected utilities has guaranteed higher future gains.

4.6 Risk management with EWMA-Lévy model

In this section we apply elliptical EWMA VaR and CVaR models to asset portfolios distributed as VG and NIG laws. The Value at Risk (VaR) and Conditional Value at Risk (CVaR) are risk measures which summarize in a single value the possible losses which could occur with a given probability in a given temporal horizon. In order to compute these risk values we have to determine the so called profit/loss distribution. For example, the RiskMetrics model (see Longerstaey and Zangari [50]) assumes that the profit/loss distribution, conditional upon the portfolio standard deviation, is Gaussian and then computes VaR and CVaR through the multiplication of the portfolio standard deviation by a constant which is function of a given confidence level. Moreover, the RiskMetrics model computes weekly, monthly, and yearly VaR and CVaR scaling daily Gaussian VaR and CVaR estimates with opportune factors. Two main critics can be assigned to the RiskMetrics model, firstly many empirical studies show the inconsistence of conditional asset returns distributed as Gaussian laws, and secondly the time rule applied to compute VaR and CVaR on different temporal horizon is valid only for independent returns. In order to exceed at least the first lack of the RiskMetrics model we propose to describe asset returns as VG and NIG distributions and thus to apply an exponential weighted moving average (EWMA) model.

Value at Risk is the maximum loss among the best $\theta\%$ cases which could occur in a given temporal horizon. Therefore, if τ denotes the temporal horizon, $S_{t+\tau}^{(w)} - S_t^{(w)}$ the profit/loss realized in the interval $[t, t+\tau]$ by the portfolio with weight vector $w = (w_1, \ldots, w_d)'$, and θ the level of confidence, then

$$\mathrm{VaR}_{\theta,[t,t+\tau]}(S_{t+\tau}^{(w)} - S_t^{(w)}) = \inf\{q | \Pr(S_{t+\tau}^{(w)} - S_t^{(w)} \leq q) > 1 - \theta\}.$$

Thus, Value at Risk is the $(1-\theta)\%$ percentile of the profit/loss distribution in the time interval $[t, t+\tau]$. But, VaR is not a coherent risk measure because it does not satisfy the subadditivity property (see Artzner *et al.* [1]), and so the recent literature (see Szegö [82]) has proposed alternative coherent measures, for example the CVaR. The Conditional Value at Risk is the expected profit/loss given that the Value at Risk has not been exceeded:

$$\text{CVaR}_{\theta,[t,t+\tau]}(S_{t+\tau}^{(w)} - S_t^{(w)}) = \frac{1}{1-\theta}\int_0^{1-\theta} \text{VaR}_{q,[t,t+\tau]}(S_{t+\tau}^{(w)} - S_t^{(w)})dq.$$

For a continuous profit/loss distribution the Conditional Value at Risk is given by

$$\text{CVaR}_{\theta,[t,t+\tau]}(S_{t+\tau}^{(w)} - S_t^{(w)}) = \text{E}[S_{t+\tau}^{(w)} - S_t^{(w)} | S_{t+\tau}^{(w)} - S_t^{(w)} \leq \text{VaR}_{\theta,[t,t+\tau]}].$$

The RiskMetrics model approximates the continuously compounded return $X_{t+1}^{(w)}$ of the portfolio w on the period $[t, t+1]$ by the convex combination

$$X_{t+1}^{(w)} = \sum_{i=1}^{d} w_i X_{t+1}^{(i)},$$

where $X_{t+1}^{(i)} = \log(S_{t+1}^{(i)}/S_t^{(i)})$ is the continuously compounded return of the i-th asset on the period [t,t+1], and $S_t^{(i)}$ is the price of the i-th asset at time t. Moreover, the random vector $X_{t+1} = [X_{t+1}^{(1)}, \ldots, X_{t+1}^{(d)}]'$ is assumed to follow a conditional joint Gaussian distribution with null mean. Therefore,

$$X_{t+1}^{(w)} = \sigma_{(w,t+1|t)} Y,$$

where $Y \sim N(0,1)$, $\sigma^2_{(w,t+1|t)} = w' Q_{(t+1|t)} w$ is the variance of the portfolio w, and $Q_{(t+1|t)} = [\sigma^2_{(ij,t+1|t)}]$ is the forecasted variance-covariance matrix. The elements of the time-dependent matrix $Q_{(t+1|t)}$ are estimated according to an exponentially weighted moving average model:

$$\sigma^2_{(ii,t+1|t)} = \text{E}[X_{t+1}^{(i)^2}] = \lambda \sigma^2_{(ii,t|t-1)} + (1-\lambda) X_t^{(i)^2}$$
$$\sigma^2_{(ij,t+1|t)} = \text{E}[X_{t+1}^{(i)} X_{t+1}^{(j)}] = \lambda \sigma^2_{(ij,t|t-1)} + (1-\lambda) X_t^{(i)} X_t^{(j)},$$

where λ is the optimal smoothing factor (see Longerstaey and Zangari [50]). Under the RiskMetrics model, the Value at Risk of the portfolio w at confidence level $\theta\%$ conditional the information available at time t, denoted as $\mathrm{VaR}_{\theta,t+1|t}$, is obtained by the forecasted volatility $\sigma_{(w,t+1|t)}$ times the $(1-\theta)\%$ percentile, $k_{1-\theta}$, of the standard Gaussian distribution, that is

$$\mathrm{VaR}_{\theta,t+1|t}(X_{t+1}^{(w)}) = k_{1-\theta}\sigma_{(w,t+1|t)}.$$

A similar formula is also valid for the Conditional Value at Risk of the portfolio w conditional the information at time t:

$$\mathrm{CVaR}_{\theta,t+1|t}(X_{t+1}^{(w)}) = c_{1-\theta}\sigma_{(w,t+1|t)},$$

where $c_{1-\theta} = \mathrm{E}[Y|Y \leq k_{1-\theta}]$ and $Y \sim N(0,1)$.

