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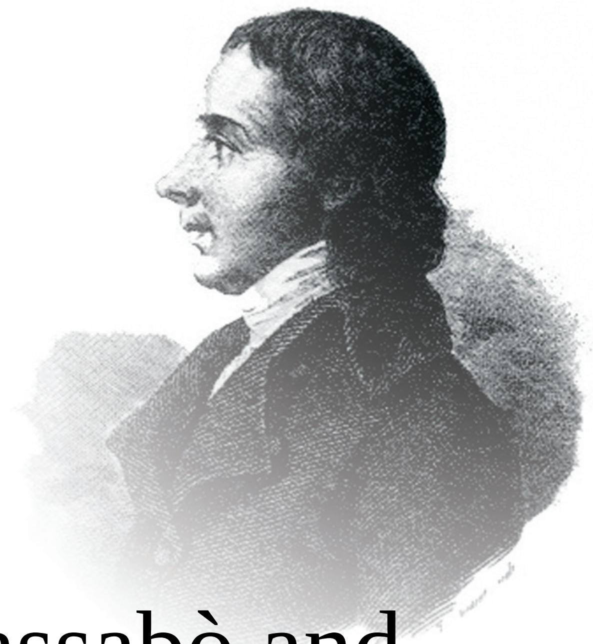

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A lattice based model for pricing equity-linked policies

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A Lattice based model for pricing equity-linked policies

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Abstract

We propose a lattice based model for computing the fair premiums of equity-linked policies with or without embedded surrender options. The model is based on a binomial lattice describing the evolution of the reference fund value where the periodical deemed contributions are invested. Since we consider the case of periodical premiums, the lattice is not recombining. In order to avoid that the evaluation process becomes computationally unmanageable, the model considers sets of representative values of the reference fund associated to each node of the tree. Then, the usual backward induction technique coupled with linear interpolation allows to define a simple and efficient method to compute the fair periodical premiums.

1 Introduction

An equity-linked policy is a policy whose payoff depends upon the performance of a reference fund made up of equities. In the case of single premium policies a unique contribution is deemed in the reference fund at the inception date but, in most cases, people prefer to pay smaller periodical premiums, typically at the beginning of each year, at the same date the deemed contributions in the reference fund are made. Since the policy payoff depends upon the performance of the reference fund, the policyholder bears the risk of a negative investment performance. To mitigate this risk, insurance companies, usually, insert into the contract a minimum guarantee providing a lower bound for the policy payoff that protect the insured investment.
The problem of computing the fair premiums for equity-linked policies with minimum guarantee has been tackled for the first time by Brennan and Schwartz [5] in 1976. The sake of their model is that the benefit at maturity may be decomposed into the guaranteed amount plus the value of an immediately exercisable call option written on the reference fund with strike price the guaranteed amount, or into the value of the reference fund plus the value of an immediately exercisable put option with strike price the guaranteed amount. Hence, the tools developed in financial

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theory to price contingent claims may be used for computing the policy premiums. In particular, Brennan and Schwartz computed the policy premiums in the Black-Scholes evaluation framework.
Many other authors were concerned in pricing equity-linked policies both with single and periodical premiums. Delbaen [8] and Aase and Persson [1] studied the periodical premium case under the assumption of deterministic interest rates. Bacinello and Ortu [4] analyzed the case of equity-linked policies with stochastic interest rates. They derived a closed form formula for the single premium case. Nielsen and Sandmann [10] used Monte Carlo simulations to compute periodical premiums under stochastic interest rates.
An interesting problem arises when a surrender option is embedded into the equity-linked contract. In this case, the policyholder may exercise the option at certain dates so receiving a pre-specified surrender value and the contract expires immediately. Otherwise, at maturity the policyholder will receive the stated benefit.
The computation of the fair premiums of a policy with an embedded surrender option has been tackled into two different ways. The first one considers the decision to surrender the policy as exogenous, hence, the early exercise of the option is treated as an independent source of risk as such as mortality risk. Given large enough statistical observations about early withdrawals, the fair policy premiums may be computed using mortality tables that consider both death and surrender probabilities.
The second approach considers the surrender decision as endogenous in the financial evaluation framework. In other words, the policyholder exercises the surrender option when it is financially convenient and the evaluation process works in the same way as in the American option case. Whenever the early withdrawal is admitted, the contract value is set equal to the maximum of two terms: the policy value in the case of no exercise and the policy value if the contract is surrendered. In this context, it is worth mentioning the contribution of Bacinello for computing the fair premiums of participating life insurance [2] and equity-linked endowment policies [3]. Periodical premiums, in a Cox-Ross-Rubinstein framework, makes the lattice describing the evolution of the reference fund value non recombining and the valuation problem becomes computationally unmanageable when the number of time steps increases.
Our contribution is devoted to overcome this obstacle by considering at each node of the tree a set of "representative values" for the reference fund. This technique is similar to that one developed for pricing path dependent options, in particular arithmetic average Asian options in a lattice framework. In this case, in fact, the number of arithmetic averages associated to each node of the tree grows exponentially when the number of time steps used for price computation increases. The problem is solved by considering sets of "representative averages" and by comput-

ing the option values via the usual backward induction scheme coupled with linear interpolation (see [9] for further details).
The remainder of the paper is organized as follows. In Section 2, we illustrate the dynamics of the reference fund value when a periodical premium is deemed in the reference fund. In Section 3, we present the evaluation model used to compute the fair periodical premiums. In Section 4, we illustrate the numerical results of the evaluation model. Finally, in Section 5, we draw conclusions.

