The value of information in multi-stage linear stochastic programming

Francesca Maggioni[§], Elisabetta Allevi[*], Marida Bertocchi[§]

Abstract. Multistage stochastic programs, which involve sequences of decisions over time, are usually hard to solve in realistically sized problems. In the two-stage case, several approaches based on different levels of available information has been adopted in literature such as the *Expected Value Problem*, EV, the *Sum of Pairs Expected Values*, $SPEV$, the *Expectation of Pairs Expected Value*, $EPEV$, solving series of sub-problems more computationally tractable than the initial one, or the *Expected Skeleton Solution Value*, $ESSV$ and the *Expected Input Value*, EIV which evaluate the quality of the deterministic solution in term of its structure and upgradability.

In this paper we generalize the definition of the above quantities to the multistage stochastic framework introducing the *Multistage Expected Value of the Reference Scenario*, $MEVRS$, the *Multistage Sum of Pairs Expected Values*, $MSPEV$ and the *Multistage Expectation of Pairs Expected Value*, $MEPEV$ by means of the new concept of auxiliary scenario and redefinition of pairs subproblems probability. Measures of quality of the average solution such as the *Multistage Loss Using Skeleton Solution*, $MLUSS^t$ and the *Multistage Loss of Upgrading the Deterministic Solution*, $MLUDS^t$ are introduced too and related to the standard *Value of Stochastic Solution*, VSS^t at stage t.
Chains of inequalities among the new quantities are proved to evaluate if it is worth the additional computations for the stochastic program versus the simplified approaches proposed. Numerical results on a simple transportation problem are shown.

Key words: Multistage stochastic programming, Expected value problem, Value of stochastic solution, Skeleton solution.

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1 Introduction

Stochastic programs, especially *multistage programs*, which involve sequences of decisions over time, are usually hard to solve in realistically sized problems. In the simpler two-stage case, several approaches and measures of levels of available information on a future realization has been adopted in literature (see for instance [1], [2], [4], [7], [8], [9], [10], [11], [14], [15], [17], [18] or [19]). The standard method is to compare, by means of the *Value of the Stochastic Solution – VSS –* [2] the expected gain from solving a stochastic model rather than its deterministic counterpart, in which random parameters are replaced by their expected values. A large *VSS* means that uncertainty is important for the optimal solution, and the deterministic solution is "bad". Bounds on VSS were introduced in [2] by means of the *Sum of Pairs Expected Values Solutions SPEV* and *Expectation of Pairs Expected Value* $EPEV$ by solving pairs subproblems much less complex than the general recourse problem; these bounds may be valuable in determining whether the additional computations for the stochastic program are warranted.

Even when VSS is high, and hence stochastic program is appropriate, in real case problem can happen that all we may have access is the deterministic solution. A qualitative understanding of the deterministic solution is then important because it could actually carries out a lot of information: in [12] the structure and upgradability of the deterministic solution has been analyzed for the two-stage case by means of the *Loss Using the Skeleton Solution* $LUSS$ and the *Loss of Upgrading the Deterministic Solution* $LUDS$ in relation to the standard *VSS*. $LUSS$ and $LUDS$ give deeper information than *VSS* on the structure of the problem and could be useful to take a fast "good" decision instead of using expensive direct techniques.

The aim of this paper is to extend to the multistage case the measures of information already adopted for the two-stage case in [3] and [12], inspired by [4], [5] and also by [16].

Because of the computational intractability of most of multistage problems, we believe it is very useful to consider especially in the multistage case, different approximations of the recourse problem and evaluations at different levels of information, of how the deterministic solution performs in the stochastic framework. This applies to algorithmic developments as well as practical use of models in management for industry and government.

An extension to multistage case of the classical VSS defined for the two-stage one, has been already introduced in [5] through a chain of values VSS^t which takes into account the information until stage t of the associated deterministic model.

In this paper approximations of the optimal stochastic solution such as the *Multistage Expected Value of the Reference Scenario*, $MEVRS$, the *Multistage Sum of Pairs Expected Values*, $MSPEV$, and the *Multistage Expectation of Pairs Expected Value,* $MEPEV$ are introduced by means of the new concept of auxiliary scenario and redefinition of pairs subproblem probability. The proposed approaches allow to bound the optimal stochastic objective function by solving less complex pairs subproblems and help to quantify if it is worth the additional computation of the former problem.

Beside the standard *Value of Stochastic Solution*, VSS^t at stage t, measures of *quality* of the average solution such as the *Multistage Loss Using Skeleton Solution*, $MLUSS^t$ and the

Multistage Loss of Upgrading the Deterministic Solution, $MLUDS^t$ are introduced.
The above measures could be useful because help to *qualitatively* understand the behavior of the deterministic solution relative to the stochastic one and reveal some general properties of the underlying problem and how the stochastic model performs when the problem is not even solvable.

As pointed out in [5] the generalization of such a measures entails several issues: first of all the decision of the variables to be fixed from the deterministic solution. The trivial case would be to fix just the first stage variables and leaving the other ones free to adapt to the particular scenario. This procedure, nevertheless, can become a paradox in some cases since it could be that the first stage deterministic solution perform better than the stochastic one since the *nonanticipativity* constraints are relaxed in later stages.
In order to update the estimation at each stage and add more information, the above classes of measures are also defined with a rolling horizon approach already considered in [4] and [16]: the *Rolling Horizon Value of Stochastic Solution*, $RHVSS$, the *Rolling Horizon Loss Using Skeleton Solution*, $RHLUSS$ and the *Rolling Horizon Loss of Upgrading the Deterministic Solution*, $RHLUDS$ are presented.

Chains of inequalities among the new quantities are proved to evaluate if it is worth the additional computations for the stochastic program versus the simplified approaches.

We finally remark that all this class of measures is often used to describe problem classes, even though, they are instance dependent. As in the two-stage case [12], we assume that if a particular performance measure is high (or low) for a given selection of instances, then it will also be high (or low) for other instances that have similar characteristics, such as larger instances of the same problem.

The paper is organized as follow: basic definitions and notations are introduced in Section 2. Section 3 contains the generalization to the multistage case of the performance measure whereas in Section 5 the measures are computed and compared on a simple logistic problem already analyzed in [13]. Section 6 concludes the paper.

2 Notations and basic definitions

We introduce the notation that we are going to use.

The following mathematical model represents a general formulation of a multistage linear stochastic program in which a decision maker has to take a sequence of decisions $x^1, x^2, \ldots, x^H$, in order to minimize (expected) costs:

$$\begin{aligned} RP &= \min_{\mathbf{x}} E_{\boldsymbol{\xi}^{H-1}} z(\mathbf{x}, \boldsymbol{\xi}^{H-1}) \\ &= \min_{x^1} c^1 x^1 + E_{\xi^1}\left[\min_{x^2} c^2 x^2\left(\boldsymbol{\xi}^1\right) + E_{\xi^2}\left[\cdots + E_{\xi^{H-1}}\left[\min_{x^H} c^H x^H\left(\boldsymbol{\xi}^{H-1}\right)\right]\right]\right] \\ & \quad \text{s.t. } Ax^1 = h^1 \ , \\ & \qquad T^1(\boldsymbol{\xi}^1)x^1 + W^2(\boldsymbol{\xi}^1)x^2(\boldsymbol{\xi}^1) = h^2(\boldsymbol{\xi}^1) \ , \\ & \qquad \vdots \\ & \qquad T^{H-1}(\boldsymbol{\xi}^{H-1})x^{H-1}(\boldsymbol{\xi}^{H-1}) + W^H(\boldsymbol{\xi}^{H-1})x^H(\boldsymbol{\xi}^{H-1}) = h^H(\boldsymbol{\xi}^{H-1}) \ , \\ & \qquad x^1 \geq 0 \ ; \quad x^t(\boldsymbol{\xi}^{t-1}) \geq 0 \ , \ t = 2, \ldots, H; \end{aligned} \tag{1}$$

with $c^1 \in \Re^{n_1}$, $h^1 \in \Re^{m_1}$, $A \in \Re^{m_1 \times n_1}$, $t = 2, \ldots, H$. E_{ξ^t} denotes the expectation with respect to a random vector ξ^t, defined on a probability space $(\Xi^t, \mathscr{A}^t, p)$ with support $\Xi^t \in \Re^{n_t}$ and given probability distribution p on the $\sigma-$algebra $\mathscr{A}^t$ (with $\mathscr{A}^t \subseteq \mathscr{A}^{t+1}$).
We denote

- $h^t \in \Re^{m_t}$, $c^t \in \Re^{n_t}$, $T^{t-1} \in \Re^{m_{t-1} \times n_{t-1}}$, $W^t \in \Re^{m_t \times n_t}$, $t = 2, \ldots, H$;
- $\boldsymbol{\xi}^t = (\xi^1, \ldots, \xi^t)$, $t = 1, \ldots, H-1$;
- $\mathbf{x} = (x^1, x^2, \ldots, x^H)$ with $x^t \in \Re^{n_t}$, $t = 1, \ldots, H$ and x^{t_j} the j-th component of x^t.

