QUANTUM LOGICS WITH BOUNDED ADDITIVE OPERATORS

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The theory of gates in quantum computation has suggested new forms of quantum logic, called *quantum computational logics*, where the meaning of a sentence is identified with a system of qubits in a pure or, more generally, mixed state. In this framework, any formula of the language gives rise to a *quantum circuit* that transforms the state associated to the atomic subformulas into the state associated to the formula and vice versa. On this bases, some holistic semantic situations can be described, where the meaning of whole determine the meaning of the parts, by non-linear and anti-unitary operators. We prove that the semantics with such operators and the semantics with unitary operators turn out to characterize the same logic.

Keywords: quantum computation; quantum logic.

1. Introduction

The theory of quantum gates has suggested new forms of quantum logic that have been called *quantum computational logics*[1]. The main difference between orthodox quantum logic (first proposed by Birkhoff and von Neumann[2]) and quantum computational logics concerns the *meanings* of the sentences of a given language. In Birkhoff and von Neumann the meanings have to be regarded as determined by convenient sets of states of quantum objects which satisfy some special closure conditions and so the elementary experimental sentences of quantum theory can be adequately interpreted as *closed subspaces* of the Hilbert space associated to the physical systems under investigation[3]. Interesting applications of orthodox quantum logic (and of its weaker variant, *orthologic*) have been recently investigated[4,5,6,7,8]. Mateus and Sernadas consider an exogenous quantum propositional logic based on superpositions of classical models as the model of the logic, leading to a natural extension of the classical language and allowing quantitative reasoning about amplitudes and probabilities[9]. Engesser et al. introduce a holistic logic where a state is a logical entity that encodes other states and also itself[10].

In quantum computational logics we use a different approach where the *meaning* of a sentence is identified with a quantum information quantity: a *quregister* or, more generally, a *mixture* of quregisters (briefly, a *qumix*)[11].

Quantum information is encoded in qubits and transformations follow the laws of quantum mechanics. This fundamental requirement implies theoretical limitations in making some transformations such as cloning and flipping. The Poincare-Bloch sphere provides a geometric representation of a qubit and De Martini et al.[12] suggest to consider the antipode point as representative of the flipped state instead of the point obtained as a π-rotation around the x or y axis. This assumption involves the adoption of the u-not operator which is anti-unitary (that is, it is not completely positive). It is exactly this property that makes the spin-flip operation so important in all criteria of inseparability for two qubit systems. Why do we have to look at anti-unitary and non-linear operators instead of the unitary forms? The reason is because unexpectedly an experimental realization of the universal optimal quantum cloning machine and of the u-not gate has been presented[13] by slightly modifying the protocol of quantum state teleportation. Anti-operators have gained increasing importance and it is interesting to understand what quantum computational logic is obtained by replacing unitary operators.

We will show that the semantics with bounded additive operators characterize the same logic.

2. Quregisters and qumixes

We will first sum up some basic concepts of quantum computation that are used in the framework of quantum computational logics. Consider the two-dimensional Hilbert space $\mathbb{C}^2$ (where any vector $|\psi\rangle$ is represented by a pair of complex numbers). Let $\mathcal{B}^{(1)} = \{|0\rangle, |1\rangle\}$ be the canonical orthonormal basis for $\mathbb{C}^2$, where $|0\rangle = \binom{1}{0}$ and $|1\rangle = \binom{0}{1}$. Recalling the Born rule, any $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ (with $|c_0|^2 + |c_1|^2 = 1$) can be regarded as an *uncertain piece of information*, where the answer *NO* has probability $|c_0|^2$, while the answer *YES* has probability $|c_1|^2$. The two basis-elements $|0\rangle$ and $|1\rangle$ are usually taken as encoding the classical bit-values 0 and 1, respectively. From a semantic point of view, they can be also regarded as the classical truth-values *Falsity* and *Truth*.

An *n-quregister* is represented by a unit vector in the n-fold tensor product Hilbert space $\mathcal{H}^{(n)} := \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n-times}$. We will use $x, y, \ldots$ as variables ranging over the set $\{0, 1\}$. At the same time, $|x\rangle, |y\rangle, \ldots$ will range over the basis $\mathcal{B}^{(1)}$. Any factorized unit vector $|x_1\rangle \otimes \ldots \otimes |x_n\rangle$ of the space $\mathcal{H}^{(n)}$ will be called an *n-configuration* (which can be regarded as a quantum realization of a classical bit sequence of length n). Instead of $|x_1\rangle \otimes \ldots \otimes |x_n\rangle$ we will also write $|x_1, \ldots, x_n\rangle$. Recall that the dimension of $\mathcal{H}^{(n)}$ is 2^n, while the set of all n-configurations $\mathcal{B}^{(n)} = \{|x_1, \ldots, x_n\rangle : x_1, \ldots, x_n \in \{0, 1\}\}$ is an orthonormal basis for the space $\mathcal{H}^{(n)}$. We will call this set a *computational basis* for the n-quregisters. Since any element of the computational basis can be labeled by a binary string which represents a natural number $j \in [0, 2^n - 1]$ in binary notation (where $j = 2^{n-1}x_1 + 2^{n-2}x_2 + \ldots + x_n$), any quregister can be briefly expressed as a superposition having the following form: $\sum_{j=0}^{2^n-1} c_j|j\rangle$, where $c_j \in \mathbb{C}$, $|j\rangle$ is the

n-configuration corresponding to the number j and $\sum_{j=0}^{2^n-1} |c_j|^2 = 1$.

For semantic aims, it is useful to distinguish the *true* from the *false* in any space $\mathcal{H}^{(n)}$. We assume the following convention (which is a natural generalization of classical semantics): any n-configuration corresponds to a classical truth-value that is determined by its last element (i.e. $x_n = 1 := true$ and $x_n = 0 := false$ or, in other words, by the parity of j, i.e. odd:=true and even:=false). Let us now decompose the Hilbert space $\mathcal{H}^{(n)}$ into its true and false subspaces $\mathcal{H}_0^{(n)}$ and $\mathcal{H}_1^{(n)}$ respectively, i.e. $\mathcal{H}^{(n)} = \mathcal{H}_0^{(n)} \oplus \mathcal{H}_1^{(n)}$, and denote by $P_1^{(n)}$ and $P_0^{(n)}$ the pertaining orthogonal projectors, $P_1^{(n)} + P_0^{(n)} = I^{(n)}$, where $I^{(n)}$ is the identity operator of $\mathcal{H}^{(n)}$. Therefore, the projectors $P_1^{(n)}$ and $P_0^{(n)}$ represent the *Truth-property* and the *Falsity-property* in $\mathcal{H}^{(n)}$, respectively. Let $\mathfrak{D}(\mathcal{H}^{(n)})$ be the set of all positive trace class operators of $\mathcal{H}^{(n)}$ and let $\mathfrak{D} := \bigcup_{n=1}^{\infty} \mathfrak{D}(\mathcal{H}^{(n)})$.

