Rapporto n. 9 2006

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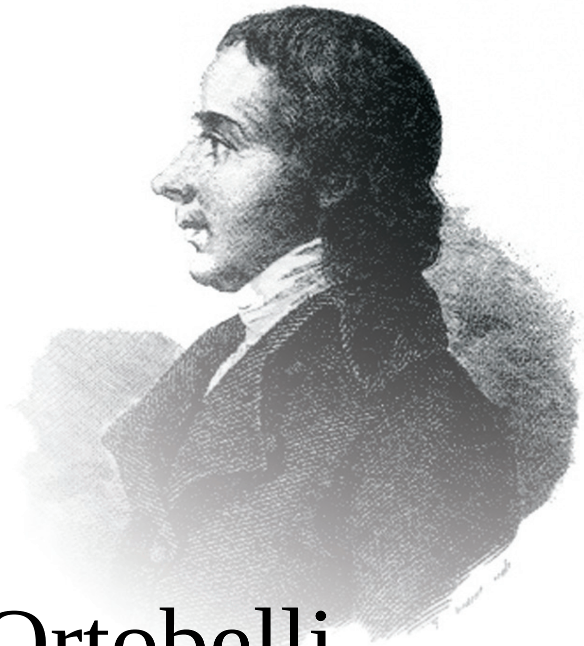

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Serie Ricerca

Dipartimento
di Matematica, Statistica,
Informatica e Applicazioni
“Lorenzo Mascheroni”

UNIVERSITÀ DEGLI STUDI DI BERGAMO

OPTION PRICING WITH NONPARAMETRIC MARKOVIAN TREES

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Abstract: In this paper we propose to use a Markov chain in order to price contingent claims. In particular, we describe a non parametric markovian approach to price American and European options. First, we discuss the risk neutral valuation of the non parametric approach. Secondly, we examine the problems of the computational complexity and of the stability with respect to the number of the states of the Markov chain. Finally, we propose an *ex post* comparison between the Markovian model and the Black and Scholes one.
Key words: Markov Chain, Risk Neutral Valuation, state dependent valuation, state independent price.

1 Introduction

After the Black-Scholes option pricing model many studies have attempted to cope with the different contradictions emerged in the empirical tests of this model. While many researchers indicate the lognormal distribution hypothesis of the financial return as not too satisfying hypothesis, many others find the constant volatility of the financial price as the great weak point of the model. There exist a wide literature on the improvements performed on this pioneer model. Many efforts have been destined to make stochastic the volatility and others to make the distributional hypothesis more realistic on the price process.

This paper shows a simple non-parametric way to model the contingent claims without assessing a distributional form a priori for the asset price and without the necessity of a valuation of parameters such as the volatility. The methodology presented enters in the class of the Markovian option pricing models. Among markovian models we essentially distinguish two categories: parametric models (see, among others, Duan and Simonato (2001); Duan, et al. (2003) and among semi-Markovian models see Limnios, Oprisan (2001), Blasi et al.(2003), D'Amico et al. (2005)), and nonparametric models. In the first category the Markovian hypothesis is used for diffusive models of the underlying returns. In the second category of models only the historical series are used to estimate the option prices. Thus nonparametric models have the main advantage in their capacity of adapting to the underlying return distributions. This paper deals with a nonparametric markovian model, that differently by the nonparametric derivatives models deal in literature (see, among others, Hutchinson, et al.(1994), Ait-Sahalia (1996, 1998), Stutzer (1996)),. explicates directly the markovian hypothesis assuming that the time evolution of the returns is described by a Markov chain. Thus our nonparametric approach is different respect to those based on the parameter estimations, those that use the Markov Chains to approximate such parameters and those that use neural network or only the historical series to approximate the option prices. With this model we are able to price American/ European and path dependent options and using the Markov chain properties we are able to simplify the computation of the derivative prices in reasonable times. Generally the resulting prices are different from those obtained with the Black and Scholes model even if this difference is strongly reduced when we use simulated log-returns. Therefore the ductility of the model suggests that one of the main applications should be for energy derivatives which are strongly influenced by the seasonality of the price.

The paper first presents the model discussing the risk neutral valuation when we consider either state dependent prices or state independent prices. Secondly we discuss the computational complexity of the algorithm and the stability of the prices with respect to the number of the states of the Markov chain. Finally we examine the empirical differences between the Black and Scholes model and the nonparametric markovian one.

2 Nonparametric Markovian Trees

Let us assume the time evolution of underlying asset return follows a Markov chain with N states. Doing so, we want to construct a multinomial recombining tree of the asset price with more degrees of freedom than the classical binomial tree. In particular, we assume that the gross return z has support on the interval $(\min_t z_t; \max_t z_t)$, where $z_t = S_{t+1}/S_t$ is the *t-th* return observation and S_t is the value of the security at time t. By convention, through all the paper, we count the states beginning from that with the greatest value. Then we build the transition matrix as follows:

1. we share in N intervals $I_i = (a_i; a_{i-1})$ (small enough) the return support $(\min_t z_t; \max_t z_t)$ where $a_0 = \max_t z_t$, $a_i = u^i \max_t z_t$, $u = \sqrt[N]{\min_t z_t / \max_t z_t}$ and $i = 1, ..., N$;
2. we assume that inside the interval I_i the return is given by the geometric mean of the extremes, i.e., $z^{(i)} := \sqrt{a_{i-1}a_i} = u^{i-0.5} \max_t z_t$;
3. we build the transition matrix $P_s = \left[p_{i,j;(s)}\right]_{1\leq i,j \leq N}$ where the probability $p_{i,j;(s)}$ points out the probability valued at time s to transit from the state $z^{(i)}$ to the state $z^{(j)}$ conditional of being in the *i-th* state.