2 The dynamics of the reference fund value

In order to compute the fair premiums of equity-linked policies we need, at first, to determine the dynamics of the reference fund value. The reference fund is generated by investing a fixed contribution, D, to acquire equities of the same kind at the beginning of each year until maturity, T. We assume that the equity value evolves in the discrete-time environment described by the Cox-Ross-Rubinstein [6] model and, as a consequence, a binomial lattice is used to describe the equity price evolution. We divide the policy lifetime into n time steps each of length $h = T/n$. Without loss of generality, we choose the number of time steps, n, equal to a multiple of T, such that $1/h = n/T$ is a positive integer. At time $t_0 = 0$ the equity value is S and at the end of the first period it may rise to Su or may decrease to Sd. After i time periods $(0 \leq i \leq n)$, the equity has registered j $(0 \leq j \leq i)$ up steps and $i - j$ down steps. The equity value at node (i, j) is $S(i, j) = Su^j d^{i-j}$. We shall denote by $\tau_{i,j} = \{(l, j_l), l = 0, \ldots, i; j_0 = 0 \quad \text{and} \quad j_i = j\}$ a generic path characterized by the equity values $S(l, j_l)$ that starts from S and reaches $S(i, j)$.
As usual, we set $u = \exp(\sigma\sqrt{h})$ and $d = 1/u$ (σ is the standard deviation of the equity rate of return). Moreover, the no-arbitrage assumption requires that $u > \exp(rh) > d$ (r is the continuously compounded risk free interest rate).
Let $t_k = k, k = 0, \ldots, T - 1$ denote the dates at which the deemed contributions in the reference fund are invested. At time $t_0 = 0$, with the first contribution, D, the insurer buys $n(0, 0) = D/S(0, 0)$ equities. At each anniversary of the contract, t_k, after $i = k/h$ time steps, when the equity value is $S(k/h, j)$, the insurer buys $n(k/h, j) = D/S(k/h, j)$ equities. Hence, the total number of equities acquired by the insurer when the equity value has followed the path $\tau_{i,j}$ is

$$N(i, j) = \sum_{k=0}^{[\frac{iT}{n}]} n(k/h, j_k), \quad 0 < i \leq n, \quad (k/h, j_k) \in \tau_{i,j}$$

where $[x]$ computes the greatest integer smaller than or equal to x. Note that $N(i, j)$ changes its value only at those time steps coinciding with the contribution dates.

Consequently, after i time steps, the reference fund value is

$$N(i,j)S(i,j) = \sum_{k=0}^{[\frac{iT}{n}]} n(k/h, j_k)S(i,j) = D\sum_{k=0}^{[\frac{iT}{n}]} \frac{S(i,j)}{S(k/h, j_k)}.$$

It is useful to remark that whenever a surrender option is embedded into the contract, at each anniversary, $t_k = k$, of the policy, just before the payment of the periodical premium, the insured has to decide whether to escape out of the contract or not. In the first case, no further contributions are invested into the reference fund and the total number of equities acquired after i time steps is

$$N(i,j) = \sum_{k=0}^{\lceil\frac{iT}{n}\rceil-1} n(k/h, j_k), \quad 0 < i \le n, \quad (k/h, j_k) \in \tau_{i,j}$$

where $\lceil x \rceil$ represents the smallest integer greater than or equal to x.

We underline that the presence of periodical deemed contributions causes a huge increment in the number of possible values of the reference fund. Indeed, the dynamics of the reference fund value is represented by a non recombining binomial tree with a number of nodes that grows exponentially when the number of time steps increases.

The following example may clarify this point. We fix $S = 100$, $T = 2$ years, $D = 100$, $\sigma = 0.25$, $r = 0.01$. We choose $n = 4$, thus $h = 0.5$, $u = 1.1934$ and $d = 0.8380$.