A *qumix* is a density operator in $\mathfrak{D}$. Needless to say, quregisters correspond to particular qumixes that are *pure states* (i.e. projections onto one-dimensional closed subspaces of $\mathcal{H}^{(n)}$). Recalling the Born rule, we can now define the *probability-value* of any qumix.

Definition 1: Probability of a density operator (qumix).
For any qumix $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$: $\mathtt{p}(\rho) = \mathtt{tr}(\rho\, P_1^{(n)})$.

$\mathtt{p}(\rho)$ is the probability that the information stocked by the qumix ρ is true. In the particular case where ρ corresponds to the 1-quregister $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, we obtain that $\mathtt{p}(\rho) = |c_1|^2$.

For any quregister $|\psi\rangle$, we will write $\mathtt{p}(|\psi\rangle)$ instead of $\mathtt{p}(P_{|\psi\rangle})$, where $P_{|\psi\rangle}$ (also indicated by $|\psi\rangle\langle\psi|$) is the density operator represented by the projection onto the one-dimensional subspace spanned by the vector $|\psi\rangle$. In particular, we have the matching of notions $P_0^{(1)} \equiv P_{|0\rangle}$ and $P_1^{(1)} \equiv P_{|1\rangle}$, with the projector also representing a pure state.

An interesting relation connects qumixes with real numbers in the interval $[0, 1]$. For any $n \in \mathbb{N}^+$, any real number $\lambda \in [0, 1]$ uniquely determines a qumix $\rho_\lambda^{(n)}$:

$$\rho_\lambda^{(n)} := (1 - \lambda) k_n P_0^{(n)} + \lambda k_n P_1^{(n)}$$

(where $k_n = \frac{1}{2^{n-1}}$ is a normalization coefficient). From an intuitive point of view, $\rho_\lambda^{(n)}$ represents a *mixture of pieces of information* that might correspond to the *Truth* with probability λ.

3. Generalized Quantum Gates

Generalized quantum gates (briefly, gates) correspond to some basic *logical operations* that admit a reversible behaviour. We will consider here the following gates: the *u-not*, the *controlled-controlled-u-not*, the *square root of the u-not* and the *square root of the identity.*

Let us first describe our gates in the framework of quregisters. For any $n \geq 1$, the u-not gate[14] on $\mathcal{H}^{(n)}$ is the anti-linear operator $\mathtt{UNot}^{(n)}$ such that for every element $|x_1,\ldots,x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$:

$$\mathtt{UNot}^{(n)}(|x_1,\ldots,x_{n-1},x_n\rangle) = (-1)^{1-x_n}|x_1,\ldots,x_{n-1},1-x_n\rangle.$$

In other words, $\mathtt{UNot}^{(n)}$ inverts the value of the last element of basis-vector of $\mathcal{H}^{(n)}$.

From the formal point of view, this anti-unitary operator satisfies the universal orthogonality (or *flipping*) condition $\exists \mathtt{U}\ \forall|\psi\rangle,\ \langle\psi|\mathtt{U}\,\psi\rangle = 0$ and the *anti self-reversibility* condition $\mathtt{U}^2 = -\mathtt{I}$, which can be reformulated as $\mathtt{U}^{-1} = -\mathtt{U}$. Starting from the orthogonality condition, recently a certain number of contributions [14,15,16,17,12] has been published about the full complementing requirement: for any arbitrary vector $|\psi\rangle$, $\mathtt{UNot}^{(1)}(|\psi\rangle) = |\psi^{\perp}\rangle$ since $\langle\psi|\mathtt{UNot}^{(1)}\,\psi\rangle = 0$ holds.

Anti–linear operators have an indispensable role in quantum field theory[18], so much so that the definition of the adjoint of such an operator can be found in a textbook on field theory by Itzykson and Zuber[19]. Basic knowledge about anti–unitary operators is due to Wigner[20].

For any $n \geq 1$ and any $m \geq 1$ the controlled-controlled-u-not (also Petri-Toffoli[21,22]) gate is the anti-linear operator $\mathtt{UT}^{(n,m,1)}$ defined on $\mathcal{H}^{(n+m+1)}$ such that for every element $|x_1,\ldots,x_n\rangle\otimes|y_1,\ldots,y_m\rangle\otimes|z\rangle$ of the computational basis $\mathcal{B}^{(n+m+1)}$:

$$\mathtt{UT}^{(n,m,1)}(|x_1,\ldots,x_n\rangle\otimes|y_1,\ldots,y_m\rangle\otimes|z\rangle) = (-1)^{x_n y_m(1-z)}|x_1,\ldots,x_n\rangle\otimes|y_1,\ldots,y_m\rangle\otimes|x_n y_m \boxplus z\rangle,$$

where $\boxplus$ represents the sum modulo 2.

One can easily show that both $\mathtt{UNot}^{(n)}$ and $\mathtt{UT}^{(n,m,1)}$ are anti-unitary operators.

Consider now the set $\mathfrak{R} = \bigcup_{n=1}^{\infty}\mathcal{H}^{(n)}$ (which contains all quregisters $|\psi\rangle$ "living" in $\mathcal{H}^{(n)}$, for an $n \geq 1$). The gates $\mathtt{UNot}$ and $\mathtt{T}$ can be uniformly defined on this set in the expected way:

$$\begin{array}{ll}\mathtt{UNot}(|\psi\rangle) := \mathtt{UNot}^{(n)}(|\psi\rangle), & \textit{if}\ |\psi\rangle\in\mathcal{H}^{(n)}\\ \mathtt{UT}(|\psi\rangle\otimes|\varphi\rangle\otimes|\chi\rangle) := \mathtt{UT}^{(n,m,1)}(|\psi\rangle\otimes|\varphi\rangle\otimes|\chi\rangle), & \textit{if}\ |\psi\rangle\in\mathcal{H}^{(n)}, |\varphi\rangle\in\mathcal{H}^{(m)}\ \textit{and}\ |\chi\rangle\in\mathcal{H}^{(1)}.\end{array}$$

On this basis, a conjunction $\mathtt{UAnd}$, a disjunction $\mathtt{UOr}$ can be defined for any pair of quregisters $|\psi\rangle$ and $|\varphi\rangle$:

$$\mathtt{UAnd}(|\psi\rangle,|\varphi\rangle) := \mathtt{UT}(|\psi\rangle\otimes|\varphi\rangle\otimes|0\rangle);$$

$$\mathtt{UOr}(|\psi\rangle,|\varphi\rangle) := \mathtt{UNot}(\mathtt{UAnd}(\mathtt{UNot}(|\psi\rangle),\mathtt{UNot}(|\varphi\rangle))).$$

Notice that our definition of $\mathtt{UAnd}$ is reversible and, as such, needs a third ancillary system. Indeed, in this framework, $\mathtt{UAnd}(|\psi\rangle,|\varphi\rangle)$ should be regarded as a metalinguistic abbreviation for $\mathtt{UT}(|\psi\rangle\otimes|\varphi\rangle\otimes|0\rangle)$. A similar observation holds for $\mathtt{UOr}$.