Since the tree recombines at each step, the number of nodes increases linearly with the number of the time steps. For this reason we can control and limit the computational complexity. Thus, after $k\Delta t$ intervals of time we have $(N-1)k+1$ nodes (i.e., the multinomial tree growths linearly with the time). Starting to count from the highest node, after k steps the j-th node has:

- value in the interval: $I_j^{(k)} = ((u^{j+\frac{k-1}{2}}) \left(\max_t z_t\right)^k, (u^{j+\frac{k-3}{2}}) \left(\max_t z_t\right)^k);$
- gross return: $z_k^{(j)} := u^{j+\frac{k}{2}-1} \left(\max_t z_t\right)^k;$
- stock price: $S_0 z_k^{(j)} \quad j = 1, \ldots, (N-1)k+1.$

Next we consider an homogeneous Markov chain with transition matrix $P = [p_{i,j}]_{1\leq i,j\leq N}$. In this case we assume the maximum likelihood estimate of probability p_{ij} which is simply given by the ratio of the count of the appropriate cells, i.e., $p_{ij} \simeq \frac{n_{ij}}{n_i}$ where n_{ij} is the number of times the return transit from the state *i-th* to the state *j-th* and $n_i = \sum_{k=1}^{N} n_{ik}$ is the number of times the return is in the *i-th* state. However, we could consider a non homogeneous Markov chain taking into account the behavior of the prices in different periods. Therefore, we can model and value differently the transition matrixes when the underlined prices change its behavior during the maturity period. For example, if we have a seasonal price, like those observed in the energy markets, we can compute different transition matrixes in order to consider the week-end effect and/or the season effect.

Once we get the transition matrix, we have to find adequate answers to the following three issues that should be object of the next sections:

a) how one obtains the risk neutral valuation starting from the market-based transition probability;

b) what we can say about the valuation procedure for European and American contingent claims and the main greek letters;

c) discuss the stability of the solutions with respect to the number of the states.

3 Risk neutral valuation

Let us assume the dividend of one unity of wealth invested in a given asset during the period $[t_0, t]$ is given by $\exp(q(t-t_0))$, where q describes the intensity of the dividend and suppose $\exp(-r(t-t_0))$ is one unity of wealth discounted at time t_0 where we assume that r defines a fixed short term interest rate. With markovian trees we can generally distinguish two possible risk neutral valuations:

1. a risk neutral price that is state independent;
2. a risk neutral price that is state dependent.

The two cases require a different valuation of the risk neutral transition matrix, that we should denote respectively with $\widehat{P}$ and $\overline{P}$.

3.0.1 State independent Risk Neutral Valuation

Let us assume no arbitrages are allowed. Then there exists a risk neutral measure such that the value "today" is equal to the expected value of the future wealth discounted with the risk-free gross return. With a Markov chain this is equivalent to write:

$$\sum_{i=1}^{N} \hat{p}_i \hat{E}(z/z \in I_i) = \exp(r-q) \tag{1}$$

where $t_0 = 0$, $\hat{E}(z/z \in I_i)$ is the risk neutral expected value of the future return conditional on being in the i-*th* state, and $\hat{p}_i$ is risk neutral probability of being in the *i-th* state. Clearly in incomplete markets could exist more than one risk neutral measure satisfying the no arbitrage criterion. One criterion proposed in literature considers the minimal entropy martingale measure (see Stutzer (1996), Frittelli (2000) and the reference therein). On the other hand, the use of the minimal entropy martingale measure can be motivated by maximum expected utility arguments (see Frittelli (2000)).

In our context, we find the minimal entropy martingale measure with respect to the unconditional probability measure $P = \{p_j p_{j,k}\}_{1 \le i,j \le N}$ where $p_j p_{j,k}$ is the unconditional probability to transit from the state j to the state k. As observed by Frittelli (2000), in order to get the minimal entropy martingale measure in the discrete case, we have to compute the value θ, unique for all the states, that is obtained as a solution of the equation:

$$\exp(r-q) = \frac{\sum_{j=1}^{N} p_j \sum_{k=1}^{N} p_{j,k} z^{(k)} \exp\left(\theta z^{(k)}\right)}{\sum_{j=1}^{N} p_j \sum_{k=1}^{N} p_{j,k} \exp\left(\theta z^{(k)}\right)}. \tag{2}$$

Then the risk neutral unconditional probability to transit from the state j to the state k is given by:

$$\hat{p}_j \hat{p}_{j,k} = \frac{p_j p_{j,k} \exp\left(\theta^* z^{(k)}\right)}{\sum_{j=1}^{N} p_j \sum_{m=1}^{N} p_{j,m} \exp\left(\theta^* z^{(m)}\right)}, \quad 1 \le i,j \le N.$$

where θ^* is the solution of equation (2). Therefore, we write the risk neutral transition matrix considering the following transition probabilities

$$\hat{p}_{j,k} = \frac{p_{j,k} \exp\left(\theta^* z^{(k)}\right)}{\sum_{m=1}^{N} p_{j,m} \exp\left(\theta^* z^{(m)}\right)} \tag{3}$$

and the probability of being in the *j-th* state is given by

$$\hat{p}_j = \frac{p_j \sum_{k=1}^{N} p_{j,k} \exp\left(\theta^* z^{(k)}\right)}{\sum_{j=1}^{N} p_j \sum_{m=1}^{N} p_{j,m} \exp\left(\theta^* z^{(m)}\right)}.$$