The RiskMetrics model can be extended in order to take into account multi-dimensional NIG and VG distributions and obtain simple formulas for conditional VaR and CVaR. Assume that $\widetilde{X}_{t+1} = [X_{t+1}^{(1)}, \ldots, X_{t+1}^{(d)}]'$, the vector of the continuously compounded returns on the period $[t, t+1]$, satisfies the equation

$$\widetilde{X}_{t+1} = Q_{(t+1|t)}^{1/2}\sqrt{Z}\widetilde{Y},$$

where $Q_{(t+1|t)}$ is a definite positive matrix, Z a positive random variable, and $\widetilde{Y}$ a d-dimensional standard Gaussian distribution independent of Z. Then, the conditional distribution of the portfolio w satisfies

$$X_{t+1}^{(w)} = \sigma_{(w,t+1|t)}\sqrt{Z}Y, \tag{4.15}$$

where $Y \sim N(0,1)$ and $\sigma_{(w,t+1|t)}^2 = w'Q_{(t+1|t)}w$. The elements of the time-dependent matrix $Q_{(t+1|t)}$ can be computed according to the EWMA model:

$$\sigma_{(ii,t+1|t)}^2 = \lambda\sigma_{(ii,t|t-1)}^2 + (1/\mathrm{E}[Z])(1-\lambda)X_t^{(i)^2}$$
$$\sigma_{(ij,t+1|t)}^2 = \lambda\sigma_{(ij,t|t-1)}^2 + (1/\mathrm{E}[Z])(1-\lambda)X_t^{(i)}X_t^{(j)},$$

where λ is the optimal smoothing factor. From equation (4.15), we can compute the Value at Risk of the portfolio w at confidence level $\theta\%$ conditional the information at time t by the formula:

$$\mathrm{VaR}_{\theta,t+1|t}(X_{t+1}^{(w)}) = \widetilde{k}_{1-\theta}\sigma_{(w,t+1|t)},$$

where $\widetilde{k}_{1-\theta}$ is the $(1-\theta)\%$ percentile of the distribution of $\sqrt{Z}Y$. Then, for the Conditional Value at Risk we have the similar formula:

$$\mathrm{CVaR}_{\theta,t+1|t}(X_{t+1}^{(w)}) = \widetilde{c}_{1-\theta}\sigma_{(w,t+1|t)},$$

where $\widetilde{c}_{1-\theta} = \mathrm{E}[\sqrt{Z}Y|\sqrt{Z}Y \leq \widetilde{k}_{1-\theta}]$. Choosing appropriately the positive random variable Z, we can define models where asset returns follow either Variance Gamma laws or Normal Inverse Gaussian laws. Specifically, if Z is a Gamma distribution with parameters $a = b = 1/\nu$, then the portfolio return $X_{t+1}^{(w)}$ follows a Variance Gamma distribution with parameters $\theta_w = 0$, $\sigma_w = \sigma_{(w,t+1|t)}$, and $\nu_w = \nu$. While, if Z is an Inverse Gaussian distribution with parameter $a = 1$ and $b > 0$, then the portfolio return $X_{t+1}^{(w)}$ follows a Normal Inverse Gaussian distribution with parameters $\alpha_w = b/\delta_w$, $\beta_w = 0$, and $\delta_w = \sigma_{(w,t+1|t)}$.

4.7 Risk management with ICA-Lévy model

In this section we suggest a generalization of the GHICA model of Chen *at al.* [17]. Given a multidimensional time series of asset prices $S(t) = (S_1(t), \ldots, S_d(t))^\top$, we assume the conditional heteroscedastic model

$$X(t) = Q_X(t)^{1/2}\xi_X(t),$$

where $X(t) = (X_1(t), \ldots, X_d(t))^\top$ is the (log) return vector, that is $X_i(t) = \log(S_i(t)/S_i(t-1))$, and $\xi_X(t) = (\xi_{X,1}(t), \ldots, \xi_{X,d}(t))^\top$ is the standardized innovation vector. The vector $\xi_X(t)$ is assumed to be predictable with respect to $\Im_{t-1}$, the σ-algebra generates by $S(0), S(1), \ldots, S(t-1)$. Using the Independent Component Analysis (ICA) we can find a linear transformation $Y(t) = KX(t)$, where K is a time constant and singular matrix, so that

$$X(t) = K^{-1}Y(t) = K^{-1}Q_Y(t)^{1/2}\xi_Y(t),$$

where now $Q_Y(t)^{1/2}$ is a diagonal matrix. Thus, the stochastic innovation $\xi_Y(t) = (\xi_{Y,1}(t), \ldots, \xi_{Y,d}(t))^\top$ can be individually modeled, for example as Lévy distributions.

The following algorithm summarizes the procedure that under the above model and with Lévy innovations permits to measure risk exposures:

1. Do ICA to find the independent components;
2. Implement the local exponential smoothing to estimate the variance of each independent component;
3. Model each independent innovation as a Lévy distribution;
4. Estimate the density of the portfolio return through the FFT technique;
5. Compute risk measures.

Independent Component Analysis

The Independent Component Analysis of a random vector X consists of estimating the following model:

$$X = KY, \tag{4.16}$$

where the latent factors Y_i in the vector $Y = (Y_1, \ldots, Y_d)^\top$ are assumed statistically independent and nongaussian, and the matrix K is nonsingular. The statistically independent requirement means that

$$f_Y(y_1, \ldots, y_d) = f_{Y_1}(y_1) \cdots f_{Y_d}(y_d),$$

where f_Y is the joint density function and f_{Y_i} is the density function of Y_i. Then, the nongaussian requirement depends of the consequence that the matrix K is not identifiable if the components are Gaussian. Another condition is on the variance of the independent component Y:

$$\mathrm{E}[YY^\top] = \mathrm{I},$$

where I is the identity matrix, otherwise there are infinite matrixes K which satisfy the model (4.16). The principle underlying the estimation of the matrix K is based on the nongaussian condition. If we multiply both sides of equation (4.16) by a matrix W we have

$$\Theta = WX = WKY = ZY,$$

where $Z = WK$. Thus, each component of Θ is a linear sum of Y. Since, from the Central Limit Theorem, the sum of independent random variable is more Gaussian than the original variables, ZY is more Gaussian than Y and is least Gaussian if it is equal to Y. Thus, we have to find the matrix W that maximizes the nongaussian property of WX, because in this case

$$\Theta = WX = WKY \approx Y,$$

being $W \approx K^{-1}$. Therefore, we find the matrix K and the latent factor Y.