In general $c^t = c^t(\xi^{t-1})$ for $t = 2, \ldots, H$. The decision x^t at stage $t = 1, \ldots, H$ depends from the history up to time t, more precisely from x^1, ξ^1, x^2, ξ^2, $\ldots$, x^{t-1}, ξ^{t-1}.
The solution x^* obtained by solving problem (1), is called the *here and now solution.*

We introduce, for later us, the form of feasible region at the stage t of problem (1).

$$K^t := \left\{ x^t \;\middle|\; \begin{array}{l} T^{t-1}(\boldsymbol{\xi}^{t-1})x^{t-1}(\boldsymbol{\xi}^{t-1}) + W^t(\boldsymbol{\xi}^{t-1})x^t(\boldsymbol{\xi}^{t-1}) = h^t(\boldsymbol{\xi}^{t-1}) \\ \\ E_{\xi^{t+1}}\left[Q^{t+1}(x^t, \xi^{t+1})\right] < +\infty \end{array} \right\}$$

with $t = 1 \ldots, H-1$, $T^0 = [0]$ is the zero-matrix and the recourse problem at stage t

$$\begin{aligned} Q^{t+1}(x^t, \xi^{t+1}) &= \min_{x^{t+1}} c^{t+1} x^{t+1}(\boldsymbol{\xi}^t) + E_{\xi^{t+2}}\left[Q^{t+2}(x^{t+1}, \xi^{t+2})\right] \\ \text{s.t.} \quad & T^t(\boldsymbol{\xi}^t)x^t(\boldsymbol{\xi}^t) + W^{t+1}(\boldsymbol{\xi}^t)x^{t+1}(\boldsymbol{\xi}^t) = h^{t+1}(\boldsymbol{\xi}^t) \ , \\ & x^{t+1}(\boldsymbol{\xi}^t) \geq 0 \quad t = 1, \ldots, H-2 \ ; \end{aligned}$$

in the last stage

$$\begin{aligned} Q^H(x^{H-1}, \xi^H) &= \min_{x^H} c^H x^H(\boldsymbol{\xi}^{H-1}) \\ \text{s.t.} \quad & T^t(\boldsymbol{\xi}^{H-1})x^{H-1}(\boldsymbol{\xi}^{H-1}) + W^H(\boldsymbol{\xi}^{H-1})x^H(\boldsymbol{\xi}^{H-1}) = h^H(\boldsymbol{\xi}^{H-1}) \ , \\ & x^H(\boldsymbol{\xi}^{H-1}) \geq 0 \ . \end{aligned}$$

If we consider the case where ξ^t is a random variable from a discrete distribution, then at each stage t it has a discrete number of atoms (nodes) n_t. In this case the probabilistic structure of the random data can be described in the form of a *scenario tree* $\mathscr{T}$. Generally, nodes at level t correspond to possible values of ξ^t that may occur. Each of them is connected to a unique node at stage $t-1$ called the ancestor node and to nodes at stage $t+1$ called the successors nodes. For each node j at stage t , we denote its ancestor with $a(j)$ and with $\pi_{a(j),j}$ the conditional probability of the random process being in node j given its history up to the ancestor node $a(j)$. We indicate with π_s the probability of scenario s passing through nodes $j_1, j_2, \ldots, j_H$ (where j_t, $t = 1, \ldots, H$ is the generic node at stage t) given by $\pi_s = \pi_{j_1,j_2} \cdot \pi_{j_2,j_3} \cdot \ldots \cdot \pi_{j_{H-1},j_H}$. We also indicate with p_j^t the probability of node j at stage t: if node j at stage t is reachable through nodes j_1 at stage 1, node j_2 at stage 2, …, node j_{t-1} at stage $t-1$, that is given by $p_j^t = \pi_{j_1,j_2} \cdot \pi_{j_2,j_3} \cdot \ldots \cdot \pi_{j_{t-1},j_t}$. We indicate x_j^t the decision in the node j at stage t. Let $\xi_1, \ldots, \xi_S$ index the possible realizations (or scenarios) of $\boldsymbol{\xi}^{H-1}$ and Ξ the support of possible scenarios and $\xi_i^{(1,j)} = (\xi_i^1, \xi_i^2, \ldots, \xi_i^j)$, $i = 1, \ldots, S$, $j = 1, \ldots, H$ with ξ_i^k is the k-stage of the i-realization, $k = 1, \ldots, j$.
The *multistage wait-and-see* problem, where the decision maker knows at the first stage the realizations of all the random variables takes the following form:

$$
\begin{aligned}
WS = E_{\xi_i} \quad \min \quad & {}_{x^1(\xi_i),\ldots,x^H(\xi_i)} c^1 x^1(\xi_i) + \ldots + c^H x^H(\xi_i) \\
\text{s.t.} \quad & A x^1(\xi_i) = h^1 \ , \\
& T^1(\xi_i^{(1,1)}) x^1(\xi_i) + W^2(\xi_i^{(1,1)}) x^2(\xi_i) = h^2(\xi_i^{(1,1)}) \ , \\
& \vdots \\
& T^{H-1}(\xi_i^{(1,H-1)}) x^{H-1}(\xi_i) + W^H(\xi_i^{(1,H-1)}) x^H(\xi_i) = h^H(\xi_i^{(1,H-1)}) \ , \\
& x^1(\xi_i) \geq 0 \ ; \quad x^t(\xi_i) \geq 0 \ , \ t = 2, \ldots, H \ , \quad i = 1, \ldots, S \ ;
\end{aligned}
\tag{2}
$$

notice that this decision process is *anticipative*, since all the decisions $x^1, x^2, \ldots, x^H$ depend on all the realization of $\boldsymbol{\xi}^{H-1}$.
The *Expected Value problem EV* is obtained by replacing all random variables by their expected values and solving a deterministic program, with $\bar{\xi} = E(\bar{\xi}^1, \bar{\xi}^2, \ldots, \bar{\xi}^{H-1}) = (E\xi^1, E\xi^2, \ldots, E\xi^{H-1})$:

$$
\begin{aligned}
EV &= \min_{\mathbf{x}} z(\mathbf{x}, \bar{\xi}) \\
&= \min_{x^1,\ldots,x^H} c^1 x^1 + \cdots + c^H x^H(\bar{\xi}^{H-1}) \\
& \quad \text{s.t.} \ \ A x^1 = h^1 \ , \\
& \quad\quad T^1(\bar{\xi}^1) x^1 + W^2(\bar{\xi}^1) x^2(\bar{\xi}^1) = h^2(\bar{\xi}^1) \ , \\
& \quad\quad \vdots \\
& \quad\quad T^{H-1}(\bar{\xi}^{H-1}) x^{H-1}(\bar{\xi}^{H-1}) + W^H(\bar{\xi}^{H-1}) x^H(\bar{\xi}^{H-1}) = h^H(\bar{\xi}^{H-1}) \ , \\
& \quad\quad x^1 \geq 0 \ ; \quad x^t(\bar{\xi}^{t-1}) \geq 0 \ , \ t = 2, \ldots, H;
\end{aligned}
\tag{3}
$$

Theorem 1 *(see Čížková [4]) For a $H-$stage problem (1) the following inequalities hold true:*

$$WS \leq RP \leq EEV \ , \tag{4}$$

where EEV denotes the optimal value of the RP model where all the decision variables until stage H are fixed at the optimal values obtained by using the average scenario. Notice that the inequality $EV \leq WS$ also holds true here when the only random elements are $h^2(\boldsymbol{\xi}^1)\dots h^H(\boldsymbol{\xi}^{H-1})$.

We introduce the *Expected result at stage t of using the Expected Value solution EEV^t*, $(t = 1, \dots, H-1)$ given by the optimal value of the RP model where the decision variables until stage t, $\mathbf{x}^{(1,t)} = (x^1, x^2, \dots, x^t)$, $t = 1, \dots, H-1$ are fixed at the optimal values obtained by the average scenario $\bar{\mathbf{x}}^{(1,t)} = (\bar{x}^1, \bar{x}^2, \dots, \bar{x}^t)$ $t = 1, \dots, H-1$. This is an alternative definition with respect with the one introduced in [5]. The *Value of the Stochastic Solution at stage t, VSS^t* is then defined as:

$$VSS^t = EEV^t - RP, \quad t = 1, \dots, H-1 \ . \tag{5}$$

Theorem 2 *(See Escudero et al. [5] (2007)) For multistage stochastic linear programs with deterministic constraint matrices and deterministic objective coefficients, the following inequalities are satisfied:*

$$VSS^t \leq EV - EEV^t, \quad t = 1, \dots, H-1 \ . \tag{6}$$

Proof
See [5].

As in [5], we notice that the problems EEV^t, $t = 1, \dots, H-1$ could be infeasible because too many variables are fixed from the deterministic problem.

Another reduced formulation of problem (1) is given by the so-called *two-stage relaxation* where the nonanticipativity constraints in the second and other stages are relaxed.
We define a new scenario tree where all random elements of stages $2, \dots, H-1$ are estimated by their expected values and solve the obtained model. We denote this new scenario tree as $\bar{\boldsymbol{\xi}}^{\,t-} = (\xi^1, \bar{\xi}^2, \dots, \bar{\xi}^t)$, $t = 2, \dots, H-1$. We can also define another scenario tree where all random elements of stages $t, \dots, H-1$, $t = 2, \dots, H-1$ are estimated by their expected values. We denote this second scenario tree as $\bar{\boldsymbol{\xi}}^{\,t+} = (\xi^1, \xi^2, \dots, \bar{\xi}^t, \dots, \bar{\xi}^{H-1})$, $t = 2, \dots, H-1$.

The following new problem is given by a two-stage model with H time periods and evaluated on scenario tree just defined $\bar{\boldsymbol{\xi}}^{\,t-}$:

$$
\begin{aligned}
TP \;=\; & \min_{x^1} c^1x^1 + E_{\bar{\boldsymbol{\xi}}^{H-1-}}\left[\min_{x^2,\ldots,x^H} c^2x^2\left(\boldsymbol{\xi}^1\right) + c^3x^3(\bar{\boldsymbol{\xi}}^{2-}) + \ldots + c^Hx^H(\bar{\boldsymbol{\xi}}^{H-1-})\right] \\
& \text{s.t. } Ax^1 = h^1 \;, \\
& \quad T^1(\boldsymbol{\xi}^1)x^1 + W^2(\boldsymbol{\xi}^1)x^2(\boldsymbol{\xi}^1) = h^2(\boldsymbol{\xi}^1) \;, \\
& \quad \vdots \\
& \quad T^{H-1}(\bar{\boldsymbol{\xi}})^{H-1-}x^{H-1}(\bar{\boldsymbol{\xi}}^{H-1-}) + W^H(\bar{\boldsymbol{\xi}})^{H-1-}x^H(\bar{\boldsymbol{\xi}}^{H-1-}) = h^H(\bar{\boldsymbol{\xi}}^{H-1-}) \;, \\
& \quad x^1 \geq 0 \;; \quad x^t(\bar{\boldsymbol{\xi}}^{\,t-1-}) \geq 0 \;,\; t = 2,\ldots,H \;.
\end{aligned}
\tag{7}
$$

3 Performance measures in multistage problems

In this section we propose performance measures for multistage stochastic linear problems. They are divided in *measures of information*, where the same problem is solved and compared with and without a piece of available information on the future, *measures of the quality of the deterministic solution* which can be identified in the class o *measures of different approaches with the same level of information* (see [4]), and *rolling horizon measures* which update the estimation and add more information at each stage.