Therefore, once we estimate the transition matrix $P = [p_{i,j}]_{1 \le i,j \le N}$, we can find the corresponding risk neutral transition matrix $\widehat{P} = [\hat{p}_{i,j}]_{1 \le i,j \le N}$ that could be used in the risk neutral valuation of contingent claims. Let $\widetilde{p} = [\hat{p}_1, ..., \hat{p}_N]$ be the row vector of risk neutral unconditional probabilities of the different states. Then if we point out with $\widetilde{z} = [z^{(1)}, ..., z^{(N)}]'$ the vector of the possible states the fundamental theorem of asset pricing after one period is simple given by

$$\widetilde{p} \widehat{P} \widetilde{z} = \exp(r-q).$$

Note that in the discrete case the minimal entropy martingale measure coincide with the minimal variance martingale measure and with the Esscher trasform risk neutral measure (see Gerber, Shiu (1994, 1996)) often used to price contingent claims with Levy processes (see Schoutens (2003)). Moreover, since we apply a risk neutral valuation that is independent on the state, we have not necessarily to correct the transition matrix as we do in the next state dependent valuation.

3.0.2 State dependent Risk Neutral Valuation

Let us assume that the gross return z at time $t_0 = 0$ is in the *i-th* state. When no arbitrage opportunities are allowed, then

$$\hat{E}(z/z \in I_i) = \exp(r - q) \tag{4}$$

where $\hat{E}(z/z \in I_i)$ is the risk neutral expected value of the future return conditional on being in the i-*th* state. However, we can find a risk neutral measure that satisfies condition (4) only if

$$\exp(r - q) \in [z^{(j_{i*})}, z^{(j_i^*)}], \tag{5}$$

where $j_{i*} = \max_{1 \le j \le N} \{j/p_{ij} > 0\}$ and $j_i^* = \min_{1 \le j \le N} \{j/p_{ij} > 0\}$. In particular, it could happen that for some extreme states we cannot guarantee condition (4) holds since we have not enough observations of these extreme states and the probability approximations in the transition matrix are not sufficiently accurate. In order to overcome this problem, we can opportunely correct the original transition matrix $P = [p_{i,j}]_{1 \le i,j \le N}$ such that condition (5) is satisfied. For example, suppose for the state "i" $\exp(r - q) \notin [z^{(j_{i*})}, z^{(j_i^*)}]$, then we correct the *i-th* row of matrix P as follows:

a) Suppose $\exp(r - q) < z^{(j_{i*})}$. We assume $p_{ij} = \begin{cases} \varepsilon_i & \text{if } j = j_* \\ p_{ij} - \frac{\varepsilon_i}{m_i} & \text{if } j : p_{ij} > 0 \\ 0 & \text{otherwise} \end{cases}$

where ε_i is an opportunely little value belonging to the interval

$$(0, \frac{\min_{1 \le j \le N} \{p_{ij}/p_{ij} > 0\}}{m_i}),$$

m_i is the number of indexes j in the *i-th* row of P such that $p_{ij} > 0$ and $j_* = \max_{1 \le j \le N} \{j/z^{(j)} < \exp(r - q)\}$.

b) Suppose $\exp(r - q) > z^{(j_i^*)}$. We assume $p_{ij} = \begin{cases} \varepsilon_i & \text{if } j = j^* \\ p_{ij} - \frac{\varepsilon_i}{m_i} & \text{if } j : p_{ij} > 0 \\ 0 & \text{otherwise} \end{cases}$

where ε_i is an opportunely little value belonging to the interval

$$(0, \frac{\min_{1 \le j \le N} \{p_{ij}/p_{ij} > 0\}}{m_i}),$$

m_i is the number of indexes j in the *i-th* row of P such that $p_{ij} > 0$ and $j^* = \min_{1 \le j \le N} \{j/z^{(j)} > \exp(r - q)\}$.

The corrected matrix (that with abuse of notation we call again P) permits to overcome some misapplications problems deriving by a non sufficiently accurate approximation of the transition matrix. Once we correct transition matrix P, condition (5) is satisfied for every state. Thus, for every state we can determine the minimal entropy martingale measure that satisfy condition (4). Hence, for every state $i = 1, ..., N$, we compute the value $\theta_{(i)}$, obtained as a solution of the equation:

$$\exp(r-q) = \frac{\sum_{k=1}^{N} p_{i,k} z^{(k)} \exp\left(\theta_{(i)} z^{(k)}\right)}{\sum_{k=1}^{N} p_{i,k} \exp\left(\theta_{(i)} z^{(k)}\right)}. \tag{6}$$

Then the risk neutral transition matrix $\overline{P} = [\overline{p}_{i,k}]$ should contain the risk neutral conditional probabilities given by:

$$\overline{p}_{i,k} = \frac{p_{i,k} \exp\left(\theta^*_{(i)} z^{(k)}\right)}{\sum_{m=1}^{N} p_{i,m} \exp\left(\theta^*_{(i)} z^{(m)}\right)}, \quad 1 \leq i, j \leq N,$$

where $\theta^*_{(i)}$ is the solution of equation (6). Using this risk neutral transition matrix we get that

$$\overline{P}\widetilde{z} = \exp(r-q)\, 1_N,$$

where $\widetilde{z} = [z^{(1)}, ..., z^{(N)}]'$ is the vector of the possible states after one period and 1_N is the unity vector column.

