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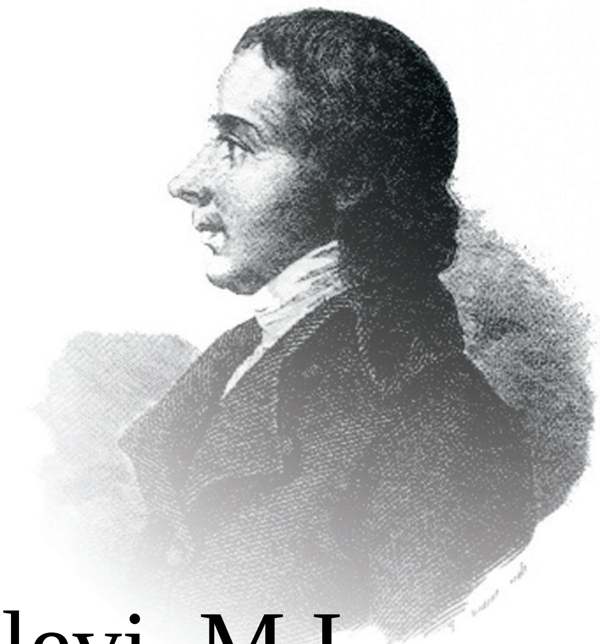

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Stochastic location aided routing model: a two stage stochastic second-order cone programming formulation

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Abstract. We study the semidefinite stochastic location-aided routing (SLAR) model described in Ariyawansa and Zhu (2006) [2], and formulate it as a two-stage stochastic second-order cone programming (SSOCP), see Alizadeh, D. Goldfarb (2003) [1] for second-order cone programming, where the main first-stage decision variables are the position of the destination node and its distance from the sender node. The random movements of the destination node are represented by ellipsoid scenarios randomly generated by a uniform distribution in a neighborhood of the starting position of the destination node. The MOSEK solver (under GAMS environment) allows to solve problems with a large number of scenarios (say 4040) versus the ten scenarios of MINOS solver. Stability results for the optimal first-stage solutions and the optimal function value are obtained.

1 Basic facts and notation

Semidefinite programming problems are a class of optimization problems that have been studied extensively during the past 15 years. Semidefinite programming is naturally related to linear programming, and both are defined using deterministic data. Semidefinite programming is concerned with choosing a symmetric matrix to minimize a linear function subject to linear constraints, and an important additional constraint that requires the matrix to be positive semidefinitive. Deterministic semidefinite programming (DSDP) generalizes deterministic linear programming (DLP). DLP has nonnegative decision variables while the decision variable in DSDP is a positive semidefinite matrix.

We use the following notation: $\Re^{n\times n}$ for the vector spaces of real $n \times n$ matrices, lower case boldface letters $\mathbf{x}, \mathbf{c}$ etc. for column vectors, and uppercase letters A, X etc. for matrices. Subscripted vectors such as $\mathbf{x_i}$ represent the i^{th} block of $\mathbf{x}$. The j^{th} component of the vectors $\mathbf{x}$ and $\mathbf{x_i}$ are indicated by x_j and x_{ij}. We use $\mathbf{0}$ and $\mathbf{1}$ for the zero vector and vectors of all ones, respectively, and 0 and I for the zero and identity matrices.

A deterministic linear programming problem (DLP) in primal standard form is

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$$\begin{array}{ll} \min \mathbf{c}^T\mathbf{x} & \\ \text{subject to} & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0} \end{array} \tag{1}$$

and its dual

$$\begin{array}{ll} \max \mathbf{b^T y} & \\ \text{subject to} & A^T\mathbf{y} \leq \mathbf{c}, \end{array} \tag{2}$$

where $A \in \Re^{n\times n}$, $\mathbf{b} \in \Re^{\mathbf{m}}$ and $\mathbf{c} \in \Re^{\mathbf{n}}$ constitute given data, and $\mathbf{x} \in \Re^{\mathbf{n}}$ is the primal variable and $\mathbf{y} \in \Re^{\mathbf{m}}$ is the dual variable.

Let $\Re^{n\times n}_s$ denotes the vector space of real $n \times n$ symmetric matrices, for $A, B \in \Re^{n\times n}_s$ we write $A \succeq 0$ ($A \succ 0$) to mean that A is positive semidefinite (positive definite) and $A \succeq B$ ($A \succ B$) to mean that $A - B \succeq 0$ ($A - B \succ 0$). For $A, B \in \Re^{n\times n}$ we denote by $A \bullet B$ the Frobenius inner product between A and B: $A \bullet B = trace(A^T B)$. A DSDP in primal standard form is

$$\begin{array}{lll} \min C \bullet X & & \\ \text{subject to} & A_i \bullet X = b_i, & i = 1, 2, \ldots, m \\ & X \succeq 0 & \end{array} \tag{3}$$

where $A_i \in \Re^{n\times n}_s$ for $i = 1, 2, \ldots, m$, $\mathbf{b} \in \Re^m$ and $C \in \Re^{n\times n}_s$ are given and $X \in \Re^{n\times n}_s$ is the variable.

A DSDP in dual standard form is

$$\begin{array}{ll} \max \mathbf{b}^T\mathbf{y} & \\ \text{subject to} & \sum_{i=1}^m \mathbf{y}_i A_i \preceq C \end{array} \tag{4}$$

where $A_i \in \Re^{n\times n}_s$ for $i = 1, 2, \ldots, m$, $\mathbf{b} \in \Re^m$ and $C \in \Re^{n\times n}_s$ are given data, and $\mathbf{y} \in \Re^m$ is the variable.

It is possible to convert a problem in the form (4) to an equivalent problem in the form (3) and vice versa.

Stochastic programming were introduced in the 1950s as a paradigm for dealing with uncertainty in data related to linear programming. Ariyawansa and Zhu (2006), [2] introduced stochastic semidefinite programs as a paradigm for dealing with uncertainty in data related to semidefinite programs.

We recall the structure of two stage stochastic linear programming problem with recourse (SLPs): a SLPs in primal standard form is

$$\begin{array}{ll} \min \mathbf{c}^T\mathbf{x} + E[Q(\mathbf{x}, \omega)] & \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \tag{5}$$

where $\mathbf{x} \in R^{n_1}$is the first-stage decision variable, $\mathbf{c} \in R^{n_1}$ is a given vector, frequently called cost vector, $\mathbf{b} \in R^{m_1}$ an other given vector, $A \in \Re^{m_1 \times n_1}$, $\mathbf{c}$, $\mathbf{b}$ and A are deterministic data. $Q(\mathbf{x}, \omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & \mathbf{q}(\omega)^T \mathbf{y} \\ \text{subject to} & T(\omega)\mathbf{x} + W(\omega)\mathbf{y} = \mathbf{h}(\omega) \\ & \qquad\qquad \mathbf{y} \geq \mathbf{0} \end{array} \tag{6}$$

and

$$E[Q(\mathbf{x}, \omega)] = \int_\Omega Q(\mathbf{x}, \omega) P(\mathrm{d}\omega) \tag{7}$$

where $\mathbf{y}(\omega) \in R^{n_2}$ is the second-stage decision vector, $\mathbf{q} \in R^{n_2}$, $T(\omega) \in \Re^{m_2 \times n_1}$ is the technology matrix, $W(\omega) \in \Re^{m_2 \times n_2}$ is the recourse matrix, $\mathbf{h} \in R^{m_2}$ and $\omega \in \Omega$ is a random outcome with a known probability distribution P.

We introduce the stochastic semidefinitive programming problem with recourse(SSDP) in primal standard form

$$\begin{array}{ll} \min & C \bullet X + E[Q(X, \omega)] \\ \text{subject to} & A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m_1 \\ & X \succeq 0 \end{array} \tag{8}$$

where $X \in R_s^{n_1 \times n_1}$ is the first-stage decision variable, $C \in R_s^{n_1 \times n_1}$ is a given matrix, $\mathbf{b} \in R^{m_1}$ an other given vector, $A \in R_s^{n_1 \times n_1}$, $\mathbf{c}$, $\mathbf{b}$ and A are deterministic data. $Q(X, \omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & Q(\omega) \bullet Y \\ \text{subject to} & T_i(\omega) \bullet X + W_i(\omega) \bullet Y = h_i(\omega) \quad i = 1, 2, \ldots, m_2 \\ & \qquad\qquad Y \succeq 0 \end{array} \tag{9}$$

and

$$E[Q(X, \omega)] = \int_\Omega Q(X, \omega) P(\mathrm{d}\omega) \tag{10}$$

where $Y(\omega) \in R_s^{n_2 \times n_2}$ is the second-stage decision vector, $Q \in R_s^{n_2 \times n_2}$, $T_i(\omega) \in R_s^{n_1 \times n_1}$, $W_i(\omega) \in \Re_s^{n_2 \times n_2}$, $\mathbf{h} \in R^{m_2}$ and $\omega \in \Omega$ is a random outcome with a known probability distribution P.

