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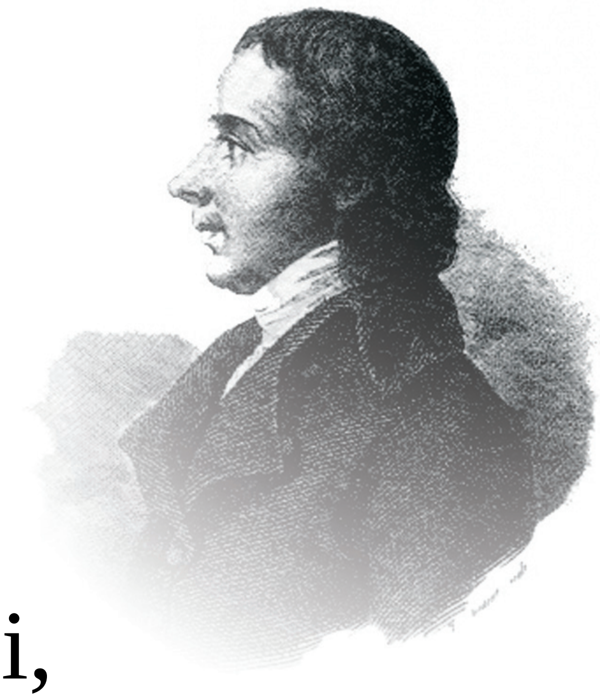

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Gauss-Seidel Method for Multi-Valued Inclusions with Z Mappings

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Gauss-Seidel Method for Multi-Valued Inclusions with Z Mappings [1]

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Abstract. We consider a problem of solution of a multi-valued inclusion on a cone segment. In the case where the underlying mapping possesses Z type properties we suggest an extension of Gauss-Seidel algorithms from nonlinear equations. We prove convergence of a modified double iteration process under rather mild additional assumptions. Some results of numerical experiments are also presented.

Key words: Multi-valued inclusions, weak Z-mappings, Gauss-Seidel algorithm.

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1 Introduction

The problem of solving a system of nonlinear equations is one of the basic and most investigated problems considered in Nonlinear Analysis; see e.g. [1] and references therein. It is also closely related to fixed point, complementarity and variational inequality problems; see e.g. [2]–[5] and references therein. However, many applications arising e.g. in Mathematical Physics and Economics require utilization of more general multi-valued mappings. Then one has to replace nonlinear equations with multi-valued inclusions; see e.g. [6, 2, 7].

Recently, in [8], Jacobi type algorithms for solving multi-valued inclusions on cone segments whose cost mappings are compositions of multi-valued Z-mappings and diagonal monotone mappings were proposed. Also, in [9], a Gauss-Seidel type algorithm for complementarity problems under similar assumptions was proposed.

In this paper, we intend to develop a Gauss-Seidel type algorithm for multi-valued inclusions on cone segments, thus extending the usual Gauss-Seidel algorithm from the single-valued case; see e.g. [1].

2 Classes of order monotone mappings

We start our considerations from recalling several order monotonicity properties of single-valued mappings. In what follows, all the inequalities for vectors are coordinate-wise; i.e. $x \geq y$ means that $x_i \geq y_i$ for every i, etc.

Definition 1 Let D be a rectangle set in $\mathbb{R}^n$. A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be

(a) *isotone* on D if for each pair of points $x', x'' \in D$ such that $x' \geq x''$, it holds that $F(x') \geq F(x'')$;

(b) *antitone* on D if the mapping $-F$ is isotone on D;

(c) a *Z-mapping* on D if for each pair of points $x', x'' \in D$ such that $x' \geq x''$, it holds that $F_k(x') \leq F_k(x'')$ for each index k with $x'_k = x''_k$.

These properties have been investigated rather well, especially, in the affine case, then they are strongly related with the corresponding classes of matrices; see e.g. [10].

We present some extensions of the concept of the Z-mapping for the multi-valued case. In what follows, $\Pi(S)$ denotes the family of all subsets of a set S.