One can easily verify that, when applied to classical bits, $\mathtt{UNot}$, $\mathtt{UAnd}$ and $\mathtt{UOr}$ behave as the standard Boolean truth-functions.

The gates we have considered so far are, in a sense, "semiclassical": a quantum logical behaviour only emerges in the case where our gates are applied to superpositions. When restricted to classical registers, such operators turn out to behave as classical (reversible) truth-functions. We will now consider two important *genuine quantum gates* that transform classical registers (elements of $\mathcal{B}^{(n)}$) into quregisters that are superpositions: the *square root of the u-not* and the *square root of the identity.*

Now, if one wants to describe a possible square root of u-not operator, i.e., some operator $\mathtt{U}$ such that $\mathtt{U} \circ \mathtt{U} = \mathtt{UNot}$, then such an operator cannot be either unitary or anti-unitary. Thus, it is necessary to seek inside non-linear operators. For any $n \geq 1$, the square root of the u-not on $\mathcal{H}^{(n)}$ is the additive operator $\sqrt{\mathtt{UNot}}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$ and for any complex number c :

$$\sqrt{\mathtt{UNot}}^{(n)}(c\,|x_1, \ldots, x_n\rangle) = |x_1, \ldots, x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}}(c\,|x_n\rangle + c^*(-1)^{1-x_n}|1 - x_n\rangle).$$

Obviously, the basic property of $\sqrt{\mathtt{UNot}}^{(n)}$ is the following:

$$\textit{for any } |\psi\rangle \in \mathcal{H}^{(n)},\ \sqrt{\mathtt{UNot}}^{(n)}(\sqrt{\mathtt{UNot}}^{(n)}(|\psi\rangle)) = \mathtt{UNot}^{(n)}(|\psi\rangle).$$

In other words, applying twice the square root of the u-not means negating.

This operator can be expressed in pure operator notation as

$$\sqrt{\mathtt{UNot}}^{(n)} = \frac{1}{\sqrt{2}}\big(I^{(n)} + \mathtt{UNot}^{(n)}\big)$$

Thus, the operator $\sqrt{\mathtt{UNot}}^{(n)}$ describes the transformation which satisfies the *universal condition* to take an *arbitrary* (unknown) qubit and to transform it into an equally superposition of the same qubit and the qubit orthogonal to it. In the Bloch-Poincaré sphere the corresponding points are antipodes. Clearly, this operator is neither homogeneous nor anti-homogeneous.

This result can be inserted in an investigation about non-linear quantum mechanics [23,24,25,26,27,28,29,30,31,32,33]. In these contributions the non-linearity is applied to the case of observables, which in general are not additive ($\exists|\psi\rangle$, $|\phi\rangle$ s.t. $A(|\psi\rangle + |\phi\rangle) \neq A|\psi\rangle + A|\phi\rangle$), but satisfies either the homogeneity ($\forall|\psi\rangle$, $\forall\alpha \in \mathbb{C}$, $A\alpha|\psi\rangle = \alpha A|\psi\rangle$), or the anti-homogeneity ($\forall|\psi\rangle$, $\forall\alpha \in \mathbb{C}$, $A\alpha|\psi\rangle = \alpha^* A|\psi\rangle$), or the absolute homogeneity ($\forall|\psi\rangle$, $\forall\alpha \in \mathbb{C}$, $A\alpha|\psi\rangle = |\alpha|A|\psi\rangle$) (see for instance [34]).

From a logical point of view, $\sqrt{\mathtt{UNot}}^{(n)}$ can be regarded as a "tentative partial negation" (a kind of "half negation") that transforms *precise pieces of information* into *maximally uncertain* ones. For, we have:

$$\mathtt{p}(\sqrt{\mathtt{UNot}}^{(1)}(|1\rangle)) = \frac{1}{2} = \mathtt{p}(\sqrt{\mathtt{UNot}}^{(1)}(|0\rangle)).$$

As expected, the square root of the u-not has no Boolean counterpart. Clearly, there exists no function $f : \{0,1\} \to \{0,1\}$ such that for any $x \in \{0,1\} : f(f(x)) = 1 - x$, since such a function in none of the possible four.

Interestingly enough, $\sqrt{\mathtt{UNot}}$ also does not have a continuous fuzzy counterpart.

Lemma 2: *There is no continuous function* $f : [0,1] \to [0,1]$ *such that for any* $x \in [0,1] : f(f(x)) = 1 - x$[35].

For any $n \geq 1$, the square root of the identity on $\mathcal{H}^{(n)}$ is the additive operator $\sqrt{\texttt{UI}}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$and for any complex number c :

$$\sqrt{\texttt{UI}}^{(n)}(c\,|x_1, \ldots, x_n\rangle) = |x_1, \ldots, x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}}(c\,(-1)^{x_n}|x_n\rangle + c^*|1 - x_n\rangle).$$

The basic property of $\sqrt{\texttt{UI}}^{(n)}$ is the following:

$$\textit{for any } |\psi\rangle \in \mathcal{H}^{(n)},\ \sqrt{\texttt{UI}}^{(n)}(\sqrt{\texttt{UI}}^{(n)}(|\psi\rangle)) = |\psi\rangle.$$

As happens in the case of $\sqrt{\texttt{UNot}}^{(n)}$, also $\sqrt{\texttt{UI}}^{(n)}$ can be regarded as a "tentative partial assertion" (a kind of "half assertion") that transforms *precise pieces of information* into *maximally uncertain* ones. Apparently, one application of $\sqrt{\texttt{UI}}^{(n)}$ to a precise information produces a *maximal disorder*, while two applications of $\sqrt{\texttt{UI}}^{(n)}$ lead back to the initial information.

This operator can be expressed as follows

$$\sqrt{\texttt{UI}}^{(n)} = \frac{1}{\sqrt{2}}\left(I^{(n)} - \texttt{UNot}^{(n)}\right)\left(I^{(n-1)} \otimes \sigma_z\right)$$

where σ_z is the Pauli operator.

Thus, the operator $\sqrt{\texttt{UI}}^{(1)}$ describes the transformation which satisfies the *condition* to take an *arbitrary* (unknown) qubit and to transform it into an equally superposition of a suitable qubit and its orthogonal. Note that if the Bloch-Poincaré representation of the pure state $|\psi\rangle$ is the triple (u_x, u_y, u_z), then the representation of the above pure state $\sigma_z|\psi\rangle$ is the triple $(-u_x, -u_y, u_z)$, i.e., the antipodal with respect to the z axis of the representation of $|\psi\rangle$. Of course, the representation of the pure state $\texttt{UNot}^{(1)}\sigma_z|\psi\rangle$ is the triple $(u_x, u_y, -u_z)$.

