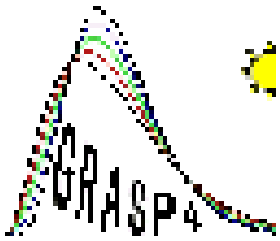
MARKOV SWITCHING AUTOREGRESSIVE MODELS: HIDDEN CHAIN RECONSTRUCTION, MISSING OBSERVATIONS RESTORATION, FORECASTING

ROBERTA PAROLI
Istituto di Statistica - Università Cattolica S.C. di Milano
rparoli@mi.unicatt.it

LUIGI SPEZIA
Dipartimento di Ingegneria - Università degli Studi di Bergamo
spezia@unibg.it



Research Group for Statistical Applications to Environmental Problems

Working Paper n. 11
December 2001

Abstract - We consider Markov switching autoregressive models to tackle the prediction problem of nonlinear time series with missing data. We propose an iterative procedure based on the EM algorithm to obtain the maximum likelihood estimators of the parameters of the model. We use a Monte Carlo procedure to reconstruct the sequence of the hidden states and then we restore the missing observations; this methodology allows us to forecast the future hidden states and the future values of the observed variable, too. This method is applied to analyse a time series of daily mean concentrations of sulphur dioxide; the different levels of the pollution are described by the dynamics of the hidden states.

Keywords - nonlinear time series, hidden Markov chain, EM algorithm, missing data.

Introduction[1]

Markov Switching AutoRegressive (MSAR) models make up a class of models for non-linear time series that present different behaviours according to the dynamics of *hidden* (or *latent*) random variables, usually known as *state variables* or *regime variables*. The MSAR models assume the dynamics of the regime variables is described by an unobservable Markov chain and the dynamics of the observed time series is modelled by an AR process, whose parameters depend on the states of the Markov chain. These models have been introduced in econometric literature by Hamilton (see for example Hamilton (1994), ch. 22 and the references therein) to study economic and financial time series with asymmetric cycles and changes in regime generated by a stochastic process (see Krozlig (1997), Kim and Nelson (1999), Franses and van Dijk (2000) for many applications and generalizations). These models are also an extension of *hidden Markov models* (MacDonald and Zucchini (1997)).

MSAR models of order (m, p) are discrete-time stochastic processes $\{Y_t; X_t\}$ such that $\{X_t\}$ is an unobservable discrete Markov chain with a finite number of states, m, while $\{Y_t\}$, given $\{X_t\}$, is an observed autoregressive process of order p, with the conditional distribution of Y_t depending on $\{X_t\}$ only through the contemporary X_t.

Optimal inference and forecasting method for MSAR models have been discussed by Hamilton (1994), but without analysing the sequence of the hidden states. In this paper we propose an iterative procedure to reconstruct the hidden states and they will be used in making inference and forecasting. According to the optimal selected model, given the maximum likelihood estimators of the parameters, the sequence of the hidden states of the Markov chain is reconstructed by means of a Monte Carlo procedure; after that the missing observations occuring within the series can be restored computing the expected value of the conditional distribution of Y_t given the p past observations and the contemporary reconstructed state. The same procedure can be adopted to forecast the values of the observed variable.

The basic model used to study univariate stationary nonlinear time series will be introduced in Section 1; in Section 2, by means of the EM algorithm, the maximum likelihood estimators of the unknown parameters will be obtained and they will be used in Section 3 to reconstruct the hidden chain, to restore the missing observations and to forecast the future hidden states and the future values of the observed process. Finally, in Section 4, an environmental application will be shown: we will examine a data set about the daily mean concentrations of sulphur dioxide, recorded by an air pollution testing station in Bergamo, where we have 21 missing values within a sequence of 1064 observations.

1. The MSAR model

Let $\{X_t\}_{t\in(1,\dots,T)\subset\mathbb{N}}$ be a discrete, homogeneous, aperiodic, irreducible Markov chain on a finite state-space $S_X = \{1, 2, \dots, m\}$. The transition probabilities matrix is $\Gamma = [\gamma_{i,j}]$, with $\gamma_{i,j} = \mathrm{P}(X_t = j \mid X_{t-1} = i)$, for any $i, j \in S_X$, and the initial distribution is $\delta = (\delta_1, \delta_2, \dots, \delta_m)'$, where $\delta_i = \mathrm{P}(X_1 = i)$, for any $i \in S_X$. Since $\{X_t\}$ is a homogeneous, irreducible Markov chain, defined on a finite state-space, it has an initial distribution δ which is stationary, that is $\delta_i = \mathrm{P}(X_t = i)$ for any $t = 1, \dots, T$; hence the equality $\delta' = \delta'\Gamma$

[1]Work partially supported by MURST 2000 grant "Statistical Methods for Environmental Analysis".

holds. Finally, the hypothesis characterizing this class of models is the unobservability of the Markov chain $\{X_t\}$: the sequence of states of the Markov chain is *hidden* in the observations. Notice that the states of the chain can have either a convenient interpretation suggested by the nature of the observed phenomenon, or be supposed only for convenience in formulating the model.

Let $\{Y_t\}_{t\in(1,\ldots,T)\subset\mathbb{N}}$ be some discrete stochastic process, on a continuous space $S_Y \equiv \mathbb{R}$. The process $\{Y_t\}$ must satisfy two conditions: (i) *contemporary dependence condition* - every observation depends on the contemporary state of the chain; (ii) *order-p dependence condition* - the present observation depends on the p past observations. Hence the equation describing MSAR(m,p) models is

$$Y_{t(i_t)} = \mu_{i_t} + \theta_{1(i_t)}Y_{t-1(i_{t-1})} + \theta_{2(i_t)}Y_{t-2(i_{t-2})} + \ldots + \theta_{p(i_t)}Y_{t-p(i_{t-p})} + E_{t(i_t)}, \qquad (1)$$

where $Y_{t(i)}$ denotes the generic variable Y_t when $X_t = i$, for any $1 \leq t \leq T$, and the autoregressive coefficients $\theta_{h(i)}$, for any $h = 1,\ldots,p$, depend on the current state i of the Markov chain. $E_{t(i)}$ denotes the gaussian random variable E_t when $X_t = i$, with zero mean and variance σ_i^2 $\left(E_{t(i)} \sim \mathcal{N}\left(0;\sigma_i^2\right)\right)$, for any $i \in S_X$, with the discrete process $\{E_t\}$, given $\{X_t\}$, satisfying the conditional independence and the contemporary dependence conditions.

As special cases of MSAR models, we have linear autoregressive models of order p, AR(p), when $m = 1$, and gaussian hidden Markov models, when $p = 0$.

From equation (1), the generic distribution of $Y_{t(i)}$, given the p past observations and the current latent state, is gaussian with mean $\mu_i + \sum_{h=1}^{p} \theta_{h(i)}y_{t-h}$ and variance σ_i^2, $(Y_t \mid y_{t-1},\ldots,y_{t-p},i) \sim \mathcal{N}\left(\mu_i + \sum_{h=1}^{p} \theta_{h(i)}y_{t-h};\sigma_i^2\right)$, with probability density function (pdf)

$$f(y_t \mid y_{t-1},\ldots,y_{t-p},i) = \frac{1}{\sqrt{2\pi}\ \sigma_i} \exp\left[-\frac{1}{2}\left(\frac{y_t - \mu_i - \sum_{h=1}^{p} \theta_{h(i)}y_{t-h}}{\sigma_i}\right)^2\right], \qquad (2)$$

for any $t = 1,\ldots,T$.