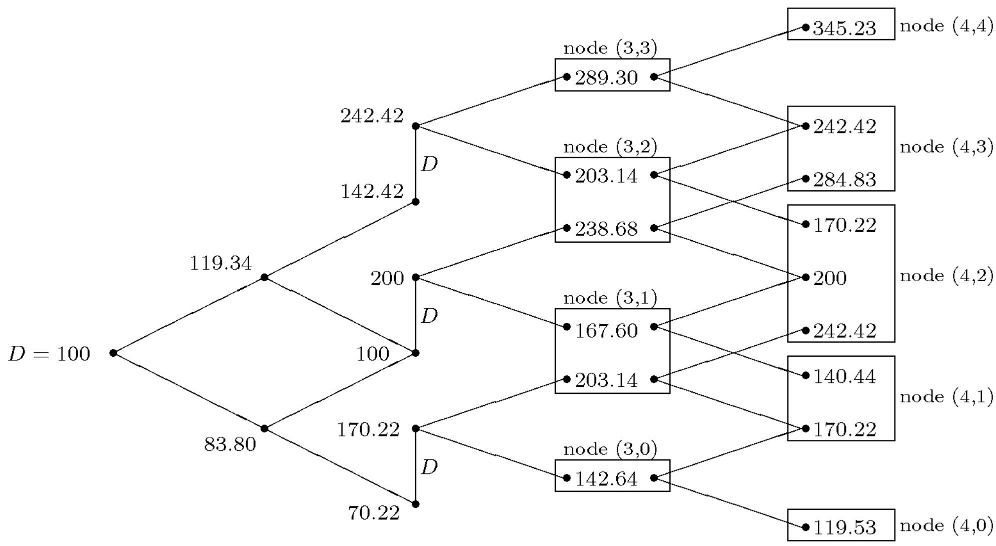

Figure 1: The dynamics of the reference fund value.

In Figure 1, we illustrate the dynamics of the value of the reference fund. Since $T = 2$ years, there are two contributions to be invested in the reference fund: the first one at time $t_0 = 0$ and the second one at time $t_1 = 1$ year, after two time steps. With the first contribution the insurer buys one equity. The number of equities purchased with the second contribution depends upon the value of the equity at time $t_1 = 1$ year. If the equity has registered two up steps during the first year, i.e., $S(2,2) = 142.4204$, the insurer buys $100/142.4204 = 0.7021$ equities and the value of the reference fund becomes 242.4204. If at time $t_1 = 1$ year only one up steps has been registered, the equity price is $S(2,1) = 100$, one equity is purchased and the value of the reference fund becomes 200. If the first two steps were down steps, the equity price has value $S(2,0) = 70.2244$, the insurer buys 1,4240 equities and the reference fund value becomes 170.2244. Let consider the situation at time $t = 1.5$ years, after two up steps and one down steps. At node $(3,2)$ the equity has price $S(3,2) = 119.34$ but the value of the reference fund depends upon the path followed by the equity price. Indeed, if the path followed by the equity has two up steps followed by a down step, the reference fund value is $242,4204 \times 0.8380 = 203.14$. On the contrary, if the path of the equity price has one up step, then one down step and, finally, one up step, the reference fund value is $200 \times 1.1934 = 238.68$. This is a key point when one tries to build up a lattice based model to evaluate equity linked policies since the binomial tree that describes the reference fund value dynamics is non recombining.

To overcome this problem, we may use a trick already applied in finance to evaluate derivative securities whose payoff depends upon a certain function of an asset value that cannot be described by a recombining tree. This is the case, for example, of arithmetic average Asian options whose payoff depends upon the arithmetic average of the underlying asset price during the option lifetime. The number of possible arithmetic averages increases exponentially when the number of time steps increases and, as a consequence, if one tries to represent them in a binomial lattice the problem becomes intractable from a computational point of view. Hull and White [9] proposed to consider a set of representative averages associated to each node of the tree describing the evolution of the underlying asset price. The option price at inception is then computed via the usual backward induction scheme coupled with linear interpolation.

The idea behind our model follows that one of Hull and White in the sense that, instead of considering all the possible values of the reference fund, we choose a set of representative values at each node of the tree representing the evolution of the equity price.

The considered values are between the minimum, $RF_{\min}(i,j)$, and the maximum, $RF_{\max}(i,j)$, value of the reference fund when the equity price is $Su^j d^{i-j}$. The first step is to describe $RF_{\max}(i,j)$ and $RF_{\min}(i,j)$ in terms of the periodical deemed contribution D. This is the content of Proposition 1 and 2, respectively.

Proposition 1. *In a binomial tree with n time steps, the maximum value of the* *reference fund when the equity price is* $Su^j d^{i-j}, 0 \leq j \leq i, 0 \leq i \leq n$ *is*

$$RF_{\max}(i,j) = \sum_{k=0}^{\lceil iT/n \rceil - 1} Du^{\min(j, i - \frac{kn}{T})} d^{\max(i - \frac{kn}{T} - j, 0)}. \quad (1)$$

Proof

We prove the proposition by forward induction on i, for all $0 \leq j \leq i$.
For $i = 1$ we have that

$$RF_{\max}(1,0) = Dd, \quad \text{and} \quad RF_{\max}(1,1) = Du.$$

On the other hand, the sum in (1) reduces to $Du^{\min(j,1)} d^{\max(i-j,0)}$, that is Dd if $j = 0$ and Du if $j = 1$. Thus (1) holds for $i = 1$.
In order to complete the proof, we start to notice that

$$RF_{\max}(i,j) = \begin{cases} [RF_{\max}(i-1, j-1) + DI_{\{i-1=q\frac{n}{T}, q \in I\!N\}}]u & \text{if} \quad j > 0 \\ [RF_{\max}(i-1, j) + DI_{\{i-1=q\frac{n}{T}, q \in I\!N\}}]d & \text{if} \quad j = 0 \end{cases}, \quad (2)$$

where $I_{\{i-1=q\frac{n}{T}, q \in I\!N\}}$ is the indicator function that is equal to one if $i - 1 = q\frac{n}{T}$ for $q \in I\!N$ and zero otherwise.