3.1 Measures of information in multistage problems

First, we intend to generalize measures introduced in [6] for the deterministic solution of the modified wait and see approach and in [3] for the stochastic two-stage ($T = 2$) case. We consider a simplified version of the stochastic program, where only the right hand side is stochastic ($h = h(\boldsymbol{\xi})$).
Instead of using a scenario given by the expected variable values, one may choose a specific realization ξ_r (the scenario $r = 1,\ldots,S$) of the random variable $\boldsymbol{\xi}^{H-1}$, called the *reference scenario*, and solve problem (1) along that one.
Let the PAIRS subproblem of scenarios ξ_r and ξ_k ($k = 1,\ldots,S$) be defined as follows:

$$
\begin{aligned}
\min z^P(\mathbf{x},\xi_r,\xi_k) = \quad & c^1x^1 + c^2\pi_r x^2(\xi_r) + (1-\pi_r)c^2x^2(\xi_k) \\
\text{s.t.} \quad & Ax^1 = h^1 \;, \\
& T_r x^1 + W_r x^2(\xi_r) = \xi_r \;, \\
& T_k x^1 + W_k x^2(\xi_k) = \xi_k \;, \\
& x^1 \geq 0 \;; \quad x^2(\xi_r) \geq 0; \quad x^2(\xi_k) \geq 0 \;.
\end{aligned}
\tag{8}
$$

In [3] the *Sum of Pairs Expected Values*, denoted by $SPEV$, is then defined as:

$$
SPEV = \frac{1}{1-\pi_r}\sum_{k=1,k\neq r}^{S} \pi_k \min z^P(\mathbf{x},\xi_r,\xi_k) \;. \tag{9}
$$

We generalize now the definition of SPEV for multistage stochastic programs. We fix an auxiliary scenario a with the following characteristics:

1. $\xi_a = \xi_r$ i.e. the values of the random parameters are the same along the nodes of this scenario;

2. If $\hat{H}$ is the first stage where scenarios r and k branch we define:

$$p_a^t = \begin{cases} 1 & \text{if } t = 1 \ldots, \hat{H}-1 \\ \pi_r & \text{if } t = \hat{H}, \ldots, H \end{cases} \qquad \pi_{j_t, j_{\hat{t}+1}} = \begin{cases} \pi_r & \text{if } t = \hat{H} \\ 1 & \text{if } t \neq \hat{H} \end{cases}$$

3. $\pi_a = \pi_r$.

Figure 1 shows an example of probabilities computation on the pair subproblem made by scenarios $a = 1$ and $k = 2$ of the four stage scenario tree of Figure 3.

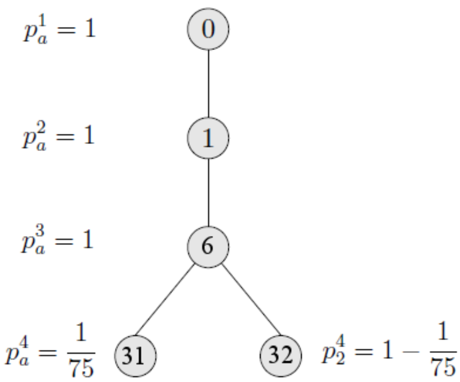

Figure 1: Pair subtree $(\xi_a, \xi_k) = (\xi_1, \xi_2)$ of scenario tree represented in Figure 3.

We then solve the following pair subproblems defined as:

$$\begin{array}{rl} \min z^P(\mathbf{x}, \xi_a, \xi_k) = & c^1 x^1 + \sum_{t=2}^{\hat{H}-1} c^t x_a^t(\xi_a) + \sum_{t=\hat{H}}^{H} \left[\pi_a c^t x_a^t(\xi_a) + (1-\pi_a) c^t x_k^t(\xi_k)\right] \\ \text{s.t.} & Ax^1 = h^1 \ , \\ & T_a^{t-1} x_a^{t-1} + W_a^t x_a^t = h_a^t \ , \\ & T_k^{t-1} x_k^{t-1} + W_k^t x_k^t = h_k^t \ , \\ & x^1 \geq 0 \ ; \quad x_a^t \geq 0 \ , \quad x_k^t \geq 0 \ , \quad t = 2, \ldots, H; \end{array} \tag{10}$$

where $\mathbf{x}_k^{(2,H)} = (x_k^2, x_k^3, \ldots, x_k^H)$. Let $\hat{\mathbf{x}}^{a,k} = (\hat{x}_k^1, \hat{\mathbf{x}}_a^{(2,H)}, \hat{\mathbf{x}}_k^{(2,H)})$ denote an optimal solution to the pair subproblem and $z^P(\hat{\mathbf{x}}^{a,k}, \xi_a, \xi_k)$ the optimal value of this problem.

Eventually, the reference scenario may not correspond to any of the given scenarios.
We define the *Multistage Sum of Pairs Expected Values* denoted by $MSPEV$ as follows:

$$MSPEV = \frac{1}{1-\pi_a} \sum_{\substack{k=1 \\ k \neq r}}^{n_H} \pi_k \min z^P(\mathbf{x}, \xi_a, \xi_k) \ . \tag{11}$$

Proposition 1 *If the scenario ξ_r is not in Ξ, then $MSPEV = WS$.*

Proof
If the scenario ξ_r is not in Ξ, then $\pi_r = 0$, consequently $\pi_a = 0$ and the pair subproblems $z^P(\mathbf{x}, \xi_a, \xi_k)$ reduce to $z(\mathbf{x}, \xi_k)$. Hence

$$\begin{aligned} MSPEV &= \sum_{\substack{k=1 \\ k \neq r}}^{n_H} \pi_k \left[c^1 x_k^1 + \sum_{t=2}^{H} \left[\pi_a c^t x_a^t(\xi_a) + (1-\pi_a) c^t x_k^t(\xi_k) \right] \right] \\ &= \sum_{\substack{k=1 \\ k \neq r}}^{n_H} \pi_k \min z\left(\mathbf{x}, \xi_k\right) = WS. \end{aligned} \tag{12}$$

In the following $\xi_r \in \Xi$.

Proposition 2 *$WS \leq MSPEV$.*

Proof
Let $\hat{\mathbf{x}}^{a,k} = (\hat{x}_k^1, \hat{\mathbf{x}}_a^{(2,H)}, \hat{\mathbf{x}}_k^{(2,H)})$ be an optimal solution to the pair subproblem of scenarios ξ_a and ξ_k, by definition (11) we have:

$$MSPEV = \frac{\sum_{\substack{k=1 \\ k \neq r}}^{n_H} \pi_k \min z^P(\mathbf{x}, \xi_a, \xi_k)}{1-\pi_a}$$

$$= \frac{\sum_{\substack{k=1 \\ k \neq r}}^{n_H} \pi_k \left[c^1 \hat{x}_k^1 + \sum_{t=2}^{\hat{H}-1} c^t \hat{x}_a^t(\xi_a) + \sum_{t=\hat{H}}^{H} \left[\pi_a c^t \hat{x}_a^t(\xi_a) + (1-\pi_a) c^t \hat{x}_k^t(\xi_k) \right] \right]}{1-\pi_a}$$

$$= \frac{\sum_{\substack{k=1 \\ k \neq r}}^{n_H} \pi_k \left[\pi_a \left[c^1 \hat{x}_k^1 + \sum_{t=2}^{\hat{H}-1} c^t \hat{x}_a^t(\xi_a) + \sum_{t=\hat{H}}^{H} c^t \hat{x}_a^t(\xi_a) \right] \right]}{1-\pi_a} +$$

$$+ \sum_{\substack{k=1 \\ k \neq r}}^{n_H} \pi_k \left[c^1 \hat{x}_k^1 + \sum_{t=2}^{\hat{H}-1} c^t \hat{x}_a^t(\xi_a) + \sum_{t=\hat{H}}^{H} c^t \hat{x}_k^t(\xi_k) \right]$$

$$\geq \frac{\sum_{\substack{k=1\\k\neq r}}^{n_H} \pi_k \pi_a z_a^*}{1-\pi_a} + \sum_{\substack{k=1\\k\neq r}} \pi_k \left[c^1 \hat{x}_k^1 + \sum_{t=2}^{\hat{H}-1} c^t \hat{x}_a^t(\xi_a) + \sum_{t=\hat{H}}^{H} c^t \hat{x}_k^t(\xi_k) \right]$$

$$= \frac{\sum_{\substack{k=1\\k\neq r}}^{n_H} \pi_k \pi_a z_a^*}{1-\pi_a} +$$

$$+ \sum_{\substack{k=1\\k\neq r}}^{n_H} \pi_k \left[c^1 \hat{x}_k^1 + \sum_{t=2}^{\hat{H}-1} c^t \hat{x}_a^t(\xi_a) + \sum_{t=2}^{\hat{H}-1} c^t \hat{x}_k^t(\xi_k) + \sum_{t=\hat{H}}^{H} c^t \hat{x}_k^t(\xi_k) - \sum_{t=2}^{\hat{H}-1} c^t \hat{x}_k^t(\xi_k) \right]$$

$$\geq WS + \sum_{\substack{k=1\\k\neq r}}^{n_H} \pi_k \left(\sum_{t=2}^{\hat{H}-1} c^t \left(\hat{x}_a^t(\xi_a) - \hat{x}_k^t(\xi_k) \right) \right) \ . \tag{13}$$

The sum in (13):

$$\sum_{\substack{k=1\\k\neq r}}^{n_H} \pi_k \left(\sum_{t=2}^{\hat{H}-1} c^t \left(\hat{x}_a^t(\xi_a) - \hat{x}_k^t(\xi_k) \right) \right) = 0 \ , \tag{14}$$

if scenarios k and a branch at stage $\hat{H} = 2$, (14) is not defined, if $2 < \hat{H} \leq H$, (14) reduces to zero because k and a are defined on the same nodes until stage $\hat{H} - 1$, consequently the optimal solutions verify $\hat{x}_a^t(\xi_a) = \hat{x}_k^t(\xi_k)$ and the thesis $MSPEV \geq WS$ is proved.