A sufficient condition for the stationarity of the process (1) is that all the m subprocesses generated by the m states of the chain are stationary, that is, for any $i \in S_X$, the roots of the auxiliary equation $z^p - \theta_{1(i)}z^{p-1} - \ldots - \theta_{p(i)} = 0$, where z is a complex variable, are all inside the unit circle.

Hence the so-defined MSAR models $\{Y_t;X_t\}_{t\in(1,\ldots,T)\subset\mathbb{N}}$ are characterized by the stationary initial distribution δ, by the transition probabilities matrix Γ and by the state-dependent pdfs $f(y_t \mid y_{t-1},\ldots,y_{t-p},i)$.

2. Parameter estimation

In MSAR(m,p) models the parameters to be estimated are the $m^2 - m$ transition probabilities $\gamma_{i,j}$, for any $i = 1,\ldots,m$; $j = 1,\ldots,m-1$ (the entries of the m^{th} column of Γ are obtained by difference, given that each row of Γ sums up to one), the m entries of the vector δ, the m parameters μ_i, entries of the vector $\mu = (\mu_1,\ldots,\mu_m)'$, the m parameters σ_i^2, entries of the vector $\sigma^2 = (\sigma_1^2,\ldots,\sigma_m^2)'$ and the mp autoregressive coefficients $\theta_{h(i)}$,

entries of the vector $\theta = (\theta_{1(1)}, \ldots, \theta_{p(1)}, \ldots, \theta_{1(m)}, \ldots, \theta_{p(m)})'$. The initial distribution δ will be estimated by the equality $\delta' = \delta'\Gamma$, after the estimation of the matrix Γ. Hence the vector of the $m(m+p+1)$ unknown parameters is

$$\phi = \left(\gamma_{1,1}, \ldots, \gamma_{1,m-1}, \ldots, \gamma_{m,1}, \ldots, \gamma_{m,m-1}, \mu', \sigma^{2'}, \theta'\right)',$$

belonging to the parameter space Φ. The estimator of the vector ϕ will be obtained by the maximum likelihood (ML) method.

Let $y^T = (y_{-p+1}, \ldots, y_0, y_1, \ldots, y_T)'$ be the vector of the observed data, that is the sequence of the realizations of the stochastic process $\{Y_t\}$, where $(y_{-p+1}, \ldots, y_0)$ are initial values, fixed for the order p dependence condition; the data vector is incomplete because the sequence $x^T = (i_1, \ldots, i_T)'$ of the states of the chain $\{X_t\}$ is unobserved; the complete data vector will be $(y^T, x^T) = (y_{-p+1}, \ldots, y_0, y_1, \ldots, y_T, i_1, \ldots, i_T)'$.

Moreover let $L_T^c(\phi)$ be the likelihood function of the complete data and $L_T(\phi)$ be that of the observed data, given the p initial values:

$$\begin{aligned}
L_T^c(\phi) &= f(y_1, \ldots, y_T, i_1, \ldots, i_T \mid y_0, \ldots, y_{-p+1}) = \\
&= \delta_{i_1} f(y_1 \mid y_0, \ldots, y_{-p+1}, i_1) \prod_{t=2}^{T} \gamma_{i_{t-1}, i_t} f(y_t \mid y_{t-1}, \ldots, y_{t-p}, i_t); \\
L_T(\phi) &= f(y_1, \ldots, y_T \mid y_0, \ldots, y_{-p+1}) = \\
&= \sum_{i_1 \in S_X} \cdots \sum_{i_T \in S_X} \delta_{i_1} \left[\prod_{t=1}^{T-1} \gamma_{i_t, i_{t+1}} \prod_{t=1}^{T} f(y_t \mid y_{t-1}, \ldots, y_{t-p}, i_t) \right].
\end{aligned}$$

The maximization of the incomplete log-likelihood function, $\ln L_T(\phi)$, is a non-linear optimization problem, so it requires numerical techniques. For the MSAR models, the most popular numerical method to obtain the ML estimators is the *Expectation-Maximization* (EM) algorithm (Dempster, Laird, Rubin (1977)) in conjunction with the iterative algorithms of Hamilton (1994) and Kim (1993) for computing the *filtered* and the *smoothed* probabilities of the hidden states given the observations.

The EM algorithm is an iterative ML estimation technique, designed for a more general class of data-incomplete problems. The iterative scheme is the following: let $\phi^{(0)}$ be some starting value of ϕ; at the first iteration, the E step requires the computation of the conditional expectation of the complete data log-likelihood, given the observed data, in $\phi = \phi^{(0)}$ and the M step provides the search for that special value $\phi^{(1)}$ which maximizes the previous function. In general, at the $(k+1)^{th}$ iteration, the E and M steps are so defined:

E step - given $\phi^{(k)}$, compute $Q\left(\phi; \phi^{(k)}\right) = \mathrm{E}_{\phi^{(k)}}\left(\ln L_T^c(\phi) \mid y^T\right)$;

M step - search for that $\phi^{(k+1)}$ which maximizes $Q\left(\phi; \phi^{(k)}\right)$, for any $\phi \in \Phi$.

The E and M steps must be repeated in an alternating way until we have the convergence of the sequence of the log-likelihood values $\left\{\ln L_T\left(\phi^{(k)}\right)\right\}$. If the convergence is reached at the $(k+1)^{th}$ iteration, the vector $\phi^{(k+1)} = \widehat{\phi}$ is then the ML estimator of ϕ. The main property of the EM algorithm is the monotonicity of the log-likelihood function for incomplete data, with respect to the iterations of the algorithm (Dempster, Laird, Rubin (1977), Theorem 1).

Nevertheless, if the likelihood surface is multimodal, the convergence of the EM algorithm to the global maximum depends on the starting value $\phi^{(0)}$. To avoid the convergence

to a stationary point which is not a global maximum, the best strategy is to start the algorithm from several different, possibly random, points in Φ and to compare the stationary points obtained at each run.

As shown in Hamilton (1994), to obtain the ML estimators of the parameters of MSAR models, the EM algorithm can be used in conjunction with the *filtering* and the *smoothing* algorithms. The basic *filtering* algorithm is an iterative procedure to compute the estimates of the probabilities of being in regime i at time t or $t+1$, given the informations available up to time t, for any $i \in S_X$. Let $y^t = (y_1, \ldots, y_t)$ be the vector of the observations up to time t, $\xi_{t+1|t}$ and $\xi_{t|t}$ respectively be the m-dimensional vectors of the conditional probabilities $P(X_{t+1} = i | y^t; \phi)$, $i = 1, \ldots, m$, $t = 1, \ldots, T-1$, and $P(X_t = i | y^t; \phi)$, $i = 1, \ldots, m$, $t = 1, \ldots, T$. These vectors are recursively computed starting from $\xi_{1|0} = \delta' = \delta'\Gamma$:

$$\xi_{t|t} = \frac{\xi_{t|t-1} \odot F_t}{1'_{(m)} \left(\xi_{t|t-1} \odot F_t\right)} \tag{3}$$

$$\xi_{t+1|t} = \Gamma' \xi_{t|t}, \tag{4}$$

where F_t denotes the m-dimensional vector containing the conditional *pdf*s (2),

$$F_t = \left(f(y_t \mid y_{t-1}, \ldots, y_{t-p}, i_t = 1), \ldots, f(y_t \mid y_{t-1}, \ldots, y_{t-p}, i_t = m)\right),$$

$1_{(m)}$ is the m-dimensional vector of ones and the symbol $\odot$ indicates the Hadamard product (Hamilton (1994)).