As before, also the gates $\sqrt{\texttt{UNot}}$, $\sqrt{\texttt{UI}}$ can be uniformly defined on the set $\mathfrak{R}$ of all quregisters. The gates considered so far can be naturally generalized to qumixes. For any qumix $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$,

$$\mathrm{G}^{(n)}(\rho) = \texttt{G}^{(n)}\rho\,\texttt{G}^{(n)\dagger},$$

where $\texttt{G}^{(n)\dagger}$ is the adjoint of $\texttt{G}^{(n)}$.

When our gates will be applied to density operators, we will use capital letters. Like in the quregister-case, the gates $\texttt{UNOT}$, $\sqrt{\texttt{UNOT}}$, $\sqrt{\texttt{U}\mathbb{I}}$, $\texttt{UAND}$ can be uniformly defined on the set $\mathfrak{D}$ of all qumixes.

Theorem 3:

(i) $\texttt{p}(\texttt{UNOT}(\rho)) = 1 - \texttt{p}(\rho)$;
(ii) $\texttt{p}(\texttt{UAND}(\rho, \sigma)) = \texttt{p}(\rho)\texttt{p}(\sigma)$;
(iii) $\texttt{p}(\sqrt{\texttt{UNOT}}(\texttt{UAND}(\rho, \sigma))) = \frac{1}{2}$;
(iv) $\texttt{p}(\sqrt{\texttt{U}\mathbb{I}}(\texttt{UAND}(\rho, \sigma))) = \frac{1}{2}$.

Proof:

(i) $\mathtt{p}(\mathtt{UNOT}(\rho)) = \mathtt{tr}(P_1^{(n)}\mathtt{UNot}^{(n)}\rho\mathtt{UNot}^{(n)\,\dagger}) = \mathtt{tr}(\mathtt{UNot}^{(n)\,\dagger}P_1^{(n)}\mathtt{UNot}^{(n)}\rho)$
$= \mathtt{tr}(-\mathtt{UNot}^{(n)}P_1^{(n)}\mathtt{UNot}^{(n)}\rho) = \mathtt{tr}(P_0^{(n)}\rho) = \mathtt{tr}((I^{(n)} - P_1^{(n)})\rho) = 1 - \mathtt{p}(\rho)$.

(ii) $\mathtt{p}(\mathtt{UAND}(\rho,\sigma)) = \mathtt{tr}(P_1^{(n+m+1)}\mathtt{UT}^{(n,m,1)}(\rho\otimes\sigma\otimes P_0^{(1)})\mathtt{UT}^{(n,m,1)\,\dagger})$
$= \mathtt{tr}(P_1^{(n)}\rho^* P_1^{(n)} \otimes P_1^{(m)}\sigma^* P_1^{(m)} \otimes P_1^{(1)}P_1^{(1)}) = \mathtt{tr}(P_1^{(n)}\rho^*)\mathtt{tr}(P_1^{(m)}\sigma^*)\mathtt{tr}(P_1^{(1)})$
$= \mathtt{p}(\rho)\mathtt{p}(\sigma)$.

(iii) $\mathtt{p}(\sqrt{\mathtt{UNOT}}(\mathtt{UAND}(\rho,\sigma)))$
$= \mathtt{tr}(P_1^{(n+m+1)}\sqrt{\mathtt{UNot}}^{(n+m+1)}\mathtt{UT}^{(n,m,1)}(\rho\otimes\sigma\otimes P_0^{(1)})\mathtt{UT}^{(n,m,1)\,\dagger}\sqrt{\mathtt{UNot}}^{(n+m+1)\,\dagger})$
$= \mathtt{tr}((I^{(n+m)} - P_1^{(n)} \otimes P_1^{(m)})(\rho^*\otimes\sigma^*)(I^{(n+m)} - P_1^{(n)}\otimes P_1^{(m)}) \otimes P_1^{(1)}\sqrt{\mathtt{UNot}}^{(1)}P_0^{(1)}$
$\sqrt{\mathtt{UNot}}^{(1)\,\dagger} + P_1^{(n)}\rho^* P_1^{(n)} \otimes P_1^{(m)}\sigma^* P_1^{(m)} \otimes P_1^{(1)}\sqrt{\mathtt{UNot}}^{(1)}P_1^{(1)}\sqrt{\mathtt{UNot}}^{(1)\,\dagger})$
$= \mathtt{tr}((I^{(n+m)} - P_1^{(n)}\otimes P_1^{(m)})(\rho^*\otimes\sigma^*))\mathtt{tr}(P_1^{(1)}\sqrt{\mathtt{UNot}}^{(1)}P_0^{(1)}\sqrt{\mathtt{UNot}}^{(1)\,\dagger})$
$+ \mathtt{tr}(P_1^{(n)}\rho^*)\mathtt{tr}(P_1^{(m)}\sigma^*)\mathtt{tr}(P_1^{(1)}\sqrt{\mathtt{UNot}}^{(1)}P_1^{(1)}\sqrt{\mathtt{UNot}}^{(1)\,\dagger})$
$= (1 - \mathtt{p}(\rho^*)\mathtt{p}(\sigma^*))\mathtt{p}(\sqrt{\mathtt{UNOT}}(P_0^{(1)})) + \mathtt{p}(\rho^*)\mathtt{p}(\sigma^*)\mathtt{p}(\sqrt{\mathtt{UNOT}}(P_1^{(1)})) = \frac{1}{2}$.

(iv) Similarly. □

An interesting preorder relation can be defined on the set $\mathfrak{D}$ of all qumixes.

Definition 4: Preorder.
$\rho \preceq \sigma$ iff the following conditions hold:

(i) $\mathtt{p}(\rho) \leq \mathtt{p}(\sigma)$;
(ii) $\mathtt{p}(\sqrt{\mathtt{UNOT}}(\sigma)) \leq \mathtt{p}(\sqrt{\mathtt{UNOT}}(\rho))$;
(iii) $\mathtt{p}(\sqrt{\mathtt{U}\,\mathbb{I}}(\rho)) \leq \mathtt{p}(\sqrt{\mathtt{U}\,\mathbb{I}}(\sigma))$.

One immediately shows that $\preceq$ is reflexive and transitive, but not antisymmetric. From an intuitive point of view, $\rho \preceq \sigma$ means that the information σ is "closer to the truth" than the information ρ.

An equivalence relation can be then defined on $\mathfrak{D}$: $\sigma \equiv \tau$ iff $\sigma \preceq \tau$ and $\tau \preceq \sigma$.

One can prove that $\equiv$ is a congruence relation with respect to the operations $\mathtt{UAND}$, $\sqrt{\mathtt{UNOT}}$, $\sqrt{\mathtt{U}\,\mathbb{I}}$. On this basis, we introduce two structures: the *quantum computational structure* and its quotient, the *contracted quantum computational structure*.