Proposition 3 *$RP \geq MSPEV + \delta$ where $\delta = \sum_{\substack{k=1\\k\neq r}}^{n_H} \pi_k \sum_{t=2}^{\hat{H}-1} c^t x_k^{t\,*}(\xi_k)$.*

Proof
Let $(x^{1*}, \mathbf{x}_k^{(2,H)*})$, be an optimal solution to the recourse problem.
Then $(x^{1*}, \mathbf{x}_a^{(2,H)*}, \mathbf{x}_k^{(2,H)*})$ is feasible for the Pairs subproblem of ξ_a, ξ_k, where this implies:

$$c^1 \hat{x}^1 + \sum_{t=2}^{\hat{H}-1} c^t \hat{x}_a^t(\xi_a) + \sum_{t=\hat{H}}^{H} \left[\pi_a c^t \hat{x}_a^t(\xi_a) + (1-\pi_a) c^t \hat{x}_k^t(\xi_k) \right]$$

$$\leq c^1 x^{1*} + \sum_{t=2}^{\hat{H}-1} c^t x_a^{t\,*}(\xi_a) + \sum_{t=\hat{H}}^{H} \left[\pi_a c^t x_a^{t\,*}(\xi_a) + (1-\pi_a) c^t x_k^{t\,*}(\xi_k) \right] \ .$$

Now, we obtain

$$\sum_{\substack{k=1\\k\neq r}}^{n_H} \pi_k \left[c^1 \hat{x}_k^1 + \sum_{t=2}^{\hat{H}-1} c^t \hat{x}_a^t(\xi_a) + \sum_{t=\hat{H}}^{H} \left[\pi_a c^t \hat{x}_a^t(\xi_a) + (1-\pi_a) c^t \hat{x}_k^t(\xi_k) \right] \right]$$
$$= (1-\pi_a) MSPEV \ .$$

On the other hand:

$$\sum_{\substack{k=1\\k\neq r}}^{n_H} \pi_k \left[c^1 x^{1^*} + \sum_{t=2}^{\hat{H}-1} c^t x_a^{t^*}(\xi_a) + \sum_{t=\hat{H}}^{H} \left[\pi_a c^t x_a^{t^*}(\xi_a) + (1-\pi_a) c^t x_k^{t^*}(\xi_k) \right] \right]$$

$$= (1-\pi_a) \left[c^1 x^{1^*} + \sum_{t=2}^{\hat{H}-1} c^t x_a^{t^*}(\xi_a) \right] +$$

$$+ \sum_{\substack{k=1\\k\neq r}}^{n_H} \pi_k \left[\sum_{t=\hat{H}}^{H} \left[\pi_a c^t x_a^{t^*}(\xi_a) + (1-\pi_a) c^t x_k^{t^*}(\xi_k) \right] \right]$$

$$= (1-\pi_a) \left[c^1 x^{1^*} + \sum_{t=2}^{\hat{H}-1} c^t x_a^{t^*}(\xi_a) + \sum_{t=\hat{H}}^{H} \pi_a c^t x_a^{t^*}(\xi_a) + \sum_{\substack{k=1\\k\neq r}}^{n_H} \pi_k \left[\sum_{t=\hat{H}}^{H} c^t x_k^{t^*}(\xi_k) \right] \right]$$

$$= (1-\pi_a)(RP - \delta)$$

and this proves the inequality $RP - \delta \geq MSPEV$ where

$$\delta = \sum_{\substack{k=1\\k\neq r}}^{n_H} \pi_k \sum_{t=2}^{\hat{H}-1} c^t x_k^{t^*}(\xi_k) \ .$$

The magnitude of δ influences the distance between RP and $MSPEV$ and it is zero in the two-stage case or in multi-stage problems with $\hat{H} = 2$.

Following [3], we introduce some upper bounds on RP for multistage problems, such as the *Multistage Expected Value of the Reference Scenario*:

$$MEVRS = E_{\boldsymbol{\xi}^{H-1}} \min_{x^H} z(\bar{\mathbf{x}}_r^{(1,H-1)}, x^H, \boldsymbol{\xi}^{H-1}) \ , \tag{15}$$

where $\bar{\mathbf{x}}_r^{(1,H-1)} = (\bar{x}_r^1, \bar{x}_r^2, \ldots, \bar{x}_r^{H-1})$ is the optimal solution until stage $H-1$ of the deterministic problem $\min_{\mathbf{x}} z(\mathbf{x}, \xi_r)$ under scenario r. The *Multistage Value of Stochastic Solution* $MVSS$ is defined as:

$$MVSS = MEVRS - RP \ . \tag{16}$$

Notice that $MVSS \geq 0$ because there are two alternatives:

1. $\bar{\mathbf{x}}_r^{(1,H-1)}$ is a feasible solution to the recourse problem;
2. $\bar{\mathbf{x}}_r^{(1,H-1)}$ is an infeasible and in this case $MEVRS = +\infty$.

The definition of $MEVRS$ can be generalized in a sequence of *Multistage Expected Value of the Reference Scenario*, $MEVRS^1$, $MEVRS^2, \ldots, MEVRS^t$ such as

$$MEVRS^t = E_{\boldsymbol{\xi}^{H-1}} \min_{\mathbf{x}^{(t+1,H)}} z(\bar{\mathbf{x}}_r^{(1,t)}, \mathbf{x}^{(t+1,H)}, \boldsymbol{\xi}^{H-1}),\ t = 1, \ldots, H-1 \ , \tag{17}$$

where $\bar{\mathbf{x}}_r^{(1,t)} = (\bar{x}_r^1, \bar{x}_r^2, \ldots, \bar{x}_r^t)$ is the optimal solution until stage t of the deterministic problem $\min_{\mathbf{x}}(\mathbf{x}, \xi_r)$ under scenario r (according to this definition $MEVRS = MEVRS^{H-1}$) and

$$MVSS^t = MEVRS^t - RP, \quad t = 1, \ldots H-1 \ . \tag{18}$$

The following relation holds true:

Proposition 4

$$MEVRS^{t+1} \geq MEVRS^t \ , \quad t = 1, \ldots, H-2 \ . \tag{19}$$

Proof
Any feasible solution of $MEVRS^{t+1}$ problem is also a solution of $MEVRS^t$ because the feasible region of $MEVRS^{t+1}$ has a set of constraints (at stage $t+1$), more than $MEVRS^t$ to be satisfied and the relation (19) holds true. If $MEVRS^t = +\infty$ the inequality is automatically satisfied.

As before let $\hat{\mathbf{x}}^{a,k} = (\hat{x}_k^1, \hat{\mathbf{x}}_a^{(2,H)}, \hat{\mathbf{x}}_k^{(2,H)})$ be optimal solutions to the pair subproblems of ξ_a and ξ_k, $k = 1, \ldots, n_H$, $k \neq r$. The *Multistage Expectation of Pairs Expected Value* is defined as:

$$MEPEV = \min_{k=1,\ldots,n_H \cup \{r\}} (E_{\boldsymbol{\xi}^{H-1}} \min_{\mathbf{x}^{(2,H)}} z(\hat{x}_k^1, \mathbf{x}^{(2,H)}, \boldsymbol{\xi}^{H-1})) \ . \tag{20}$$

Proposition 5

$$RP \leq MEPEV \leq MEVRS^1 \ , \tag{21}$$

Proof
We denote by $K = \{\mathbf{x} | x^t \in K^t \ t = 1, \ldots, H-1\}$ the feasibility set of RP, $K \cap \{\hat{x}_k^1, \ k = 1, \ldots, n_H \cup \{r\}\}$ the feasibility set of $MEPEV$ and $K \cap \bar{x}_r^1 = \hat{x}_r^1$ the one of $MEVRS^1$, which are obviously smaller and smaller and the thesis is proved.

As a consequence of Proposition (4) it follows

$$RP \leq MEPEV \leq MEVRS^1 \leq \cdots \leq MEVRS^{H-1} \ . \tag{22}$$

Putting the previous relations together it holds:

Theorem 3

$$\begin{aligned} 0 \ &\leq \ MEVRS^t - MEPEV \leq MVSS^t \leq \\ &\leq \ MEVRS^t - MSPEV + \delta \leq MEVRS^t - WS + \delta, \quad t = 1, \ldots, H-1 \ . \end{aligned} \tag{23}$$

3.2 Measures of the quality of deterministic solution in multistage problems

Measures of the structure and upgradeability of the deterministic solution for the two-stage case, such as the *Loss Using the Skeleton Solution* $LUSS$ and the *Loss of Upgrading the Deterministic Solution* $LUDS$ has been introduced in [12], in relation to the standard VSS. The aim of the measures is to find out, even when VSS is large, if the deterministic solution carries useful information for the stochastic case.

We recall the definition of $LUSS$ and $LUDS$ for the two-stage case. Let $\mathcal{J}$ be the set of indices for which the components of the expected value solution $\bar{x}(\bar{\xi})$ are at zero or at their lower bound. Then let $\hat{x}$ be the solution of:

$$\begin{array}{cl} \min_{x \in X} & E_{\boldsymbol{\xi}} z\left(x, \boldsymbol{\xi}\right) \\ \text{s.t.} & x_j = \bar{x}_j(\bar{\xi}), \;\; j \in \mathcal{J}. \end{array} \tag{24}$$

We then compute the *Expected Skeleton Solution Value*

$$ESSV = E_{\boldsymbol{\xi}}\left(z\left(\hat{x}, \boldsymbol{\xi}\right)\right), \tag{25}$$

and we compare it with RP by means of the *Loss Using the Skeleton Solution*

$$LUSS = ESSV - RP. \tag{26}$$

Consider the expected value solution $\bar{x}(\bar{\xi})$ as a starting point (input) to the stochastic two-stage model and compare it, in terms of objective functions, without such input. So we test if the expected value solution can improve (if not optimal) in the stochastic setting. This is equivalent to adding in the former problem the constraint $x \geq \bar{x}(\bar{\xi})$ and hence solve the following problem with solution $\widetilde{x}$:

$$\begin{array}{cl} \min_{x \in X} & E_{\boldsymbol{\xi}} z\left(x, \boldsymbol{\xi}\right) \\ \text{s.t.} & x \geq \bar{x}(\bar{\xi}). \end{array} \tag{27}$$

We then compute the *Expected Input Value*

$$EIV = E_{\boldsymbol{\xi}}\left(z\left(\widetilde{x}, \boldsymbol{\xi}\right)\right) \tag{28}$$

and we compare it with RP, by means of the *Loss of Upgrading the Deterministic Solution*:

$$LUDS = EIV - RP. \tag{29}$$

We extend the above definitions to the multistage-case by considering the *Multistage Loss Using the Skeleton Solution until stage t* $MLUSS^t$ and the *Multistage Loss of Upgrading the Deterministic Solution until stage t* $MLUDS^t$ in relation to VSS^t, $t = 1, \ldots H$ defined by (5).

The computation of $MLUSS^t$ is based on the following procedure: we fix at zero (or at the lower bound) all the variables which are at zero (or at the lower bound) in the expected value solution until stage t, and then solve the stochastic program.