Furthermore, semidefinite programming (SDP) includes second-order cone programming (SOCP) as a special case. SOCP problems consist in convex optimization problems in which a linear function is minimized over the intersection of an affine set and the product of second-order (Lorentz) cones:

$$\mathscr{K}_n := \{\mathbf{x} = (x_0; \bar{\mathbf{x}}) \in \Re^n : x_0 \geq \|\bar{\mathbf{x}}\|\} \ , \tag{11}$$

where $\|\cdot\|$ refers to the standard Euclidean norm and n the dimension of $\mathscr{K}_n$ (see Alizadeh and Goldfarb, (2003) [1]) .

The second-order cone can be embedded in the cone of positive semidefinite matrices since a second-order cone constraint is equivalent to a linear matrix inequality according to the following relation:

$$Arw(\mathbf{x}) := \begin{pmatrix} x_0 & -\bar{\mathbf{x}}^T \\ -\bar{\mathbf{x}} & x_0 I \end{pmatrix} \succeq 0 \;\Leftrightarrow\; x_0 \geq \|\bar{\mathbf{x}}\| \; . \tag{12}$$

In fact $Arw(\mathbf{x}) \succeq 0$ if and only if either $\mathbf{x} = \mathbf{0}$, or $x_0 > 0$ and the Shur Complement $x_0 - \bar{\mathbf{x}}^T(x_0 I)^{-1}\bar{\mathbf{x}} \geq 0$.

Notice that the computational effort per iteration required by interior point methods to solve SOCP problems is less of that required to solve SDP's problems of similar size and structure. In fact the number of iterations to decrease the duality gap to a constant fraction of itself using the primal dual method, is bounded above by $O(\sqrt{N})$, where N is the number of second-order constraints, for the SOCP algorithm, and by $O(\sqrt{\sum_{i=1}^N n_i})$, where n_i is the dimension of each second-order cone constraint $i = 1, \ldots, N$, for the SDP algorithm (see Nesterov and Nemirovsky (1994) [8]). Furthermore, each iteration is much faster: in the SOCP algorithm is $O(n^2 \sum_{i=1}^N n_i)$ and in the SDP $O(n^2 \sum_{i=1}^N n_i^2)$ where n is the dimension of the optimization variable $\mathbf{x}$.

2 Stochastic semidefinite program for modeling networks with moving nodes

In this section we recall the semidefinite stochastic location-aided routing (SLAR) model described in Ariyawansa and Zhu (2006) [3]. A sender node S needs to find a route to a destination node D through broadcasting to its neighbors a route request. Once D receives the route request (and this should happen within a time-out interval t_1, otherwise the route request has to be reinitiated), D responds by reversing the path followed by the route request received by D. In this model, while node D is supposed to move at a random speed, S is supposed to be static. Notice that the communication is successful when the reply message is sent back to the source node.

Consider an origin node S that needs to find a route to an other destination node D. We assume that:

- The source node S knows the location $\boldsymbol{l}, \boldsymbol{l} \in \Re^n$ of the destination node D at time t_0 and vice versa the node D knows the location, that for simplicity we suppose in the origin $\mathbf{0}$ of node S at time t_0;
- The nodes in the network are uniformly distributed.
- The node D moves at a random speed $v(\omega_1)$, which depends on an underlying outcome ω_1 in an event space Ω_1 with a known probability distribution P_1;

- The node D moves towards a (normalized) random direction $\mathbf{d}(\omega_2)$, which depends on an underlying outcome ω_2 in an event space Ω_2 with a known probability distribution P_2;
- P_1 and P_2 are both discrete;
- Let $\left\{\left(v^{(k)}, \mathbf{d}^{(k)}\right) : k = 1, \ldots, K'\right\}$ be the possible realizations of the couple $(v(\omega_1), \mathbf{d}(\omega_2))$ given with probability $p_k := P\left(\left(v\left(\omega_1\right), \mathbf{d}\left(\omega_2\right)\right) = \left(v^k, \mathbf{d}^k\right)\right)$, $k = 1, \ldots, K'$;
- At time $t_1 > t_0$ the node D will be at location $\boldsymbol{l} + (t_1 - t_0)v^{(k)}\mathbf{d}^{(k)}$ with probability p_k;
- The K ellipsoids

$$E_k = \{\mathbf{x} \in \Re^n : \mathbf{x}^T H_k \mathbf{x} + 2\mathbf{g}_k{}^T \mathbf{x} + v_k \leq 0\}, \qquad k = 1, 2, \ldots, K \tag{13}$$

  are the realizations of the random ellipsoid $\tilde{E} = \{\mathbf{x} \in \Re^n : \mathbf{x}^T \tilde{H} \mathbf{x} + 2\tilde{\mathbf{g}}^T \mathbf{x} + \tilde{v} \leq 0\}$, where $\tilde{H} \in \Re^{n \times n}_s$, $\tilde{H}_k \succ 0$ $\tilde{g}(\omega) \in \Re^n$ and $\tilde{v}(\omega) \in \Re$, for $k = 1, \ldots, K$ are random data depending on the outcome ω in an event space Ω with a known probability P, and $H_k \in \Re^{n \times n}_s$, $H_k \succ 0$, $\mathbf{g}_k \in \Re^n$ and $v_k \in \Re$ for $k = 1, 2, \ldots, K$;
- At time t_1 the node D is in E_k with probability p_k for $k = 1, 2, \ldots, K$.

Knowing the location of D at time t_0, in order to determine the new location of D at time t_1 Ariyawansa and Zhu use the following procedure:

Stage 1 Pick a disk C

$$C = \{\mathbf{x} \in \Re^n : \mathbf{x}^T \mathbf{x} - 2\tilde{\mathbf{x}}^T \mathbf{x} + \gamma \leq 0\} \tag{14}$$

with center in $\tilde{\mathbf{x}}$ and radius $\sqrt{\tilde{\mathbf{x}}^T \tilde{\mathbf{x}} - \gamma}$ which contains the disk C_0 centered in $\boldsymbol{l}$ with radius $v(t_1 - t_0)$, where v is the minimum speed the node D is supposed to move.

Stage 2 If happens that node D is in C, no further action is needed; otherwise D is in E_k for some k, thus we pick a new disk C_k^*

$$C_k^* = \{\mathbf{x} \in \Re^n : \mathbf{x}^T \mathbf{x} - 2\tilde{\mathbf{x}}^T \mathbf{x} + \tilde{\gamma}_k \leq 0\} \tag{15}$$

with center in $\tilde{\mathbf{x}}$ and radius $\sqrt{\tilde{\mathbf{x}}^T \tilde{\mathbf{x}} - \tilde{\gamma}_k}$ which contains the ellipsoids E_k for each $k = 1, 2, \ldots, K$. To be close to reality, we fix an upper bound on the difference $\gamma - \tilde{\gamma}_k$. We are sure that at the cost of enlarging the radius we can pick up the new position of D.