Definition 5: The quantum computational structure.
The structure

$$(\mathfrak{D}, \mathtt{UAND}, \sqrt{\mathtt{UNOT}}, \sqrt{\mathtt{U}\,\mathbb{I}}, P_0^{(1)}, P_1^{(1)}, \rho_{1/2}^{(1)}),$$

where $P_0^{(1)}, P_1^{(1)}, \rho_{1/2}^{(1)}$ represent respectively the *Falsity*, the *Truth* and the *indeterminate truth-value*, is called the *quantum computational structure*. The structure

$$([\mathfrak{D}]_\equiv, \mathtt{UAND}, \sqrt{\mathtt{UNOT}}, \sqrt{\mathtt{U}\,\mathbb{I}}, [P_0^{(1)}]_\equiv, [P_1^{(1)}]_\equiv, [\rho_{1/2}^{(1)}]_\equiv),$$

where the operations $\mathtt{UAND}$, $\sqrt{\mathtt{UNOT}}$, $\sqrt{\mathtt{U}\,\mathbb{I}}$ are defined on the equivalence classes belonging to the quotient $[\mathfrak{D}]_\equiv$ in the expected way, is called the *contracted quantum computational structure*.

4. Poincaré Sphere Considerations

Let $\vec{u} = (u_x, u_y, u_z)$ be a *fixed* vector on the unit surface $S_1(R^3)$, with polar representation $\vec{u} = (\sin\vartheta\,\cos\varphi,\,\sin\vartheta\,\sin\varphi,\,\cos\vartheta)$, and let us consider the orthonormal basis of $\mathbb{C}^2$

$$|{\uparrow_{\vec{u}}}\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix} \qquad |{\downarrow_{\vec{u}}}\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \\ -e^{i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \end{pmatrix}$$

These are eigenvectors corresponding respectively to the eigenvalues $+1$ and -1 of the spin 1/2 observable along the $\vec{u}$ direction

$$\sigma_{\vec{u}} = \sigma(\vartheta,\varphi) = u_x\,\sigma_x + u_y\,\sigma_y + u_z\,\sigma_z = \begin{pmatrix} \cos\vartheta & e^{-i\varphi}\sin\vartheta \\ e^{i\varphi}\sin\vartheta & -\cos\vartheta \end{pmatrix}$$

Denoting the first eigenvector also as $|\vartheta,\,\varphi\rangle := |{\uparrow_{\vec{u}}}\rangle$ then trivially $|{\downarrow_{\vec{u}}}\rangle = |\vartheta - \pi,\,\varphi\rangle$ i.e., the antipodal of the original unit vector $\vec{u}$. Thus, as to the 1/2 spin interpretation we have that $|{\downarrow_{\vec{u}}}\rangle = |{\uparrow_{-\vec{u}}}\rangle$, the spin down eigenvector along $\vec{u}$ coincides with the spin up eigenvector along its antipodal $-\vec{u}$.

In this context, the quantum realization of the classical Not gate is given by the unitary operator depending from the polar angles ϑ,φ:

$$\texttt{Not}(\vartheta,\,\varphi) := \begin{pmatrix} \sin\vartheta & -e^{-i\varphi}\cos\vartheta \\ -e^{i\varphi}\cos\vartheta & -\sin\vartheta \end{pmatrix}$$

Indeed, if one set $|0\rangle := |{\uparrow_{\vec{u}}}\rangle$ and $|1\rangle := |{\downarrow_{\vec{u}}}\rangle$, then this operator realizes the transitions required to the quantum Not gate. In particular, we have $\texttt{Not}(0,0) = -\sigma_x$.

From the unitary point of view, the operator more similar to UNot is the following

$$\texttt{Not}_1(\varphi) := \begin{pmatrix} 0 & e^{-i\varphi} \\ -e^{i\varphi} & 0 \end{pmatrix}$$

In particular, we have $\texttt{Not}_1(0) = i\sigma_y$ which applied to a generic vector $|\vartheta,\varphi\rangle$ produces the transition $|\vartheta,\varphi\rangle \rightarrow |\vartheta-\pi,-\varphi\rangle$. The outgoing vector is the antipodal of the incoming one, not with respect to the origin of the space $\mathbb{R}^3$ in which the Poincaré sphere is embedded, but with respect to its y axis. The inner product $<\vartheta,\varphi|\vartheta-\pi,-\varphi>= i\sin\varphi\sin\vartheta$ is trivially 0 (orthogonality) under the condition $\varphi = 0$, i.e., for any pure state whose Poincaré surface representation is on the xz plane. Setting $\vartheta = 2\alpha$, the vector $|\alpha\rangle = |2\alpha,0\rangle$ describes the quantum (pure) state of light linear polarized along direction α with respect to the reference axis $\mathbf{A}_1$ of the analyzer Nicol prism which constitute the preparation part. In this interpretation the unitary realization $i\sigma_y$ of the classical Not gate performs an antipodal transformation of *all* possible pure states of linearly polarizations light. These considerations can be extended to the case of states obtained as mixture of linear polarized pure states.

Of course, they are not unitary descriptions of a gate which satisfies the universal orthogonality condition, but it is possible to unitary flip an arbitrary state whose Poincaré representation is on a known plane. Indeed, let us consider a vector $(\sin\vartheta\,\cos\varphi,\,\sin\vartheta\,\sin\varphi,\,\cos\vartheta)$ and suppose the Poincaré representation of an arbitrary state $\rho(r,\omega)$ lies on a plane determined by the orthogonal vector $r(-\cos\vartheta\,\cos\varphi\,\sin\kappa+\sin\varphi\,\cos\kappa,\,-\cos\vartheta\,\sin\varphi\,\sin\kappa-\cos\varphi\,\cos\kappa,\,\sin\vartheta\,\sin\kappa)$:

$$\rho(r,\omega)=\frac{1}{2}\begin{pmatrix}1+r(\cos\omega\cos\vartheta-\sin\omega\sin\vartheta\cos\kappa) & re^{-i\varphi}(\cos\omega\sin\vartheta+\sin\omega(\cos\vartheta\cos\kappa+i\sin\kappa))\\ re^{i\varphi}(\cos\omega\sin\vartheta+\sin\omega(\cos\vartheta\cos\kappa-i\sin\kappa)) & 1-r(\cos\omega\cos\vartheta+\sin\omega\sin\vartheta\cos\kappa)\end{pmatrix}$$

where $\omega\in[0,2\pi)$ and $r\in[0,1]$. The unitary operator that realizes the partial universal flipping condition is the following:

$$\mathtt{Not_1}(\vartheta,\,\varphi,\,\kappa)=\begin{pmatrix}-i\sin\vartheta\sin\kappa & e^{-i\varphi}(\cos\kappa+i\cos\vartheta\sin\kappa)\\ -e^{i\varphi}(\cos\kappa-i\cos\vartheta\sin\kappa) & i\sin\vartheta\sin\kappa\end{pmatrix}$$