Let $\mathcal{J}^t$, $t = 1, \dots, H-1$ be the set of indices for which the components of the expected value solution $\bar{\mathbf{x}}_r^{(1,t)}$ are at zero or at their lower bound. Then let $\tilde{\mathbf{x}}^{(1,t)}$ be the solution of:

$$\begin{aligned} \min_{\mathbf{x}} \quad & E_{\boldsymbol{\xi}^{H-1}} \; z\left(\mathbf{x}, \boldsymbol{\xi}^{H-1}\right) \\ \text{s.t.} \quad & x^{t_j} = \bar{x}^{t_j}(\bar{\boldsymbol{\xi}}^t), \; j \in \mathcal{J}^t. \end{aligned} \tag{30}$$

We then compute the *Multistage Expected Skeleton Solution Value at stage t*

$$MESSV^t = E_{\boldsymbol{\xi}^{H-1}} \min_{\mathbf{x}^{(t+1,H-1)}} z\left(\tilde{\mathbf{x}}^{(1,t)}, x^{(t+1,H-1)}, \boldsymbol{\xi}^{H-1}\right), \; t = 1, \dots, H-1 \; , \tag{31}$$

and we compare it with RP by means of *Multstistage Loss Using Skeleton Solution until stage t*

$$MLUSS^t = MESSV^t - RP \; , \quad t = 1, \dots, H-1 \; . \tag{32}$$

Notice that $MLUSS^t \geq 0$, $t = 1, \dots, H-1$ because $\tilde{\mathbf{x}}^{(1,t)}$ is a feasible solution to the recourse problem or infeasible such that $MLUSS^t = +\infty$. The case $MLUSS^t$ close to zero means that the variables chosen by the deterministic solution until stage t are good but their values may be off. We have:

Proposition 6

$$MLUSS^{t+1} \geq MLUSS^t, \quad t = 1, \dots, H-2 \; . \tag{33}$$

Proof

Any feasible solution of $MLUSS^{t+1}$ problem is also a solution of $MLUSS^t$ because the feasible region of $MLUSS^{t+1}$ has a set of constraints $x^{(t+1)_j} = \bar{x}^{(t+1)_j}(\bar{\boldsymbol{\xi}}^{t+1})$ with $j \in \mathcal{J}^{t+1}$, larger than $MLUSS^t$ to be satisfied and the relation (33) holds true. If $MLUSS^t = +\infty$ the inequality is automatically satisfied.

We have:

$$RP \leq MESSV^t \leq EEV^t \; , \tag{34}$$

and consequently,

$$VSS^t \geq MLUSS^t \geq 0 \; . \tag{35}$$

For multistage stochastic linear programs with deterministic constraint matrices and deterministic objective coefficients, the following inequalities are satisfied (see Escudero et al. [5] (2007)):

$$EEV^t - EV \geq VSS^t \; . \tag{36}$$

Notice that the case $MLUSS^t = 0$ (i.e. $MESSV^t = RP$) corresponds to the *perfect skeleton solution until stage* t in which the condition $x^{t_j} = \bar{x}^{t_j}(\bar{\boldsymbol{\xi}}^t), \; j \in \mathcal{J}^t$ is satisfied by the stochastic solution even without being enforced by the set of constraints.

$MLUDS^t$, $t = 1, \ldots, H-1$ measures if the expected value solution $\bar{\mathbf{x}}^t = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^t)$ until stage t can be considered as a starting point (if not optimal) in the stochastic setting. This is equivalent to adding in problem (1) the constraint $x^t \geq \bar{x}^t(\bar{\boldsymbol{\xi}}^t)$ and hence solve the following problem obtaining the solution $\bar{\bar{x}}^t$:

$$\begin{array}{ll} \min_{\mathbf{x}} & E_{\boldsymbol{\xi}^{H-1}}\, z\left(\mathbf{x}, \boldsymbol{\xi}^{H-1}\right) \\ \text{s.t.} & x^t \geq \bar{x}^t(\bar{\boldsymbol{\xi}}^t)\ . \end{array} \tag{37}$$

We then compute the *Multistage Expected Input Value until stage t*

$$MEIV^t = E_{\boldsymbol{\xi}^{H-1}} \min_{x^{(t+1,H-1)}} z\left(\bar{\bar{x}}^t, \boldsymbol{\xi}^{H-1}\right) \tag{38}$$

and we compare it with RP, by means of the *Multistage Loss of Upgrading the Deterministic Solution until stage t*:

$$MLUDS^t = MEIV^t - RP\ . \tag{39}$$

As in the case of $MLUSS^t$ the following inequalities hold true:

Proposition 7

$$MLUDS^{t+1} \geq MLUDS^t, \quad t = 1, \ldots, H-2\ , \tag{40}$$

$$EEV^t \geq MEIV^t \geq RP, \quad t = 1, \ldots, H-1\ , \tag{41}$$

$$EEV^t - EV \geq VSS^t \geq MLUDS^t \geq 0, \quad t = 1, \ldots, H-1\ . \tag{42}$$

Proof
See the proof of Proposition 6.

Notice that the case $MLUDS^t = 0$ (i.e. $MEIV^t = RP$) corresponds to the case where the conditions $\mathbf{x}^t \geq \bar{\mathbf{x}}^t(\bar{\boldsymbol{\xi}}^t)$ are satisfied by the stochastic solution even without being enforced by these constraints (under the assumption that the stochastic first-stage decision is unique).

3.3 Rolling horizon measures in multistage problems

Multistage problems such as $MEVRS^t$, $MESSV^t$ and $MEIV^t$ ($t = 1, \ldots, H-1$) are often infeasible because they require to fix too many variables from the mean or reference scenario.

An alternative approach is to consider a *rolling of time horizon* procedure (see [4] and [16]) in order to update the estimations and add more information to the model. We propose the following methodology for the evaluation of the reference scenario; in [13] it has been adopted for the deterministic solution.

1. Solve the reference scenario r and store the first stage decision variables $\bar{x}_r^1$;

2. Define a new scenario tree $\mathscr{T}^{2,ev}$ where all random elements of stages $2,\ldots,H-1$ are estimated by their expected values $\bar{\boldsymbol{\xi}}^{\,2+} = (\xi^1, \bar{\xi}^2, \ldots, \bar{\xi}^{H-1})$ and solve the obtained model with $x^1 = \bar{x}^1_r$. Store all the second stage variables $\bar{x}^2_{r,ev}$.

3. At stage t ($t = 2,\ldots,H-1$) define a new scenario tree $\mathscr{T}^{t+1,ev}$ with all random elements of stages $t{+}1,\ldots,H{-}1$ estimated by their expected value $\bar{\boldsymbol{\xi}}^{\,t+1+} = (\xi^1, \xi^2, \ldots, \bar{\xi}^{t+1}, \ldots, \bar{\xi}^{H-1})$, $t = 2,\ldots,H-1$ and solve the associated model with

$$x^{(1,t)} = (x^1, x^2, \ldots, x^t) = (\bar{x}^1_r, \bar{x}^2_{r,ev}, \ldots, \bar{x}^t_{r,ev}) = \bar{\mathbf{x}}^{(1,t)}_{r,ev} \ .$$

   Store all the $t+1$ stage variables $\bar{x}^{t+1}_{r,ev}$.

4. Finally, solve the model on the initial scenario tree $\mathscr{T}$ with all the t^{th} variables ($t = 1,2,\ldots,H-1$) fixed to the stored values $x^{(1,H-1)} = \bar{\mathbf{x}}^{(1,H-1)}_{r,ev}$.

We denote the *Rolling Horizon Value of the Reference Scenario*:

$$RHVRS = E_{\boldsymbol{\xi}^{H-1}} \min_{x^H} z(\bar{\mathbf{x}}^{(1,H-1)}_{r,ev}, x^H, \boldsymbol{\xi}^{H-1}) \ , \tag{43}$$

and *Rolling Horizon Value of Stochastic Solution* by:

$$RHVSS = RHVRS - RP \ . \tag{44}$$

In a similar way, the *Rolling Horizon Expected Skeleton Solution Value RHESSV* can be obtained as follows:

1. Solve the expected value problem and store the first stage decision variables $\bar{x}^{1_j},\ j \in \mathcal{J}^1$ which are at zero or at their lower bound;

2. Define a new scenario tree $\mathscr{T}^{2,ev}$ where all random elements of stages $2,\ldots,H-1$ are estimated by their expected values and solve the obtained model with $x^{1_j} = \bar{x}^{1_j},\ j \in \mathcal{J}^1$. Store all the second stage variables which are at zero or at their lower bound $\bar{x}^{2_j},\ j \in \mathcal{J}^2$.

3. At stage t ($t = 2,\ldots,H-1$) define a new scenario tree $\mathscr{T}^{t+1,ev}$ with all random elements of stages $t+1,\ldots,H-1$ estimated by their expected value and solve the associated model with $x^{(1,t)_j} = (x^{1_j}, x^{2_j}, \ldots, x^{t_j}) = (\bar{x}^{1_j}, \bar{x}^{2_j}, \ldots, \bar{x}^{t_j}) = \bar{\mathbf{x}}^{(1,t)_j},\ j \in \mathcal{J}^t$. Store all the $t+1$ stage variables which are at zero or at their lower bound $\bar{x}^{(t+1)_j},\ j \in \mathcal{J}^{t+1}$.

4. Finally solve the model on the initial scenario tree $\mathscr{T}$ with all the $j-$components, $j \in \mathcal{J}^t$ at stage t ($t = 1,2,\ldots,H-1$) fixed to zero or at their lower bound: $x^{(1,H-1)_j} = (x^{1_j}, x^{2_j}, \ldots, x^{(H-1)_j}) = (\bar{x}^{1_j}, \bar{x}^{2_j}, \ldots, \bar{x}^{(H-1)_j}) = \bar{\mathbf{x}}^{(1,H-1)_j},\ j \in \mathcal{J}^t,\ (t = 1,2,\ldots,H-1)$.

We denote the *Rolling Horizon Expected Skeleton Solution Value*:

$$RHESSV = E_{\boldsymbol{\xi}^{H-1}} \min_{x^H} z(\bar{\mathbf{x}}^{(1,H-1)_j}, x^H, \boldsymbol{\xi}^{H-1}),\ j \in \mathcal{J}^t,\ t = 1, \ldots, H-1\ , \tag{45}$$

and *Rolling Horizon Loss Using Skeleton Solution* by:

$$RHLUSS = RHESSV - RP\ . \tag{46}$$

Starting from the definition of $MEIV^t$, we can analogously define the *Rolling Horizon Expected Input Value RHEIV* and *Rolling Horizon Loss of Upgrading the Deterministic Solution*:

$$RHLUDS = RHEIV - RP\ . \tag{47}$$

4 Case study: a multistage stochastic optimization model for a single-sink transportation problem

We consider a real case of *clinker* replenishment in Sicily, provided by the primary italian cement producer. The problem has been already analyzed in detail in [13]. The logistics system is organized as follows: in Catania there is a warehouse to be replenished by *clinker* produced by four plants located in Palermo (PA), Agrigento (AG), Cosenza (CS) and Vibo Valentia (VV). The demand of the single customer at Catania as well as the production capacities of the four plants are stochastic.