We may rewrite (2) more conveniently, as

$$RF_{\max}(i,j) = \{RF_{\max}[i-1, \max(j-1, 0)] + DI_{\{i-1=q\frac{n}{T}, q \in I\!N\}}\} u^{\min(j,1)} d^{\max(1-j,0)}. \quad (3)$$

The key observation that allows to justify (2) and (3) is that the maximum value of the reference fund at a given node (i, j) is obtained from those trajectories of the reference fund value that reach the node (i, j) from below. This is because when a deemed contribution is invested to buy equities, less is the equity value more is the number of equities that the insurer has to insert in the reference fund.
Indeed, when $i - 1$ is a time step where no contribution arises, that is the indicator function is equal to zero, then the maximum value of the reference fund at time i, $RF_{\max}(i, j)$, is obtained from the maximum value of the reference fund at the previous time step, $RF_{\max}(i-1, j-1)$ multiplied by u if $j > 0$ or $RF_{\max}(i-1, j)$ multiplied by d if $j = 0$. Hence (3) holds or equivalently, we have that

$$RF_{\max}(i,j) = RF_{\max}[i-1, \max(j-1, 0)] u^{\min(1,j)} d^{\max(1-j,0)}.$$

Otherwise, when $i - 1$ is a time step coinciding with a contribution date, then the indicator function is equal to one and the maximum value of the reference fund at the next time step, $RF_{\max}(i, j)$, is obtained from the maximum value of the reference fund at the previous time step, $RF_{\max}(i-1, \max(j-1, 0))$ plus the amount D of the deemed contribution multiplied by $u^{\min(1,j)} d^{\max(1-j,0)}$, that is

$$RF_{\max}(i,j) = \{RF_{\max}[i-1, \max(j-1, 0)] + D\} u^{\min(1,j)} d^{\max(1-j,0)}.$$

Thus (3) holds.
Now we can complete our proof of (1) by induction. Indeed, we assume that (1) is valid at a generic time step $i-1>0$, i.e.,

$$RF_{\max}[i-1,\max(j-1,0)] = \sum_{k=0}^{\lceil\frac{(i-1)T}{n}\rceil-1} Du^{\min[i-1-\frac{kn}{T},\max(j-1,0)]}d^{\max[i-1-\frac{kn}{T}-\max(j-1,0),0]}.$$

Then, using (3) and the inductive assumption, we have that

$$RF_{\max}(i,j) = \sum_{k=0}^{\lceil\frac{(i-1)T}{n}\rceil-1} Du^{\min[i-1-\frac{kn}{T},\max(j-1,0)]}d^{\max[i-1-\frac{kn}{T}-\max(j-1,0),0]}u^{\min(1,j)}d^{\max(1-j,0)}+$$

$$+DI_{\{i-1=q\frac{n}{T},q\in I\!N\}}u^{\min(1,j)}d^{\max(1-j,0)} =$$

$$= \sum_{k=0}^{\lceil\frac{(i-1)T}{n}\rceil-1} Du^{\min(i-\frac{kn}{T},j)}d^{\max(i-\frac{kn}{T}-j,0)} + DI_{\{i-1=q\frac{n}{T},q\in I\!N\}}u^{\min(1,j)}d^{\max(1-j,0)}. \qquad (4)$$

Now, if $(i-1)\neq qn/T$ with $q\in I\!N$ then, the last addendum in (4) is equal to zero since $\lceil (i-1)T/n\rceil = \lceil iT/n\rceil$, hence (4) reduces to (1).
If $(i-1)=qn/T$ with $q\in I\!N$, we have that $(i-1)T/n = \lceil iT/n\rceil - 1$. Clearly, all the terms in the first addendum of the sum described in (4) coincide with the first $(\lceil iT/n\rceil-2)$-th terms in (1) and the last addendum in (4),

$$DI_{\{i-1=q\frac{n}{T},q\in I\!N\}}u^{\min(1,j)}d^{\max(1-j,0)},$$

is the $(\lceil iT/n\rceil-1)$-th addendum in (1) since $i-(\lceil iT/n\rceil-1)n/T=1$ and the indicator function is equal to one. Consequently, (1) holds for the i-th time step and Proposition 1 is completely justified. □ The proof of Proposition 2 below is like

that of Proposition 1 and we shall omit it.