The *smoothing* algorithm allows us to compute the estimates of the probabilities of being in regime i at time t, given all the available informations, y^T, for any $i \in S_X$. Let $\xi_{t|T}$ be the m-dimensional vector of the conditional probabilities $P(X_t = i | y^T; \phi)$, $i = 1, \ldots, m$, $t = 1, \ldots, T$; these vectors are recursively computed backward starting from $\xi_{T|T}$, obtained by (3):

$$\xi_{t|T} = \xi_{t|t} \odot \left[\Gamma \left(\xi_{t+1|T} \div \xi_{t+1|t}\right)\right], \tag{5}$$

for $t = T-1, T-2, \ldots, 1$, where the symbol $\div$ indicates element-by-element division (Kim (1993)).

Finally, the incomplete log-likelihood function can be computed:

$$\ln L_T(\phi) = \sum_{t=1}^{T} \ln f(y_1, y_2, \ldots, y_T \mid y_0, \ldots, y_{-p}) = \sum_{t=1}^{T} \ln \left(1'_{(m)} \left(\xi_{t|t-1} \odot F_t\right)\right) \tag{6}$$

(Hamilton (1994)).

Now the two steps of the EM algorithm at the $(k+1)^{th}$ iteration are analysed into details, remembering that at the k^{th} iteration the vector of estimates $\phi^{(k)}$ has been obtained,

$$\phi^{(k)} = \left(\gamma_{1,1}^{(k)}, \ldots, \gamma_{1,m-1}^{(k)}, \ldots, \gamma_{m,1}^{(k)}, \ldots, \gamma_{m,m-1}^{(k)}, \mu^{(k)\prime}, \sigma^{2(k)\prime}, \theta^{(k)\prime}\right)',$$

and the filtered and smoothed probabilities have been computed. Henceforth the superscript (k) will denote a quantity obtained at the k^{th} iteration as a function of the vector $\phi^{(k)}$; notice that $\delta^{(k)}$ is the left eigenvector of the matrix $\Gamma^{(k)} = \left[\gamma_{i,j}^{(k)}\right]$, associated with the eigenvalue one, such that $\delta'^{(k)} = \delta'^{(k)}\Gamma^{(k)}$.

At the E step of the $(k+1)^{th}$ iteration, the analytic closed-form of the function $Q\left(\phi;\phi^{(k)}\right)$ is

$$\begin{aligned} Q(\phi;\phi^{(k)}) &= E_{\phi^{(k)}}\left\{\ln L_T^c(\phi) \mid y^T\right\} = \sum_i \frac{f^{(k)}(y_1,\ldots,y_T,X_1=i)}{f^{(k)}(y_1,\ldots,y_T)} \ln \delta_i + \\ &+ \sum_i \sum_j \left[\frac{\sum_{t=1}^{T-1} f^{(k)}(y_1,\ldots,y_T,X_t=i,X_{t+1}=j)}{f^{(k)}(y_1,\ldots,y_T)} \ln \gamma_{i,j} \right] + \\ &+ \sum_i \left[\frac{\sum_{t=1}^{T} f^{(k)}(y_1,\ldots,y_T,X_t=i)}{f^{(k)}(y_1,\ldots,y_T)} \ln f(y_t \mid y_{t-1},\ldots,y_{t-p},i) \right]. \end{aligned}$$

Using the filtered and smoothed probabilities, computed at the k^{th} iteration, we obtain

$$\begin{aligned} Q(\phi;\phi^{(k)}) &= \sum_i \xi_{1|T}^{(k)}(i) \ln \delta_i + \sum_i \sum_j \left\{ \sum_{t=1}^{T-1} \gamma_{i,j}^{(k)} \xi_{t+1|T}^{(k)}(i) \frac{\xi_{t|t}^{(k)}(i)}{\xi_{t+1|t}^{(k)}(j)} \right\} \ln \gamma_{i,j} + \\ &+ \sum_i \sum_{t=1}^{T} \xi_{t|T}^{(k)}(i) \ln f(y_t \mid y_{t-1},\ldots,y_{t-p},i). \end{aligned}$$

At the M step of the $(k+1)^{th}$ iteration, to obtain $\phi^{(k+1)}$, the function $Q\left(\phi;\phi^{(k)}\right)$ must be maximized with respect to the m^2-m parameters $\gamma_{i,j}$, for any $i=1,\ldots,m; j=1,\ldots,m-1$, the m parameters μ_i, the m parameters σ_i^2, for any $i \in S_X$, and the mp parameters $\theta_{h(i)}$, for any $i=1,\ldots,m; h=1,\ldots,p$. We shall not maximize $Q\left(\phi;\phi^{(k)}\right)$ with respect to the m parameters δ_i, for any $i \in S_X$, because, as we said previously, the initial distribution δ will be estimated by the equality $\delta' = \delta'\Gamma$, after the estimation of the matrix Γ. But, by the stationarity assumption, δ contains informations about the transition probabilities matrix Γ, since $\delta_j = \sum_{i \in S_X} \delta_i \gamma_{i,j}$, for any $j \in S_X$. However, for large T, the effect of δ is negligible; so the first term of the function $Q(\phi;\phi^{(k)})$ can be ignored searching for the maximum likelihood estimator of $\gamma_{i,j}$, for any i,j (Basawa and Prakasa Rao (1980), pp. 53-54).

In practice the M step can be performed in two ways: numerically, using a standard numerical optimization procedure (like Newton-Raphson method) or, when possible, analitically, obtainig the closed-form solutions of the parameter estimators.

Notice that the analytic expression of $Q\left(\phi;\phi^{(k)}\right)$ is the sum of three terms: the first two are functions only of the parameters of the Markov chain, while the third is a function only of the parameters of the *pdf*s (2). This separation of parameters makes the maximization problem into a simple closed-form, that gives the explicit solutions of the estimators of $\gamma_{i,j}, \sigma_i^2$, while for the other estimators, the entries of the vectors μ and θ, the solutions must be obtained using the Newton-Raphson method. The explicit expression of the estimators of the parameters $\gamma_{i,j}$, $i=1,\ldots,m$, $j=1,\ldots,m-1$, at the $(k+1)^{th}$ iteration is

$$\gamma_{i,j}^{(k+1)} = \frac{\sum_{t=1}^{T-1} \gamma_{i,j}^{(k)} \xi_{t+1|T}^{(k)}(i) \frac{\xi_{t|t}^{(k)}(i)}{\xi_{t+1|t}^{(k)}(j)}}{\sum_{t=1}^{T-1} \xi_{t|T}^{(k)}(i)}$$

while that of the estimators of the parameters σ_i^2, $i = 1, \ldots, m$, is

$$\sigma_i^{2(k+1)} = \frac{\sum_{t=1}^{T} \xi_{t|T}^{(k)}(i) \left(y_t - \mu_i^{(k+1)} - \sum_{h=1}^{p} \theta_{h(i)}^{(k+1)} y_{t-h} \right)^2}{\sum_{t=1}^{T} \xi_{t|T}^{(k)}(i)},$$

performed after having numerically obtained the values of $\mu_i^{(k+1)}$and $\theta_{h(i)}^{(k+1)}$, for $i = 1, \ldots, m$ and $h = 1, \ldots, p$.