All the vehicles must be booked in advance from an external transportation company, before the demand and production capacities are revealed. We assume that the transportation company has an unlimited fleet and that only full load shipments are allowed. When the demand and the production capacity are revealed, there is an option to cancel some of the reservations against a cancellation fee. If the quantity delivered from the four suppliers is not enough to satisfy the demand, the residual quantity is purchased from an external company at a higher price b. The problem is to determine, for each supplier, the number of vehicles to book in order to minimize the total costs, given by the sum of the transportation costs (including the cancellation fee for vehicles booked but not used) and the costs of the product purchased from the external company.
The notation adopted is the following:

Sets:

$$\mathscr{I} = \{i : i = 1, \ldots, I\} \quad : \quad \text{set of suppliers (AG, CS, PA, VV);}$$
$$\mathscr{J}^t = \{j : j = 1, \ldots, n_t\} \quad : \quad \text{set of ordered nodes of the tree at stage } t = 1 \ldots, H;$$

and n_t is the number of nodes at stage t.

Parameters:

t_i : unit transportation costs of supplier $i \in \mathscr{I}$;
b : buying cost from an external source;
q : vehicle capacity;
g : unloading capacity at the customer;
l_0 : initial inventory level at the customer;
$l_{\max}$: storage capacity at the customer;
p_j : probability of node $j \in \mathscr{J}^t,\ t = 1\ldots,H$;
$v_{i,j}$: production capacity of supplier $i \in \mathscr{I}$ in node $j \in \mathscr{J}^t,\ t = 2,\ldots,H$;
d_j : customer demand at node $j \in \mathscr{J}^t,\ t = 2,\ldots,H$;
α : cancellation fee;
$\mathscr{J}^1 = \{0\}$: root of the tree;
$a(j)$: ancestor of the node $j \in \mathscr{J}^t,\ t = 2,\ldots,H$ in the scenario tree.

Notice that b is fixed on the basis of the known production and transportation costs of each producers. In our case we suppose $b > \max_i(t_i + c_i))$ where c_i is the unit production costs of supplier $i \in \mathscr{I}$.

Variables:

$x_{i,j} \in \mathbb{N}$: number of vehicles booked from supplier $i \in \mathscr{I},\ j \in \mathscr{J}^t,\ t = 1,\ldots,H-1$;
$z_{i,j} \in \mathbb{N}$: number of vehicles actually used from $i \in \mathscr{I},\ j \in \mathscr{J}^t,\ t = 2,\ldots,H$;
$y_j \in \mathbb{R}$: product to purchase from an external source in $j \in \mathscr{J}^t,\ t = 2,\ldots,H$;
$l_j \in \mathbb{R}$: inventory level of the customer at node j :

$$l_j = l_{a(j)} + q\sum_{i=1}^{I} z_{i,j} + y_j - d_j\,, \quad j \in \mathscr{J}^t,\ t = 2,\ldots,H\,; \tag{48}$$

The multistage model can be then formulated as follows:

$$\min \sum_{t=1}^{H-1}\sum_{j=1}^{n_t} p_j\left[q\sum_{i=1}^{I} t_i x_{i,j}\right] + \sum_{t=2}^{H}\sum_{j=1}^{n_t} p_j\left[b\,y_j - (1-\alpha)q\sum_{i=1}^{I} t_i\big(x_{i,a(j)} - z_{i,j}\big)\right] \tag{49}$$

subject to

$$q\sum_{i=1}^{I} x_{i,j} \leq g\ , \quad j \in \mathscr{J}^t, \quad t = 1,\ldots,H-1 \tag{50}$$

$$l_{a(j)} + q\sum_{i=1}^{I} z_{i,j} + y_j - d_j \geq 0\ , \quad j \in \mathscr{J}^t, \quad t = 2,\ldots,H \tag{51}$$

$$l_{a(j)} + q\sum_{i=1}^{I} z_{i,j} + y_j - d_j \leq l_{\max}\ , \quad j \in \mathscr{J}^t, \quad t = 2,\ldots,H \tag{52}$$

$$z_{i,j} \leq x_{i,a(j)}\ , \quad i \in \mathscr{I}, \quad j \in \mathscr{J}^t, \quad t = 2,\ldots,H \tag{53}$$

$$qz_{i,j} \leq v_{i,j}\ , \quad i \in \mathscr{I}, \quad j \in \mathscr{J}^t, \quad t = 2,\ldots,H \tag{54}$$

$$x_{i,j} \in \mathbb{N}\ , \quad i \in \mathscr{I}, \quad j \in \mathscr{J}^t, \quad t = 1,\ldots,H-1 \tag{55}$$

$$y_j \geq 0\ , \quad j \in \mathscr{J}^t, \quad t = 2,\ldots,H \tag{56}$$

$$z_{i,j} \in \mathbb{N}\ , \quad i \in \mathscr{I}, \quad j \in \mathscr{J}^t, \quad t = 2,\ldots,H\ . \tag{57}$$

The first sum in the objective function (49) is the booking costs of the vehicles, while the second sum represents the recourse actions, consisting of buying extra clinker (y_j^t) and canceling unwanted vehicles. Constraint (50) guarantees that the total quantity delivered from the suppliers to the customer is not greater than the customer's unloading capacity g, inducing thus an upper bound on the total number of vehicles. Constraints (51) and (52) ensure that the storage levels are between zero and $l_{\max}$. Constraint (53) guarantees that the number of vehicles servicing supplier i is at most equal to the number booked in advance and (54) controls that the quantity of clinker delivered from supplier i does not exceed its production capacity $a_{i,j}^t$. Finally, (55)–(57) define the decision variables of the problem.

5 Comparison measures for "a single sink transportation problem"

We compute the performance measures described in Sections 2 and 3 on the single sink transportation problem described in Section 4. For simplicity we analyze first the two-stage case and then the multi-stage stochastic one. We refer to [13] for the data used in the simulation.

5.1 Two-stage case

Tables 1 shows the optimal number of booked vehicles for each supplier, the total optimal costs and the values (see Table 2) assumed by the performance measures in the two-stage case.

We first observe that both the deterministic cases, using the mean (EV) and the worst scenario, underestimate the stochastic optimal cost and the model will always book the exact number of vehicles needed in the next time period (see [13]). The deterministic model sorts the suppliers according to the transportation costs and books much less vehicles than the stochastic one with a resulting cost lower than RP solution. However, EEV is much higher that the deterministic cost (€ 495 788 instead of the predicted cost of € 294 898) resulting in

$$\text{VSS} = 495\,788 - 438\,304 = 57\,384\,,$$

which shows that we can save about 12% of the cost by using the stochastic model, compared to the deterministic one. EVRS is still higher (€ 522 877 instead of the predicted cost of € 427 374). Notice also that the inequalities of Theorem 1

$$WS = 319\,100 \leq RP = 438\,304 \leq EEV = 495\,788\ ,$$

are satisfied.

Fixing as auxiliary scenario the average scenario we get a value for $SPEV$ equal to 319 100 which is exactly the wait and see solution (WS) (see Proposition 1), while choosing as auxiliary scenario the worst one ξ_{10} we get a worst value for $SPEV$ of 343 626. The series of inequalities (see Proposition 7 in [3]):

$$WS = 319\,100 \leq SPEV = 343\,626 \leq RP = 438\,304\ ,$$

are satisfied and shows the advantage of a deeper information on the future. For details on the optimal values of the pair subproblems ξ_{10} and ξ_k, $k = 1, \ldots, 14$ with respect to the worst scenario ξ_{10} see Table 3. The value of $EPEV$ is determined by the optimal first stage solution of the pair subproblem ξ_{10} and ξ_{14} performed into the stochastic model and it satisfies the inequality in Proposition 5:

$$RP = 438\,304 \leq EPEV = 485\,875 \leq EVRS = 522\,877\ .$$

In order to understand the reason of the badness of the deterministic solution quantified by the high value of $VSS = 57\,384$, we compute now the *Expected Skeleton Solution Value* $ESSV$, following the skeleton from the deterministic model, not allowing to book vehicles from CS and VV. $ESSV$ is € 462 214, still higher than RP with a consequent *Loss Using the Skeleton Solution* of

$$LUSS = 462\,214 - 438\,304 = 23\,910\ ,$$

which measures the loss by booking vehicles coming only from suppliers AG and PA as suggested by the deterministic model. We can conclude that the deterministic solution is bad because it books the wrong number of vehicles from the wrong suppliers.

The *Expected Input Value* is computed by taking the number of vehicles booked in the deterministic solution as input in the stochastic model and checking if the solution can be upgraded in a second run. Notice that for all the four suppliers the stochastic solution is higher than in the deterministic one (see Table 2) with $LUDS = 0$.

The measures defined in [12] allow us to conclude that the deterministic solution does not perform well in a stochastic environment because of the too low number of vehicles booked at the fist stage (736 instead of 1080) just considering AG and PA as possible suppliers. However, the deterministic solution should be considered as a lower bound for the stochastic case.

Table 1: Optimal solutions and comparison measures for the two-stage case of the "single-sink transportation problem" . The table shows optimal number of booked vehicles for each supplier and total optimal costs.

	AG	CS	PA	VV	Objective value (€)
deterministic (mean scenario = $\bar{\xi}$)	206	0	530	0	294 898 = EV
deterministic (worst scenario = ξ_{10})	433	33	366	0	427 374
stochastic	400	0	563	117	438 304 = RP
	206	0	530	0	495 788 = EEV
	433	33	366	0	522 877 = EVRS (w.r.t. ξ_{10})
	400	0	637	0	462 214 = ESSV
	400	0	563	117	438 304 = EIV
					319 100 = WS
					343 626 = SPEV (w.r.t. ξ_{10})
					319 100 = SPEV (w.r.t. $\bar{\xi}$)
pair subproblem (ξ_{10}, ξ_{14})	300	0	370	110	485 875 = EPEV (w.r.t. ξ_{10})

Table 2: Performance measures for the two-stage case of the "single-sink transportation problem".