Proposition 2. *In a binomial tree with n time steps, the minimum value of the reference fund when the equity price is $Su^jd^{i-j}, 0\le j\le i, 0\le i\le n$ is*

$$RF_{\min}(i,j) = \sum_{k=0}^{\lceil iT/n\rceil-1} Dd^{\min(i-j,i-\frac{kn}{T})}u^{\max(j-\frac{kn}{T},0)}. \qquad (5)$$

Now we can describe the set of representative values of the reference fund corresponding to each node of the tree describing the evolution of the equity price. The smallest value in the set associated to the node (i,j), denoted by $RF(i,j,1)$, is chosen equal to $RF_{\min}(i,j)$ and the greatest value is equal to $RF_{\max}(i,j)$. The other reference fund representative values are of the form $RF(i,j,k+1) = RF_{\min}(i,j)e^{ak}$ where a is a positive real number and k assumes all the integer values in the interval $[1, k_{\max}(i,j)]$ in such a way that $RF_{\min}(i,j)e^{ak_{\max}(i,j)} < RF_{\max}(i,j) \leq RF_{\min}(i,j)e^{a[k_{\max}(i,j)+1]}$. In this way, at each node (i,j) we associate $k_{\max}(i,j)+1$ values.

3 The evaluation model

We start by considering the simple case of a term policy with periodical premiums typically paid at the beginning of each year either if the insured is alive or if he dies before the policy maturity, T. The policy forces the insurer to pay at maturity a capital $C(T)$. For the moment, we consider the case of no surrender option embedded into the contract; hence the insured has no chance of early withdrawal.
We consider the case

$$C(T) = \max[RF(T), G(T)],$$

where:

- $RF(T)$ is the value at time T of the reference fund made up of equities of the same kind that the insurer buys at the beginning of each year by investing a fixed component, D, of the annual premium;

- $G(T)$ is the value at maturity of the minimum guarantee inserted into the contract to protect the policyholder's investment against a negative performance of the reference fund.

Among the different possible types of minimum guarantees, we consider

$$G(T) = \sum_{k=1}^{T} De^{k\delta} = De^{\delta}\frac{e^{T\delta}-1}{e^{\delta}-1},$$

where $\delta > 0$ is the minimum guaranteed continuously compounded interest rate (the case $\delta = 0$ is trivial). In other words, the insurer is forced to pay at least the deemed contributions D invested at the annual interest rate δ.
Following Brennan e Schwartz [5], the payoff at maturity of the equity-linked policy may be decomposed into two different forms as

$$C(T) = RF(T) + \max[G(T) - RF(T), 0], \tag{6}$$

or as

$$C(T) = G(T) + \max[RF(T) - G(T), 0]. \tag{7}$$

Equation (6) characterizes the payoff at maturity of the equity-linked policy as the sum of the value of the reference fund and the value of a put option written on the reference fund with strike price $G(T)$. Conversely, equation (7) characterizes the payoff at maturity as the sum of a fixed amount, $G(T)$, and the payoff of a call option written on the reference fund with strike price $G(T)$.
Our goal is to compute the fair value at time $t_0 = 0$ of the policy payoff at maturity. Using equation (6) it is equal to

$$V_0[C(T)] = V_0[RF(T)] + V_0\{\max[G(T) - RF(T), 0]\}.$$

Clearly, the put option value represents the cost of the minimum guarantee embedded into the contract.
In a risk-neutral evaluation framework, this is equal to

$$De^r\left(\frac{1-e^{-rT}}{e^r-1}\right) + e^{-rT}\tilde{E}\{\max[G(T) - RF(T), 0]\},$$

where $\tilde{E}$ represents the expectation under the risk-neutral probability measure and r is the risk-free continuously compounded interest rate. The second term in the above sum is the price at time $t_0 = 0$ of a put option written on the reference fund with strike price $G(T)$.
If we consider equation (7), the value at inception of the policy payoff is given by

$$V_0[C(T)] = V_0[G(T)] + V_0[\max[RF(T) - G(T), 0]] =$$
$$= De^{\delta - rT}\left(\frac{e^{T\delta}-1}{e^{\delta}-1}\right) + e^{-rT}\tilde{E}\{\max[RF(T) - G(T), 0]\}.$$

The second addendum of the above sum is the value at time $t_0 = 0$ of a call option written on the reference fund with strike price $G(T)$.
In order to compute the fair value of the policy at inception, we need to evaluate the put option embedded into the contract according to (6) or the call option if we prefer to decompose the policy payoff according to (7).
We tackle the pricing problem within the Cox, Ross and Rubinstein [6] binomial model. Hence, the dynamics of the reference fund value is that one described in Section 2.
We label $O(i,j,k)$ the k-th option value (for the call or for the put) when the equity price has reached the node (i,j) and the reference fund value is $RF(i,j,k)$. We compute $O(i,j,k)$ via the usual backward-induction scheme, i.e.,

$$O(i,j,k) = e^{-rh}[pO(i+1,j+1,k_u) + qO(i+1,j,k_d)]$$

where:
p is the risk neutral probability of an up step and $q = 1 - p$;