Definition 2 Let D be a rectangle set in $\mathbb{R}^n$. A multi-valued mapping $G : \mathbb{R}^n \to \Pi(\mathbb{R}^n)$ is said to be

(a) a *Z-mapping* on D if for each pair of points $x', x'' \in D$ such that $x' \geq x''$, $x' \neq x''$, it holds that $g'_k \leq g''_k$ for all $g' \in G(x')$, $g'' \in G(x'')$ and for each index k such that $x'_k = x''_k$;

(b) an *upper* (a *lower*) *Z-mapping* on D if for each pair of points $x', x'' \in D$ such that $x' \geq x''$ and for each $g' \in G(x')$ there exists $g'' \in G(x'')$ (respectively, for each $g'' \in G(x'')$ there exists $g' \in G(x')$) such that $g'_k \leq g''_k$ for every index k such that $x'_k = x''_k$;

(c) a *weak Z-mapping* on D if it is both an upper and a lower Z-mapping.

Note that the additional condition $x' \neq x''$ can not be dropped in (a) since otherwise the Z-mapping becomes single-valued.

Definition 3 A mapping $G : \mathbb{R}^n \to \Pi(\mathbb{R}^n)$ is said to be

(a) *diagonal* if $G(x) = \prod_{i=1}^{n} G_i(x_i)$;

(b) *quasi-diagonal* [11] if $G(x) = \prod_{i=1}^{n} G_i(x)$.

Clearly, (a)$\Longrightarrow$(b). Moreover, each single-valued mapping is quasi - diagonal. Next, observe that each diagonal single-valued mapping is Z, but this is not the case if it is multi-valued. Hence, various compositions of multi-valued diagonal and Z-mappings may not possess the Z property as well. Hence, the streamlined extension of the Z-mapping given in Definition 2 (a) may appear too restrictive. For this reason, it seems more suitable to utilize weaker concepts of multi-valued Z-mappings given in Definition 2, (b)–(c), which contain arbitrary diagonal multi-valued mappings. More detailed discussions of order monotonicity properties for multi-valued mappings can be found in [12, 8].

We recall also the known continuity and monotonicity type properties for multi-valued mappings.

Definition 4 A mapping $G : \mathbb{R}^n \to \Pi(\mathbb{R}^n)$ is said to be

(a) *monotone* on $D \subseteq \mathbb{R}^n$, if for each pair of points $x', x'' \in D$ and for all $g' \in G(x'), g'' \in G(x'')$, it holds that

$$\langle g' - g'', x' - x'' \rangle \geq 0;$$

(b) a *Kakutani-mapping* (K-mapping) on $D \subseteq \mathbb{R}^n$ if it is upper semicontinuons and has nonempty, convex, and compact image sets on D.

3 Statement of the problem and the Gauss-Seidel algorithm

Let us consider the problem of finding a point $x^* \in \mathbb{R}^n$ such that

$$0 \in G(x^*) \tag{1}$$

under the following standing assumptions.

(A1) *The mapping* $G : \mathbb{R}^n \to \Pi(\mathbb{R}^n)$ *is of the form*

$$G(x) = \sum_{s=1}^{l} F^{(s)} \circ H^{(s)}(x), \tag{2}$$

where $F^{(s)} : \mathbb{R}^n \to \Pi(\mathbb{R}^n)$ *is a quasi-diagonal, weak* Z*-, and* K*-mapping on some rectangle containing* $H^{(s)}(D)$*,* $H^{(s)} : \mathbb{R}^n \to \Pi(\mathbb{R}^n)$ *is a diagonal monotone* K*-mapping on* D *for each* $s = 1, \dots, l$*,* D *is a rectangle set in* $\mathbb{R}^n$*.*

(A2) *There exist points* $x^0,\ y^0 \in D$*,* $x^0 < y^0$*, such that*

$$g' \leq 0 \leq g'' \quad \textit{for some} \quad g' \in G(x^0) \quad \textit{and} \quad g'' \in G(y^0). \tag{3}$$

We now describe a double iteration Gauss-Seidel algorithm for the above problem.

Algorithm (DGS). Starting from the points $x^0,\ y^0 \in D$, $x^0 < y^0$, construct sequences $\{x^k\}$ and $\{y^k\}$ in conformity with the following rules.