4 Valuation procedure for European and American contingent claims and computational complexity

Given an asset with gross return z, then we can build the tree of the underlying price. Thus starting from a price in a generic node, we could generate N possible future prices (depending on the N possible future states). On the other hand, the original price should be conditioned from the state of provenance (N possible backward states). This aspect is fundamental in the state dependent valuation because the procedure must take into account of the previous steps. The state dependent and the state independent risk neutral valuations allow us to determine a backward computation of contingent claims particularly useful for American derivatives. However, for European contingent claims, we can also propose an alternative state dependent forward valuation that is generally different from the previous ones. Thus, we can generally consider two different types of valuation procedures: forward and backward. The first one is used for European contingent claims, whilst the second one is a much more versatile approach that can be used for American, European and path dependent derivatives.

4.1 State dependent forward valuation of European contingent claims

Recent studies have proposed a simple algorithm to determine the return distribution function on a recombining markovian tree after k periods of time (see Iaquinta and Ortobelli (2006)). Therefore an easy way to compute the value of European contingent claims consists in using the Iaquinta and Ortobelli's recursive algorithm that presents computational complexity of $O(N^3k^2)$ order. In this framework we apply the same algorithm to the transition matrix P of an homogeneous Markov chain in order to obtain the distribution after k periods of time. The forward procedure of the algorithm builds a sequence of matrixes Q_k of dimension $((N\text{-}1)k{+}1)\times N$ such that, after k periods of time, the return probabilities in the $(N\text{-}1)k{+}1$ nodes of the tree are given by the vector $Q_k 1_N$ where 1_N is the unity vector column. Note that each node of the tree is simultaneously achievable from different states. Thus each node could be in different states and this depends on the provenance state. In particular $Q_k = [q_{(k)j,i}]_{\substack{1\leq j\leq (N-1)k+1 \\ 1\leq i\leq N}}$, where $q_{(k)j,i}$ is the probability of being in *i-th* state and in the *j-th* node (counting from the highest node) after k periods of time. Therefore, if we suppose the initial state is the *i-th* one, then the first transition matrix is the diagonal matrix with the discounted probabilities corresponding to the i-th row of P on the diagonal, i.e., $Q_1 = diag(p_{i1},...,p_{iN})$. Instead, the other matrixes are given by $Q_k = diagM(Q_{k-1}P)$, where the $diagM$ operator performs a *diagonalization process* consisting in the following two operations applied to $Q_{k-1}P$:

1. shift below the s-th column of s-1 spaces for s=2,...,N, creating a new matrix $((N\text{-}1)k{+}1)\times N$;
2. fill all the new spaces with zeros.

In order to get the minimal entropy martingale measure that is risk neutral with respect the distribution after k periods of time, we have to compute the unique value θ_k, solution of the equation:

$$\exp\left((r-q)\,k\right) = \frac{\sum_{j=1}^{(N-1)k+1}\sum_{i=1}^{N} q_{(k)j,i} z_k^{(j)} \exp\left(\theta_k z_k^{(j)}\right)}{\sum_{j=1}^{(N-1)k+1}\sum_{i=1}^{N} q_{(k)j,i} \exp\left(\theta_k z_k^{(j)}\right)}. \tag{7}$$

Then the risk neutral unconditional probability of being in the j-th node after k steps is given by:

$$\widetilde{q}_{(k),j} = \frac{\sum_{i=1}^{N} q_{(k)j,i} \exp\left(\theta_k^* z_k^{(j)}\right)}{\sum_{j=1}^{(N-1)k+1}\sum_{i=1}^{N} q_{(k)j,i} \exp\left(\theta_k^* z_k^{(j)}\right)}, \quad 1\leq j\leq (N-1)k+1$$

where θ_k^* is the solution of equation (7). Let $f_{(T)} = [f_{(T),1}, ..., f_{(T),(N-1)T+1}]'$ the vector of contingent claim value at maturity T. Then the price of the European contingent claim is simply given by:

$$\exp\left(-\left(r-q\right)T\right)\widetilde{q}'_{(T)} f_{(T)}.$$

This is a forward risk neutral valuation of the price with computational complexity of $O(N^3 T^2)$ order.

In this valuation we do not correct the transition matrix as we suggest in the previous state dependent valuation, since we implicitly assume that $\exp\left(\left(r-q\right)T\right)$ belongs to the support of the gross return after T periods of time i.e.:

$$z_T^{(k_*)} \leq \exp\left(\left(r-q\right)T\right) \leq z_T^{(k^*)},$$

where

$$k_* = \max_{1 \leq j \leq (N-1)T+1} \left\{ j/a_j = \sum_{i=1}^{N} q_{(T)j,i} > 0 \right\}$$

and

$$k^* = \min_{1 \leq j \leq (N-1)T+1} \left\{ j/a_j = \sum_{i=1}^{N} q_{(T)j,i} > 0 \right\}.$$

This inequality is generally verified when T is big enough. Even for this reason we could expect some differences in the price valuations when we do not correct the transition matrix before applying the recursive algorithm to compute the return distribution after T periods of time.

4.2 The backward valuation procedure for American and European contingent claims

The backwards pricing of derivatives proceeds as any other standard backward process, distinguishing the state dependent and state independent valuations.

State dependent backward valuation:

In the state dependent valuation each node represents different values in dependence on the previous provenance state. It is the case to observe that this seemingly complication of a node with multiply values, due to the recombining purpose of the tree, allows a great advantage in order to save computational time and the memory usage.

Since the tree is multinomial, the single node considered has N possible final nodes representing the final payoff of the derivative. A single backward step in the expected discounted process consists of the matrix multiplication between the discounted transition matrix transformed (as previously explained) and the vector of the final payoff. The result is a vector of N elements which represent the different values of the node in dependence of the provenance state.