The decision variables are given by:

$$\mathbf{x} = [d_1, d_2, \tilde{\mathbf{x}}, \gamma, \tau]^T \ , \tag{16}$$

$$\mathbf{y} = [\mathbf{z}, \tilde{\boldsymbol{\gamma}}\ , \boldsymbol{\delta}]^T \ , \tag{17}$$

where $\mathbf{x}$ is the first stage decision variable with components:

- d_1: is an upper bound on the distance between the center of the disk
$$C = \{\mathbf{x} \in \Re^n : \mathbf{x}^T\mathbf{x} - 2\tilde{\mathbf{x}}^T\mathbf{x} + \gamma \leq 0\}$$
and the source node ($S = \mathbf{0}$);
- d_2: is an upper bound on square of the radius of the disk C;
- $\tilde{\mathbf{x}} \in \Re^n$: is the center of disk C;
- γ: is a coefficient in the equation of disk C;
- τ: is a nonnegative parameter (see Vandenberghe and Boyed (1996), [11]; Sun and Freund (2004) [10]).

and $\mathbf{y}$ is the second stage decision variables whose components are:

- $\mathbf{z} \in \Re^K$: is the vector of the upper bounds for scenario k on the distance between the coefficients γ and $\tilde{\gamma}_k$ in
$$C_k^* = \{\mathbf{x} \in \Re^n : \mathbf{x}^T\mathbf{x} - 2\tilde{\mathbf{x}}^T\mathbf{x} + \tilde{\gamma}_k \leq 0\};$$
- $\tilde{\boldsymbol{\gamma}} \in \Re^K$: is the vector of the coefficients $\tilde{\gamma}_k$ of the second stage circles
$$C_k^* = \{\mathbf{x} \in \Re^n : \mathbf{x}^T\mathbf{x} - 2\tilde{\mathbf{x}}^T\mathbf{x} + \tilde{\gamma}_k \leq 0\} \qquad k = 1, \ldots, K;$$
- $\boldsymbol{\delta} \in \Re^K$: is a vector of nonnegative parameters (see Vandenberghe and Boyed (1996) [11]; Sun and Freund (2004), [10]).

The unit cost vectors are given by:

$$\begin{aligned} \mathbf{c} &= [\tilde{c}, \alpha, \mathbf{0}, 0, 0]^T \ , & (18) \\ \mathbf{q} &= [\boldsymbol{\beta}, \mathbf{0}, \mathbf{0}]^T \ ; & (19) \end{aligned}$$

Then the SLAR model is given by

$$\begin{aligned} &\min \ \mathbf{c}^T\mathbf{x} + E[Q(\mathbf{x}, \omega)] \\ &\text{subject to} \quad \begin{pmatrix} I & -\tilde{\mathbf{x}} \\ -\tilde{\mathbf{x}}^T & \gamma \end{pmatrix} \preceq \tau \begin{pmatrix} I & -\boldsymbol{l} \\ -\boldsymbol{l}^T & \|\boldsymbol{l}\|^2 - (t_1 - t_0)^2 v^2 \end{pmatrix} \\ &\qquad\qquad 0 \leq \tau \\ &\qquad\qquad 0 \preceq \begin{pmatrix} d_1 I & \tilde{\mathbf{x}} \\ \tilde{\mathbf{x}}^T & d_1 \end{pmatrix} \\ &\qquad\qquad 0 \preceq \begin{pmatrix} I & \tilde{\mathbf{x}} \\ \tilde{\mathbf{x}}^T & d_2 + \gamma \end{pmatrix} \ , \end{aligned} \qquad (20)$$

where $Q(\mathbf{x},\omega)$ is the minimum of the problem

$$
\begin{aligned}
&\min\ \mathbf{q}^T\mathbf{y}\\
&\text{subject to} \quad \begin{pmatrix} I & -\tilde{\mathbf{x}} \\ -\tilde{\mathbf{x}}^T & \tilde{\gamma}_k \end{pmatrix} \preceq \delta_k \begin{pmatrix} H_k & \mathbf{g}_k \\ \mathbf{g}_k{}^T & v_k \end{pmatrix}\\
&\qquad\qquad 0 \le \delta_k\ , \qquad k=1,\dots,K\\
&\qquad\qquad 0 \le \gamma - \tilde{\gamma}_k \le z_k\ , \qquad k=1,\dots,K\ .
\end{aligned}
\tag{21}
$$

3 Stochastic Second-order cone model for SLAR

As mentioned, before from a computational point of view the effort per iteration required by interior-point method to solve SOCP problems is less than that required to solve SDP's of similar size and structure. The aim of this section is thus to formulate the semidefinite stochastic location-aided routing (SLAR) problem presented in the previous section as a stochastic second-order cone SSOCP problem.

The constraint

$$
\begin{pmatrix} I & -\tilde{\mathbf{x}} \\ -\tilde{\mathbf{x}}^T & \gamma \end{pmatrix} \preceq \tau \begin{pmatrix} I & -\boldsymbol{l} \\ -\boldsymbol{l}^T & \|\boldsymbol{l}\|^2 - (t_1-t_0)^2 v^2 \end{pmatrix} \tag{22}
$$

is equivalent to

$$
0 \preceq \begin{pmatrix} \tau I - I & -\tau\,\boldsymbol{l} + \tilde{\mathbf{x}} \\ -\tau\,\boldsymbol{l}^T + \tilde{\mathbf{x}}^T & \tau\,\|\boldsymbol{l}\|^2 - \tau\,(t_1-t_0)^2\,v^2 - \gamma \end{pmatrix} \tag{23}
$$

and it holds if and only if, by Schur Complements, $\tau I - I > 0$, i.e. $\tau > 1$ (or if $\tau = 1$, $-\tau\,\boldsymbol{l}^T + \tilde{\mathbf{x}}^T = 0$), and

$$
\tau\,\|\boldsymbol{l}\|^2 - \tau\,(t_1-t_0)^2\,v^2 - \gamma - \left(-\tau\,\boldsymbol{l}^T + \tilde{\mathbf{x}}^T\right)(\tau I - I)^{-1}\,(-\tau\,\boldsymbol{l} + \tilde{\mathbf{x}}) \ge 0\ , \tag{24}
$$

or equivalently

$$
\left(\tau\,\|\boldsymbol{l}\|^2 - \tau\,(t_1-t_0)^2\,v^2 - \gamma\right)(\tau-1) - \left(-\tau\,\boldsymbol{l}^T + \tilde{\mathbf{x}}^T\right)(-\tau\,\boldsymbol{l} + \tilde{\mathbf{x}}) \ge 0\ , \tag{25}
$$

that is

$$
\tau\,\|\boldsymbol{l}\|^2 - \tau\,(t_1-t_0)^2\,v^2 - \gamma - \sum_{j=1}^{n} \frac{(-\tau l_j + \tilde{x}_j)^2}{(\tau-1)} \ge 0\ . \tag{26}
$$

If we define $\boldsymbol{r} = (r_1,\dots,r_n)$, where $r_j = \dfrac{(-\tau l_j + \tilde{x}_j)^2}{(\tau-1)}$ for all j such that $\tau > 1$ and $r_j = 0$ otherwise (see Alizadeh and Goldfarb (2003), [1]), then (26) is equivalent to

$$
\tau\,\|\boldsymbol{l}\|^2 - \tau\,(t_1-t_0)^2\,v^2 - \mathbf{1}^T\,\boldsymbol{r} \ge \gamma\ . \tag{27}
$$

Since we are minimizing the radius of the circle C, $\sqrt{\tilde{\mathbf{x}}^T\tilde{\mathbf{x}}-\gamma}$, we can relax the definition of r_j replacing it by $(\tau l_j - \tilde{x}_j)^2 \le r_j\,(\tau-1)$, $j=1,\dots,n$. Combining all the above constraints, (22) is equivalent to the following formulation involving only linear and restricted hyperbolic first-stage constraints:

$$\begin{aligned} (\tau l_j - \tilde{x}_j)^2 &\le r_j\,(\tau-1)\ , \qquad j=1,\dots,n\ , & (28)\\ \gamma &\le \tau\,\|\,\boldsymbol{l}\|^2 - \tau\,(t_1-t_0)^2\,v^2 - \mathbf{1}^T\,\boldsymbol{r}\ , & (29)\\ \tau &\ge 1\ . & (30)\end{aligned}$$

Notice that the restricted hyperbolic constraint (28) is equivalent to the following n 3-dimensional second-order cone inequalities:

$$\left\|\begin{pmatrix} 2\,(\tau l_j - \tilde{x}_j) \\ r_j - \tau + 1 \end{pmatrix}\right\| \le r_j + \tau - 1 \Leftrightarrow \begin{pmatrix} 2\,(\tau l_j - \tilde{x}_j) \\ r_j - \tau + 1 \\ r_j + \tau - 1 \end{pmatrix} \in \mathscr{K}_3\ , \quad j=1,\dots,n; \tag{31}$$

and each of the linear constraints (29) and (30) are 1-dimensional second-order cone constraints.