At the k-th iteration, $k = 0, 1, \dots$, we have points $x^k,\ y^k \in D$ such that $x^0 \leq x^k \leq y^k \leq y^0$ and that there exist $g^k \in \sum\limits_{s=1}^{l} F^{(s)}(h^{(s,k,x)})$ and

$q^k \in \sum_{s=1}^{l} F^{(s)}(h^{(s,k,y)})$ for some $h^{(s,k,x)} \in H^{(s)}(x^k)$, $h^{(s,k,y)} \in H^{(s)}(y^k)$, satisfying the conditions:

$$g^k \leq 0 \leq q^k \text{ and } h^{(s,0,x)} \leq \; h^{(s,k,x)} \leq h^{(s,k,y)} \leq \; h^{(s,0,y)}, \; s = 1, \dots, l; \qquad (4)$$

where $h^{(s,0,x)} \in H^{(s)}(x^0)$ and $h^{(s,0,y)} \in H^{(s)}(y^0)$.

In the sequel we will use the notation:

$$\left(h_{-i}^{(s,k+1,k,x)}, p_i^{(s)}\right) = \left(h_1^{(s,k+1,x)}, \dots, h_{i-1}^{(s,k+1,x)}, p_i^{(s)}, h_{i+1}^{(s,k,x)}, \dots, h_n^{(s,k,x)}\right),$$

with $p_i^{(s)} \in \mathbb{R}$.

Now, for each separate index $i = 1, \dots, n$, we determine numbers x_i^{k+1}, $p_i^{(1)}, \dots, p_i^{(l)}$ such that

$$\begin{array}{c} x_i^k \leq x_i^{k+1} \leq y_i^k, p_i^{(s)} \in H_i^{(s)}(x_i^{k+1}), \\ h_i^{(s,k,x)} \leq p_i^{(s)} \leq h_i^{(s,k,y)} \quad \text{for} \quad s = 1, \dots, l, \end{array} \qquad (5)$$

and

$$\exists \tilde{g}_i^k \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,x)}, p_i^{(s)}), \quad \tilde{g}_i^k = 0; \qquad (6)$$

with the help of the bisection type Procedure A below. Afterwards, set $h_i^{(s,k+1,x)} = p_i^{(s)}$ for $s = 1, \dots, l$.

Next, for each separate index $i = 1, \dots, n$, we determine numbers y_i^{k+1}, $t_i^{(1)}, \dots, t_i^{(l)}$ such that

$$\begin{array}{c} x_i^{k+1} \leq y_i^{k+1} \leq y_i^k, t_i^{(s)} \in H_i^{(s)}(y_i^{k+1}), \\ h_i^{(s,k+1,x)} \leq t_i^{(s)} \leq h_i^{(s,k,y)} \quad \text{for} \quad s = 1, \dots, l, \end{array} \qquad (7)$$

and

$$\exists \tilde{q}_i^k \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,y)}, t_i^{(s)}), \quad \tilde{q}_i^k = 0; \qquad (8)$$

with the help of the bisection type Procedure B below. Afterwards, set $h_i^{(s,k+1,y)} = t_i^{(s)}$ for $s = 1, \dots, l$. If $i = n$, go to the $(k+1)$-th iteration.

Procedure A. It is applied when the indices k and i are fixed and consists of the following sequence of steps.

Step 1: Choose $p_i^{(s)} = h_i^{(s,k,x)} \in H_i^{(s)}(x_i^k)$ for $s = 1, \dots, l$ and $\bar{g}_i^k \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,x)}, p_i^{(s)})$. If $\bar{g}_i^k \geq 0$, then set $x_i^{k+1} = x_i^k$ and stop. Otherwise set $x_i' = x_i^k, \alpha_i^{(s)} = h_i^{(s,k,x)}$, $x_i'' = y_i^k$, $\beta_i^{(s)} = h_i^{(s,k,y)}$ for $s = 1, \dots, l$.

Step 2: Generate a sequence of inscribed segments $[x_i', x_i'']$ contracting to a point z_i by choosing $u_i = \frac{1}{2}(x_i' + x_i'')$, $\gamma_i^{(s)} \in H_i^{(s)}(u_i)$ and setting $x_i' = u_i$, $\alpha_i^{(s)} = \gamma_i^{(s)}$ if $\tilde{g}_i^k \leq 0$ for some $\tilde{g}_i \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,x)}, \gamma_i^{(s)})$ or $x_i'' = u_i$, $\beta_i^{(s)} = \gamma_i^{(s)}$ otherwise, i.e. when $\tilde{g}_i > 0$.

Step 3: Set $x_i^{k+1} = z_i$ and compute numbers $p_i^{(s)} \in H_i^{(s)}(z_i)$ for $s = 1, \dots, l$, such that conditions (5), (6) are satisfied.

Procedure B. It is applied when the indices k and i are fixed and consists of the following sequence of steps.