A way to measure the nongaussianity of a random vector Y is to use the negentropy which is based on the quantity of entropy. The entropy described how much randomness is a random variable, and it is shown that the Gaussian variable has the largest entropy among all random variables of equal variance (see Cover and Thomas [23]). For a continuous random variable or vector the entropy is called differential entropy and is defined as

$$H(Y) = -\int f_Y(y) \log\left(f_Y(y)\right) dy.$$

Then, the negentropy is defined as

$$J(Y) = H(Y_0) - H(Y),$$

where Y_0 is standard multinormal distribution. Negentropy is always nonnegative and is zero if and only if the random vector is Gaussian. Thus, to maximize the nongaussianity is equivalent to maximize the negentropy. However, it is not easy to use the negentropy and generally the following approximation is considered (see Hyvärinen [42]):

$$J_g(Y) = \sum_{i=1}^{d} k_i \left(\mathrm{E}[G(Y_i)] - \mathrm{E}[G(Y_{0,i})]\right)^2,$$

where k_i is a positive constant, $Y_{0,i}$ is a standard normal distribution and G is a nonquadratic function. Practical choices of G are:

$$G_1(x) = \tfrac{1}{a}\log(\cosh(ax)), \quad G_2(x) = -\exp(-x^2/2) \;,$$

where $1 \leq a \leq 2$. The symmetric FastICA algorithm (see Hyvärinen, Karhunen, and Oja [43]) can be use to maximize the approximated negentropy:

1. Choose initial vectors $\widehat{w}_i^{(1)}$ for $W = (w_1, \ldots, w_d)'$ with unit norm.

2. Loop:

    - At step n, calculate

    $$\widehat{w}_i^n = \mathrm{E}\left[X(t)^\top G'\left(\widehat{w}_i^{(n-1)\top} X(t)\right)\right] - \mathrm{E}\left[G''\left(\widehat{w}_i^{(n-1)\top} X(t)\right)\right] \widehat{w}_i^{(n-1)},$$

    where G' is the first derivative of G, G'' the second derivative, and $\mathrm{E}[\cdot]$ is approximated by the sample mean.

    - Do a symmetric orthogonalization of the estimated matrix $\widehat{W}^{(n)}$:

    $$\widehat{W}^{(n)} = \left(\widehat{W}^{(n)}\widehat{W}^{(n)\top}\right)^{-1/2} \widehat{W}^{(n)}.$$

    - If not converged, that is $\det\left[\widehat{W}^{(n)} - \widehat{W}^{(n-1)}\right] \neq 0$, go back to 2. Otherwise, the algorithm stops.

3. The last estimate is the final estimate of $\widehat{W}$.

Local exponential smoothing

Given the independent components Y and the transformation matrix W, then the covariance matrixes of Y and X are respectively:

$$\begin{aligned} Q_Y(t) &= \operatorname{diag}\left(\sigma_{Y_1}(t)^2, \ldots, \sigma_{Y_d}(t)^2\right), \\ Q_X(t) &= W^{-1} Q_Y(t) W^{-1\top}, \end{aligned}$$

where $\sigma_{Y_i}(t)$ is the heteroscedastic volatility of the i-th independent component. The next step is to estimate the diagonal matrix $Q_Y(t)$ and generally the local exponential smoothing procedure is adopted.

For each $i = 1, \ldots, d$ we have the univariate heteroscedastic model

$$Y_i(t) = \sigma_{Y_i}(t)\xi_{Y_i}(t),$$

where

$$\mathrm{E}[\xi_{Y_i}(t)|\Im_{t-1}] = 0, \quad \mathrm{E}[\xi_{Y_i}(t)^2|\Im_{t-1}] = 1,$$

and $\sigma_{Y_i}(t)$ is predictable with respect to $\Im_{t-1}$. Under the assumption of normal distributed innovation and using the exponentially decreasing weights $\{\eta^{t-s}\}_{s\leq t}$, the local maximum likelihood estimate (MLE) of $\sigma_{Y_i}(t)$ (see Polzehl and Spokoiny [67]) is given by

$$\widetilde{\sigma}_{Y_i}(t) = \left(N_t^{-1} \sum_s Y_{s-1}^2 \eta^{t-s}\right)^{1/2}, \tag{4.17}$$

where $N_t = \sum_s \eta^{t-s}$. Removing the assumption of normal innovation, if the exponential of the squared innovation $\mathrm{E}[\exp(\rho\xi_{Y_i}(t)^2)]$ exists, then we can still use the estimate (4.17). In order to guarantee the existence of the exponential moment, it is possible to apply a power transformation (see Chen *et al.* [17]):

$$\begin{aligned} Y_{p,i}(t) &= \operatorname{sign}(Y_i(t))|Y_i|^p, \\ \theta_{Y_i}(t) &= \operatorname{Var}[Y_{p,i}|\Im_{t-1}] \\ &= \sigma_{Y_i}(t)^{2p}\mathrm{E}[|\xi|^{2p}|\Im_{t-1}], \end{aligned}$$

where $0 \le p < 1/2$. Observe that $\theta_{Y_i}(t)$ is one-to-one correspondence to $\sigma_{Y_i}(t)$ and can be estimated by

$$\widetilde{\theta}_{Y_i}(t) = N_t^{-1} \sum_s |Y_i(s-1)|^{2p} \eta^{t-s}.$$

Suppose now that a finite set $\{\eta_k : k = 1, \ldots, K\}$ of values of smoothing parameter is given. Then, for each value η_k we have the local MLE

$$\widetilde{\theta}_{Y_i}^{(k)}(t) = \left(N_t^{-1} \sum_s |Y_i(s-1)|^{2p} \eta_k^{t-s} \right)^{1/2},$$

and the spatial stagewise aggregation (SSA) method (see Belomestny and Spokoiny [8], and Chen *et al.* [17]) can be implemented to estimate $\theta_{Y_i}(t)$. According to this method we can choose an aggregation estimate $\widetilde{\theta}_{Y_i}(t)$ which summarizes all the estimates $\widetilde{\theta}_{Y_i}^{(k)}(t)$.

Independent innovations as Lévy distributions

In their model Chen *et al.* propose to model the independent components as NIG distributions. Instead, we suggest to implement the methodology assuming that every component can follow the Lévy distribution that better describes it according to some statistics. By the independent component analysis, we have

$$Y(t) = KX(t),$$

where $K = W^{-1}$, and thus we can use the observe data to conjecture the Lévy distribution which better describes a particular component Y_i. The choice could be done taking into account semi-heavy and heavy tails distributions, for example GH, NIG, VG, and α-stable distributions, and then selecting that one which has the smaller values of the Kolmogorov-Smirnov statistic, or of the Anderson-Darling statistic, or of the product of these two statistics.