The corresponding unitary realization of the square root of the u-not and of the square root of the identity is the following:

$$\sqrt{\mathtt{Not_1}}(\vartheta,\,\varphi,\,\kappa)=\frac{1}{\sqrt{2}}\begin{pmatrix}1-i\sin\vartheta\sin\kappa & e^{-i\varphi}(\cos\kappa+i\cos\vartheta\sin\kappa)\\ -e^{i\varphi}(\cos\kappa-i\cos\vartheta\sin\kappa) & 1+i\sin\vartheta\sin\kappa\end{pmatrix}$$

$$\sqrt{\mathtt{I}}(\vartheta,\,\varphi,\,\kappa)=\frac{1}{\sqrt{2}}\begin{pmatrix}\cos\vartheta-\sin\vartheta\cos\kappa & e^{-i\varphi}(\cos\vartheta\cos\kappa+\sin\vartheta+i\sin\kappa)\\ e^{i\varphi}(\cos\vartheta\cos\kappa+\sin\vartheta-i\sin\kappa) & -\cos\vartheta+\sin\vartheta\cos\kappa\end{pmatrix}$$

As particular cases, one can find the Hadamard transformations for an unknown qubit chosen either from the polar or equatorial great circles[36] as well as the unitary operators for the square root of the not and of the identity[35]. In particular, for states of linearly polarizations light, we have

$$\sqrt{\mathtt{Not_1}}(0,0,0)=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\quad \sqrt{\mathtt{Not_1}}(0,0,\pi)=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}$$

$$\sqrt{\mathtt{I}}(0,0,0)=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\quad \sqrt{\mathtt{I}}(0,0,\pi)=-\frac{1}{\sqrt{2}}\begin{pmatrix}-1&1\\1&1\end{pmatrix}$$

and for states whose Poincaré representation is on the yz plane, we obtain

$$\sqrt{\mathtt{Not_1}}(0,0,-\frac{\pi}{2})=\frac{e^{-i\frac{\pi}{4}}}{2}\begin{pmatrix}1+i&1-i\\1-i&1+i\end{pmatrix}\quad \sqrt{\mathtt{Not_1}}(0,0,\frac{\pi}{2})=\frac{e^{i\frac{\pi}{4}}}{2}\begin{pmatrix}1-i&1+i\\1+i&1-i\end{pmatrix}$$

$$\sqrt{\mathtt{I}}(0,0,-\frac{\pi}{2})=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-i\\i&-1\end{pmatrix}\quad \sqrt{\mathtt{I}}(0,0,\frac{\pi}{2})=\frac{1}{\sqrt{2}}\begin{pmatrix}1&i\\-i&-1\end{pmatrix}$$

Moreover, by using the unitary operators $\sqrt{\mathtt{Not_1}}(\vartheta,\varphi,0)$ and $\sqrt{\mathtt{I}}(\vartheta,\varphi,0)$ one can realize the following transformations of the vector $\vec{v}=r(\sin\vartheta\cos\varphi,\sin\vartheta\sin\varphi,\cos\vartheta)$ of the Poincaré sphere:

$$(v_x,v_y,v_z)\rightarrow^{\sqrt{\mathtt{Not_1}}}\begin{cases}(-v_z,0,0) & \text{if } v_x=v_y=0\\ \frac{\text{sign}(v_x)}{\sqrt{v_x^2+v_y^2}}(-v_zv_x,-v_zv_y,v_x^2+v_y^2) & \text{otherwise}\end{cases}$$

$$(v_x,v_y,v_z)\rightarrow^{\sqrt{\mathtt{I}}}\begin{cases}(v_z,0,0) & \text{if } v_x=v_y=0\\ \frac{\text{sign}(v_x)}{\sqrt{v_x^2+v_y^2}}(v_zv_x,v_zv_y,v_x^2+v_y^2) & \text{otherwise}\end{cases}$$

Note that the probability of the qumix ρ univocally associated to $\vec{v}$ is $\mathtt{p}(\rho)=\frac{1-v_z}{2}$.

5. Quantum Trees

We consider a *minimal quantum propositional language* $\mathcal{L}$ that contains a privileged atomic formula $\mathbf{f}$ (whose intended interpretation is the *Falsity*) and the following primitive connectives: the *square root of the negation* $\sqrt{\neg}$, the *square root of the identity* $\sqrt{id}$, a ternary *conjunction* $\bigwedge$ (which corresponds to the Petri-Toffoli gate). In this framework, the negation $\neg\alpha$ and the usual conjunction $\alpha\wedge\beta$ are dealt with as metalinguistic abbreviation for $\sqrt{\neg}\sqrt{\neg}\alpha$ and the ternary conjunction $\bigwedge(\alpha,\beta,\mathbf{f})$ of $\mathcal{L}$ respectively. We will use the following metavariables: $\mathbf{q},\mathbf{r},\ldots$ for atomic formulas and $\alpha,\beta,\ldots$ for formulas. The connective disjunction ($\vee$) is supposed to be defined via the *de Morgan law* ($\alpha\vee\beta := \neg(\neg\alpha\wedge\neg\beta)$).

Any formula α of $\mathcal{L}$ can be naturally decomposed into its parts, giving rise to the *syntactical tree* of α and describes a *quantum circuit* that can be applied to an input, represented by a qumix living in a Hilbert space $\mathcal{H}^{(At(\alpha))}$ whose dimension depends on the number $At(\alpha)$ of occurrences of atomic formulas in α. The syntactical tree of α uniquely determines the *qubit tree*, a sequence of gates $(G_1^\alpha,\ldots,G^\alpha_{Height(\alpha)-1})$ that are all defined on the semantic space of α, where $Height(\alpha)$ is the number of levels. Each j-th node of the i-th level $Level_i^j(\alpha)$ can be naturally associated to an operator Op_i^j, according to the following *operator-rule*:

$$Op_i^j := \begin{cases} \mathtt{I}^{(1)} \text{ if } Level_i^j(\alpha) \text{ is an atomic formula;} \\ \sqrt{\mathtt{UNot}}^{(r)} \text{ if } Level_i^j(\alpha) = \sqrt{\neg}\beta \text{ and } At(\beta)=r; \\ \sqrt{\mathtt{UI}}^{(r)} \text{ if } Level_i^j(\alpha) = \sqrt{id}\,\beta \text{ and } At(\beta)=r; \\ \mathtt{UT}^{(r,s,1)} \text{ if } Level_i^j(\alpha) = \bigwedge(\beta,\gamma,\mathbf{f}),\ At(\beta)=r \text{ and } At(\gamma)=s. \end{cases}$$

On this basis, one can associate a gate G_i^α to each $Level_i(\alpha)$ (such that $1\le i < Height(\alpha)$): $G_i^\alpha := \bigotimes_{j=1}^{|Level_i(\alpha)|} Op_i^j$, where $|Level_i(\alpha)|$ is the length of the sequence $Level_i(\alpha)$. The notion of qubit tree can be naturally generalized to qumixes defining the following sequence of functions: $\mathrm{G}_i^\alpha(\rho) = G_i^\alpha\,\rho\, G_i^{\alpha\dagger}$.