Step 1: Choose $t_i^{(s)} = h_i^{(s,k,y)} \in H_i^{(s)}(y_i^k)$ for $s = 1, \dots, l$ and $\bar{q}_i^k \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,y)}, t_i^{(s)})$. If $\bar{q}_i^k \leq 0$, then set $y_i^{k+1} = y_i^k$ and stop. Otherwise set $y_i' = x_i^{k+1}, \alpha_i^{(s)} = h_i^{(s,k+1,x)}$, $y_i'' = y_i^k$, $\beta_i^{(s)} = h_i^{(s,k,y)}$ for $s = 1, \dots, l$.

Step 2: Generate a sequence of inscribed segments $[y_i', y_i'']$ contracting to a point $\tilde{z}_i$ by choosing $v_i = \frac{1}{2}(y_i' + y_i'')$, $\gamma_i^{(s)} \in H_i^{(s)}(v_i)$ and setting $y_i' = v_i$, $\alpha_i^{(s)} = \gamma_i^{(s)}$ if $\tilde{q}_i^k \leq 0$ for some $\tilde{q}_i \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,y)}, \gamma_i^{(s)})$ or $y_i'' = v_i$, $\beta_i^{(s)} = \gamma_i^{(s)}$ otherwise, i.e. when $\tilde{q}_i > 0$.

Step 3: Set $y_i^{k+1} = \tilde{z}_i$ and compute numbers $p_i^{(s)} \in H_i^{(s)}(\tilde{z}_i)$ for $s = 1, \dots, l$, such that conditions (7), (8) are satisfied.

4 Convergence

We are now in a position to establish a convergence result for the Gauss-Seidel algorithm described.

Theorem 1 *Let assumptions* **(A1)** *and* **(A2)** *be fulfilled. Then the Gauss-Seidel algorithm with the bisection procedures A and B is well defined and generates the sequences* $\{x^k\}$ *and* $\{y^k\}$ *converging to the solutions* x^* *and* y^* *of problem (1) such that* $x^0 \leq x^* \leq y^* \leq y^0$.

Proof. First we note that (4) holds for $k = 0$ due to **(A2)** and to the monotonicity of $H^{(s)}$.

Next, we show that Procedure A is well defined. Suppose that (5)–(6) are true for each index $1, \dots, i-1$. Then termination at Step 1 gives $x_i^{k+1} = x_i^k$ and $p_i^{(s)} = h_i^{(s,k,x)} \in H_i^{(s)}(x_i^k)$ for $s = 1, \dots, l$, i.e. (5) holds. By definition, $g_i^k \leq 0$ for some $g_i^k \in \sum\limits_{s=1}^{l} F_i^{(s)}(h^{(s,k,x)})$ but

$$h^{(s,k,x)} = (h_{-i}^{(s,k,x)}, p_i^{(s)}) \leq (h_{-i}^{(s,k+1,k,x)}, p_i^{(s)})$$

and by the weak Z property of $F^{(s)}$ there exists $g_i \in \sum\limits_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,x)}, p_i^{(s)})$ such that $g_i \leq g_i^k \leq 0$. Since $\bar{g}_i^k \geq 0$ where $\bar{g}_i^k \in \sum\limits_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,x)}, p_i^{(s)})$ there exists $\tilde{g}_i^k \in \sum\limits_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,x)}, p_i^{(s)})$ such that $\tilde{g}_i^k = 0$ and (6) holds.

In Step 2, by construction, we have $\alpha_i^{(s)} \leq \beta_i^{(s)}$ for $s = 1, \dots, l$ and $\bar{g}_i^k \leq 0$ where $\bar{g}_i^k \in \sum\limits_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,x)}, \alpha_i^{(s)})$. Note that $\beta_i^{(s)} = h_i^{(s,k,y)}$ implies

$$(h_{-i}^{(s,k+1,k,x)}, \beta_i^{(s)}) = (h_{-i}^{(s,k+1,k,x)}, h_i^{(s,k,y)}) \leq h^{(s,k,y)}$$

and by the weak Z property of $F^{(s)}$ there exists $g_i \in \sum\limits_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,x)}, \beta_i^{(s)})$ such that $g_i \geq q_i^k \geq 0$.