The description of the entire European option pricing process is offered through its algorithm form. Let consider a recombining multinomial price tree composed by M time steps and N branches for each node. Then we can build the tree of the contingent claim.

1. Suppose we have the final payoff at M-th step (the j-th payoff from above is given by $f_{M:j}$). Starting to count from the highest node then at the j-th node (for $j = 1, ..., (N-1)(M-1)+1$) we consider the vector of payoffs $\tilde{f}_{M:j} = [f_{M:j}, ..., f_{M:j+N-1}]^I$. Thus, at the $(M-1)$-th step we consider the $(N-1)(M-1)+1$ vectors of discounted possible prices:

$$\tilde{\tilde{f}}_{M-1:j} = \exp(q-r)\,\overline{P}\tilde{f}_{M:j}.$$

However, in this step we get more prices than those we have in the tree. In order to eliminate the prices which are not on the tree, we have to reorder the vectors that should be discounted in the backward process.

2. We build the new vectors $\tilde{f}_{M-1:k} = \left[\tilde{\tilde{f}}^{(1)}_{M-1:k}, ..., \tilde{\tilde{f}}^{(N)}_{M-1:k+N-1}\right]^I$ for $k = 1, \ldots, (N-1)(M-2)+1$ where $\tilde{\tilde{f}}^{(i)}_{M-1:s}$ is the i-*th* component of vector:

$$\tilde{\tilde{f}}_{M-1:s} = \exp(q-r)\,\overline{P}\tilde{f}_{M:s}.$$

3. After s steps we have at the *j-th* node (starting from above) the vector:

$$\tilde{\tilde{f}}_{s:j} = \exp(q-r)\,\overline{P}\tilde{f}_{s+1:j}$$

and the new recombining $(N-1)(s-1)+1$ vectors $\tilde{f}_{s:k} = \left[\tilde{\tilde{f}}^{(1)}_{s:k}, ..., \tilde{\tilde{f}}^{(N)}_{s:k+N-1}\right]^I$ $k = 1, \ldots, (N-1)(s-1)+1$.

4. At the first step we have only one vector $\tilde{f}_{1:1} = \left[\tilde{\tilde{f}}^{(1)}_{1:1}, ..., \tilde{\tilde{f}}^{(N)}_{1:N}\right]^I$. The value of the contingent claim depends on the state I_m we begin from and it is given by the m-th component of $\exp(q-r)\,\overline{P}\tilde{f}_{1:1}$.

The complexity of this algorithm is the same of the state dependent forward valuation (i.e., of $O(N^3k^2)$ order). As a matter of fact, in the backward procedure the algorithm above can be summarized as follows. We build a sequence of matrixes of payoffs $F_k = [\tilde{f}_{k:1}, ..., \tilde{f}_{k:(N-1)(k-1)+1}]$ of dimension $N \times ((N\text{-}1)(k\text{-}1)+1)$. Thus, given the final payoff matrix F_M the other matrixes are given by $F_k = reductM(\exp(q-r)\,\overline{P}F_{k+1})$, where the *reductM* operator performs a *reduction process* consisting in the following two operations applied to $\exp(q-r)\,\overline{P}F_{k+1}$:

1. at the s-th row, cancel the first s-1 values for s=2,...,N and the last N-s for s=1,...,N-1 ;

2. shift on the left the s-th row of s-1 spaces for s=2,...,N, creating a new matrix $((N\text{-}1)(k\text{-}1)+1) \times N$ (without considering the cancelled spaced of the first operation).

Finally the contingent claim price is given by the m-th component of $\exp(q-r)\,\overline{P}F_1$ when we suppose that the initial state is the *m-th* one. Observe that this reduction process is in some sense the inverse operation of the diagonalization process and it has the same computational complexity. This algorithm can be easily adapted to compute American options. For example, if we value an American put with exercise price X for every s less than the time to maturity (i.e., $s \leq M-1$), in the backward procedure, we have to consider the vector

$$\tilde{f}_{s:k} = \left[\max\left(\tilde{\tilde{f}}^{(1)}_{s:k}, X - S_0 z_s^{(k)}\right), ..., \max\left(\tilde{\tilde{f}}^{(N)}_{s:k+N-1}, X - S_0 z_s^{(k+N-1)}\right)\right]^I,$$

$k = 1, \ldots, (N-1)(s-1)+1$. Moreover, even if this approach is not parametric we can also approximate the greek letters which are often used to hedge the investors' positions.. However, in this case we take into account the incremental ratios with their risk neutral probability. Suppose at time zero we are on the *m-th* state, then after one period we have the vector of contingent claims $\tilde{f}_{1:1}$ whose the *k-th* component is realized with the risk neutral probability $\overline{p}_{m,k}$. In order to estimate the delta ($\Delta = \frac{\Delta f}{\Delta S}$) of the option, we have $\binom{N}{2}$ incremental ratios $\overline{\Delta}_{ij} = \frac{1}{S_0}\frac{\tilde{\tilde{f}}^{(i)}_{1:i} - \tilde{\tilde{f}}^{(j)}_{1:j}}{z^{(i)} - z^{(j)}}$ with probability estimates $q_{ij} = \frac{\overline{p}_{m,i} + \overline{p}_{m,j}}{N-1}$ for $i = 1, ..., N-1$, $j = i+1, ..., N$. Thus, an estimate of delta after one period conditioned by the starting state m is given by the average

$$\Delta^{(m)} = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} q_{ij} \overline{\Delta}_{ij}.$$