On the other hand

$$0 \preceq \begin{pmatrix} d_1 I & -\tilde{\mathbf{x}} \\ -\tilde{\mathbf{x}}^T & d_1 \end{pmatrix} \Leftrightarrow d_1 \ge \sqrt{\tilde{\mathbf{x}}^T\tilde{\mathbf{x}}} \Leftrightarrow \begin{pmatrix} d_1 \\ \tilde{\mathbf{x}} \end{pmatrix} \in \mathscr{K}_{n+1}\ , \tag{32}$$

and

$$0 \preceq \begin{pmatrix} I & -\tilde{\mathbf{x}} \\ -\tilde{\mathbf{x}}^T & d_2+\gamma \end{pmatrix} \Leftrightarrow d_2+\gamma \ge \tilde{\mathbf{x}}^T\tilde{\mathbf{x}} \Leftrightarrow \begin{pmatrix} \sqrt{d_2+\gamma} \\ \tilde{\mathbf{x}} \end{pmatrix} \in \mathscr{K}_{n+1}\ ; \tag{33}$$

the second stage constraint

$$\begin{pmatrix} I & -\tilde{\mathbf{x}} \\ -\tilde{\mathbf{x}}^T & \tilde{\gamma}_k \end{pmatrix} \preceq \delta_k \begin{pmatrix} H_k & \mathbf{g}_k \\ \mathbf{g}_k{}^T & v_k \end{pmatrix}\ , \quad k=1,\dots,K\ , \tag{34}$$

is equivalent to

$$M_k := \begin{pmatrix} \delta_k H_k - I & \delta_k \mathbf{g}_k + \tilde{\mathbf{x}} \\ \delta_k \mathbf{g}_k^T + \tilde{\mathbf{x}}^T & \delta_k v_k - \tilde{\gamma}_k \end{pmatrix} \succeq 0 \qquad k=1,\dots,K\ . \tag{35}$$

Following Alizadeh and Goldfarb (2003), [1], let $H_k = Q_k \Lambda_k Q_k^T$ be the spectral decomposition of H_k, $\Lambda_k = Diag\,(\lambda_{k1};\dots;\lambda_{kn})$ and $\mathbf{h}_k = Q_k^T\,(\delta_k \mathbf{g}_k + \tilde{\mathbf{x}})$, for $k=1,\dots,K$. Then

$$\bar{M}_k := \begin{pmatrix} Q_k^T & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} M_k \begin{pmatrix} Q_k & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \delta_k \Lambda_k - I & \mathbf{h}_k \\ \mathbf{h}_k^T & \delta_k v_k - \tilde{\gamma}_k \end{pmatrix} \succeq 0 \tag{36}$$

for $k=1\dots,K$, and $M_k \succeq 0$ if and only if $\bar{M}_k \succeq 0$. It holds if and only if $\delta_k \ge \frac{1}{\lambda_{min}(\Lambda_k)}$, i.e. $\delta_k\lambda_{kj} - 1 \ge 0\ \forall\, k,j$, $h_{kj}=0$ if $\delta_k\lambda_{kj}-1=0$ and the Shur complement of the columns and rows of $\bar{M}_i$ that are not zero

$$\delta_k v_k - \tilde{\gamma}_k - \sum_{\delta_k\lambda_{kj}>1} \frac{h_{kj}^2}{\delta_k\lambda_{kj}-1} \ge 0\ . \tag{37}$$

If we define $\boldsymbol{s_k} = (s_{k1}; \ldots; s_{kn})$, where $s_{kj} = \frac{h_{kj}^2}{\delta_k \lambda_{kj} - 1}$, for all j such that $\delta_k \lambda_{kj} > 1$ and $s_{kj} = 0$, otherwise, then (37) is equivalent to

$$\tilde{\gamma}_k \leq \delta_k v_k - \mathbf{1}^T \ \boldsymbol{s_k} \ . \tag{38}$$

Since we are minimizing the radius of the circle C^*, $\sqrt{\tilde{\mathbf{x}}^T \tilde{\mathbf{x}} - \tilde{\gamma}_k}$, we can relax the definition of s_{kj} replacing it by $h_{kj}^2 \leq s_{kj} \left(\delta_k \lambda_{kj} - 1\right)$, $k = 1, \ldots, K$, $j = 1, \ldots, n$. Combining all of the above constraints (34) is equivalent to the following formulation involving only linear and restricted hyperbolic second-stage constraints:

$$\begin{aligned}
\mathbf{h}_k &= Q_k^T \left(\delta_k \mathbf{g}_k + \tilde{\mathbf{x}}\right) \ , \qquad k = 1, \ldots, K \ , && (39)\\
h_{kj}^2 &\leq s_{kj} \left(\delta_k \lambda_{kj} - 1\right) \ , \qquad k = 1, \ldots, K, \quad j = 1, \ldots, n \ , && (40)\\
\tilde{\gamma}_k &\leq \delta_k v_k - \mathbf{1}^T \ \boldsymbol{s_k} \ , \qquad k = 1, \ldots, K, && (41)\\
\delta_k &\geq \frac{1}{\lambda_{min}\left(\Lambda_k\right)} \ , \qquad k = 1, \ldots, K \ . && (42)
\end{aligned}$$

Notice that the linear constraint (39) is equivalent to $2nK$ 1-dimensional second-order cone inequalities given by

$$\begin{aligned}
Q_k^T \left(\delta_k \mathbf{g}_k + \tilde{\mathbf{x}}\right) - \mathbf{h}_k &\geq 0 \ , \qquad k = 1, \ldots, K \ , && (43)\\
-Q_k^T \left(\delta_k \mathbf{g}_k + \tilde{\mathbf{x}}\right) + \mathbf{h}_k &\geq 0 \ , \qquad k = 1, \ldots, K \ ; && (44)
\end{aligned}$$

the restricted hyperbolic constraint (40) is equivalent to the following nK 3-dimensional second-order cone inequalities:

$$\left\| \begin{pmatrix} 2h_{kj} \\ s_{kj} - \delta_k \lambda_{kj} + 1 \end{pmatrix} \right\| \leq s_{kj} + \delta_k \lambda_{kj} - 1 \Leftrightarrow \begin{pmatrix} 2h_{kj} \\ s_{kj} - \delta_k \lambda_{kj} + 1 \\ s_{kj} + \delta_k \lambda_{kj} - 1 \end{pmatrix} \in \mathscr{K}_3 \ , \quad k = 1, \ldots K, \quad j = 1, \ldots, n, \tag{45}$$

and each of the linear constraints (41) and (42) are K 1-dimensional second-order cone constraints. In conclusion the SLAR model (20) and (21) can be formulate as a stochastic second-order cone SSOCP problem with two (n+1)-dimensional second-order cone constraints (see eqs. (32), (33)), $n(K+1)$ 3-dimensional second-order cone constraints (see eqs. (31) and (45)) and with all the other constraints linear, in the following way:

$$
\begin{aligned}
&\min\ \mathbf{c}^T\mathbf{x} + E[Q(\mathbf{x},\omega)] \\
&\text{subject to} \quad \begin{pmatrix} 2\,(\tau l_j - \tilde{x}_j) \\ r_j - \tau + 1 \\ r_j + \tau - 1 \end{pmatrix} \in \mathscr{K}_3\ , \quad j = 1,\ldots,n\ , \\
&\qquad\qquad \gamma \le \tau \,\|\, \boldsymbol{l}\|^2 - \tau\,(t_1 - t_0)^2\, v^2 - \mathbf{1}^T\ \boldsymbol{r}\ , \\
&\qquad\qquad 1 \le \tau\ , \\
&\qquad\qquad \begin{pmatrix} d_1 \\ \tilde{\mathbf{x}} \end{pmatrix} \in \mathscr{K}_{n+1}\ , \\
&\qquad\qquad \begin{pmatrix} \sqrt{d_2 + \gamma} \\ \tilde{\mathbf{x}} \end{pmatrix} \in \mathscr{K}_{n+1}\ ,
\end{aligned}
\tag{46}
$$

where $Q(\mathbf{x},\omega)$ is the minimum of the problem

$$
\min\ \mathbf{q}^T\mathbf{y} \tag{47}
$$

$$
\text{subject to} \qquad \begin{pmatrix} 2h_{kj} \\ s_{kj} - \delta_k\lambda_{kj} + 1 \\ s_{kj} + \delta_k\lambda_{kj} - 1 \end{pmatrix} \in \mathscr{K}_3\ , \quad k = 1,\ldots K, \quad j = 1,\ldots,n\ ,
$$