At the point z_i, we define a multi-valued mapping $\Phi : \mathbb{R}^l \to \Pi(\mathbb{R})$ on the rectangle $[\alpha_i^{(1)}, \beta_i^{(1)}] \times \dots \times [\alpha_i^{(l)}, \beta_i^{(l)}]$ as follows

$$\Phi(p_i) = \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,x)}, p_i^{(s)}) \quad \text{with} \quad p_i = (p_i^{(1)}, \dots, p_i^{(l)}) \in \mathbb{R}^l.$$

Since $F^{(s)}$ are K-mappings, so is Φ. Then, by construction, $-\Phi(\alpha_i) \bigcap \mathbb{R}_+ \neq \emptyset$ and $\Phi(\beta_i) \bigcap \mathbb{R}_+ \neq \emptyset$ for $\alpha_i = (\alpha_i^{(1)}, \dots, \alpha_i^{(l)})$ and $\beta_i = (\beta_i^{(1)}, \dots, \beta_i^{(l)})$. Hence, there exists a number $\lambda \in [0,1]$ such that $0 \in \Phi(p_i^{(s)})$ for the point $p_i^{(s)} = \lambda \alpha_i + (1-\lambda)\beta_i \in \mathbb{R}^l$. Since $\alpha_i^{(s)}$, $\beta_i^{(s)} \in H_i^{(s)}(z_i)$ and each $H_i^{(s)}$ has convex images, it follows that $p_i^{(s)} \in H_i^{(s)}(z_i)$ for $s = 1, \dots, l$. Then all the relations in (5), (6) are satisfied and Procedure A is well defined.

Next, since

$$(h_{-i}^{(s,k+1,k,x)}, h_i^{(s,k+1,x)}) \leq h^{(s,k+1,x)}$$

by the weak Z property of $F^{(s)}$ for each $\tilde{f}_i^{(s)} \in F_i^{(s)}(h_{-i}^{(s,k+1,k,x)}, h_i^{(s,k+1,x)})$, there exists $f_i^{(s)} \in F_i^{(s)}(h^{(s,k+1,x)})$ such that $\tilde{f}_i^{(s)} \geq f_i^{(s)}$ for $i = 1, \dots, n$. We now conclude that

$$0 = \tilde{g}_i^k = \sum_{s=1}^{l} \tilde{f}_i^{(s)} \geq \sum_{s=1}^{l} f_i^{(s)} = g_i^{k+1}$$

for $i = 1, \dots, n$, hence the ascent process is well defined.

Similarly, we can prove that Procedure B is well defined. Suppose that (7)–(8) are true for each index $1, \dots, i-1$. Then termination at Step 1 gives $y_i^{k+1} = y_i^k$ and $p_i^{(s)} = h_i^{(s,k,y)} \in H_i^{(s)}(y_i^k)$ for $s = 1, \dots, l$, i.e. (7) holds. By definition, $q_i^k \geq 0$ for some $q_i^k \in \sum_{s=1}^{l} F_i^{(s)}(h^{(s,k,y)})$ but

$$h^{(s,k,y)} = (h_{-i}^{(s,k,y)}, t_i^{(s)}) \geq (h_{-i}^{(s,k+1,k,y)}, t_i^{(s)})$$

and by the weak Z property of $F^{(s)}$ there exists $q_i \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,y)}, t_i^{(s)})$ such that $q_i \geq q_i^k \geq 0$. Since $\bar{q}_i^k \leq 0$ where $\bar{q}_i^k \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,y)}, t_i^{(s)})$ there exists $\tilde{q}_i^k \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,y)}, t_i^{(s)})$ such that $\tilde{q}_i^k = 0$ and (8) holds.

In Step 2, by construction, we have $\alpha_i^{(s)} \leq \beta_i^{(s)}$ for $s = 1, \dots, l$ and $\bar{q}_i^k \geq 0$ where $\bar{q}_i^k \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,y)}, \beta_i^{(s)})$. Note that $\alpha_i^{(s)} = h_i^{(s,k+1,x)}$ implies

$$(h_{-i}^{(s,k+1,k,y)}, \alpha_i^{(s)}) = (h_{-i}^{(s,k+1,k,y)}, h_i^{(s,k+1,x)}) \geq h^{(s,k+1,x)}$$

and by the weak Z property of $F^{(s)}$ there exists $q_i \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,y)}, \alpha_i^{(s)})$ such that $q_i \leq q_i^{k+1} \leq 0$.