Performance measures	**Value (€)**
VSS	57 384
EVRS − RP	84 573
LUSS	23 910
LUDS	0

5.2 Multi-stage case

In this section we compute the performance measures described in Sections 2 and 3 on the four-stage case of single sink transportation problem (see Section 4). For this purpose, we consider the scenario tree from Figure 3: this is a four-stage tree with 5 branches from the root, 5 from each of the second-stage nodes, and three from each of the third-stage nodes, resulting in $S = 5 \times 5 \times 3 = 75$ scenarios and 106 nodes. We declare it as a benchmark to evaluate the cost of optimal solutions obtained using the other reduced scenario trees (see Figures 2). The results are presented in Table 5 and Figure 4 and performance measures in Table 6. From Table 5 we see that a better description of the stochasticity leads to larger bookings in the first stage. Actually, in the four-stage scenario tree, the total number of booked vehicles is equal to 1260, that is the customer's unloading capacity. This is due to the low initial inventory level $l_0 = 2000$ at the customer (the actual case from real data).

Anyway the total costs from the two deterministic models, mean scenario (EV) (see Table 4) and worst scenario ξ_{44}, the recourse problem (RP) and the two-stage relaxation (TP) (see Table 5) are not directly comparable. The optimal solutions are then compared on the scenario tree of Figure 3 used as a benchmark. First of all we evaluate the optimal solutions of the

Table 3: Pair subproblems first stage solutions and total costs with respect to the worst case scenario ξ_{10}.

pair subproblem	AG	CS	PA	VV	Objective value (€)
(ξ_{10}, ξ_1)	303	0	297	0	258 185
(ξ_{10}, ξ_2)	0	0	757	0	365 796
(ξ_{10}, ξ_3)	0	97	533	116	427 930
(ξ_{10}, ξ_4)	56	0	638	0	331 009
(ξ_{10}, ξ_5)	433	0	334	0	302 193
(ξ_{10}, ξ_6)	133	0	577	0	325 037
(ξ_{10}, ξ_7)	136	0	333	281	379 316
(ξ_{10}, ξ_8)	303	0	126	298	361 825
(ξ_{10}, ξ_9)	316	0	438	0	313 279
(ξ_{10}, ξ_{10})	433	33	366	0	522 877
(ξ_{10}, ξ_{11})	433	0	237	0	270 515
(ξ_{10}, ξ_{12})	0	170	563	0	447 357
(ξ_{10}, ξ_{13})	40	0	680	0	344 212
(ξ_{10}, ξ_{14})	300	0	370	110	340 483

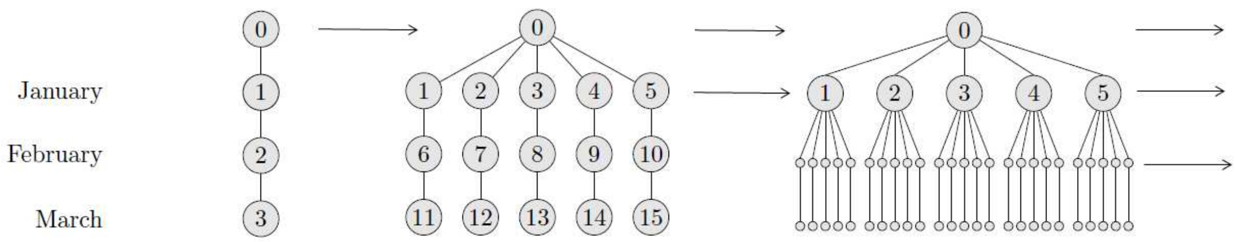

Figure 2: Reduced scenario trees respectively considered for the mean scenario model (EV), for the two-stage relaxation (TP) and the computations of the rolling horizon values reported in Tables 4 and 5 for the single-sink transportation problem.

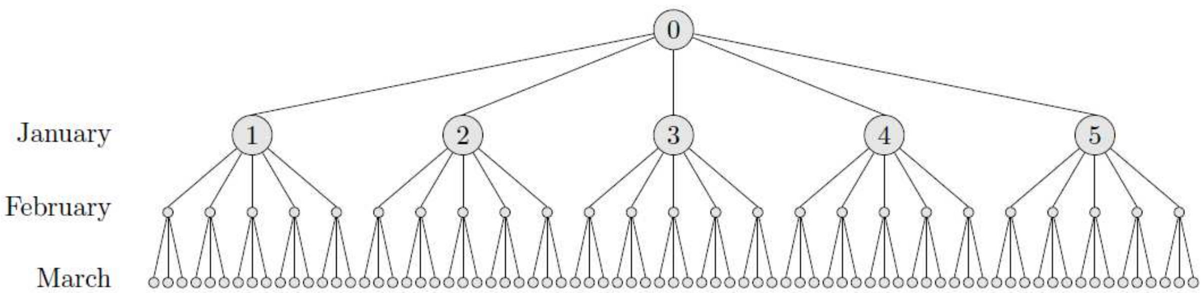

Figure 3: Four-stages scenario tree considered for the single-sink transportation problem.

average scenario model by fixing just the number $x_{i,j}$ of vehicles booked from supplier $i \in \mathscr{I}$ in nodes $j \in \mathscr{J}^t$ until stage t $(t = 1, 2, 3)$ in the stochastic framework by means of EEV^t:

$$EEV^1 = 1364343.58 < EEV^2 = 1431896.24 < EEV^3 = 1579825.92 \ .$$

The associated chain

$$VSS^1 = 91\,267.32 < VSS^2 = 185\,002.22 < VSS^3 = 299\,449.62 \ ,$$

show the losses by booking the number of vehicles suggested by the deterministic solution (see Table 4). The low first stage deterministic booking is compensated at each of second stage nodes of the stochastic framework, by a reservation almost equal to the customer's unloading capacity and by buying extra clinker at a higher price.

The evaluation of the deterministic solution on a rolling-horizon basis (see Figure 2), allows to update the estimations and add more information step by step as measured by:

$$RHEEV = 1\,540\,248.22 < EEV^3 = 1\,579\,825.92 \ .$$

Notice that by fixing all the decision variables $x_{i,j}$, y_j and $z_{i,j}$ from the average scenario model until the second stage, we get $EEV^2 = \infty$ and consequently $EEV^3 = \infty$ $(EEV^2 \leq EEV^3)$ concluding a badness of the deterministic solution. We will try to understand later by means of $MLUSS^t$ and $MLUDS^t$ the reason of its infeasibility.

The same considerations can be applied by evaluating the worst scenario model ξ_{44} in the stochastic framework by means of $MEVRS^t$. In particular by fixing just the number $x_{i,j}$ of vehicles booked from supplier $i \in \mathscr{I}$ in nodes $j \in \mathscr{J}^t$ until stage t, we get

$$MEVRS^1 = 1\,372\,285.48 < MEVRS^2 = 1\,433\,501.32 < MEVRS^3 = 1\,436\,997.78 \ .$$

The evaluation of the worst case solution on a rolling-horizon basis is measured by $RHVRS = 1\,535\,476.52$.

$MESSV^t$ allows the evaluation of the *structure* of the deterministic solution until stage t. We do not allow to book vehicles from CS in all the three stages and from VV in the root and at stage $t = 3$ (see Table 4) and we get:

$$MESSV^1 = 1\,299\,327.68 \leq MESSV^2 = 1\,301\,017.28 \leq MESSV^3 = 1\,404\,215.16 \ ,$$

with a consequent chain of measures:

$$MLUSS^1 = 26\,282.62 \leq MLUSS^2 = 27\,863.9 \leq MLUSS^3 = 131\,140.86 \ ,$$

which measure the loss by booking vehicles coming only from the suppliers as suggested by the deterministic model. We can conclude that the deterministic solution $x_{i,j}$ is bad because it books the wrong number of vehicles from the wrong suppliers already from the first stage.

The evaluation of the deterministic skeleton solution on a rolling-horizon basis, allows to update the estimations as suggested by:

$$RHESSV = 1\,302\,705.36 \leq MESSV^3 = 1\,404\,215.16 \ .$$

As before, by fixing at zero also the clinker purchased y_j and the vehicles actually used $z_{i,j}$ we get an infeasibility already at the second stage ($MESSV^2 = MESSV^3 = \infty$) concluding a badness of the structure of the full deterministic solution.

We then consider the vehicles booked in the average scenario model as an *input* in the stochastic setting and we check if the solution can be upgraded. Notice that in the root for all the four suppliers the booked number of vehicles in the stochastic solution is higher than in the deterministic one with $MLUDS^1 = 0$. The condition is no longer satisfied at stage 2 for suppliers PA and VV with $MLUDS^2 = 2707.4$ and at stage 3 for suppliers AG and PA with $MLUDS^3 = 27552.22$. Notice that the chain (41) in Proposition 7:

$$MLUDS^1 = 0 \leq MLUDS^2 = 2707.4 \leq MLUDS^3 = 27552.22 \ ,$$

holds true. We can conclude that the deterministic solution can be taken as input in the stochastic model only in the first stage.

An alternative approach to the deterministic solution is to solve pairs subproblems of the initial stochastic program with respect to the worst scenario ξ_{44}.

The best pair subproblem is given by the couple (ξ_{44}, ξ_2) with $MEPEV = 1\,313\,983.3$ which satisfies the chain of inequalities (22):

$$\begin{aligned} RP = 1\,273\,074.3 < MEPEV = 1\,313\,983.3 &< MEVRS^1 = 1\,372\,285.48 < \\ &< MEVRS^2 = 1\,433\,501.32 < \\ &< MEVRS^3 = 1\,436\,997.78 \ . \end{aligned}$$

This means that the optimal first stage solution of the pair subproblem (ξ_{44}, ξ_2) performs better than the deterministic one (mean or worst scenario), and it should be chosen for large scale problems instead of solving them, in case we have more information on the future.
When the auxiliary scenario does not belong to the scenario tree, $MSPEV = WS = 1037820$ as proved in Proposition 1.
If the auxiliary scenario (the worst one) belongs to the scenario tree, then $WS = 1037820 < 1041627.34 = MSPEV$ (see Proposition 2). Notice that the sum (14) is zero: if $k = 1, \ldots, 30$ or $k = 46, \ldots, 75$, scenarios k and the auxiliary a, branch at stage $\hat{H} = 2$ and (14) is not defined, if $k = 31 \ldots, 43$ scenarios k and a branch at $\hat{H} = 3$ and at stage 2 are both defined on node 3 with $\hat{x}^2_{44}(\xi_{44}) = \hat{x}^2_k(\xi_k)$. The same arguments can be applied for scenarios 43 and 46 which branch with scenario 44 at $\hat{H} = 4$ and are defined on the same node 20 at stage 3.