In the case of convergence of the algorithm at the $(k+1)^{th}$ iteration, $\left(\phi^{(k+1)}; \ln L_T\left(\phi^{(k+1)}\right)\right)$ is a stationary point of $\ln L_T(\phi)$, and therefore $\phi^{(k+1)}$ is the ML estimator of the unknown parameter ϕ. Assuming $\sigma_i \in [\varepsilon, 1/\varepsilon]$, for any $i \in S_X$ and for some small $\varepsilon > 0$, this fundamental result holds because the four regularity conditions on the convergence of the EM algorithm to a stationary point (Wu (1983), conditions (5), (6), (7), p. 96; (10), p. 98) are satisfied.

As Douc, Moulines, Rydén (2001) proved, the maximum likelihood estimator of ϕ is strongly consistent and asymptotically normal.

3. Missing restoration and forecasting

One of the major aims in time series analysis is forecasting the future values of an observed process. Computing forecasts for nonlinear time series is generally more complex than for the linear ones, because the non linearity of the function that defines the model makes the optimal predictor not to be linear, but the case of one-step-ahead forecasting. Hence we performe the h-step-ahead ($h > 0$) forecasting in the "naive" way, iterating h times the one-step-ahead forecasting and using every time the forecasted value as the last observation.

For the MSAR models the one-step-ahead forecast of the future values of the observed series can be decomposed into the forecast of the conditional observed variable given the p past observations and the current state of the Markov chain and the estimated probabilities of occurrence of each state. Let $\widehat{y}_{T+1}$ be the one-step-ahead forecasting value:

$$\begin{aligned} \widehat{y}_{T+1} &= E(Y_{T+1} \mid y_T, \ldots, y_{T-p+1}) = \\ &= \sum_{i=1}^{m} E(Y_{T+1} \mid y_T, \ldots, y_{T-p+1}, X_{T+1} = i) P(X_{T+1} = i \mid y^T), \end{aligned} \tag{7}$$

where the forecast of the conditional variable Y_{T+1} given the p past observations and the current state of the Markov chain is the expected value

$$E(Y_{T+1} \mid y_T, \ldots, y_{T-p+1}, X_{T+1} = i) = \mu_i + \sum_{h=1}^{p} \theta_{h(i)} y_{T-h+1},$$

whereas $P(X_{T+1} = i \mid y^T)$, for any $i \in S_X$, are given by the filtered probabilities (Hamilton (1994)).

We now introduce an alternative forecasting method based on the reconstruction of the sequence of the hidden states. The sequence $x^T = (i_1, \ldots, i_T)$ of the hidden states can be

generated by Monte Carlo simulations through the *forward filtering-backward sampling* (*ff-bs*) algorithm (Frühwirth-Schnatter (1994)), based on the equality

$$P\left(X_1 = i_1, \ldots, X_T = i_T \mid y^T\right) = P\left(X_T = i_T \mid y^T\right) \prod_{t=1}^{T-1} P\left(X_t = i_t \mid X_{t+1} = i_{t+1}, y^t\right).$$

Given $\widehat{\phi}$, the vector of the ML estimators of the parameters, the iterative scheme of the algorithm is the following:

(*i*) for $t = 1, \ldots, T$, compute the vectors of the filtered probabilities $\widehat{\xi}_{t|t} = P\left(X_t = i \mid y^t, \widehat{\phi}\right)$, for any $i \in S_X$;

(*ii*) compute i_T as the mode of a sufficiently large number of values generated from $\widehat{\xi}_{T|T}$;

(*iii*) for $t = T-1, \ldots, 1$, compute the vector

$$\widehat{\xi}_t = \frac{\widehat{\xi}_{t|t} \odot \widehat{\Gamma}_{\cdot i_{t+1}}}{1'_{(m)}\left(\widehat{\xi}_{t|t} \odot \widehat{\Gamma}_{\cdot i_{t+1}}\right)}$$

and i_t as the mode of a sufficiently large number of values generated from $\widehat{\xi}_t$, where $\widehat{\Gamma}_{\cdot i_{t+1}}$ represents the column of $\widehat{\Gamma}$ corresponding to the state previously generated.

To obtain the state of the hidden chain for the future time $T+1$ and consequently to forecast the future observation y_{T+1}, we run the *ff-bs* algorithm up to $T+1$, with the same iterative scheme, replacing the vector F_{T+1} in (3) with the unit vector. If we repeat this procedure h times, using the previously obtained one-step-ahead forecast as the last observation, we can have the h-step-ahead "naive" forecast.

Now we introduce the method we use to analyse Markov switching time series with missing observations; it is based on the iteration of three consecutive procedures: EM algorithm for the estimation of the parameters, reconstruction of the hidden chain, restoration of the missing observations as forecasting based on the best model estimated through the subseries bounded by the missing values. Consider for example that two observations, $y_{t_1^*}$ and $y_{t_2^*}$, are missing at times t_1^* and t_2^* $(1 < t_1^* < t_2^* < T)$. First we consider the subseries $(y_1, \ldots, y_{t_1^*-1})$: the parameters vector $\widehat{\phi}$ is estimated by means of the EM algorithm; the sequence of hidden states is reconstructed up to t_1^* by the *ff-bs* algorithm; $\widehat{y}_{t_1^*}$ is forecasted given $i_{t_1^*}$:

$$\widehat{y}_{t_1^*} = E(Y_{t_1^*} \mid y_{t_1^*-1}, \ldots, y_{t_1^*-p}, X_{t_1^*} = i_{t_1^*}) = \hat{\mu}_{i_{t_1^*}} + \sum_{h=1}^{p} \hat{\theta}_{h(i_{t_1^*})} y_{i_{t_1^*}-h}. \tag{8}$$

Then we consider the subseries $(y_1, \ldots, y_{t_2^*-1})$ in which the missing value $y_{t_1^*}$ is filled with $\widehat{y}_{t_1^*}$: the parameters vector $\widehat{\phi}$ is estimated; the sequence of hidden states is reconstructed up to t_2^*; $y_{t_1^*}$ is updated and finally $\widehat{y}_{t_2^*}$ is computed, according to (8). Now the iterative procedure can restart using all the series $(y_1, \ldots, y_T)$, to find the final estimates of ϕ and to make forecasts.

In the forthcoming section we will apply this methodology to a real data set with missing observations that will be restored. Then using the restored sequence of observations we will make forecasting.