Consider now a formula α and let $(\mathrm{G}_1^\alpha,\ldots,\mathrm{G}_{k-1}^\alpha)$ be the qumix tree of α. Any choice of a qumix ρ in $\mathcal{H}^{(At(\alpha))}$ determines a sequence $(\rho_k,\ldots,\rho_1)$ of qumixes of $\mathcal{H}^{(At(\alpha))}$, where $\rho_k=\rho$ and $\rho_{i-1} = \mathrm{G}_{i-1}^\alpha(\rho_i)$ with $1<i\le k$. The qumix ρ_k can be regarded as a possible *input-information* concerning the atomic parts of α, while ρ_1 represents the *output-information* about α, given the input-information ρ_k. Each ρ_i corresponds to the *information* about $Level_i(\alpha)$, given the input-information ρ_k. Apparently, all qumix gates are bijections and represent reversible information processes. Thus, any choice of a qumix ρ_1 determines a sequence $(\rho_k,\ldots,\rho_1)$ of qumixes and in particular the input-information ρ_k, given the output-information about α.

How to determine an information about the parts of α under a given state ρ_1? It is natural to consider $red^j(\rho_i)$, the *reduced state* of ρ_i with respect to the j-th subsystem. From a semantic point of view, this state can be regarded as a *contextual information* about the subformula of α occurring at the j-th position of the i-th level under the *global information* ρ_1.

An interesting situation arises when the qumix ρ_1 is an *entangled* pure state, representing a global information about the atomic parts of α. As an example, consider the formula $\alpha = \neg \bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$ (which represents an example of the *non-contradiction principle* formalized in the quantum propositional language). The global information might be the following entangled state:

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}|110\rangle + \frac{1}{\sqrt{2}}|000\rangle \;\rightsquigarrow\; Level_4(\alpha) = (\mathbf{q}, \mathbf{q}, \mathbf{f})$$

The reduced states turn out to be the following:

$$red^1(P_{|\psi_4\rangle}) = \frac{1}{2}P_0^{(1)} + \frac{1}{2}P_1^{(1)} = red^2(P_{|\psi_4\rangle}),\; red^3(P_{|\psi_4\rangle}) = P_0^{(1)}$$

Hence, the contextual information about both occurrences of $\mathbf{q}$ is the (proper) mixture $\frac{1}{2}P_0^{(1)} + \frac{1}{2}P_1^{(1)}$. At the same time, the contextual information about $\mathbf{f}$ is projection $P_0^{(1)}$ (representing the *Falsity*). Hence, the information about the *whole* is more precise than the information about the *parts*.

6. Compositional and holistic quantum computational semantics

Two kinds of quantum computational semantics have been investigated[35]: a *compositional* and a *holistic* semantics. In the compositional semantics, the meaning of a molecular formula is determined by the meanings of its parts (like in classical logic). In this framework, the input-information about the top level of the syntactical tree of a formula α is always associated to a factorized state $\rho_1 \otimes \ldots \otimes \rho_{At(\alpha)}$, where $\rho_1, \ldots, \rho_{At(\alpha)}$ are qumixes of $\mathbb{C}^2$. As a consequence, the meaning of a molecular α cannot be a pure state, if the meanings of some atomic parts of α are proper mixtures.

We can now introduce the basic definitions of the holistic semantics. The main concept is the notion of *holistic model*: a function `UHol` that assigns to any formula α of the quantum propositional language a *global meaning*, which cannot be generally inferred from the meanings of the parts of α. Of course, the function `UHol` shall respect the logical form of α like in the standard semantic approaches.

Definition 6: Holistic model.
An *holistic model* is a map `UHol` that associates a qumix to any formula α of $\mathcal{L}$, satisfying the following conditions:

(1) $\mathtt{UHol}(\alpha)$ is a density operator in $\mathcal{H}^{(At(\alpha))}$;
(2) Let $Level_i^j(\alpha)$ and $Level_h^k(\alpha)$ be two nodes of the syntactical tree of α, ρ_i and ρ_h the corresponding information about $Level_i(\alpha)$ and $Level_h(\alpha)$ determined by $\rho_1 = \mathtt{UHol}(\alpha)$ and by the qumix tree. Then,
(2.1) if $Level_i^j(\alpha) = \mathbf{f}$, then $red^j(\rho_i) = P_0$;
(2.2) if $Level_i^j(\alpha)$ and $Level_h^k(\alpha)$ are two occurrences in α of the same formula, then $red^j(\rho_i) = red^k(\rho_h)$.

Conditions (2.1) and (2.2) guarantee that $\texttt{UHol}(\alpha)$ is well behaved: the contextual meaning of $\mathbf{f}$ is always the *Falsity*, while two different occurrences in α of the same atomic formula have the same contextual meaning. Given a formula α, $\texttt{UHol}$ determines the *contextual meaning* of any occurrence of a subformula β in α with respect to the context $\texttt{UHol}(\alpha)$. Let β be a subformula of α occurring at the j-th position of the i-th level of the syntactical tree of α. Let $(\rho_k, \ldots, \rho_1)$ be the sequence of qumixes determined by $\rho_1 = \texttt{UHol}(\alpha)$ and by the qumix tree of α. The contextual meaning of β with respect to the context $\texttt{UHol}(\alpha)$ is defined as follows:

$$\texttt{UHol}^{\alpha}(\beta) = red^{j}(\rho_i).$$

Apparently, $\texttt{UHol}^{\alpha}$ is a partial function that only assigns meanings to the subformulas of α. Given a formula α, we will call the partial function $\texttt{UHol}^{\alpha}$ a *contextual holistic model* of the language. Suppose now that β is a subformula of two different formulas γ and δ. Generally, we have $\texttt{UHol}^{\gamma}(\beta) \neq \texttt{UHol}^{\delta}(\beta)$. In other words, formulas may receive different contextual meanings in different contexts!

From the computational point of view, the unique possibility to determine the contextual meanings of the subformulas of a given formula is by non-linear and anti-unitary operators.

In this framework, compositional models can be described as limit-cases of holistic models.

Definition 7: Compositional model.
Let $(\rho_k, \ldots, \rho_1)$ be the sequence of qumixes determined by $\texttt{UHol}(\alpha)$ and by the qumix tree of α. A model $\texttt{UHol}$ is called *compositional* iff the following condition is satisfied for any formula α: $\rho_k = \texttt{UHol}(\mathbf{q}_1) \otimes \ldots \otimes \texttt{UHol}(\mathbf{q}_t)$, where $\mathbf{q_1}, \ldots, \mathbf{q_t}$ are the atomic formulas occurring in α.

As expected, unlike holistic models, compositional models are context-independent. Suppose that β is a subformula of two different formulas γ and δ. We have $\texttt{UHol}^{\gamma}(\beta) = \texttt{UHol}^{\delta}(\beta) = \texttt{UHol}(\beta)$.

Moreover, in the compositional context one can use an exact unitary strategy in order to compute the output-information about α, given the input-information about the atomic parts of α, by setting the rotation axis of the devices which implement the gates. In particular, for arbitrary states of linearly polarizations light, one can use the unitary operators $\sqrt{\texttt{Not}_1}(0,0,0)$, $\sqrt{\texttt{I}}(0,0,0)$ and the usual $\texttt{CCNot}$[35].