In order to implement the ICA-Lévy model we have to be sure that the mentioned Lévy distributions satisfy the scaling property, that is if X is distributed according to a particular law then the law of cX has to be of the same class. Looking at the characteristic functions of the GH, VG, and α-stable laws, we have that if $X \sim \text{GH}(x; \mu, \alpha, \beta, \delta, \nu)$ then $Y = cX$ satisfies

$$Y \sim \text{GH}(y; c\mu, \alpha/|c|, \beta/c, |c|\delta, \nu),$$

if $X \sim \text{VG}(x; \mu, \sigma, \nu, \theta)$ then

$$Y \sim \text{VG}(y; c\mu, |c|\sigma, \nu, c\theta),$$

if $X \sim \text{S}_\alpha(\sigma, \beta, \mu)$ then

$$Y \sim S_\alpha(|c|\sigma, \text{sign}(c)\beta, c\mu),$$

and, finally, the NIG distribution satisfies the scaling property being a special case of the GH one with $\nu = 0$. The notation $S_\alpha(\sigma, \beta, \mu)$ represents an α-stable distribution with parameters $\alpha \in (0, 2]$, $\sigma \geq 0$, $\beta \in [-1, -1]$, and $\mu \in \mathbb{R}$.

Estimate of the density function of the portfolio return (FFT algorithm)

Given an asset portfolio at time $t-1$ with weight vector $w(t) = (w_1 \ldots, w_d(t))^\top$, under the ICA-Lévy model the portfolio return follows

$$R(t) = w(t)^\top W^{-1} Q_Y(t)^{1/2} \xi_Y(t),$$

where the components of the innovation vector $\xi_Y(t)$ are Lévy distributed. Setting

$$(v_1, \ldots, v_d) = w(t)^\top W^{-1} Q_Y(t)^{1/2},$$

then the random variable $\zeta_i = v_i \xi_i$ follows a Lévy distribution of the same class of ξ_i. Moreover, the characteristic function of the portfolio return $R(t) = \sum_{i=1}^d \zeta_i$ is given by

$$\phi_R(z) = \prod_{i=1}^d \phi_{\zeta_i}(z),$$

and the density function can be approximated by the Fourier transformation:

$$
\begin{aligned}
f_R(r) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-izr)\phi_R(z)dz \\
&\approx \frac{1}{2\pi}\int_{-a}^{a}\exp(-izr)\phi_R(z)dz.
\end{aligned}
$$

The density function $f_R(r)$ can be numerically computed by the FFT algorithm which evaluates the discrete Fourier transformation (DFT) efficiently (see, among the others, Menn and Rachev [56], Chen *et al.* [17]). The DFT is a special mapping of a vector $y = (y_0, \dots, y_{N-1}) \in \mathbb{C}^N$ onto a vector $s = (s_0, \dots, s_{N-1}) \in \mathbb{C}^N$:

$$
s_l = \mathrm{DFT}(y)_l = \sum_{j=0}^{N-1} y_j e^{-il\frac{2\pi j}{N}}, \quad l = 0, \dots, N-1.
$$

The following algorithm can be implemented to approximate $f_R(r)$:

1. Let $N = 2^m$ with $m \in \mathbb{N}$ and define an equidistance grid over the integral interval $[-a, a]$ by setting $h = 2a/N$ and the grid points $z_j = -a + jh$ with $j = 0, \dots, N$.

2. The input of the DFT are $y_j = (-1)^j \phi_R(z_j^*)$, where $z_j^* = 0.5(z_j + z_{j+1})$ are the middle points.

3. We have the density $f_R(r) = \frac{1}{2\pi} C_k DFT(y)_k$ for $r = -\frac{N\pi}{2a} + \frac{\pi k}{a}$, where $C_k = \frac{2a}{N}(-1)^k e^{-i(\pi/N)k}$ and $k = 0, \dots, N-1$.

Risk measures

Given the density function $f_R(r)$ of the portfolio return $R(t)$ we can compute the risk measures VaR and CVaR at confidence level $\theta\%$ according to the formulas:

$$
\begin{aligned}
\mathrm{VaR}_{\theta,[t-1,t]}(R(t)) &= \{q|\Pr(R(t) \le q) = 1-\theta\}, \\
\mathrm{CVaR}_{\theta,[t-1,t]}(R(t)) &= \mathrm{E}[R(t)|R(t) \le \mathrm{VaR}_{\theta,[t-1,t]}].
\end{aligned}
$$

Chapter 5

Concluding remarks

In this dissertation three main topics have been studied: option pricing, portfolio selection, and risk management. The underlying randomness has been modeled by Lévy processes, which are stochastic processes with some remarkable properties, such as independent and stationary increments, and right continuous paths with left limits. For example, the same Brownian motion is an element of this class. Chapter 1 has been an introduction to Lévy processes and exponential-Lévy models. We have shown the deep link between infinitely divisible distributions and Lévy processes, and further we have stated some important results, such as the Lévy-Ito decomposition and Lévy-Khintchine representation. Since financial applications of Lévy processes often exploit their Markov property, we have dedicated a section to the correspondence between Lévy and Markov processes. In financial applications important Lévy processes are those obtained by subordination, i.e., it is defined a subordinator which represents the business time. A subordinator is a stochastic process with positive and non-decreasing paths and, thus, it can be used as time-change for other stochastic processes. In our introduction to Lévy processes we have reported some results about the theory of subordinated processes and shown how to construct two important subordinated Lévy processes, the Normal Inverse Gaussian and Variance Gamma processes. Moreover, we have studied the Generalized Hyperbolic, CGMY, and Meixner processes, giving their definitions and characteristics. In the last section of chapter 1 we have studied the exponential-Lévy model, which is the market model generally assumed under Lévy processes. Precisely, the log return or continuously compound return of the risky asset is modeled by a Lévy process. The resulting market model is not complete and thus, given a contingent claim, there is not a unique risk-neutral probability. So, under an exponential-Lévy model there is the problem to select the equivalent martingale measure which better summarizes the investors' choices. In our introduction to exponential-Lévy models we have explained how to construct three possible equivalent martingale measures, the mean correcting, Esscher transform, and minimal entropy martingale measures.