$O(i+1,j+1,k_u)$ is the option value at node $(i+1,j+1)$ where the reference fund has value $[RF(i,j,k)+DI_{\{i=qn/T,q\in I\!N\}}]u$;
$O(i+1,j+1,k_d)$ is the option value at node $(i+1,j)$ where the reference fund has value $[RF(i,j,k)+DI_{\{i=qn/T,q\in I\!N\}}]d$.
Both $[RF(i,j,k)+DI_{\{i=qn/T,q\in I\!N\}}]u$ and $[RF(i,j,k)+DI_{\{i=qn/T,q\in I\!N\}}]d$ may not be in the set of the representative values of the reference fund at node $(i+1,j+1)$ and $(i+1,j)$, respectively. To overcome this obstacle, $O(i+1,j+1,k_u)$ can be computed by linear interpolating between the option values $O(i+1,j+1,k_1)$ and $O(i+1,j+1,k_2)$ where k_1 and k_2 are chosen in a way that $RF(i+1,j+1,k_1)$ is the greatest reference fund value smaller than $[RF(i,j,k)+DI_{\{i=qn/T,q\in I\!N\}}]u$ and $RF(i+1,j+1,k_2)$ is the smallest value of the reference fund greater than $[RF(i,j,k)+DI_{\{i=qn/T,q\in I\!N\}}]u$. $O(i+1,j,k_d)$ is computed analogously.
Once the policy fair value at inception has been computed, the periodical premiums to be paid, typically, at the beginning of each year, can be calculated by following the approach presented in [5] simply by annualizing at the riskless interest rate the policy value at inception.

Now we consider the case of a term policy with periodical premiums and an embedded surrender option under the assumption that the insured may escape out of the contract at the beginning of each year, just before the payment of the premium. If the insured decides to surrender the policy, he receives a sum that depends upon the current value of the reference fund and/or on the minimum guarantee evaluated at the surrender time. We label $SV(i,j,k)$ the surrender value at time $t=ih$, corresponding to the reference fund value $RF(i,j,k)$. Among others, we consider the cases

$$SV(i,j,k)=RF(i,j,k),$$

i.e., the insured receives the accrued reference fund value,

$$SV(i,j,k)=G(ih)=\sum_{l=0}^{\lceil i\frac{T}{n}\rceil-1} De^{(ih-l)\delta},$$

i.e., the insured receives the minimum guarantee given by the deemed contributions paid by the policyholder after i time-steps evaluated at the minimum guaranteed interest rate and, finally,

$$SV(i,j,k)=\max[RF(i,j,k),G(ih)]$$

i.e., the insured receives the maximum between the reference fund and the minimum guarantee evaluated after i time periods.
In order to evaluate the periodical premium, P, we define $V(i,j,k)$ as the value of the policy in the state of nature (i,j,k). At the maturity of the contract, after n

time steps,

$$V(n,j,k) = \max[RF(n,j,k), G(T)].$$

At time step i $(0 < i < n)$, we must distinguish between two cases. If i is a time step coinciding with an anniversary of the contract, i.e., $i = qn/T, q \in I\!N$, we have

$$V(i,j,k) = \max\{e^{-rh}[pV(i+1,j+1,k_u) + qV(i+1,j,k_d)] - P, SV(i,j,k)\}.$$

Otherwise, if i is a time step that does not coincide with an anniversary of the policy

$$V(i,j,k) = e^{-rh}[pV(i+1,j+1,k_u) + qV(i+1,j,k_d)].$$

$V(i+1,j,k_d)$ and $V(i+1,j+1,k_u)$ are, respectively, the values of the policy corresponding to the reference fund value $[RF(i,j,k)+DI_{\{i=qn/T,q\in I\!N\}}]d$ and $[RF(i,j,k)+DI_{\{i=qn/T,q\in I\!N\}}]u$. Again, $[RF(i,j,k)+DI_{\{i=qn/T,q\in I\!N\}}]d$ and $[RF(i,j,k)+DI_{\{i=qn/T,q\in I\!N\}}]u$, may not be in the set of the representative values of the reference fund at time step $i+1$. As before, we compute $V(i+1,j+1,k_u)$ using linear interpolation between the option values $V(i+1,j+1,k_1)$ and $V(i+1,j+1,k_2)$ where k_1 and k_2 are chosen in a way that $RF(i+1,j+1,k_1)$ is the greatest reference fund value smaller than $[RF(i,j,k) + DI_{\{i=qn/T,q\in I\!N\}}]u$ and $RF(i+1,j+1,k_2)$ is the smallest value of the reference fund greater than $[RF(i,j,k) + DI_{\{i=qn/T,q\in I\!N\}}]u$. $V(i+1,j,k_d)$ is computed in the same way.
At time $t = 0$, in order to compute the fair periodical premium, P, we have to solve the following equation with respect to P

$$V(0,0,1) = e^{-rh}[pV(1,1,k_u) + qV(1,0,k_d)] - P = 0. \tag{8}$$

We solve (8) numerically and the solution represents the periodical premium to be paid at the beginning of each year by the insured until he will surrender the contract [2].