$$
\begin{aligned}
\mathbf{h}_k &= Q_k^T\,(\delta_k \mathbf{g}_k + \tilde{\mathbf{x}})\ , \quad k = 1,\ldots,K\ , \\
\tilde{\gamma}_k &\le \delta_k v_k - \mathbf{1}^T\ \boldsymbol{s_k}\ , \quad k = 1,\ldots,K\ , \\
\delta_k &\ge \frac{1}{\lambda_{min}\,(\Lambda_k)}\ , \quad k = 1,\ldots,K\ , \\
0 &\le \delta_k\ , \quad k = 1,\ldots,K\ , \\
0 &\le \gamma - \tilde{\gamma}_k \le z_k\ , \quad k = 1,\ldots,K\ ,
\end{aligned}
$$

or equivalently

$$
\begin{array}{ll}
\min \ \mathbf{c}^T\mathbf{x} + \sum_{k=1}^{K} p_k \mathbf{q}^T\mathbf{y} & \\
\text{subject to} & \begin{pmatrix} 2\left(\tau l_j - \tilde{x}_j\right) \\ r_j - \tau + 1 \\ r_j + \tau - 1 \end{pmatrix} \in \mathscr{K}_3 \ , \quad j = 1, \ldots, n \ , \\
& \gamma \leq \tau \parallel \boldsymbol{l} \|^2 - \tau \left(t_1 - t_0\right)^2 v^2 - \mathbf{1}^T \ \boldsymbol{r} \ , \\
& 1 \leq \tau \ , \\
& \begin{pmatrix} d_1 \\ \tilde{\mathbf{x}} \end{pmatrix} \in \mathscr{K}_{n+1} \ , \\
& \begin{pmatrix} \sqrt{d_2 + \gamma} \\ \tilde{\mathbf{x}} \end{pmatrix} \in \mathscr{K}_{n+1} \ , \\
& \begin{pmatrix} 2h_{kj} \\ s_{kj} - \delta_k \lambda_{kj} + 1 \\ s_{kj} + \delta_k \lambda_{kj} - 1 \end{pmatrix} \in \mathscr{K}_3 \ , \quad k = 1, \ldots K, \quad j = 1, \ldots, n \ , \\
& \begin{array}{rcl}
\mathbf{h}_k & = & Q_k^T \left(\delta_k \mathbf{g}_k + \tilde{\mathbf{x}}\right) \ , \quad k = 1, \ldots, K \ , \\
\tilde{\gamma}_k & \leq & \delta_k v_k - \mathbf{1}^T \ \boldsymbol{s_k} \ , \quad k = 1, \ldots, K \ , \\
\delta_k & \geq & \dfrac{1}{\lambda_{min}\left(\Lambda_k\right)} \ , \quad k = 1, \ldots, K \ , \\
0 & \leq & \delta_k \ , \quad k = 1, \ldots, K \ , \\
0 & \leq & \gamma - \tilde{\gamma}_k \leq z_k \ , \quad k = 1, \ldots, K \ .
\end{array}
\end{array}
\tag{48}
$$

We observe that in the implementation the constraint

$$
\begin{pmatrix} \sqrt{d_2 + \gamma} \\ \tilde{\mathbf{x}} \end{pmatrix} \in \mathscr{K}_{n+1} \ , \tag{49}
$$

has been treated as a rotated quadratic cone (or hyperbolic constraint)

$$
\mathscr{K}_{n+2} = \left\{ \mathbf{u} \in \Re^{n+2} : 2u_1u_2 \geq \sum_{j=3}^{n+2} u_j^2, u_1, u_2 \geq 0 \right\} \tag{50}
$$

with $u_2 = d_2 + \gamma$, $u_j = x_{j-2}, \ j = 3, \ldots, n+2$ intersected with the hyperplane $u_1 = 1/2$.

At first we implement the following model:

$$
\begin{aligned}
&\min \ \mathbf{c}^T\mathbf{x} + \textstyle\sum_{k=1}^{K} p_k \mathbf{q}^T\mathbf{y} = \\
&\text{subject to} \\
&\left(\tau \parallel \boldsymbol{l}\|^2 - \tau\,(t_1 - t_0)^2\, v^2 - \gamma\right)(\tau - 1) \geq \|\tau\ \boldsymbol{l} + \tilde{\mathbf{x}}\|^2 \ , \\
&\qquad \tau \geq 1 \ , \\
&\qquad d_1^2 \geq \|\tilde{\mathbf{x}}\|^2 \ , \\
&\quad d_2 + \gamma \geq \|\tilde{\mathbf{x}}\|^2 \ , \\
&(s_{kj} + \delta_k\lambda_{kj} - 1)^2 \geq \left\| \begin{pmatrix} 2h_{kj} \\ s_{kj} - \delta_k\lambda_{kj} + 1 \end{pmatrix} \right\|^2 \ , \\
&\begin{array}{rcl}
\mathbf{h}_k & = & Q_k^T\left(\delta_k\mathbf{g}_k + \tilde{\mathbf{x}}\right) \ , \quad k = 1, \ldots, K \ , \\
\tilde{\gamma} & \leq & \delta_k v_k - \mathbf{1}^T\mathbf{s}_k \ , \quad k = 1, \ldots, K \ , \\
\delta_k & \geq & \dfrac{1}{\lambda_{min}\left(\Lambda_k\right)} \ , \quad k = 1, \ldots, K \ , \\
\delta_k & \geq & 0 \ , \quad k = 1, \ldots, K \ , \\
\gamma - \tilde{\gamma}_k & \geq & 0 \ , \quad k = 1, \ldots, K \ , \\
\gamma - \tilde{\gamma}_k & \leq & z_k \ , \quad k = 1, \ldots, K \ , \\
d_1 & \geq & 0 \ , \\
d_2 & \geq & 0 \ .
\end{array}
\end{aligned}
\tag{51}
$$

4 Ellipsoid scenarios generation

In this section we consider the generation of random ellipsoids

$$
E_k = \{\mathbf{x} \in \Re^n : \mathbf{x}^T H_k \mathbf{x} + 2\mathbf{g}_k{}^T\mathbf{x} + \nu_k \leq 0\}, \quad k = 1, 2, \ldots, K \ . \tag{52}
$$

In our computational experiment we have considered the case $n = 2$, that is we have generated real ellipses in the plane $\Re^2$.
The general algebraic equation for a second-order curve is of the type:

$$
e_{11}x_1^2 + 2e_{12}x_1x_2 + e_{22}x_2^2 + 2e_{13}x_1 + 2e_{23}x_2 + e_{33} = 0 \ , \tag{53}
$$

or equivalently, in matricial notation

$$\left(\begin{array}{cc} x_1 & x_2 \end{array} \right) \left(\begin{array}{cc} e_{11} & e_{12} \\ e_{21} & e_{22} \end{array} \right) \left(\begin{array}{c} x_1 \\ x_2 \end{array} \right) + 2 \left(\begin{array}{cc} e_{13} & e_{23} \end{array} \right) \left(\begin{array}{c} x_1 \\ x_2 \end{array} \right) + e_{33} = 0$$

In order to be an ellipse, the coefficients of the simmetrix matrix E associated to eq. (53):

$$E = \left(\begin{array}{ccc} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{array} \right) ,$$

have to satisfy the following conditions on the sign of the invariants (see e.g. Ilyin and Poznyak (1981), [5]):

$$I_2 = \left| \begin{array}{cc} e_{11} & e_{12} \\ e_{21} & e_{22} \end{array} \right| > 0 , \quad I_3 = \left| \begin{array}{ccc} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{array} \right| < 0 . \tag{54}$$

We note that for each scenario k the coefficient H, $\mathbf{g}$ and ν of eq. (52) correspond to

$$H = \left(\begin{array}{cc} e_{11} & e_{12} \\ e_{21} & e_{22} \end{array} \right) , \quad \mathbf{g} = \left(\begin{array}{cc} e_{13} & e_{23} \end{array} \right) , \quad \nu = e_{33} ,$$

and in order to satisfy the condition $H \succ 0$ we have to consider also

$$e_{11} > 0 . \tag{55}$$

The center O, $\mathbf{x}^0 = (x_1^0, x_2^0)$ of the second-order curve (53) is obteined as solution of the following system:

$$\begin{cases} e_{11}x_1^0 + e_{12}x_2^0 + e_{13} = 0 , \\ e_{12}x_1^0 + e_{22}x_2^0 + e_{23} = 0 , \end{cases} \tag{56}$$

the angle φ between the x_1-axis and the main axis of the conic is such that

$$\cot 2\varphi = \frac{e_{11} - e_{12}}{2e_{12}} , \tag{57}$$

or equivalently is given by

$$\varphi = \left(\frac{\pi}{4} - \frac{1}{2} \arctan\left(\frac{e_{11} - e_{12}}{2e_{12}}\right)\right) , \tag{58}$$

and the semiaxes s_{x_1} and s_{x_2} of the ellipse are equal to

$$\begin{cases} s_{x_1} = \sqrt{\dfrac{-I_3}{I_2 \left(e_{12} \sin 2\varphi + \frac{1}{2} (e_{11} - e_{22}) \cos 2\varphi + \frac{1}{2} (e_{11} + e_{22})\right)}} \\ s_{x_2} = \sqrt{\dfrac{-I_3}{I_2 \left(-e_{12} \sin 2\varphi - \frac{1}{2} (e_{11} - e_{22}) \cos 2\varphi + \frac{1}{2} (e_{11} + e_{22})\right)}} \end{cases} \tag{59}$$