In chapter 3 we have studied option pricing under Lévy processes. We have started explaining three possible ways to price European options: density function, Fourier transform, and PIDE method. Density function method is the simpler one and consists in a closed formula. But, it can be applied only when the Lévy process admits density function in explicit form. Fourier transform method is more general and does not require the knowledge of the density function. Given the characteristic function of the Lévy process, it finds the Fourier transform of the option price and proposes the Fast Fourier method as pricing procedure. Finally, PIDE method consists in solving numerically a differential equation with integral component. These methods can be used only when the underlying Lévy process satisfies some specific conditions, and they can require numerical algorithms of not simple implementation. Then, pricing under Lévy processes becomes even more complex when the option is path-dependent, such as barrier and lookback options. For these reasons option pricing under Lévy processes is often based on simulation techniques and Monte Carlo methods.

In the remaining part of chapter 3 we have introduced the markovian approach and we have shown how to price European, American, compound, barrier, and lookback options. Markovian approach is a very simple methodology which allows to price options without particular difficulties. It consists in to construct a sequence of Markov chains converging weakly to the underlying Lévy process, thus the problem to price options is set in a Markov chain framework. European and American options can be priced easily and the price convergence is very fast. We have displayed numerical results concerning these options, and, in the European case, we have compared the results with the density function method. Compound, barrier, and lookback options can be priced under a Markov chain framework. We have shown numerical results, and, in the case of compound options, we have computed some prices under Brwonian motion and compared the results with the Geske's formula. Original contributions of this dissertation have been to extend the markovian approach from the Brownain motion to Lévy processes and to explain, under a Markov chain framework, option pricing for compound and lookback options. Our numerical applications are been performed under the assumption of three particular Lévy processes, the NIG, VG, and Meixner processes. Thus, possible future developments of this methodology could concern other Lévy processes, such as CGMY, Generalized Hyperbolic, and α-stable processes, and further it would be worth to extend the markovian approach to other path-dependent instruments such as asian options.

In chapter 4 we have studied portfolio selection under subordinated Lévy processes and compared these models with the Markowitz one. Our first empirical comparison has concerned investor's choices of portfolios composed by the market portfolio and riskless asset. The NIG model has given the better performance in terms of expected utility and final wealth, therefore it has been the more conservative one, given that in the year of our ex-post analysis financial markets were marked by very big losses. In the subsequent sections we have studied multidimensional Lévy processes in order to model financial markets. Specifically, we have analyzed the possibility to model asset log returns as subordinated Lévy models characterized by the same subordinator. In this contest, we have faced the problem of parameter estimates and suggested some maximum likelihood procedures. Then, we have tested the validity of our distributional assumptions performing graphical and statistical comparisons (qq-plot, Kolmogorv-Smirnov distance). We have obtained that subordinated Lévy models has been able to describe observed data better than Normal distribution. A first empirical analysis under multidimensional Lévy processes has regarded five market indexes in the year 2006. We have considered two possible markets, one without transaction costs, and the other one with proportional and constant transaction costs. In our results the presence of transaction costs have not determined a significant difference, that is, in both cases there has been a first part of the year where the NIG model has given better results with respect to the VG and Normal models and a second part of the year where the VG model has been the better one. A second empirical analysis has been on multi-period portfolio selection problem under subordinated Lévy processes. We have shown the formulas of mean, variance, skewness, and kurtosis of the final wealth under different distributional assumptions and we have performed an empirical comparison supposing three possible evolutions of the term structure. Our results have consisted in an ex-ante analysis based on the expected utility and one ex-post based on the final wealth. The VG model has been able to guarantee the higher expected utilities and final wealths for investors characterized by different utility functions and risk aversions. In chapter 4 we have also proposed empirical comparisons under subordinated processes with time-dependent means. Furthermore, we have shown the possibility to estimate very quickly the correlation structure of the financial market, obtaining a description of observed data better than Normal model in terms of qq-plot and Kolmogorov-Smirnov distance. Our analysis has been based on a three-dimensional efficient frontier, because we have explained how to define an objective function which took into account the skewness of portfolios. The empirical comparisons have shown a better performance of the NIG and VG models with respect to the Normal one. In particular, we have studied daily portfolios of investors with exponential utility during the year 2006 and we have obtained greater gains under NIG and VG distributions. Possible future researches could concern other multidimensional subordinated models, for example Generalized Hyperbolic distributions. Then, it would be worth to define and solve optimization problems where the kurtosis of portfolios is minimized. Finally, our model considers only time-dependent means, but, applying the independent component analysis, one could define models with time-dependent variance-covariance matrixes.

In chapter 4 we have also proposed a risk management analysis with Lévy distributions. Specifically, we have shown two possible modeling, EWMA-Lévy and ICA-Lévy models. The first one is an extension of the EWMA RiskMetrics model, where the conditional asset returns are either VG distributed or NIG distributed. An important consequence of this modeling is the same computational complexity of the RiskMetrics model, that is, VaR and CVaR can be computed with very simple formulas. The second modeling is a generalization of the GHICA model of Chen *et al.* [17]. In particular, we have shown how to model each stochastic innovation through the Lévy distribution which better describes it. These two proposed modeling, EWMA-Lévy and ICA-Lévy models, have been exhibited only by a theoretical point of view, thus future researches will be empirical analyses and comparisons with other benchmark models.

Appendix A

Simulation of Lévy processes

In this appendix we introduce some methods about the simulation of Lévy processes. Section A.1 is just a briefly explanation of the method of exponential spacings, which allows to simulate Poisson processes. Then, Section A.2 focuses on the compound Poisson approximation which is a technique applicable to any Lévy process. Finally, Section A.3 shows how to simulate Normal Inverse Gaussian and Variance Gamma processes exploiting their expression as subordinated Brownian motion with drift.

A.1 The method of exponential spacings

From the construction of the Poisson process (see Sato [73], Theorem 3.2), we have

$$N_t(\omega) = n \qquad \text{iff} \qquad W_n(\omega) \leq t < W_{n+1}(\omega),$$

where $\{W_n : n = 0, 1, \ldots\}$ is a random walk, defined on a probability space $(\Omega, \Im, \mathrm{P})$, such that $T_n = W_n - W_{n-1}$ is a exponential random variable $\mathrm{Exp}(\lambda)$ with mean λ^{-1}. Now, a random number e_n from $\mathrm{Exp}(\lambda)$ can be obtained drawing a random number u_n from an uniform distribution on $(0, 1)$ and then setting

$$e_n = -\log(u_n)/\lambda.$$

Thus, a simulation of the random walk $\{W_n\}$ is given by

$$w_0 = 0, \quad w_n = w_{n-1} + e_n, \quad n \geq 1.$$

and then a simulation of the Poisson process $\{N_t\}$ on the time points $\{n\Delta t : n = 0, 1, \ldots\}$ is given by

$$N_0, \quad N_{n\Delta t} = \sup\{k : w_k \leq n\Delta t\}, \quad n \geq 1.$$

Figure A.1 exhibits a sample path of a Poisson process with intensity $\lambda = 30$ obtained with this technique.