To determine gamma ($\Gamma = \frac{\partial^2 f}{\partial S^2}$) note that we have $\binom{N}{2}$ estimates of delta $\overline{\Delta}_{ij}$ after one period. Therefore after one period we have $\binom{N}{2}\left(\binom{N}{2} - 1\right)$ estimates of gamma $\Gamma_{ijsk} = \frac{\overline{\Delta}_{ij} - \overline{\Delta}_{sk}}{0.5 S_0\left(z^{(i)} + z^{(j)} - z^{(s)} - z^{(k)}\right)}$ with probability estimates $q_{ijsk} = \frac{q_{ij} + q_{sk}}{2\left(\binom{N}{2} - 1\right)}$ for $i, s = 1, ..., N-1$, $j = i+1, ..., N$, $k = s+1, ..., N$ and $i, j \neq s, k$. Thus, an estimate of gamma after one period conditioned by the starting state m is given by the average:

$$\Gamma^{(m)} = \sum_{s=1}^{N-1} \sum_{k=s+1}^{N} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} q_{ijsk} \overline{\Gamma}_{ijsk},$$

where we have not considered $i, j \neq s, k$ since when $i = s \wedge j = k$ we get $\Gamma_{ijij} = 0$.

State independent backward valuation:

With the state independent valuation we get a price at each step instead of a vector of prices since we have not to take into account the state of provenance. Let consider a recombining multinomial price tree composed by M time steps and N branches for each node. Then we can build the tree of the contingent claim.

1. Suppose we have the final payoff at M-th step (the j-th payoff from above is given by $f_{M:j}$). Starting to count from the highest node then at the j-th

node (for $j = 1, ..., (N-1)(M-1)+1$) we consider the vector of payoffs $\tilde{f}_{M:j} = [f_{M:j}, ..., f_{M:j+N-1}]^I$. Thus, at the $(M-1)$-th step we consider the $(N-1)(M-1)+1$ prices:

$$f_{M-1:j} = \exp(q-r)\, \widetilde{p}\widehat{P}\tilde{f}_{M:j},$$

and we build the new vectors $\tilde{f}_{M-1:k} = [f_{M-1:k}, ..., f_{M-1:k+N-1}]^I$ for $k = 1, \ldots, (N-1)(M-2)+1$.

2. Thus, after s steps we have at the *j-th* node (starting from above) the price:

$$f_{s:j} = \exp(q-r)\, \widetilde{p}\widehat{P}\tilde{f}_{s+1:j}$$

and we build the new $(N-1)(s-1)+1$ vectors $\tilde{f}_{s:k} = [f_{s:k}, ..., f_{s:k+N-1}]^I$ $k = 1, \ldots, (N-1)(s-1)+1$.

3. At the first step we have only one vector $\tilde{f}_{1:1} = [f_{1:1}, ..., f_{1:N}]^I$ and the value of the contingent claim is given by $\exp(q-r)\, \widetilde{p}\widehat{P}\tilde{f}_{1:1}$.

Observe that the complexity of this algorithm is the same of the state dependent one (i.e., $O(N^3k^2)$ order) and, even in this case, we can easily adapt the algorithm to value an American contingent claim. So in order to value an American put with exercise price X for every s less than the time to maturity (i.e., $s \leq M-1$), in the backward procedure, we have to consider the vector

$$\tilde{f}_{s:k} = \left[\max\left(f_{s:k}, X - S_0 z_s^{(k)}\right), ..., \max\left(f_{s:k+N-1}, X - S_0 z_s^{(k+N-1)}\right)\right]^I,$$

$k = 1, \ldots, (N-1)(s-1)+1$. Similarly to the state dependent valuation of greek letters we get the $\binom{N}{2}$ incremental ratios $\widehat{\Delta}_{ij} = \frac{1}{S_0}\frac{f_{1:i}-f_{1:j}}{z^{(i)}-z^{(j)}}$ with probability estimates $\widehat{q}_{ij} = \sum_{m=1}^{N} \widehat{p}_m \frac{\widehat{p}_{m,i}+\widehat{p}_{m,j}}{N-1}$. Thus, an estimate state independent of delta after one period is given by the average

$$\Delta = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \widehat{q}_{ij}\widehat{\Delta}_{ij}$$

In order to compute a state independent valuation of Gamma we use the $\binom{N}{2}\left(\binom{N}{2}-1\right)$ estimates of gamma $\widehat{\Gamma}_{ijsk} = \frac{\widehat{\Delta}_{ij}-\widehat{\Delta}_{sk}}{0.5S_0\left(z^{(i)}+z^{(j)}-z^{(s)}-z^{(k)}\right)}$ with probability estimates $\widehat{q}_{ijsk} = \frac{\widehat{q}_{ij}+\widehat{q}_{sk}}{2\left(\binom{N}{2}-1\right)}$ for $i, s = 1, ..., N-1$, $j = i+1, ..., N$, $k = s+1, ..., N$ and $i, j \neq s, k$. Thus, an estimate of gamma after one period is given by the average:

$$\Gamma = \sum_{s=1}^{N-1} \sum_{k=s+1}^{N} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \widehat{q}_{ijsk}\widehat{\Gamma}_{ijsk}.$$

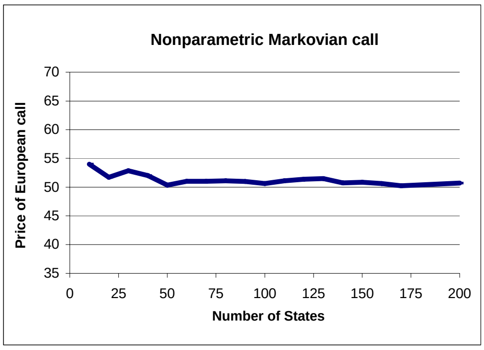

Figure 1: This figure reports the values of an European call on the S&P500 computed with the nonparametric markovian trees varying the number of states of the Markov chain.