Thus the parametric equation of the ellipse is given by

$$\begin{cases} x_1 = s_{x_1} \cos\varphi \cos\vartheta - s_{x_2} \sin\varphi \sin\vartheta + x_1^0 \ , & \vartheta \in [0, 2\pi] \\ x_2 = s_{x_1} \sin\varphi \cos\vartheta + s_{x_2} \cos\varphi \sin\vartheta + x_2^0 \ , & \vartheta \in [0, 2\pi] \ . \end{cases} \tag{60}$$

We construct the coefficient e_{ij}^k of random ellipsoid E_k corresponding to scenario k in this way:

- we extract the coefficient e_{11}^k by a uniform distribution in the interval $(0, 1]$, the coefficients $e_{12}^k = e_{21}^k$, $e_{13}^k = e_{31}^k$, and $e_{23}^k = e_{32}^k$ by a uniform distribution in the interval $[-1, 1]$;
- we fix $\text{treshold1}^k := \dfrac{(e_{12}^k)^2}{e_{11}}$ and in order to satisfy the condition (54) on I_2 we impose $e_{22}^k > \text{treshold1}^k$; thus we extract e_{22}^k by a uniform distribution in the interval $\left(\text{treshold1}^k, 11\ \text{treshold1}^k\right]$;
- we fix $\text{treshold2}^k := \left(-e_{12}^k e_{23}^k e_{13}^k - e_{13}^k e_{12}^k e_{23}^k + (e_{13}^2)^k e_{22}^k + (e_{23}^2)^k e_{11}^k\right)/I_2$ and in order to satisfy the condition on I_3 of eq. (54) we need $e_{33}^k < \text{treshold2}^k$; thus we extract e_{33}^k by a uniform distribution in the interval $\left(-\text{treshold2}^k, \text{treshold2}^k\right)$ if $\text{treshold2}^k > 0$ and in $\left(\text{treshold2}^k, -\text{treshold2}^k\right)$ otherwise.

Furthermore, in order to simulate a real case, the ellipsoid scenarios are generated in such a way they belong to an area closed to the location $\boldsymbol{l}$ of the destination node D at time t_0 and by imposing upper bounds $s_{x_1}^{max}$ and $s_{x_2}^{max}$ on the length of the main semiaxis $s_{x_1}^k$ and $s_{x_2}^k$ of the ellipsoid, according to the following conditions:

$$\begin{cases} 0 \le \sqrt{(\mathbf{x}^{0,k} - \boldsymbol{l})^T (\mathbf{x}^{0,k} - \boldsymbol{l})} \le d^{max} \ , & \forall\, k = 1, \ldots, K \ , \\ 0 < s_{x_1}^k \le \ s_{x_1}^{max} \ , \quad \forall\, k = 1, \ldots, K \ , \\ 0 < s_{x_2}^k \le \ s_{x_2}^{max} \ , \quad \forall\, k = 1, \ldots, K \ , \end{cases} \tag{61}$$

where d^{max} is an upper bound on the distance between $\boldsymbol{l}$ and the center $\mathbf{x}^0$ of the ellipsoid.

5 Numerical results

In this section we present numerical results obtained for the semidefinite stochastic location-aided routing (SLAR) problem presented in section 2 and stochastic second order cone SSOCP presented in section 3. The simulation is based on the scenarios randomly generated under MATLAB 7.4.0 framework, according to the method described in the previous section where we have set $d^{max} = 3$ and $s_{x_1}^{max} = s_{x_2}^{max} = 3$. The problem was implemented in GAMS 22.5 framework. At first we solved the model (51) using the Minos solver, then the stochastic second order cone SSOCP (48) with the Mosek one.

In our computational experiment we have supposed that the scenarios are equiprobable; furthermore we have fixed the location $\boldsymbol{l} = (1, 1)$ of the node D at initial time $t_0 = 0$, its average speed $v = 1$, and the final time $t_1 = 1$ so that the disk C_0 is described by the equation:

$$C_0 = \{\mathbf{x} \in \Re^2 : x_1^2 + x_2^2 - 2x_1 - 2x_2 + 1 = 0\} \tag{62}$$

with center in $\boldsymbol{l} = (1, 1)$ and unitary radius.
The first and second stage costs $\mathbf{c}$ and $\mathbf{q}$ are given by:

$$\mathbf{c} = [0.1, 0.5, \mathbf{0}, 0, 0]^T , \tag{63}$$
$$\mathbf{q} = [\mathbf{0.5}, \mathbf{0}, \mathbf{0}]^T . \tag{64}$$

Table 1 refers to the centres (x_1^0, x_2^0), the angle φ between the x_1-axis and the main axis of the conic and to semiaxes s_{x_1} and s_{x_2} of five ellipsoids E_k, $k = 1, \ldots, 5$ randomly generated according to the procedure described in the previous section.

k	x_1^0	x_2^0	φ	s_{x_1}	s_{x_2}
1	-0.7602	0.0344	1.4161	0.2005	0.6919
2	-1.4598	-0.3381	0.1535	1.8714	0.6180
3	-1.3677	-0.0986	0.2077	1.9617	0.8421
4	-0.56036	-0.1551	1.2429	0.5319	0.9667
5	-1.5149	-0.3629	0.1391	2.2671	0.6893

Table 1: Centre (x_1^0, x_2^0), angle φ between the x_1-axis and the main axis of the conic and semiaxes s_{x_1} and s_{x_2} of five ellipsoids E_k, $k = 1, \ldots, 5$ randomly generated according to the procedure described in the previous section.

At first we have considered a sensitivity analysis of the solution according to different lower bounds of the second stage variable z_k; the relative results are reported in Table 2 and represented in Figure 1 in the case of one scenario (see the first line of Table 1 for the values of one scenario parameters).

K	$\tilde{x}_1$	$\tilde{x}_2$	d_1	d_2	γ	$\tilde{\gamma}_1$	τ	z_1	obj. value
1	0.27	0.68	0.73	3.24	-2.7	-2.7	1.8	$z_1 \geq 0$	1.69
1	0.28	0.69	0.74	3.19	-2.64	-2.74	1.79	$z_1 \geq 0.1$	1.72
1	0.33	0.71	0.78	2.99	-2.38	-2.88	1.73	$z_1 \geq 0.5$	1.83

Table 2: Solution in the case of one scenario according to the increasing value for lower bound on the second stage decision variable z_1.

As we can see in Figure 1(a), because we are in the deterministic case, and the future is completely known, the first stage disc C with centre in $(0.27, 0.68)$ and radius $\sqrt{\tilde{x}_1^2 + \tilde{x}_2^2 - \gamma} =$

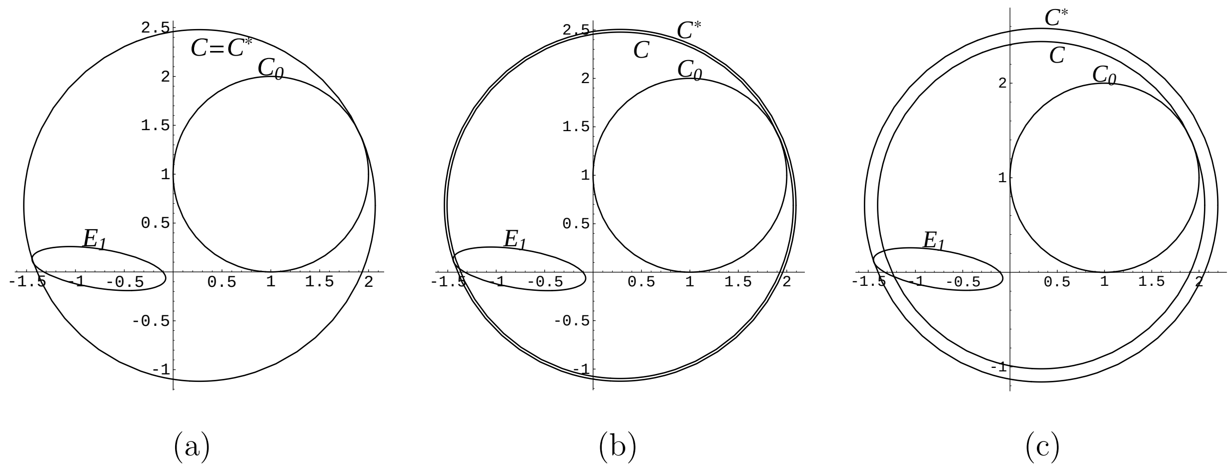

Figure 1: Solution in the case of one scenario with lower bound on z_1 respectively given by (a) $z_1 = 0$, (b) $z_1 = 0.1$ and (c) $z_1 = 0.5$.