4. Application to air pollution data

Air quality control includes the study of data sets recorded by air pollution testing stations. We consider one of the five stations located in Bergamo[2]. It records seven types of pollutants and each hour we have the mean concentration of every pollutant. We are interested in the analysis of the dynamics of daily mean concentrations of sulphur dioxide (SO_2). SO_2, measured in $\mu g/m^3$, is a gas with a characteristic pungent and choking smell: it is produced by the combustion of substances containing sulphur (coal, fuel oil, diesel oil). The principal sources of SO_2 are the industries that need a lot of energy (refineries, steelworks, thermoelectric power stations), the domestic heating (even if the use of methane drastically reduced the SO_2 emissions), the traffic of the heavy transport (diesel vehicular traffic). SO_2 is responsible, with the nitrogen oxides (NO_X), for the acid rains. It can cause the onset and the worsening of the respiratory tract diseases (persistent cough, bronchitis, sinusitis); very high concentrations can cause tissues destruction, giving rise to emphysema.

We analised the time series of the daily mean concentrations of SO_2 recorded by the air pollution testing station placed in Via San Giorgio, Bergamo, a medium density of population and intense traffic area, during the period 1.1.1997-30.11.1999.

Figure 1 shows the behaviours of the series of 1064 observations. A seasonal pattern is evident: it depends on the unobserved levels representing the different climates along the year. As in finite mixture analysis, plotting a histogram of the values in Figure 1, we have a rough indication of the number of the different unobserved levels: in Figure 2 we can guess the presence of some hidden states, noticing a multi-peaks distribution.

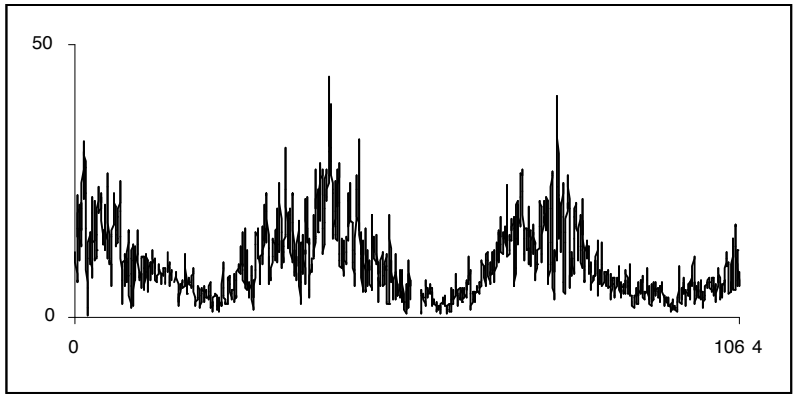


Figure 1: *Series of the daily mean concentrations of SO_2, recorded by the air pollution testing station placed in Via San Giorgio, Bergamo, from 1.1.1997 to 30.11.1999*

[2]The data set has been provided by the Bergamo ARPA.

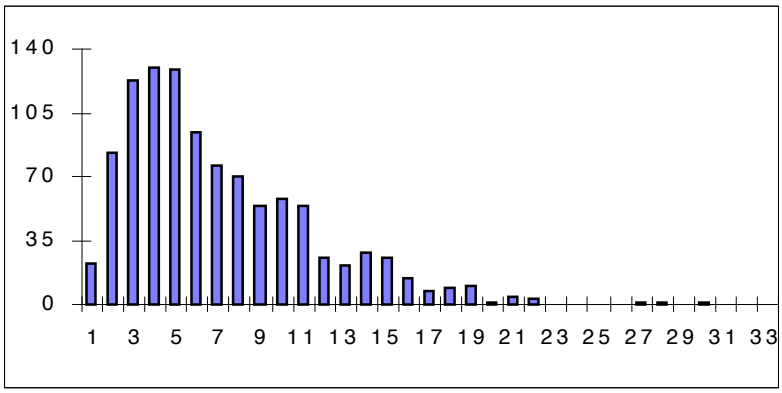


Figure 2: *Histogram of the daily mean concentrations of SO_2, obtained by the data plotted in Figure 1*

Notice that in Figure 1 some observations are missing, either because the station must be stopped for automatic calibration, because of occasional mechanical failure, ordinary maintenance, or because some data are clearly anomalous and so are removed by technicians. There are 21 missing values within the series: the observation y_{147} and the groups of consecutive observations $(y_{156}, \ldots, y_{159})$, (y_{312}, y_{313}), $(y_{539}, \ldots, y_{552})$. So the estimation procedure we propose will be iteratively repeated for the four subseries $y^{146} = (y_1, \ldots, y_{146})$, $y^{155} = (y_1, \ldots, y_{155})$, $y^{311} = (y_1, \ldots, y_{311})$, $y^{538} = (y_1, \ldots, y_{538})$ and for the whole series $y^{1064} = (y_1, \ldots, y_{1064})$.

In the following analysis the sample period 1.1.1997-23.11.1999 (y^{1057}) has been used as the fit period for the model selection, while the observations of the last week, from 24.11.1999 to 30.11.1999 $(y_{1058}, \ldots, y_{1064})$, have been left to evaluate and compare the one-step- ahead "naive" forecasts performance.

The procedures introduced in Sections 2 and 3 have been implemented in a GAUSS code to analyse several models, for a range of values of the model order m, the number of states of the Markov chain, and p, the autoregressive order. For every analised subseries, we select the optimal model for $m = 1, 2, 3, 4$ and $p = 0, 1, \ldots, 5$, according to the *Akaike Information Criterion* (AIC) and the *Bayesian Information Criterion* (BIC) or the minimum Mean Squared Error (MSE).

To avoid reporting local maxima of the log-likelihood surface, the EM algorithm runs several times with different initial values. Among the acceptable solutions of the parameters (i.e. those such that the Markov chain is irriducible and the autoregressive coefficients satisfy the stationarity condition), we choose the stationary point which corresponds to the largest log-likelihood value.

The variance-covariance matrix of the parameters estimates are obtained from the inverse of a numerical approximation of the Hessian matrix with reverse sign.

The results of model selection for the five steps of the iterative procedure are summarized in Tables A1-A5 in Appendix, where the values of the maximized log-likelihood and the corresponding AIC, BIC and MSE are given. The values obtained by $m = 4$ and $p = 5$ are not included in the tables because they give rise to reducible hidden chains or nonstationary models.

First we consider the subseries $y^{146} = (y_1, \ldots, y_{146})$. We choose as best model the MSAR(3,4), as in Table A1: three states of the hidden chain are selected by two criteria, while four autoregressive components are selected by all the criteria. The estimates of the

parameters of the hidden Markov chain are (standard errors in brackets)

$$\Gamma^{(92)} = \begin{bmatrix} 0.9401 & 0.0001 & 0.0597 \\ (0.1261) & (0.1270) & (0.0354) \\ 0.0001 & 0.9119 & 0.0880 \\ (0.0692) & (0.0071) & (0.0334) \\ 0.2794 & 0.6473 & 0.0733 \\ (0.1424) & (0.1560) & (0.0928) \end{bmatrix},$$

from which we have the estimate of the stationary initial distribution,

$$\delta^{(92)} = (0.3590; 0.5643; 0.0767)',$$

while those of the parameters of the three gaussian *pdf*s are

i	$\mu_i^{(92)}$	$\sigma_i^{2(92)}$	$\theta_{1(i)}^{(92)}$	$\theta_{2(i)}^{(92)}$	$\theta_{3(i)}^{(92)}$	$\theta_{4(i)}^{(92)}$
1	6.6002	23.3265	0.4066	0.1650	0.1240	−0.0706
	(2.3561)	(4.1969)	(0.1222)	(0.1262)	(0.1257)	(0.1181)
2	9.4890	3.2689	0.4536	−0.5276	0.0316	−0.0822
	(0.9340)	(0.5479)	(0.0924)	(0.0975)	(0.0989)	(0.0681)
3	10.5535	0.0047	0.6268	−0.3155	0.1389	−0.1738
	(0.0437)	(0.0020)	(0.0046)	(0.0116)	(0.0081)	(0.0093)

.