At the point $\tilde{z}_i$, we define a multi-valued mapping $\Phi : \mathbb{R}^l \to \Pi(\mathbb{R})$ on the rectangle $[\alpha_i^{(1)}, \beta_i^{(1)}] \times \cdots \times [\alpha_i^{(l)}, \beta_i^{(l)}]$ as follows

$$\Phi(t_i) = \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s,k+1,k,y)}, t_i^{(s)}) \quad \text{with} \quad t_i = (t_i^{(1)}, \dots, t_i^{(l)}) \in \mathbb{R}^l.$$

Since $F^{(s)}$ are K-mappings, so is Φ. Then, by construction, $-\Phi(\alpha_i)\bigcap\mathbb{R}_+ \neq \emptyset$ and $\Phi(\beta_i)\bigcap\mathbb{R}_+ \neq \emptyset$ for $\alpha_i = (\alpha_i^{(1)}, \dots, \alpha_i^{(l)})$ and $\beta_i = (\beta_i^{(1)}, \dots, \beta_i^{(l)})$. Hence, there exists a number $\lambda \in [0, 1]$ such that $0 \in \Phi(t_i^{(s)})$ for the point $t_i^{(s)} = \lambda\alpha_i + (1-\lambda)\beta_i \in \mathbb{R}^l$. Since $\alpha_i^{(s)}$, $\beta_i^{(s)} \in H_i^{(s)}(\tilde{z}_i)$ and each $H_i^{(s)}$ has convex images, it follows that $t_i^{(s)} \in H_i^{(s)}(\tilde{z}_i)$ for $s = 1, \dots, l$. Then all the relations in (7), (8) are satisfied and Procedure B is well defined.

Next, since

$$(h_{-i}^{(s,k+1,k,y)}, h_i^{(s,k+1,y)}) \geq h^{(s,k+1,y)}$$

by the weak Z property of $F^{(s)}$ for each $\tilde{f}_i^{(s)} \in F_i^{(s)}(h_{-i}^{(s,k+1,k,y)}, h_i^{(s,k+1,y)})$, there exists $f_i^{(s)} \in F_i^{(s)}(h^{(s,k+1,y)})$ such that $\tilde{f}_i^{(s)} \leq f_i^{(s)}$ for $i = 1, \dots, n$. We now conclude that

$$0 = \tilde{q}_i^k = \sum_{s=1}^{l} \tilde{f}_i^{(s)} \leq \sum_{s=1}^{l} f_i^{(s)} = q_i^{k+1}$$

for $i = 1, \dots, n$, hence the descent process is well defined.

On account of (5) and (7), the sequence $\{x^k\}$ is non-decreasing and bounded from above and the sequence $\{y^k\}$ is non-increasing and bounded from below. Therefore, the sequence $\{x^k\}$ converges to a point x^* and the sequence $\{y^k\}$ converges to a point y^* such that $x^0 \leq x^* \leq y^* \leq y^0$. Analogously, for each s the sequence $\{h^{(s,k+1,k,x)}\}$ is non-decreasing and bounded and $\{h^{(s,k+1,k,y)}\}$ is non-increasing and bounded, hence, by the K property,

$$\lim_{k\to\infty} h^{(s,k+1,k,x)} = h^{(s,x)}$$

for some $h^{(s,x)} \in H^{(s)}(x^*)$ and

$$0 = \lim_{k\to\infty} \tilde{g}_i^k = g_i^* \in \sum_{s=1}^{l} F_i^{(s)}(h^{(s,x)}),$$

i.e. $0 \in G(x^*)$. Analogously it is possible to verify that $0 \in G(y^*)$. The proof is complete. □

5 Numerical experiments

In this section we present some numerical examples tested with the help of the following computer environment OS 32 bit: Windows XP Pro; CPU:

Intel (R) Core (TM)2 Duo CPU 1.66 GHz; Memory: 2 GB; OS Software: Matlab. For each numerical example we applied the Jacobi and Gauss-Seidel algorithms with the same input values and the same criteria. The Jacobi algorithm was constructed in conformity with [8].

As noticed in [12], the choice of the elements $p_i^{(s)}$ and $t_i^{(s)},\ s = 1, \ldots, l$ such that relations (5) and (7) hold, can be much easily done by taking an arbitrary value in $H_i^{(s)}(\cdot)$. In Step 2 of Procedures A and B, the values $\gamma_i^{(s)}$ were chosen as the middle point of the segment for both Jacobi and Gauss-Seidel algorithms.