1.8 coincides with the second-stage one C_1^* ($\gamma = \tilde{\gamma}_1$, see sixth and seventh columns of Table 2). Figures 1(b)-(c) refer to the cases where the lower bounds are given by $z_1 = 0.1$ and $z_1 = 0.5$ respectively. As expected in all the cases considered the disk C_1^* contains the ellipse E_1, the disks C and C_0 (see eq. (62)).

Furthermore, we want to analyze the convergence of optimal first-stage solutions and function value as the number of scenarios increases. In this test we set the lower bound for the second-stage decision variable $z_1 = 0$. Table 3 reports the first-stage decision variables according to the increasing number of scenarios $k = 1, \ldots, 5$. In particular Figure 2 refers to the case of five scenarios reported in Table 1. In this case the second-stage decision variables $\tilde{\gamma}_k$, $k = 1, \ldots, 5$ obtained, related to the radius of the second-stage disks C_k^* with centre in $(\tilde{x}_1, \tilde{x}_2) = (-0.64, 0.33)$ are: $\tilde{\gamma}_1 = \tilde{\gamma}_3 = \tilde{\gamma}_4 = -7.18$, $\tilde{\gamma}_2 = -7.52$ and $\tilde{\gamma}_5 = -10.23$. As expected each second-stage disk C_k^*, $k = 1, \ldots, 5$ contain the disks C_0, C and the ellipse E_k of the corresponding scenario k.
Notice that the starting points given to the first-stage decision variables d_1, d_2, $\tilde{x}_1$, $\tilde{x}_2$ and τ, by using the Minos solver, are: $d_1 = 0.8$, $d_2 = 2.3$, $\tilde{x}_1 = -0.5$, $\tilde{x}_2 = 0.3$ and $\tau = 1.5$.

K	$\tilde{x}_1$	$\tilde{x}_2$	d_1	d_2	γ	τ	obj. value
1	0.27	0.68	0.73	3.24	-2.7	1.8	1.69
2	0.82	0.8	1.15	1.6	-0.28	1.27	0.92
5	-0.64	0.33	0.72	7.70	-7.18	2.77	4.26

Table 3: First stage decision solutions according to the increasing number of scenarios $k = 1, \ldots, K$.

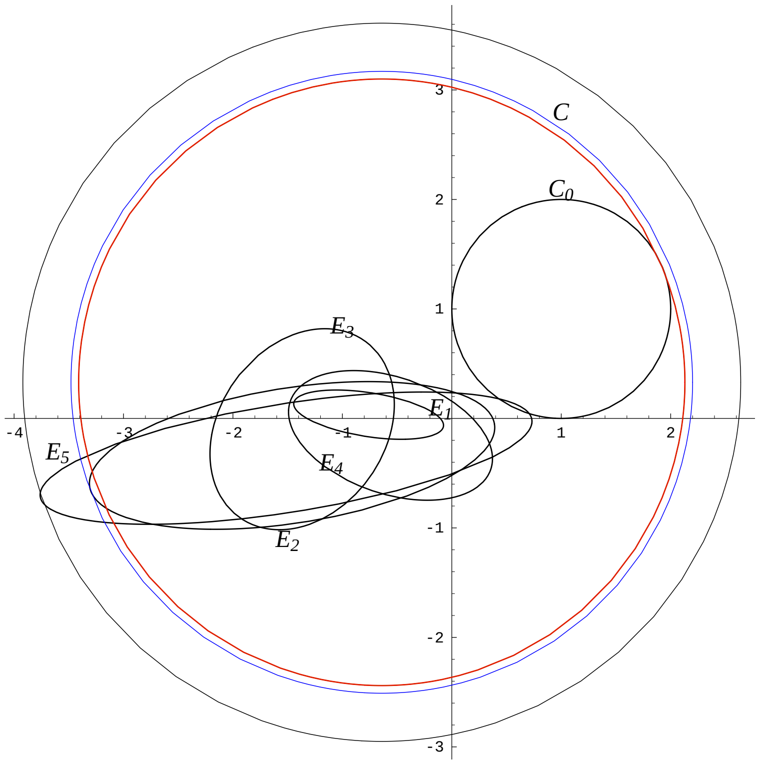

Figure 2: Solution in the case of five scenarios $k = 1, \dots, 5$.

The problem we faced to, by using the Minos solver was that, being it a local non linear solver, by increasing the number of scenarios, the dimension of the initial starting points of the variables, increases too and it becomes difficult to identify the right points. On the contrary, the advantage to use Mosek is that we transform non linear conic constraints in linear ones. Therefore we don't need to fix initial starting points. Furthermore it allows to solve problems with a large number of scenarios (say 4040) versus few scenarios of Minos solver (see Table 4 for the optimal first-stage solutions and the optimal function value obtained by using the Mosek solver). Notice that, in the case of five scenarios, the execution time by using both Minos5 and Mosek solvers is 0.016 seconds (cpu time 0.07); in the first case we have 14 blocks of equations, 52 single equations, 8 block of variables, 42 single variables and the number of iterations is 1037; on the contrary in the second case we have 25 blocks of equations, 94 single equations, 19 block of variables and 84 single variables and the optimal solutions is obtained after just 10 iterations. Notice that the execution time by using the Mosek solver, for the largest case considered of 4040 scenarios is of 0.718 seconds (cpu time 3.26), it is composed by 25 blocks of equations, 60619 single equations, 19 block of variables and 52539 single variables, the number of iterations is 47.

By the results shown in Table 4 we deduce that the model gives an *in-sample stability*, that is whichever number of scenarios we consider, the optimal objective values are approximately the same (for a definition of in-sample stability see Kaut and Wallace, (2007) [7]). In particular,

Figure 3 shows the convergence of the optimal profit value as the number of scenarios increases. However, we have to remember that the values presented in Table 4 represent *in-sample*

K	$\tilde{x}_1$	$\tilde{x}_2$	d_1	d_2	γ	τ	obj. value
50	0.64	0.48	0.8	2.67	-2.03	1.63	3.24
60	0.46	0.5	0.67	3.03	-2.57	1.74	3.37
70	0.64	0.61	0.89	2.33	-1.54	1.53	3.09
80	0.76	0.58	0.96	2.2	-1.28	1.48	3.38
90	0.55	0.65	0.85	2.47	-1.75	1.57	3.22
100	0.5	0.51	0.72	2.89	-2.38	1.7	3.78
150	0.74	0.53	0.92	2.34	-1.50	1.53	3.72
200	0.53	0.58	0.78	2.66	-2.05	1.63	3.44
250	0.66	0.51	0.84	2.53	-1.82	1.59	3.61
300	0.6	0.48	0.77	2.75	-2.16	1.66	3.43
350	0.66	0.54	0.85	2.47	-1.74	1.57	3.85
400	-0.62	0.49	0.79	2.69	-2.07	1.64	3.6
450	0.53	0.53	0.75	2.76	-2.19	1.66	3.75
500	0.68	0.53	0.87	2.45	-1.7	1.57	3.94
1000	0.59	0.58	0.82	2.54	-1.86	1.59	3.56
1500	0.65	0.55	0.85	2.47	-1.75	1.57	3.64
2000	0.61	0.52	0.8	2.62	-1.98	1.62	3.57
2670	0.63	0.55	0.84	2.50	-1.79	1.58	3.57
3580	0.62	0.53	0.82	2.57	-1.90	1.60	3.52
3800	0.60	0.54	0.81	2.59	-1.94	1.61	3.59
4040	0.63	0.54	0.83	2.54	-1.86	1.59	3.58

Table 4: First stage decision solutions according to the increasing number of scenarios $k = 1, \ldots, 4040$ and optimal profit value.

values, so the costs are not directly comparable. To be able to estimate the effect of using a better scenario tree, we have to compare the *out-of-sample* costs (see again Kaut and Wallace, (2007), [7]). For this purpose, we declare the tree composed by 4040 scenarios to be the true representation of the real world and use it as a *benchmark* to evaluate the cost of optimal solutions obtained using other trees with a smaller number of branches or scenarios. We report in Table 5 some of the results of the out-of-sample analysis with benchmark the case of 4040 scenarios.