The missing observation y_{147} is now restored by means of the procedure introduced in the previous section and its value results 8.4321.

Then we consider the second subseries $y^{155} = (y_1, \ldots, y_{155})$. We choose as best model the MSAR(3,4), as in Table A2: three states of the hidden chain are selected by two criteria, while four autoregressive components are selected by all the criteria. The estimates of the parameters of the hidden Markov chain are

$$\Gamma^{(187)} = \begin{bmatrix} 0.9403 & 0.0001 & 0.0596 \\ (0.1216) & (0.1230) & (0.0358) \\ 0.0001 & 0.9239 & 0.0760 \\ (0.0000) & (0.0439) & (0.0296) \\ 0.2818 & 0.6444 & 0.0738 \\ (0.1498) & (0.1624) & (0.0938) \end{bmatrix},$$

from which we have the estimate of the stationary initial distribution,

$$\delta^{(187)} = (0.3326; 0.5970; 0.0704)',$$

while those of the parameters of the three gaussian *pdf*s are

i	$\mu_i^{(187)}$	$\sigma_i^{2(187)}$	$\theta_{1(i)}^{(187)}$	$\theta_{2(i)}^{(187)}$	$\theta_{3(i)}^{(187)}$	$\theta_{4(i)}^{(187)}$
1	6.5960	23.3232	0.4066	0.1651	0.1240	−0.0705
	(2.3504)	(4.1962)	(0.1223)	(0.1262)	(0.1258)	(0.0186)
2	9.6846	3.3015	0.4218	−0.5024	0.0332	−0.0982
	(0.9133)	(0.5206)	(0.0904)	(0.0933)	(0.0950)	(0.0665)
3	10.5559	0.0047	0.6266	−0.3154	0.1387	−0.1737
	(0.0439)	(0.0021)	(0.0045)	(0.0118)	(0.0082)	(0.0101)

.

Given that we have more than one missing value, i.e. $y_{156}, \ldots, y_{159}$, the procedure used to restore the missing values, described at Sections 2 and 3, must be modified in the following way:

(i) compute $\widehat{\xi}_{1|0} = \widehat{\delta}' = \widehat{\delta}'\widehat{\Gamma}$;

(ii) compute

$$\widehat{\xi}_{t|t} = \frac{\widehat{\xi}_{t|t-1} \odot F_t}{1'_{(3)}\left(\widehat{\xi}_{t|t-1} \odot F_t\right)}$$

and

$$\widehat{\xi}_{t+1|t} = \widehat{\Gamma}\widehat{\xi}_{t|t},$$

for any $t = 1, \ldots, 155$;

(iii) compute

$$\widehat{\xi}_{156|156} = \frac{\widehat{\xi}_{156|155} \odot 1_{(3)}}{1'_{(3)}\left(\widehat{\xi}_{156|155} \odot 1_{(3)}\right)} = \frac{\widehat{\xi}_{156|155}}{1'_{(3)}\widehat{\xi}_{156|155}};$$

(iv) compute i_{156} as the mode of a sufficiently large number of values generated from $\widehat{\xi}_{156|156}$;

(v) compute $\widehat{y}_{156}$, according to (8);

(vi) for any $t = 155, 154 \ldots$, compute

$$\widehat{\xi}_t = \frac{\widehat{\xi}_{t|t} \odot \widehat{\Gamma}_{\cdot i_{t+1}}}{1'_{(3)}\left(\widehat{\xi}_{t|t} \odot \widehat{\Gamma}_{\cdot i_{t+1}}\right)}$$

and i_t as the mode of a sufficiently large number of values generated from $\widehat{\xi}_t$. This sixth step runs up to the special t such that the generated state is equal to the previously reconstructed state;

(vii) compute (ii) for $t = 156$;

$(viii)$ to obtain i_{157} and $\widehat{y}_{157}$, repeat (iii), (iv), (v), (vi) for $t = 157$. Notice that formula (8) must be reapplied to all the restored values corresponding to the states that are different from those previously generated;

(ix) to obtain i_{158} and $\widehat{y}_{158}$, repeat (vii) for $t = 157$ and $(viii)$ for $t = 158$;

(x) to obtain i_{159} and $\widehat{y}_{159}$, repeat (vii) for $t = 158$ and $(viii)$ for $t = 159$.

By this procedure the restored missing values are (7.7807, 9.1870, 9.0142, 8.4585), while the missing value y_{147}, restored by the new set of parameters, is 8.4609.

For the third subseries $y^{311} = (y_1, \ldots, y_{311})$ we choose as best model the MSAR(3,4) again, as in Table A3: three states of the hidden chain and four autoregressive components are selected by two criteria. The estimates of the parameters of the hidden Markov chain are

$$\Gamma^{(337)} = \begin{bmatrix} 0.9645 & 0.0001 & 0.0354 \\ (0.0184) & (0.0269) & (0.0256) \\ 0.0001 & 0.9731 & 0.0268 \\ (0.0000) & (0.0181) & (0.0196) \\ 0.0573 & 0.0374 & 0.9053 \\ (0.0324) & (0.0345) & (0.0471) \end{bmatrix},$$

from which we have the estimate of the stationary initial distribution,

$$\delta^{(337)} = (0.4034; 0.3471; 0.2495)',$$

while those of the parameters of the three gaussian *pdf*s are

i	$\mu_i^{(337)}$	$\sigma_i^{2(337)}$	$\theta_{1(i)}^{(337)}$	$\theta_{2(i)}^{(337)}$	$\theta_{3(i)}^{(337)}$	$\theta_{4(i)}^{(337)}$
1	1.0046	1.9815	0.8001	−0.0681	0.0060	0.0636
	(0.3455)	(0.3091)	(0.0927)	(0.1036)	(0.1250)	(0.0840)
2	4.4789	21.7986	0.5016	0.0722	0.1662	−0.0377
	(1.8748)	(3.2731)	(0.1015)	(0.1049)	(0.1084)	(0.1073)
3	11.3052	6.4493	0.2191	−0.2645	−0.1091	−0.1089
	(1.3834)	(2.1330)	(0.1141)	(0.1223)	(0.1434)	(0.0883)

.

The missing observations (y_{312}, y_{313}) are now restored by means of the procedure used for the second subseries, updated for $t = 312, 313$. We obtain (11.8267, 11.7593), while the missing values y_{147}, $(y_{156}, \ldots, y_{159})$, restored by the new set of parameters, are 6.2452, (6.2912, 7.2307, 8.4254, 8.0725).