5 Stability of the price valuation

From the analysis of the backward valuation procedure we understand that the stability of the price depend on the opportune number of states N used in the pricing valuation. From a simple empirical analysis we could observe that the price of contingent claims do not substantially change with N greater than 50. In particular, we consider historical data from January 1995 to August 2005 of Dow Jones Industrials, S&P500 and Nasdaq and we compute the price of several European puts and calls changing the number of the states (from 10 to 200) the strikes (five in the money and five out the money) and the time to maturity (7) for a total of 210 possible options. We compute the prices with a state dependent valuation and with a state independent valuation. While there exist differences in the prices, we generally do not observe differences in stability between the two procedures. Moreover, for all the experiments we obtain the stability of the price with N around 40, while, for N lower than 40, we not always have a stable price.

Figure 1 and 2 summarize two of these experiments for a call and a put on the S&P500. The graphs show clearly how increasing the number of the states the prices tend to be stable and it makes sense to consider at least 50 states.

On the other hand, the valuation of the price of contingent claims with the above algorithms requires few seconds using a notebook dual centrino with one Gb of Ram. As a matter of fact, Figure 3 reports the graphs with the seconds necessary to compute the price of an European call with a backward state dependent valuation and considering the mean of 10 prices with maturity 20, 40, 60, 90, 120 trading days and states varying between (41-50), (51-60) (61-70) (71-80).

In view of this simple empirical analysis, next we consider as contingent

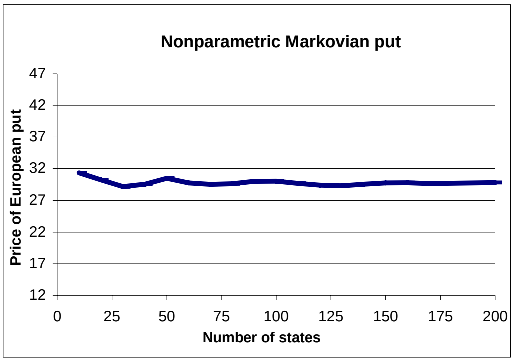

Figure 2: This figure reports the values of an European put on the S&P500 computed with the nonparametric markovian trees varying the number of states of the Markov chain.

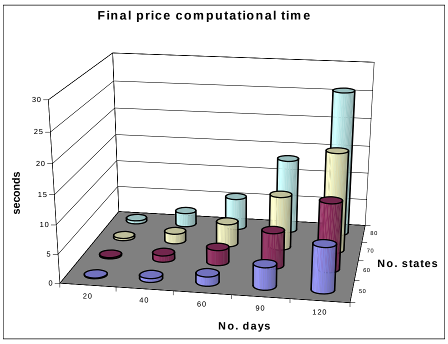

Figure 3: This figure reports the numbers of seconds necessary to compute the mean of 10 prices of European calls with different maturities computed varying the number of the states of the Markov chain.

claim price, the average of the prices obtained with transition matrixes from 40 to 60 states.

6 An empirical comparison

In this section we propose a comparison between nonparametric markovian option prices and the prices obtained under the Black and Scholes assumptions. First of all we compute the differences of valuation when we assume the hypotheses of the Black and Scholes model holds. Then we describe the differences of prices computed using real data.

In order to value the performance of our model when we assume the same hypotheses of the Black and Scholes model, we propose a MonteCarlo simulation comparison. In particular, we generate 10000 Gaussian scenarios N(0.002,0.03) of log returns. We assume that the risk free rate is 4% and the price of the stock today is 50 USD. Then we compute the prices of call options with 20, 40, 60 days to maturity considering different exercise prices X (X=42, 44, 46, 48, 50, 52, 54, 56, 58). For all the options we compute the real Black and Scholes price the Black and Scholes estimated price, the price estimated with the backward state dependent and state independent valuation. Then we compute the average of the differences observed by estimated models and the real Black and Scholes prices. We observe that the estimated Black and Scholes price differs in average of about 0.0001 USD from the real one, while both the backward state dependent and state independent valuation differ in average of about 0.00025 USD. Therefore, this analysis confirms that the nonparametric markovian models well fit the underline distribution and we do not observe significative differences between the state dependent and state independent valuations. On the other hand, it is well known that log returns are not Gaussian distributed (see Rachev and Mittnik (2000)). In an analysis of long time distributions Iaquinta and Ortobelli (2006) have recently shown that the approximation of the long time return distributions with a nonparametric markovian tree presents much better fit than that obtained assuming log-normal distributed returns. Therefore we expect that the prices computed with markovian trees are more precise than those obtained with the Black and Scholes model. Using historical data from January 1995 to August 2005 of Dow Jones Industrials, S&P500 and Nasdaq we compute some of these differences in Table 1. In particular Table 1 reports the values of European calls and puts valued in different weeks between July and August 2005. We assume maturity $T = 60$; exercise price $X = E(S_T)$ and risk-free rate equal to the Treasury Bill 3 months. Since we have not observed significative differences between the backward state dependent and state independent valuation, in this table we consider only the state dependent one.