The notion of logical consequence in the framework of the holistic quantum computational semantics represents a reasonable variant of the standard notions of logical consequence. Let us first define the notion of *consequence in a given contextual model.*

Definition 8: Consequence in a given contextual model and Logical consequence.
A formula β is a consequence of a formula α in a given contextual model $\texttt{UHol}^{\gamma}$ $\alpha \models_{\texttt{UHol}^{\gamma}} \beta$) iff α and β are subformulas of γ and $\texttt{UHol}^{\gamma}(\alpha) \preceq \texttt{UHol}^{\gamma}(\beta)$ (where $\preceq$

is the preorder relation defined in 4). A formula β is a consequence of a formula α in the holistic semantics iff for any formula γ such that α and β are subformulas of γ and for any $\mathtt{UHol}$, $\alpha \models_{\mathtt{UHol}^\gamma} \beta$.

We call **UHQCL** the logic that is semantically characterized by the logical consequence relation and **UCQCL** the logic that is semantically characterized by the class of all compositional quantum computational models. One is dealing with a nonstandard forms of *unsharp quantum logic*, where the noncontradiction principle breaks down ($\not\models \neg(\alpha \wedge \neg\alpha)$), while conjunction is not idempotent ($\alpha \not\models \alpha \wedge \alpha$). Interestingly enough, distributivity is here violated "in the wrong direction" with respect to orthodox quantum logic. For, $\alpha \wedge (\beta \vee \gamma) \models (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$, but not the other way around!

We will now prove that **UHQCL** is strictly weaker than **UCQCL**.

Theorem 9: $\alpha \models_{UHQCL} \beta \curvearrowright \alpha \models_{UCQCL} \beta$, but not the other way around.

Proof: $\alpha \models_{UHQCL} \beta \curvearrowright \alpha \models_{UCQCL} \beta$, because compositional models are special examples of holistic models.

$\alpha \models_{UHQCL} \beta \not\curvearrowright \alpha \models_{UCQCL} \beta$, Consider the following counterexample. Let $\alpha = \bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f}), \beta = \bigwedge(\sqrt{id}\mathbf{f}, \sqrt{id}\mathbf{f}, \mathbf{f}), \gamma = \bigwedge(\alpha, \beta, \mathbf{f})$. We define a holistic $\mathtt{UHol}$ that assigns to the top level of the syntactical tree of γ a pure state, whose first component is entangled.

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \otimes |00000\rangle \;\leftrightsquigarrow\; Level_4(\gamma) = (\mathbf{q}, \mathbf{q}, \mathbf{f}, \mathbf{f}, \mathbf{f}, \mathbf{f}, \mathbf{f})$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle) \otimes \frac{1}{2}(|000\rangle + |010\rangle + |100\rangle + |110\rangle) \otimes |0\rangle \;\leftrightsquigarrow\; Level_3(\gamma) = (\mathbf{q}, \neg\mathbf{q}, \mathbf{f}, \sqrt{id}\mathbf{f}, \sqrt{id}\mathbf{f}, \mathbf{f}, \mathbf{f})$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \otimes \frac{1}{2}(|000\rangle + |010\rangle + |100\rangle + |111\rangle) \otimes |0\rangle \;\leftrightsquigarrow\; Level_2(\gamma) = (\alpha, \beta, \mathbf{f})$$

$$|\psi_1\rangle \;\leftrightsquigarrow\; Level_1(\gamma) = \bigwedge(\alpha, \beta, \mathbf{f})$$

Hence, $\mathtt{p}(\mathtt{UHol}^\gamma(\alpha)) = \frac{1}{2}, \mathtt{p}(\mathtt{UHol}^\gamma(\beta)) = \frac{1}{4}$. Consequently, $\alpha \not\models_{\mathtt{UHol}^\gamma} \beta$. At the same time, one can easily show that for any compositional model $\mathtt{UHol}$, $\alpha \models_{\mathtt{UHol}} \beta$. □

The counterexample clearly shows how entanglement is responsible for the creation of somewhat pathological holistic models in comparison with the compositional semantics.

In Dalla Chiara et al.[35] the following unitary operators were used instead of $\sqrt{\mathtt{UNot}}$, $\sqrt{\mathtt{UI}}$ and $\mathtt{UT}$:
For any $n \geq 1$, the square root of the not on $\mathcal{H}^{(n)}$ is the linear operator $\sqrt{\mathtt{Not}}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$:

$$\sqrt{\mathtt{Not}}^{(n)}(|x_1, \ldots, x_n\rangle) = |x_1, \ldots, x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}}((1+i)|x_n\rangle + (1-i)|1-x_n\rangle).$$

For any $n \geq 1$, the square root of the identity on $\mathcal{H}^{(n)}$ is the linear operator $\sqrt{\mathtt{I}}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$:

$$\sqrt{\mathtt{I}}^{(n)}(|x_1, \ldots, x_n\rangle) = |x_1, \ldots, x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}}((-1)^{x_n}|x_n\rangle + |1 - x_n\rangle).$$

For any $n \geq 1$ and any $m \geq 1$ the controlled-controlled-not gate is the linear operator $\mathtt{T}^{(n,m,1)}$ defined on $\mathcal{H}^{(n+m+1)}$ such that for every element $|x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle \otimes |z\rangle$ of the computational basis $\mathcal{B}^{(n+m+1)}$:

$$\mathtt{T}^{(n,m,1)}(|x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle \otimes |z\rangle) = |x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle \otimes |x_n y_m \boxplus z\rangle,$$

where $\boxplus$ represents the sum modulo 2.

Lemma 10: *Let* UHol *be a model and let* Hol *be a model with the above unitary operators*[35] *such that for any atomic sentence* **q**: $\mathtt{UHol}(\mathbf{q}) = \mathtt{Hol}(\mathbf{q})$. *Then, for any sentence* α *of* $\mathcal{L}$:

$$\mathtt{p}(\mathtt{UHol}(\alpha)) = \mathtt{p}(\mathtt{Hol}(\alpha)).$$

Proof: The proof is by induction on the length (i.e. the number of connectives) of α. □

Corollary 11:
(i) For any model UHol*, there exists a model* Hol *such that for any* α *of* $\mathcal{L}$: $\mathtt{p}(\mathtt{UHol}(\alpha)) = \mathtt{p}(\mathtt{Hol}(\alpha))$.
(ii) For any model Hol *there exists a model* UHol *such that for any* α *of* $\mathcal{L}$: $\mathtt{p}(\mathtt{Hol}(\alpha)) = \mathtt{p}(\mathtt{UHol}(\alpha))$.

Theorem 12: $\alpha \models_{UHQCL} \beta$ iff $\alpha \models_{HQCL} \beta$

Proof: The theorem is a direct consequence of the above Corollary. □

Hence, each **UHQCL** and its corresponding **HQCL** with the above unitary operators[35] are the same logic.

7. Acknowledges

We wish to thank F. De Martini, G. Cattaneo, M.L. dalla Chiara and R. Giuntini for very useful suggestions and discussions this paper originated from.

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