For the fourth subseries $y^{538} = (y_1, \ldots, y_{538})$ we choose as best model the MSAR(3,3), as in Table A4: three states of the hidden chain and three autoregressive components are selected by two criteria. The estimates of the parameters of the hidden Markov chain are

$$\Gamma^{(119)} = \begin{bmatrix} 0.9536 & 0.0463 & 0.0001 \\ (0.0290) & (0.0238) & (0.0325) \\ 0.0302 & 0.9619 & 0.0079 \\ (0.0163) & (0.0180) & (0.0082) \\ 0.0001 & 0.0097 & 0.9902 \\ (0.0031) & (0.0082) & (0.0062) \end{bmatrix},$$

from which we have the estimate of the stationary initial distribution,

$$\delta^{(119)} = (0.2645; 0.4061; 0.3294)',$$

while those of the parameters of the three gaussian *pdf*s are

i	$\mu_i^{(119)}$	$\sigma_i^{2(119)}$	$\theta_{1(i)}^{(119)}$	$\theta_{2(i)}^{(119)}$	$\theta_{3(i)}^{(119)}$
1	0.9955	1.6283	0.8624	−0.1522	0.0864
	(0.3586)	(0.2374)	(0.0942)	(0.1143)	(0.0825)
2	4.9824	11.5426	0.2751	0.0563	0.0650
	(0.8519)	(1.3748)	(0.0811)	(0.0800)	(0.0739)
3	5.3720	27.8386	0.5377	0.1368	−0.0109
	(1.0616)	(2.6125)	(0.0651)	(0.0727)	(0.0666)

.

The missing observations $(y_{539}, \ldots, y_{552})$ are now restored by the usual procedure for $t = 539, \ldots, 552$. Their values are (6.5991, 7.4203, 7.5984, 7.9194, 8.0711, 8.1425, 8.1916, 8.2190, 8.2339, 8.2427, 8.2478, 8.2506, 8.2523, 8.2532), while the missing values y_{147}, $(y_{156}, \ldots, y_{159})$, (y_{312}, y_{313}), restored by the new set of parameters, are 6.1673, (6.1121, 6.1406, 6.8641, 7.7046), (12.1249, 13.2061).

Finally, for the whole series $y^{1057} = (y_1, \ldots, y_{1057})$ the final best model is MSAR(3,4) as in Table A5: three states of the hidden Markov chain and four autoregressive components are selected by two criteria. The estimates of the parameters of the hidden Markov chain

are

$$\Gamma^{(76)} = \begin{bmatrix} 0.9633 & 0.0366 & 0.0001 \\ (0.0221) & (0.0139) & (0.0090) \\ 0.0705 & 0.9181 & 0.0114 \\ (0.0339) & (0.0326) & (0.0093) \\ 0.0001 & 0.0090 & 0.9909 \\ (0.0000) & (0.0054) & (0.000) \end{bmatrix},$$

from which we have the estimate of the stationary initial distribution,

$$\delta^{(76)} = (0.4598; 0.2388; 0.3014)',$$

while those of the parameters of the three gaussian *pdf*s are

i	$\mu_i^{(76)}$	$\sigma_i^{2(76)}$	$\theta_{1(i)}^{(76)}$	$\theta_{2(i)}^{(76)}$	$\theta_{3(i)}^{(76)}$	$\theta_{4(i)}^{(76)}$
1	0.7286	2.2774	0.6404	0.0047	0.0890	0.1190
	(0.2182)	(0.2448)	(0.0595)	(0.0746)	(0.0661)	(0.0495)
2	5.3030	11.2926	0.2234	0.2090	0.0100	0.0962
	(0.9970)	(1.3905)	(0.0870)	(0.0770)	(0.0612)	(0.0728)
3	6.4221	26.3704	0.5733	0.0427	−0.0263	0.0044
	(0.9274)	(1.9306)	(0.0522)	(0.0588)	(0.0785)	(0.0711)

.

The missing observations y_{147}, $(y_{156}, \ldots, y_{159})$, (y_{312}, y_{313}), $(y_{539}, \ldots, y_{552})$, restored by the new set of parameters, are 6.6081, (6.4333, 6.2957, 6.2497, 6.5651), (12.9059, 13.6257), (4.5367, 6.2918, 6.6621, 7.1050, 7.9268, 8.5677, 9.0788, 9.5212, 9.9262, 10.2802, 10.5886, 10.8584, 11.0956, 11.3035).

The one-step-ahead "naive" forecasts for the period from 24.11.99 to 30.11.99 are now obtained by means of the usual procedure for $t = 1058, \ldots, 1064$. Their values are 9.7455, 8.3064, 7.9654, 8.6002, 8.4116, 8.6410, 8.1725 and they are plotted in Figure 3 with the series of the fitted values by the best model, for $t = 5, \ldots, 1057$. Furthermore Figure 4 shows the final sequence of the reconstructed states of the hidden Markov chain, representing the three different levels of pollution occured during the analysed period.

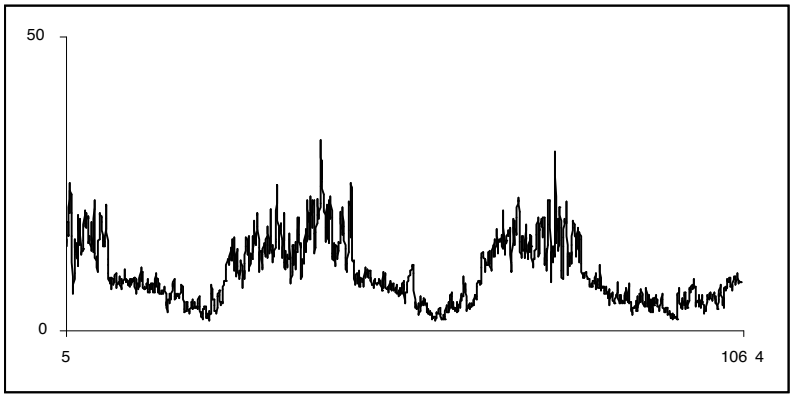


Figure 3: *Series of the fitted values from time 5 to time 1057 and the forecasted values from time 1058 to time 1064.*

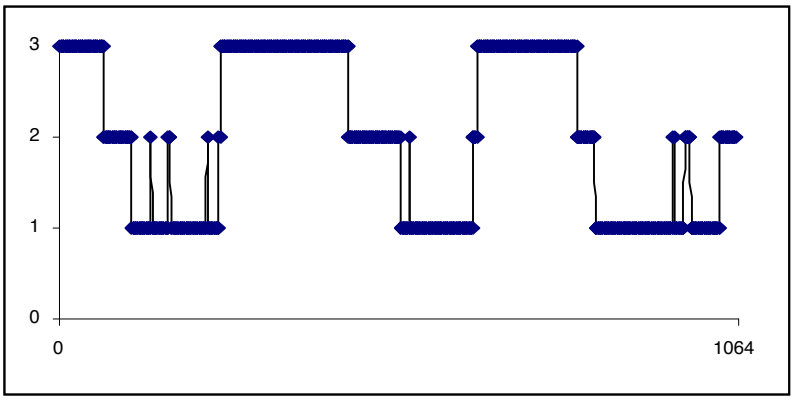


Figure 4: *Reconstructed series of the hidden states.*

To evaluate the forecasting performance of our method, the one-step-ahead "naive" forecasts are compared with the actual values through some descriptive statistics: the mean squared error (MSE), the mean absolute error (MAE) and the correlation coefficient (CORR) between actual and forecasted values. The values of these statistics for the forecasts are reported in Table 1, left column. They are also compared with those computed by means of Hamilton's method (Formula 7), right column, which is now applied to the complete series, where the missing observations have been restored through the procedure based on the *ff-bs* algorithm.