If we choose as auxiliary scenario the best one ξ_1 (the one that gives the minimum cost over all the scenarios in the tree) $WS = 1037820 < MSPEV = 1039068.91$.
Finally, the value of $\delta = 154147.7$ measures the distance between $MSPEV$ and the value of the stochastic model $RP = 1273074.3$ (see Proposition 3). Notice that in our example the value of δ is 66% of the real distance between RP and $MSPEV$. Theorem 3 is verified.

By the analyzed measures we can conclude that the deterministic model performs bad in the multistage stochastic environment because of the too low number of booked vehicles, already from the first stage. The positive values of $MLUSS^t$ mean that the badness of the deterministic solution is partially in its structure, booking vehicles from the wrong suppliers.

However the determistic solution should be considered as a lower bound for the first stage stochastic one. The rolling-horizon approach should be also considered as an useful alternative to the standard methods allowing to update the estimation at each time period. A better option than choosing the deterministic solution is also given by the best pair subproblem solution with performance measured by $MEPEV$, under the assumption of a better information about the future.

Table 4: Optimal solution for the deterministic (mean scenario) four-stage case of the "single-sink transportation problem" . The table shows optimal number of booked vehicles (equal to the optimal number of used vehicles at node $j+1$) for each supplier, the clinker purchased y_j at each stage $j = 1, 2, 3$ and partial optimal costs.

node	AG	CS	PA	VV	y_j	Costs (€)
0	158	0	647	0	-	327 717
1	264	0	416	143	0	345 621
2	315	0	518	0	0	326 844
3	-	-	-	-	0	0

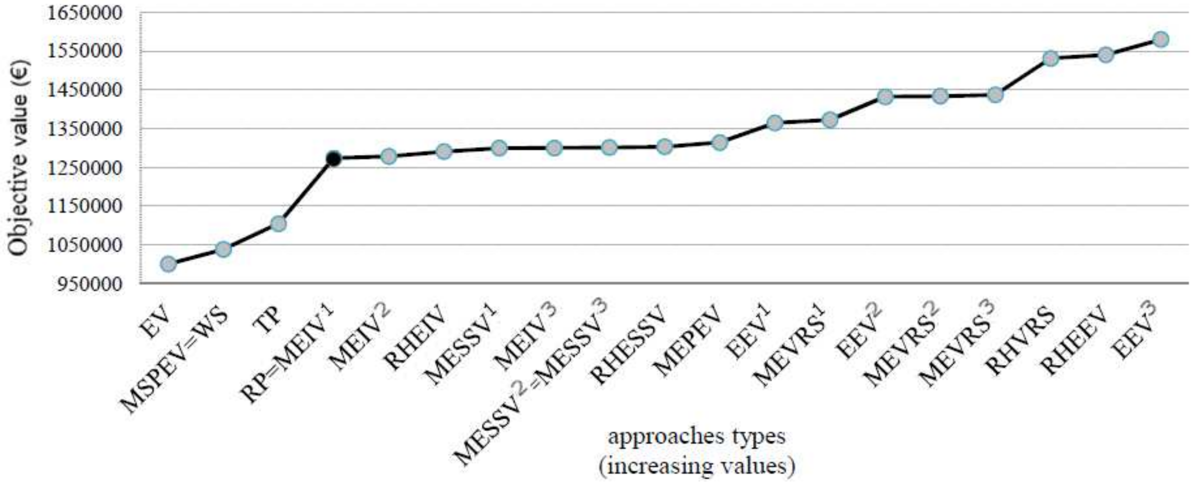

Figure 4: Comparison of objective functions of different approaches as in Table 5, reported for increasing values. The black circle denotes the multistage stochastic recourse problem RP.

6 Conclusions

The paper extends classical measures to value different approaches and levels of information for two-stage stochastic problems to the multistage case. We generalize bounds of *Value of Stochastic Solution* VSS to the multistage case through the *Multistage Sum of Pairs of*

Table 5: Optimal solutions and comparison measures for the four-stage case of the "single-sink transportation problem" . The table shows optimal number of booked vehicles for each supplier and total optimal costs.

	AG	CS	PA	VV	Objective value (€)
EV (mean scenario = $\bar{\xi}$)	158	0	647	0	1 000 182
deterministic (worst scenario = ξ_{44})	0	299	533	116	1 342 803
RP	389	0	755	116	1 273 074.30
TP	401	0	641	116	1 104 279.60
EEV^1 (w.r.t. $\bar{\xi}$)	158	0	647	0	1 364 343.58
EEV^2 (w.r.t. $\bar{\xi}$) fixing $x_{i,j}$	158	0	647	0	1 431 896.24
EEV^2 fixing all 1^{st} and 2^{nd} stage var.	158	0	647	0	∞
EEV^3 (w.r.t. $\bar{\xi}$) fixing $x_{i,j}$	158	0	647	0	1 579 825.92
EEV^3 fixing all 1^{st}, 2^{nd} and 3^{rd} stage var.	158	0	647	0	∞
$MEVRS^1$ (w.r.t. ξ_{44})	0	299	533	116	1 372 285.48
$MEVRS^2$ (w.r.t. ξ_{44}) fixing $x_{i,j}$	0	299	533	116	1 433 501.32
$MEVRS^2$ fixing $x_{i,j}$ and y_j	0	299	533	116	∞
$MEVRS^2$ fixing all 1^{st} and 2^{nd} stage var.	0	299	533	116	∞
$MEVRS^3$ (w.r.t. ξ_{44}) fixing $x_{i,j}$	0	299	533	116	1 436 997.78
$MEVRS^3$ fixing $x_{i,j}$ and y_j	0	299	533	116	∞
$MEVRS^3$ fixing all 1^{st}, 2^{nd} and 3^{rd} stage var.	0	299	533	116	∞
$MESSV^1$	389	0	871	0	1 299 327.68
$MESSV^2$ fixing $x_{i,j}$	389	0	871	0	1 301 017.28
$MESSV^2$					∞
$MESSV^3$ fixing $x_{i,j}$	389	0	871	0	1 301 017.28
$MESSV^3$					∞
$MEIV^1$	389	0	755	116	1 273 074.30
$MEIV^2$ lower bound on $x_{i,j}$	401	0	743	116	1 278 053.08
$MEIV^2$					∞
$MEIV^3$ lower bound on $x_{i,j}$	401	0	743	116	1 299 784.80
$MEIV^3$					∞
WS					1 037 820
$MEPEV$ (pair subproblem (ξ_{44}, ξ_2))	303	0	516	116	1 313 983.30
$MSPEV$ (w.r.t. ξ_{44})					1 041 627.34
$MSPEV$ (w.r.t. ξ_1)					1 039 068.91
$MSPEV$ (w.r.t. $\bar{\xi}$)					1 037 820
$RHEEV$ (w.r.t. $\bar{\xi}$ fixing $x_{i,j}$)	158	0	647	0	1 540 248.22
$RHVRS$ (w.r.t ξ_{44} fixing $x_{i,j}$)	0	299	533	116	1 531 020.74
$RHESSV$ (fixing $x_{i,j}$)	401	0	859	0	1 302 705.36
$RHEIV$ (lower bound on $x_{i,j}$)	401	0	743	116	1 290 604.32

Table 6: Performance measures for the four-stages case of the "single-sink transportation problem".

Performance measures	**Value (€)**
VSS^1	91 267.32
VSS^2 fixing $x_{i,j}$	185 002.22
VSS^2	∞
VSS^3 fixing $x_{i,j}$	299 449.62
VSS^3	∞
$MEVRS^1 - RP = MVSS^1$	103 463.38
$MEVRS^2 - RP = MVSS^2$ fixing $x_{i,j}$	164 122.76
$MEVRS^2 - RP = MVSS^2$	∞
$MEVRS^3 - RP = MVSS$ fixing $x_{i,j}$	168 187.02
$MEVRS^3 - RP = MVSS$	∞
$MLUSS^1$	26 282.62
$MLUSS^2$ fixing $x_{i,j}$	27 863.90
$MLUSS^2$	∞
$MLUSS^3$ fixing $x_{i,j}$	27 863.90
$MLUSS^3$	∞
$MLUDS^1$	0
$MLUDS^2$ lower bound on $x_{i,j}$	2 707.40
$MLUDS^2$	∞
$MLUDS^3$ lower bound on $x_{i,j}$	27 552.22
$MLUDS^3$	∞
$RHVSS$ (w.r.t $\bar{\xi}$)	267 173.92
$RHVSS$ (w.r.t ξ_{44})	262 402.22
$RHLUSS$	29 631.06
$RHLUDS$	17 530.02
δ	154 147.7

Expected Value $MSPEV$ and *Multistage Expectation of Pairs Expected Value* $MEPEV$ by solving a series of sub-problems more computationally tractable than the initial one under the assumption that a piece of information on the future development of a random variable is available. This extension has been done by introducing the new concept of auxiliary scenario and redefinition of probability of pairs subproblem. The results show that a better alternative than choosing the deterministic solution is given by the best pair subproblem solution as measured by $MEPEV$ in case we have more information about the future.
We also extend to the multistage case the *Expected Value of the Reference Scenario* $MEVRS$ and measures of quality of the expected value solution in terms of structure and upgradeability such as *Multistage Loss Using the Skeleton Solution* $MLUSS^t$ and *Multistage Loss of Upgrading the Deterministic Solution* $MLUDS^t$ and related with the standard *Value of Stochastic Solution* VSS^t at stage t. Such measures can help to understand the behavior of the deter-

ministic solution with respect to the stochastic and the reason of its badness/goodness. The above measures are also defined in a *rolling horizon* framework by means of the *Rolling Horizon Value of Stochastic Solution* $RHVVS$, the *Rolling Horizon Loss Using Skeleton Solution* and *Rolling Horizon Loss of Upgrading the Deterministic Solution* $RHLUDS$. The results show that they should be considered as an useful alternative to the standard methods allowing to update the estimation at each time period.
Chains of inequalities among the new measures are proved and tested on a stochastic multistage single-sink transportation problem. Differences among the values in the chains indicate the distance, at stage t, of the proposed approach to the stochastic one and give insight of what is potentially wrong with a solution coming from the deterministic or the approximated method considered.

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