As we can observe from the table there exist significative differences between the pricing models much higher that those observed under the Black and Sc-

Table 1

This table summarizes some of the differences we observe on European options valued for the Dow Jones Industrials, Nasdaq and S&P500 when we consider or the non-parametric Markovian model or the Black and Scholes one.

Price of Options with T=60 days to maturity and Strike price equal to $E(S_T)$						
		Dow Jones Industrials				
		1st week	**2nd week**	**3rd week**	**4th week**	**5th week**
Nonparametric MKV	**Call**	374.0771	375.2497	375.2559	376.3665	376.4187
	Put	374.0771	375.2497	375.2559	376.3665	376.4187
Black and Scholes	**Call**	363.3221	364.4487	364.5591	365.516	365.5856
	Put	362.7763	363.8909	363.9958	364.9513	365.0102
		Nasdaq				
Nonparametric MKV	**Call**	118.6815	119.543	119.456	120.5268	120.4506
	Put	124.6223	125.524	125.4358	126.5597	126.4811
Black and Scholes	**Call**	117.4822	118.3118	118.2607	119.2999	119.2681
	Put	117.3741	118.2009	118.1488	119.187	119.1531
		S&P500				
Nonparametric MKV	**Call**	41.96457	42.23176	42.2164	42.39177	42.42173
	Put	41.96457	42.23176	42.2164	42.39177	42.42173
Black and Scholes	**Call**	41.69187	41.93954	41.929	42.10166	42.13395
	Put	41.63079	41.87694	41.8659	42.03822	42.06927

holes assumptions. Therefore it makes sense to consider this modelization as alternative to the classic Black and Scholes one.

7 Concluding remarks

We have proposed a Markovian model to price contingent claims. The model is nonparametric, ductile and it presents a reasonable computational complexity. Using the minimal entropy martingale measure as risk neutral valuation, we have studied the stability of the price with respect to the number of the states. Moreover we have proposed an ex-post empirical comparison with the Black and Scholes model showing the ductility of the model with respect to the underline distribution.

The model here proposed consider only a homogeneous Markov chain to value European and American derivatives. However, it can be easily extended assuming non homogeneous Markov chains to value plain vanilla and path dependent options. We also observe that the transition probability matrix associated with the Markov chain is usually sparse. It means that many elements of this matrix are numerically negligible. This property is important because it deeply reduces the computational cost of the algorithm (see Zlatev (1991) and Broyden, Vespucci (2004)). Therefore we believe that the computational time of $O(N^3k^2)$ order could be further reduced taking into account this fact.

References

[1] Ait-Sahalia Y., Lo A.W. 1998. Nonparametric estimation of state price densities implicit in financial asset prices, Journal of Finance 53, 499-549.

[2] Ait-Sahalia Y.1996. Nonparametric pricing of interest rate derivative securities, Econometrica 64, 527-560.

[3] Blasi, A., Janssen, J., Manca, R., 2003. Numerical treatment of homogeneous and non-homogeneous reliability semi-Markov models. Communications in Statistics, Theory and Models.

[4] Broyden, C. G., Vespucci, M. T., 2004. Krylov Solvers for Linear Algebraic Systems. Elsevier, New York.

[5] D'Amico, G., Janssen, J., Manca, R., June 2005. Non-homogeneous backward semi-Markov reliability approach to downward migration credit risk problem. Proceedings of the 8^{th} Italian Spanish Meeting on Financial Mathematics, Verbania, June 2005.

[6] Duan, J., Dudley, E., Gauthier, G., Simonato, J., 2003. Pricing discretely monitored barrier options by a Markov chain. Journal of Derivatives 10, 9–23.

[7] Duan, J., Simonato, J., 2001. American option pricing under GARCH by a Markov chain approximation. Journal of Economic Dynamics and Control 25, 1689–1718.

[8] Frittelli M., 2000. The minimal entropy martingale measure and the valuation problem in incomplete markets, Mathematical Finance 10, 39-52.

[9] Gerber H.U., Shiu, E.S.W., 1994. Option pricing by Esscher transform, *Transactions of the Society of Actuaries* 46, 99-140; *Discussion* 141-191.

[10] Gerber H.U., Shiu, E.S.W., 1996. Actuarial bridges to dynamic hedging and option pricing. Insurance: Mathematics and Economics 18, 183-218.

[11] Hutchinson, J.M., Lo A. W., Poggio T., 1994. A non parametric approach to the pricing and hedging of derivative securities via learning networks, Journal of Finance 49, 851-889.

[12] Iaquinta G. and Ortobelli S., 2006. Distributional Approximation of Asset Returns with Nonparametric Markovian Trees, International Journal of Computer Science and Network Security, VOL.6 No.10.

[13] Limnios, N., Oprisan, G., 2001. Semi-Markov processes and reliability modeling. World Scientific, Singapore.

[14] Rachev, S., Mittnik, S., 2000. Stable Paretian Models in Finance. John Wiley & Sons, New York.

[15] Schoutens W. 2003. Lèvy Processes in Finance: Pricing Financial Derivatives, Chichester, John Wiley & Sons.

[16] Stutzer, M. J. 1996. A Simple Nonparametric Approach to Derivative Security Valuation, Journal of Finance 51, 1633-1652.

[17] Zlatev, Z., 1991. Computational Methods for General Sparse Matrices. Kluwer, Boston.

Redazione

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La Redazione ottempera agli obblighi previsti dall'art. 1 del D.L.L. 31.8.1945, n. 660 e successive modifiche

Stampato nel 2006
presso la Cooperativa
Studium Bergomense a r.l.
di Bergamo