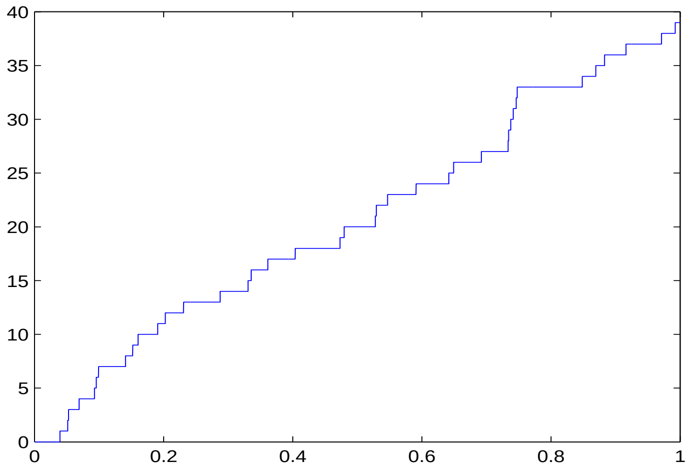

Figure A.1: *Sample path of a Poisson process with intensity* $\lambda = 30$.

A.2 The compound Poisson approximation

Consider a Lévy process $\{X_t\}$ with Lévy triplet $[\gamma, \sigma^2, \nu(dx)]$, then, from the Lévy-Ito decomposition, we know that

$$X_t = \gamma t + \sigma W_t + X_t^l + \lim_{\varepsilon \downarrow 0} \widetilde{X}_t^\varepsilon,$$

where $\{X_t^l\}$ is a compound Poisson process with jumps greater or equal to 1, and $\{\widetilde{X}_t^\varepsilon\}$ is a compensated compound Poisson process with jumps between ε and 1.

Thus, a way to simulate $\{X_t\}$ is to approximate $\{X_t^l\}$ and $\{\widetilde{X}_t^\varepsilon\}$ by a sum of independent Poisson processes:

$$X_t^d = \gamma t + \sigma W_t + \sum_{i=1}^{d} c_i(N_t^{(i)} - \lambda_i 1_{|c_i|<1}).$$

Specifically, given $\epsilon \in (0,1)$, we partition $\mathbb{R} \setminus [-\epsilon, \epsilon]$ by

$$a_0 < a_1 < \cdots < a_k = -\epsilon, \quad \epsilon = a_{k+1} < a_{k+2} < \cdots < a_{d+1},$$

where, generally $d = 2k$ and

$$\begin{cases} a_{i-1} = -\alpha i^{-1} & 1 \leq i \leq k+1 \\ a_{2k+2-i} = \alpha i^{-1} & 1 \leq i \leq k+1, \end{cases}$$

with $\alpha > 0$. Then, we specify d independent Poisson processes $\{N_t^i\}$, $i = 1, \ldots, d$. In particular, $\{N_t^i\}$ is defined so that its intensity λ_i is equal to the Lévy measure on the interval $[a_{i-1}, a_i)$, if $1 \leq i \leq k$, or $[a_i, a_{i+1})$, if $k+1 \leq i \leq d$, that is

$$\lambda_i = \begin{cases} \nu([a_{i-1}, a_i)) & \text{for} \quad 1 \leq i \leq k, \\ \nu([a_i, a_{i+1})) & \text{for} \quad k+1 \leq i \leq d. \end{cases} \tag{A.1}$$

Further, the jump size c_i of $\{N_t^{(i)}\}$ is chosen so that the variance of $\{N_t^{(i)}\}$ is equal to the variance of $\{X_t^l + \lim_{\varepsilon \downarrow 0} \widetilde{X}_t^\varepsilon\}$ on the interval $[a_{i-1}, a_i)$, if $1 \leq i \leq k$, or $[a_i, a_{i+1})$, if $k+1 \leq i \leq d$, that is

$$c_i = \begin{cases} -\sqrt{\lambda^{-1} \int_{a_{i-1}}^{a_i-} x^2 \nu(dx)} & \text{for} \quad 1 \leq i \leq k, \\ \sqrt{\lambda^{-1} \int_{a_i}^{a_{i+1}-} x^2 \nu(dx)} & \text{for} \quad k+1 \leq i \leq d. \end{cases} \tag{A.2}$$

So far, we have described how to approximate the jumps of $\{X_t\}$ belonging to $\mathbb{R} \setminus [-\epsilon, \epsilon]$, but an important improvement is to include even the small jumps. The component of the small jumps can be approximated by a Brownian motion with volatility

$$\sigma^2(\epsilon) = \int_{|x|<\epsilon} x^2 \nu(dx).$$

Thus, the process $\{X_t\}$ is approximated by

$$X_t^d = \gamma t + \widetilde{\sigma} W_t + \sum_{i=1}^{d} c_i(N_t^{(i)} - \lambda_i 1_{|c_i|<1}),$$

where

$$\widetilde{\sigma}^2 = \sigma^2 + \sigma^2(\epsilon),$$

and λ_i and c_i are given by (A.1) and (A.2), respectively. Asmussen and Rosiński [4] show that the approximation of small jumps by a Brownian component is valid if and only if for each $k > 0$

$$\lim_{\epsilon \to 0} \frac{\sigma(k\sigma(\epsilon) \wedge \epsilon)}{\sigma(\epsilon)} = 1. \tag{A.3}$$

Observe that the condition (A.3) is implied by $\lim_{\epsilon\to 0} \sigma(\epsilon)/\epsilon = \infty$, and thus the approximation of small jumps by a Brownian component can be used for both the Normal Inverse Gaussian and Meixner processes, because for these processes we have

$$\sigma(\epsilon) \sim \sqrt{2\alpha\delta/\pi}\epsilon^{1/2}.$$

Figure A.2 exhibits a sample path of a Meixner process obtained with this technique.