We made all the calculations with double precision and chose the following implementation setting:

1. The zero tolerance is 10^{-10}.

2. The stopping criteria of the dichotomy procedure is $|x_i' - x_i''| < 10^{-6}$ and of the main procedure is $\|x^{(k+1)} - x^{(k)}\| < 10^{-5}$ or the number of iterations are equal to MAXITER.

We considered two examples.

Example 1: We chose the mapping

$$G(x) = x + A \circ E(x) - C(x),$$

which is a particular case of that in (2), where $l = 3$, $F^{(1)} = I$, $F^{(2)} = A$, $F^{(3)} = I$, $H^{(1)} = I$, $H^{(2)} = E$, $H^{(3)} = -C$.

For numerical tests we set $A(x) = Mx$ where

$$m_{ij} \begin{cases} = \text{rand}(0,1) & \text{if } \ i = j, \\ \in (-10^{-k}\text{rand}(0,0.5), -10^{-k}\text{rand}(0.5,1)) & \text{if } \ i \neq j; \end{cases}$$

with $k = 0, 1, 2$.

$C_i(x_i) = [\alpha_i x_i, \beta_i x_i],\ x_i \in [-10, 10]$, $\alpha_i = ((i-1)/n)10^{-2}$ and $\beta_i = (i/n)10^{-2}$, $i = 1, \ldots, n$. We also set $E_i(x_i) = [\gamma_i x_i, \delta_i x_i],\ x_i \in [0, 10]$, $\gamma_i = ((i-1)/n)10^{-2}$ and $\delta_i = (i/n)10^{-2},\ i = 1, \ldots, n$.

We observe that A is a quasi-diagonal, weak Z-, and K-mapping.

The initial values were generated randomly as $x_i^0 \in (-10, 0)$ and $y_i^0 \in (0, 10)$ with $i = 1, \ldots, n$. A comparison of the average CPU time for the Jacobi and Gauss-Seidel algorithms is shown in Table 1. From the results of numerical tests we observe that the computational precision had no essential influence on these two algorithms.

Method	n=10	n=100	n=150	n=200	n=500
Jacobi	0.47	7.19	11.67	16.34	71.60
Gauss-Seidel	0.47	7.15	11.59	16.13	70.45

Table 1: Example 1: Average of CPU time (sec)

Example 2: We chose the mapping

$$G(x) = Mx + b + \Phi(x) + \Psi(x),$$

where M is an $n \times n$ matrix with nonpositive off-diagonal entries, Φ is a nonsmooth and continuous mapping, and Ψ is a multi-valued K-mapping.

For the experiments we determined the matrix M as

$$m_{ij} = \begin{cases} -|\sin(i)\cos(j)| & \text{if } i \neq j; \\ 1 + \sum\limits_{j \neq i} |m_{ij}| & \text{if } i = j; \end{cases} \quad i, j = 1, \dots, n;$$

the vector b as

$$b_i = \sin(i)/i, \quad i = 1, \dots, n.$$

the mappings Φ and Ψ as

$$\Phi(x) = \prod_{i=1}^{n} \Phi_i(x_i), \quad \Phi_i(x_i) = \max\left\{x_i^2 - 1/\sin(i), 0\right\}, \quad i = 1, \dots, n;$$

$$\Psi(x) = \prod_{i=1}^{n} \Psi_i(x_i), \quad \Psi_i(x_i) = \partial\psi_i(x_i),$$

$$\psi_i(x_i) = \alpha_i|x_i - \beta_i|, \alpha_i = (1+i)/i, \beta_i = 1/\cos(i), \quad i = 1, \dots, n.$$

Then G is a particular case of that in (2), where $l = 3$, $F^{(1)} = Mx + b$, $F^{(2)} = I$, $F^{(3)} = I$, $H^{(1)} = I$, $H^{(2)} = \Phi$, $H^{(3)} = \Psi$.

A comparison of the average CPU time for the Jacobi and Gauss-Seidel algorithms is shown in Table 2. From the results of numerical tests we observe that the performance of the Gauss-Seidel algorithm is better.

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Method	n=10	n=100	n=150	n=200	n=500
Jacobi	1.80	29.92	51.79	81.42	273.27
Gauss-Seidel	1.61	25.51	41.54	77.91	258.48

Table 2: Example 2: Average of CPU time (sec)

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