The previous analysis may be easily extended including mortality risk. To do this, we now consider an endowment policy with maturity T years. The insurer pays a certain capital if the insured dies during the T years, otherwise, if the insured is alive at time T, the insurance company pays a prespecified sum. To buy the policy, the insured pays a fixed premium, P, at the beginning of each year as long as he is alive. Moreover, we consider an embedded surrender option that gives the insured the chance to escape the contract at the beginning of each year just before the payment of the annual premium.
We label ${}_tp_x$ the probability that an individual of age x will survive at least t years while ${}_tq_x = 1 -_t p_x$ will denote the probability of the individual death during the next t years. We assume independence between mortality risk and financial risk.

[2] As already proved by Bacinello [3], equation (8) admits a unique solution

The evaluation framework is easily extended from that one developed for term policies. At the beginning of each year, if the insured is still alive and has not previously surrendered the contract, the periodical premium, P, is paid and a contribution D is deemed in the reference fund to buy equities. The dynamics of the equity value and that one of the reference fund value are given by the same binomial processes described in Section 2.

We divide the policy lifetime, T, into n time intervals each of length $h = T/n$ with n/T a positive integer. We assume that, if the insured dies during the time interval $((i-1)h, ih)$, the company pays at time $t = ih$ a capital given by $f_D(i,j,k)$ where $f_D(i,j,k)$ is a function specifying the sum paid by the insurer in case of death when the state of nature is (i,j,k). Among the possibilities, we may consider $f_D(i,j,k) = RF(i,j,k)$, $f_D(i,j,k) = G(ih)$, $f_D(i,j,k) = \max[RF(i,j,k), G(ih)]$. Even more complicated functions specifying the capital to be paid in the case of death may be easily managed in this evaluation framework.

Furthermore, if the insured is alive at maturity T, the capital paid is $\max[RF(n,j,k), G(T)]$ if the state of nature is (n,j,k).

Now, our goal is to compute the fair periodical premium of the policy.

At time step $n-1$, given that the policyholder is still alive and the contract is still in force, if $(n-1) \neq qn/T$, $q \in I\!N$, the policy value in the state of nature $(n-1,j,k)$ (when the reference fund value is $RF(n-1,j,k)$) is

$$V(n-1,j,k) = e^{-rh}\{{}_hp_{x+(n-1)h}[p\max[RF(n,j+1,k_u), G(T)]+$$

$$+q\max[RF(n,j,k_d), G(T)] +_h q_{x+(n-1)h}[pf_D(n,j+1,k_u) + (1-p)f_D(n,j,k_d)]\}.$$

We remark that the function $f_D(i,j,k_u)$ ($f_D(i,j,k_d)$) determines the sum paid by the insurer in case of death when the reference fund value is $RF(i,j,k_u)$ ($RF(i,j,k_d)$). As before, $RF(i,j,k_u)$ and $RF(i,j,k_d)$ are computed, respectively, as $RF(i-1,j-1,k)u$ and $RF(i-1,j,k)d$.

In particular, if the sum paid in the case of death is given by the minimum guarantee, the evaluation formula reduces to

$$V(n-1,j,k) = e^{-rh}\{{}_hp_{x+(n-1)h}[p\max[RF(n,j+1,k_u), G(T)]+$$

$$q\max[RF(n,j,k_d), G(T)]] +_h q_{x+(n-1)h}G(T)\}.$$

Going backwards along the tree, at the i-th time step, assuming that the insured is still alive and that the contract is still in force, we have to distinguish two cases. Consider at first the possibility that $i \neq qn/T, q \in I\!N$, i.e., i is a time step not coinciding with an anniversary of the contract, then

$$V(i,j,k) = e^{-rh}\{{}_hp_{x+ih}[pV(i+1,j+1,k_u) + qV(i+1,j,k_d)]+$$

$$+{}_hq_{x+ih}[pf_D(i+1,j+1,k_u) + qf_D(i+1,j,k_d)]\}.$$

On the contrary, if $i = qn/T, q \in I\!N$, i.e., i is a time step coinciding with an anniversary of the contract, then

$$V(i,j,k) = \max(e^{-rh}\{{}_hp_{x+ih}[pV(i+1,j+1,k_u) + qV(i+1,j,k_d)]+$$
$$+{}_hq_{x+ih}[pf_D(i+1,j+1,k_u) + qf_D(i+1,j,k_d)]\} - P, SV(i,j,k)).$$

Once the backward induction scheme reaches $t = 0$, in oder to compute the fair periodical premium, P, we need to solve, as in [3] the following equation

$$V(0,0,1) = e^{-rh}\{{}_hp_x[pV(1,1,k_u)+qV(1,0,k_d)]+{}_hq_x[pf_D(1,1,k_u)+qf_D(1,0,k_d)]\}-P = 0. \tag{9}$$

We solve (9) numerically and the solution represents the periodical premium to be paid at the beginning of each year by the insured until he is alive and has not exercised the surrender option to escape out of the contract.