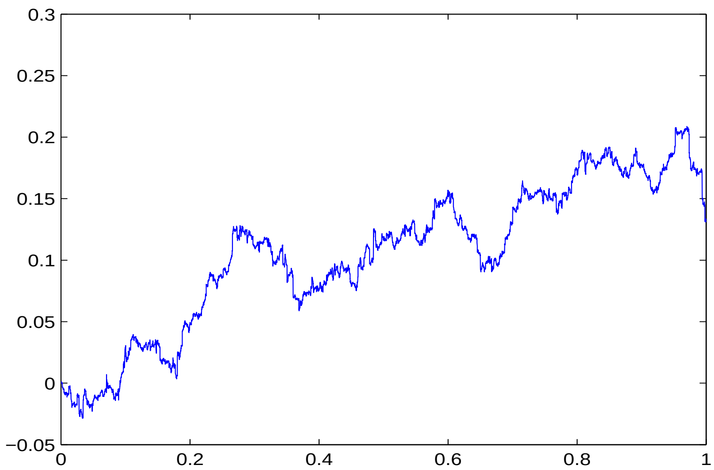

Figure A.2: *Sample path of a Meixner process with parameters* $\mu = 0.002$ $\alpha = 0.015$, $\beta = 0.12$, *and* $\delta = 94$.

A.3 Simulation of NIG and VG processes

For the Variance Gamma and Normal Inverse Gaussian processes we can exploit their expression as subordinated Lévy processes:

$$X_t = \mu t + \alpha Z_t + \beta W_{Z_t}, \tag{A.4}$$

where $\{Z_t\}$ is a subordinator and $\{W_t\}$ is a Brownian motion. Since $\{Z_t\}$ can be either a Gamma process or an Inverse Gaussian process, then, first of all, we have to explain how to generate random numbers from these two distributions.

A Gamma distribution with parameters $a > 0$ and $b > 0$, $\mathrm{G}(a, b)$, satisfies the following scaling property: if X is $\mathrm{G}(a, b)$, then, for $c > 0$, cX is $\mathrm{G}(a, b/c)$. Thus, it is sufficient to generate random numbers from $\mathrm{G}(a, 1)$. The following algorithm is the so-called Johnk's Gamma generator and can be used when $a \leq 1$, which is the case generally met with applications (see Schoutens [76], Section 8.4):

1. Generate two independent random numbers u_1 and u_2 from an uniform distribution.
2. Set $x = u_1^{1/a}$ and $y = u_2^{1/(1-a)}$.
3. If $x + y \leq 1$ goto step 4, else goto step 1.
4. Generate a random number z from an exponential distribution with mean 1.
5. Return $zx/(x + y)$ as a random number from a $\mathrm{G}(a, 1)$.

In order to sample random numbers from an Inverse Gaussian distribution with parameters $a > 0$ and $b > 0$, $\mathrm{IG}(a, b)$, we can use the generator proposed by Michael *et al.* [58]:

1. Generate a random number ν from a standard Normal distribution.
2. Set $y = \nu^2$.

3. Set $x = (a/b) + y/(2b^2) - \sqrt{4aby + y^2}/(2b^2)$.

4. Generate a random number u from an uniform distribution.

5. If $u \leq a/(a + xb)$, then return x as a random number from a $\mathrm{IG}(a, b)$, else return $a^2/(b^2 x)$ as a random number from a $\mathrm{IG}(a, b)$.

Now, using equation (A.4), we can simulate sample paths from VG and NIG processes.

Let $\{X_t : t \geq 0\}$ be a VG process with parameters μ, θ, σ, and ν. A sample path on time points $\{k\Delta t : k = 0, 1, \ldots, n\}$ can be generated with the following procedure:

1. Generate n independent random numbers $\{G_k : k = 1, \ldots, n\}$ from a $\mathrm{G}(\Delta t/\nu, \nu)$.

2. Generate n independent random numbers $\{Z_k : k = 1, \ldots, n\}$ from a standard Normal distribution.

3. A sample path on time points $\{k\Delta t : k = 0, 1, \ldots, n\}$ is given by

$$X_0 = 0, \quad X_{k\Delta t} = X_{(k-1)\Delta t} + \mu\Delta t + \theta G_k + \sigma\sqrt{G_k} Z_k, \quad 1 \leq k \leq n.$$

When, instead, $\{X_t : t \geq 0\}$ is a Normal Inverse Gaussian process with parameters μ, α, β, and δ, then a sample path on time points $\{k\Delta t : k = 0, 1, \ldots, n\}$ can be generated with the following procedure:

1. Generate n independent random numbers $\{I_k : k = 1, \ldots, n\}$ from a $\mathrm{IG}(\Delta t, b)$, where $b = \delta\sqrt{\alpha^2 - \beta^2}$.

2. Generate n independent random numbers $\{Z_k : k = 1, \ldots, n\}$ from a standard Normal distribution.

3. A sample path on time points $\{k\Delta t : k = 0, 1, \ldots, n\}$ is given by

$$X_0 = 0, \quad X_{k\Delta t} = X_{(k-1)\Delta t} + \mu\Delta t + \beta\delta^2 I_k + \delta\sqrt{I_k} Z_k, \quad 1 \leq k \leq n.$$

Figure A.3 exhibits two sample paths, on the left, of a VG process and, on the right, of a NIG process.

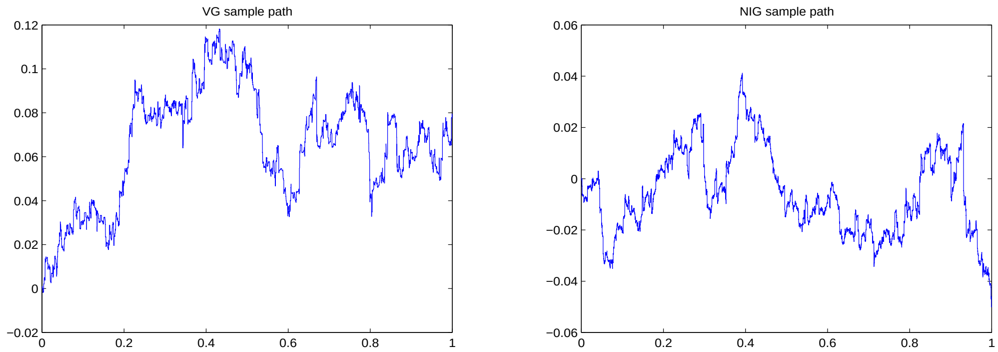

Figure A.3: *Sample path of a VG process with parameters* $\mu = 0.005$, $\theta = 0.08$, $\sigma = 0.1$, *and* $\nu = 0.0025$, *and sample path of a NIG process with parameters* $\mu = 0.003$, $\alpha = 150$, $\beta = 5$, *and* $\delta = 1$.