Furthermore, to check the importance of modelling the randomness of the parameters, we compare the optimal solutions and objective value of the stochastic model with those

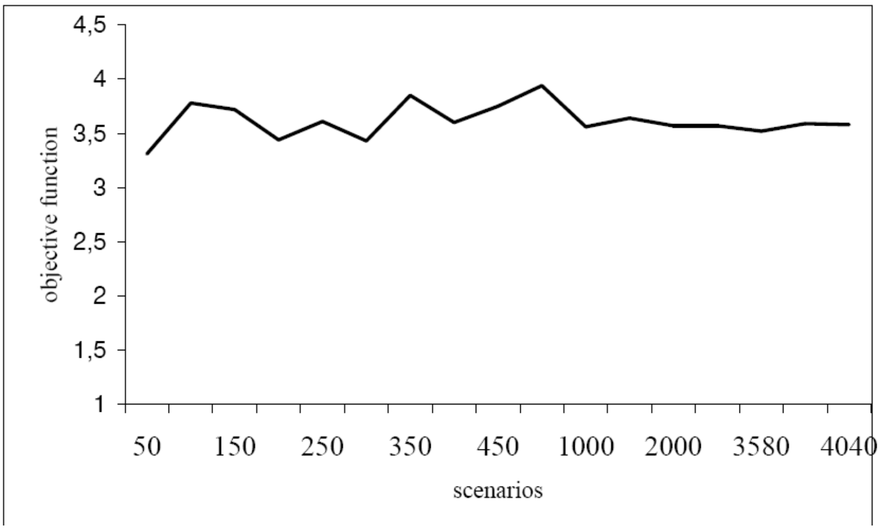

Figure 3: Convergence of the optimal function value for an increasing number of ellipsoid scenarios.

K in 4040	*obj. value*
50 in 4040	3.58
100 in 4040	3.60
200 in 4040	3.59
300 in 4040	3.58
400 in 4040	3.58
1000 in 4040	3.59

Table 5: Out-of-sample objective value obtained by substituting the first-stage solutions in the case of $K = 50, 100, 200, 300, 400, 1000$ scenarios, into the benchmark tree made by 4040 branches.

obtained from the corresponding *deterministic* model, where is considered an unique scenario represented by an ellipse with the centre $(x_1^{mean}, x_2^{mean}) = (0.2485, 0.063)$, the angle $\varphi^{mean} = 0.786$, the semiaxis $s_{x_1}^{mean} = 0.7618$, $s_{x_2}^{mean} = 0.7771$ given respectively by the mean of the centres, of the angles and of the semiaxes of the ellipses E_k, $k = 1, \ldots, 4040$; its parametric parametric equation is given by

$$\begin{cases} x_1 = 0.7618 \cos(0.786) \cos\vartheta - 0.7771 \sin(0.786) \sin\vartheta + 0.2485 \ , & \vartheta \in [0, 2\pi] \ , \\ x_2 = 0.7618 \sin(0.786) \cos\vartheta + 0.7771 \cos(0.786) \sin\vartheta + 0.063 \ , & \vartheta \in [0, 2\pi] \ . \end{cases} \tag{65}$$

In literature, this kind of problem is called *Expected value problem* or *Mean value problem*, (see Birge and Louveaux, (1997) [4] and Kall and Wallace (1994) [6]).

Solutions to the deterministic model are reported in Table 6 and shown in Figure 4.

K	$\tilde{x}_1$	$\tilde{x}_2$	d_1	d_2	γ	τ	obj. value
1	0.71	0.65	0.97	2.10	-1.17	1.45	1.15

Table 6: First stage decision solutions and optimal profit value in the case of *Expected value problem* with one scenario given by eq. (65).

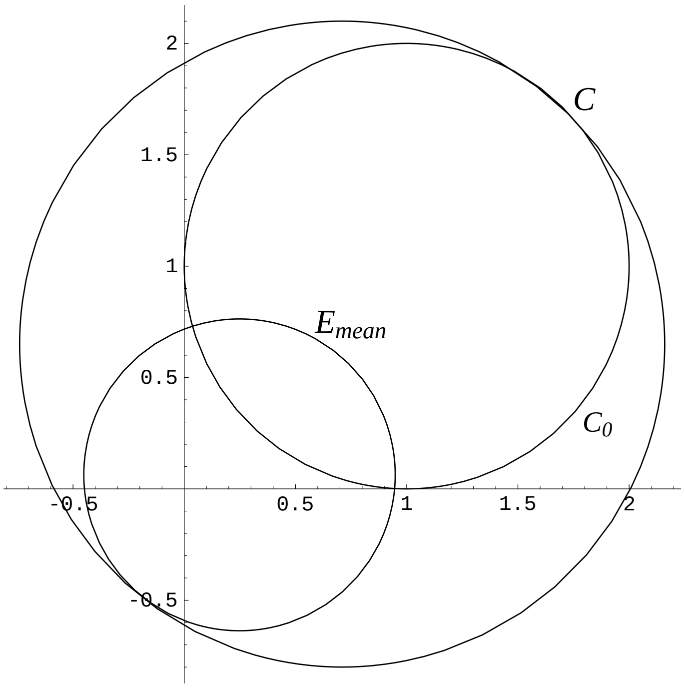

Figure 4: Solution in the case of *Expected value problem* with one scenario ellipse given by eq. (65).

Because in a deterministic problem the future is completely known, the first stage disk C coincides with the second stage one C_1^* ($\gamma = \tilde{\gamma} = -1.17$) and consequently the total cost is much smaller than in the stochastic case. However, we have to remember that this is an *in-sample* objective value (using the terminology from Kaut and Wallace, (2007) [7]) and the true cost of the solution—or the *out-of-sample* objective value—is likely to be higher. To see how much, we can solve the stochastic model with 4040 scenarios and the first-stage variables fixed to the deterministic solution. The result is a total cost of 3.60, much higher than the predicted (in-sample) cost of 1.15. Furthermore, we see that the resulting total cost is higher than the optimal solution for the benchmark tree with 4040 branches. The difference is known as the *Value of stochastic solution* (VSS), (see e.g. Birge and Louveaux, (1997) [4]). In our case, it is

$$\begin{aligned} VSS &= \textit{obj. val. (det. sol. on benchmark tree)} - \textit{obj. val.(opt. sol. of benchmark tree)} \\ &= 3.60 - 3.58 = 0.02 \; . \end{aligned}$$

The low VSS indicates that, because of the high computational effort to solve the stochastic model, it is safe to save time by rather solving the mean value problem.

Another measure of the role of the randomness of the parameters in the model is given by the *Expected value of perfect information* (EVPI) (see again e.g. Birge and Louveaux, (1997) [4]), given by the difference between the optimal objective value of the stochastic model with 4040 scenarios, also called *here-and-now* solution and the expected value of the *wait-and-see* solution (WS), calculated by finding the optimal solution for each possible realization of the random variables, as follow:

$$\begin{aligned} EVPI &= obj.\ val.\ (opt.\ sol.\ of\ benchmark\ tree) - obj.\ val.(WS) \\ &= 3.58 - 1.91 = 1.67\ . \end{aligned}$$

This means that we should be ready to pay 1.67 in return for complete information before, about the direction and velocity of the destination node D. The large value obtained for EVPI means that the randomness plays an important role in the problem.

6 Conclusions

We have proposed a two-stage stochastic second order cone programming for the stochastic location aided routing model. For the uncertainty representation through scenarios, we have used a simple approach of sampling from a uniform distribution. A convergence both of the optimal value of the objective function and of the first stage decision variables has been proved. The computed expected value of perfect information (EVPI) shows that a good estimate of the stochastic parameters plays an important role in the problem even if the value of stochastic solution (VSS) is still limited. A possible extension of the work should include a more sophisticated representation of the uncertainty.

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