4 Numerical results

We tested the pricing model depicted in Section 3 by computing the fair premiums of different equity linked policies. Each policy benefit is linked to a reference fund accrued by investing at the beginning of each year a fixed amount $D = 100$. All the premiums computed with the binomial model used $n = 30$ time steps and $a = 0.0001$.
In Table 1 we reported the fair present value (PVB) and the fair periodical premiums (PB) computed with the binomial algorithm for a term policy without surrender option for different maturities and a minimum guaranteed interest rate $\delta = 0$. The risk-free continuously compounded interest rate is $r = 0.04$ and the volatility is $\sigma = 0.1358$. The results are compared with those ones computed by Brennan and Schwartz ($PVBrS$ and $PBrS$).

Table 1

Fair premiums for term policies
without surrender option

T	1	5	10	15
PVB	103.53	477.29	863.89	1176.25
PVBrS	103.5	477.1	863.8	1176.4
PB	103.53	103.2432	102.747	102.2221
PBrS	103.5	103.2	102.7	102.2

In Table 2 we report the fair periodical premiums for equity linked term policies with an embedded surrender option. At the beginning of each year the policyholder has the right to escape out of the contract. In this case, he receives the surrender value $\max[RF(i,j,k), G(ih)]$ if the state of nature is (i,j,k). Otherwise, he pays the annual premium and a contribution $D = 100$ is deemed into the reference fund. No mortality risk is yet considered and the volatility is equal to $\sigma = 0.1358$

Table 2

Fair premiums for term policies
with surrender option

T	r=0.04 $\delta = 0$	r=0.04 $\delta = 0.02$	r=0.06 $\delta = 0.02$
1	103.5292	104.4635	103.6043
5	105.1015	106.7734	105.0780
10	105.6214	108.1607	105.5145
15	105.9325	109.0168	105.6956

In Table 3, 4, 5 we report the fair periodical premiums for equity-linked endowment policies with surrender option for individuals with different initial age. At the beginning of each year, if the policyholder is still alive, he decides to surrender the contract receiving the benefit $\max[RF(i,j,k), G(ih)]$ or to pay the premium and, in this case, a contribution $D = 100$ is deemed into the reference fund. The company is forced to pay a capital $\max[RF(i,j,k), G(ih)]$ at the end of the time interval of death if the insured dies before maturity or a capital if the insured is alive at that date.

We modeled the mortality risk by considering Italian Statistics for Male mortality in 2002. In this table, there are quoted the annual probabilities of death. It means that, if we consider an individual of age x, the table quotes the probability ${}_1q_x = q_x$, i.e., the probability that the individual dies before age $x+1$. In our evaluation framework, we face the problem to value death probabilities on period of time smaller than one year, ${}_hq_{x+ih}$. We solve this problem by invoking the assumption of uniformity of deaths, in the sense that in any fraction of width z of one year it is expected the same fraction z of the deaths related to that age. Hence, the death probability on a fraction z of one year, ${}_zq_x = zq_x$. In our case, $z = h$ and ${}_hq_{x+ih} = hq_{x+[ih]}$, where $[ih]$ computes the largest integer smaller than or equal to ih. Clearly, ${}_hp_{x+ih} = 1 -_h q_{x+ih}$.

Table 3

Fair premiums for endowment policies with surrender option

	$x = 40 \quad \sigma = 0.1358$		
	r=0.04	r=0.04	r=0.06
T	$\delta = 0$	$\delta = 0.02$	$\delta = 0.02$
1	103.5281	104.4617	103.6031
5	105.0960	106.7654	105.0730
10	105.6103	108.1376	103.5035

Table 4

Fair premiums for endowment policies with surrender option

	$x = 50 \quad \sigma = 0.1358$		
	r=0.04	r=0.04	r=0.06
T	$\delta = 0$	$\delta = 0.02$	$\delta = 0.02$
1	103.5265	104.4593	103.6014
5	105.0880	106.7532	105.0653
10	105.5294	108.1001	105.4857

Table 5

Fair premiums for endowment policies with surrender option

	$x = 40 \quad \sigma = 0.2716$		
	r=0.04	r=0.04	r=0.06
T	$\delta = 0$	$\delta = 0.02$	$\delta = 0.02$
1	108.6516	109.7526	108.7607
5	113.2604	115.5214	113.2692
10	115.7685	119.4469	115.8029

5 Conclusions

We proposed an evaluation model to compute fair periodical premiums of equity-linked policies with minimum guarantees and an embedded surrender option that allows the policyholder to escape out of the contract at certain prespecified dates. We assumed that a fixed component of the periodical premiums are invested in a reference fund whose evolution is described within the binomial model of Cox, Ross and Rubinstein. The periodical contributions makes the tree non recombining with a huge complexity from a computational point of view. We propose to overcome this obstacle by choosing sets of representative values of the reference fund associated to each node of the tree. The periodical premiums are computed by solving an equation obtained via the usual backward induction technique. Finally, in order to test the proposed algorithm, numerical results are presented in Section 4.

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