Appendix B

Special functions

In this appendix we give a brief description of special functions recalled in some mathematical results during the dissertation. These functions are the Gamma and Bessel functions (fundamental reference for these and many other functions is Abramowitz and Stegun ()). The Gamma function was introduced by the Swiss mathematician Leonard Euler (1707-1783) who generalized the factorial function to non integer values. Later, other eminent mathematicians, such as Adrien-Marie Legendre (1752-1833), Carl Friedrich Gauss (1777-1855), Karl Weierstrass (1815-1897), and many others, studied this special function. The Gamma function appears in many mathematical areas, such as asymptotic series, definite integration, number theory, and so on.

The Bessel functions was studied by the mathematicians Daniel Bernoulli (1700-1782) and Friedrich Bessel (1784-1846). These functions are canonical solutions of Bessel's differential equation and are especially important for problems of wave propagation, static potential, and so on.

B.1 Gamma function

The Gamma function can be considered an extension of the factorial function to real numbers, and for positive real numbers it can be defined as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0.$$

Let us verify that actually the function $\Gamma(x)$ is an extension of the factorial one. Using integration by parts,

$$\Gamma(x+1) = x\Gamma(x), \tag{B.1}$$

then $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$, and so the factorial function is a special case of the Gamma one:

$$\Gamma(n+1) = n\Gamma(n) = \cdots = n!\Gamma(1) = n!.$$

Exploiting the functional equation (B.1), we can extend the Gamma function to whole real axis except on the negative integer (0,-1,-2,...). As a matter of fact, for $-1 < x < 0$ we can set

$$\Gamma(x) = \frac{\Gamma(x+1)}{x},$$

thus reiterating this identity we have the definition

$$\Gamma(x) = \frac{\Gamma(x+n)}{x(x+1)\cdots(x+n-1)}, \quad -n < x < -n+1.$$

Let us solve an integral whose solution represents the Laplace exponent of the tempered stable subordinator and where the Gamma function on negative values appears. Specifically, we have to solve the integral

$$\begin{aligned} l(u) &= \int_0^\infty (e^{ux}-1)\frac{ce^{-\lambda x}}{x^{\alpha+1}}dx \\ &= c\int_0^\infty \left(e^{(u-\lambda)x} - e^{-\lambda x}\right) x^{-\alpha-1}dx, \end{aligned}$$

where $u \leq 0$, $c, \lambda > 0$, and $0 < \alpha < 1$. Using integration by parts, we can write

$$c\int_0^\infty \left(e^{(u-\lambda)x} - e^{-\lambda x}\right) x^{-\alpha-1}dx = \frac{c}{\alpha}\int_0^\infty \left((u-\lambda)e^{(u-\lambda)x} + \lambda e^{-\lambda x}\right) x^{-\alpha}dx,$$

and thus

$$\begin{aligned} l(u) &= \frac{c}{\alpha}\int_0^\infty \left((u-\lambda)e^{(u-\lambda)x} + \lambda e^{-\lambda x}\right) x^{-\alpha} dx \\ &= \frac{c}{\alpha}\int_0^\infty (u-\lambda)e^{(u-\lambda)x} x^{-\alpha} dx + \frac{c}{\alpha}\int_0^\infty \lambda e^{-\lambda x} x^{-\alpha} dx. \end{aligned}$$

Finally, using integration by substitution, we obtain

$$l(u) = c\left((\lambda-u)^\alpha - \lambda^\alpha\right)\Gamma(-\alpha).$$

B.2 Bessel and modified Bessel functions

Bessel functions of the first kind $J_{\pm\nu}(z)$, of the second kind $N_\nu(z)$, and of the third kind $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are solutions to the differential equation:

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2-\nu^2)w = 0.$$

All these functions are holomorphic functions of z throughout the z-plane cut along the negative real axis, and for fixed $z \neq 0$ they are entire function of ν. Moreover, if $\nu = \pm n$ then $J_\nu(z)$ has no branch point and is an entire function of z. The function $J_\nu(z)$ can be written as the series

$$J_\nu(z) = (z/2)^\nu \sum_{k=0}^\infty \frac{(-z^2/4)^k}{k!\Gamma(\nu+k+1)},$$

and there exists the relation

$$N_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)},$$

where the right side is replaced by its limiting value if ν is an integer or zero. Furthermore,

$$\begin{aligned} H_\nu^{(1)}(z) &= J_\nu(z) + iN_\nu(z), \\ H_\nu^{(2)}(z) &= J_\nu(z) - iN_\nu(z). \end{aligned}$$

Modified Bessel functions of the first kind $I_{\pm\nu}(z)$ and of the third kind $K_\nu(z)$ are solutions to the differential equation:

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} - (z^2+\nu^2)w = 0.$$

Even these functions are holomorphic functions of z throughout the z-plane cut along the negative real axis, and for $z \neq 0$ they are entire functions of z. For $\nu = \pm n$, $I_\nu(z)$ is an entire function of z. The function $I_\nu(z)$ can be written as the series

$$I_\nu(z) = (z/2)^\nu \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!\Gamma(\nu+k+1)},$$

and there exists the relation

$$K_\nu(z) = \frac{\pi}{2}\frac{I_\nu(z) - I_{-\nu}(z)}{\sin(\nu\pi)},$$

where the right side is replaced by its limiting value if ν is an integer or zero. The Bessel function $K_\nu(z)$ admits the integral form

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t-\frac{z^2}{4t}} t^{-\nu-1} dt.$$

Useful properties are:

$$\begin{aligned}
K_\nu(z) &= K_{-\nu}(z), \\
K_{\nu+1}(z) &= \frac{2\nu}{z}K_\nu(z) + K_{\nu-1}(z), \\
K_{1/2}(z) &= \sqrt{\pi/2}z^{-1/2}\exp(-z), \\
K'_\nu(z) &= -\frac{\nu}{z}K_\nu(z) - K_{\nu-1}(z).
\end{aligned}$$

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