Table 1 - Predictions statistics

MSE	53.0423	45.5853
MAE	6.9492	6.4039
CORR	-0.0589	0.0307

The results show that the two methods approximatively have the same forecasting ability, but it must be noticed that the advantage in using the *ff-bs* algorithm is that we can reconstruct the sequence of the hidden states.

Conclusions

We recurred to *Markov Switching AutoRegressive* (MSAR) models to analyse the series of 1064 daily mean concentrations of sulphur dioxide, recorded by an air pollution testing station, placed in Bergamo. MSAR models are sequences of observed variables depending on the past observations and on the realization of a contemporary unobserved variable which is the current state of a hidden Markov chain. We performed the EM algorithm, jointly with the *filtering* and the *smoothing* algorithms, to obtain the maximum likelihood estimators of the parameters of the model; we also reconstructed the sequence of the hidden state through the *forward filtering-backward sampling* algorithm, restored the missing observations and forecasted the future values. We compared the different models, varing for both the order of the autoregressive component and the cardinality of the state-space of the hidden Markov chain. The comparisons have been made by means of the *Akaike Information Criterion*, the *Bayesian Information Criterion* and the minimum Mean Squared Error.

Appendix

Table A1 - Results of the model selection for the first subseries $y^{146}=(y_1,...,y_{146})$.

		p				
m		0	1	2	3	4
1	$\ln L_T(\phi)$	-469.509	-425.321	-421.856	-414.583	-411.543
	AIC	-471.509	-428.321	-425.856	-419.583	-417.543
	BIC	-474.492	-432.786	-431.796	-426.990	-426.411
	MSE	36.621	20.811	20.622	19.442	19.403
2	$\ln L_T(\phi)$	-418.194	-402.117	-392.923	-386.024	-381.062
	AIC	-424.194	-410.117	-402.923	-398.024	-395.062
	BIC	-433.144	-422.024	-417.772	-415.801	-415.753
	MSE	14.848	19.421	14.269	13.593	13.729
3	$\ln L_T(\phi)$	-407.963	-399.165	-379.618	-369.619	-356.322
	AIC	-419.963	-414.165	-397.618	-390.619	-380.322
	BIC	-437.864	-436.490	-424.346	-421.729	-415.792
	MSE	14.789	19.330	13.146	12.455	12.115

Table A2 - Results of the model selection for the second subseries $y^{155}=(y_1,...,y_{155})$.

		p				
m		0	1	2	3	4
1	$\ln L_T(\phi)$	-496.274	-448.437	-444.881	-437.324	-434.306
	AIC	-498.274	-451.437	-448.881	-442.324	-440.306
	BIC	-501.317	-455.992	-454.942	-449.884	-449.358
	MSE	35.592	19.934	19.733	18.596	18.560
2	$\ln L_T(\phi)$	-437.482	-420.328	-412.037	-405.003	-399.394
	AIC	-443.482	-428.328	-422.037	-417.003	-413.394
	BIC	-452.613	-440.476	-437.189	-435.146	-434.515
	MSE	21.101	18.467	13.630	12.999	13.304
3	$\ln L_T(\phi)$	-426.158	-417.546	-402.005	-388.582	-375.469
	AIC	-438.158	-432.546	-420.005	-409.582	-399.469
	BIC	-456.418	-455.323	-447.279	-441.332	-435.677
	MSE	14.113	18.254	12.521	11.9142	11.596

Table A3 - Results of the model selection for the third subseries y^{311}=(y_1,...,y_{311}).

		p				
m		0	1	2	3	4
1	$\ln L_T(\phi)$	-988.320	-853.005	-846.448	-834.305	-831.873
	AIC	-990.320	-856.005	-850.448	-839.305	-837.873
	BIC	-994.060	-861.610	-857.914	-848.630	-849.054
	MSE	33.822	14.373	14.010	13.193	13.215
2	$\ln L_T(\phi)$	-865.225	-784.129	-779.005	-771.723	-767.321
	AIC	-871.225	-792.129	-789.005	-783.723	-781.321
	BIC	-882.444	-807.076	-807.672	-806.103	-807.409
	MSE	16.970	13.137	13.098	12.465	12.347
3	$\ln L_T(\phi)$	-812.631	-765.842	-761.823	-755.093	-751.251
	AIC	-824.631	-780.842	-779.823	-776.093	-775.251
	BIC	-847.070	-808.866	-813.423	-815.259	-819.973
	MSE	13.963	11.693	12.099	10.144	10.109

Table A4 - Results of the model selection for the fourth subseries y^{538}=(y_1,...,y_{538}).

		p				
m		0	1	2	3	4
1	$\ln L_T(\phi)$	-1794.574	-1565.986	-1549.280	-1539.886	-1535.870
	AIC	-1796.574	-1568.986	-1553.280	-1544.886	-1541.870
	BIC	-1800.867	-1575.415	-1561.848	-1555.592	-1554.711
	MSE	46.305	19.974	18.971	18.517	18.434
2	$\ln L_T(\phi)$	-1596.147	-1482.186	-1471.348	-1464.332	-1459.814
	AIC	-1602.147	-1490.186	-1481.348	-1476.332	-1473.814
	BIC	-1615.011	-1507.330	-1502.768	-1502.026	-1503.777
	MSE	25.601	19.134	18.400	18.084	18.051
3	$\ln L_T(\phi)$	-1522.272	-1455.469	-1448.867	-1442.825	-1440.650
	AIC	-1534.272	-1470.469	-1466.867	-1463.825	-1464.650
	BIC	-1559.999	-1502.615	-1505.425	-1508.789	-1516.015
	MSE	15.906	17.647	17.108	16.885	16.588

Table A5 - Results of the model selection for the whole series y^{1057}=(y_1,...,y_{1057}).

		p				
m		0	1	2	3	4
1	$\ln L_T(\phi)$	-3476.812	-2957.361	-2935.961	-2920.827	-2911.997
	AIC	-3478.812	-2960.361	-2939.961	-2925.827	-2917.997
	BIC	-3483.775	-2967.805	-2949.884	-2938.228	-2932.875
	MSE	42.170	15.850	15.315	14.961	14.789
2	$\ln L_T(\phi)$	-2978.014	-2716.584	-2705.131	-2693.166	-2683.910
	AIC	-2984.014	-2724.584	-2715.131	-2705.166	-2697.910
	BIC	-2998.903	-2744.433	-2739.937	-2734.928	-2732.626
	MSE	21.257	14.640	14.454	14.246	14.168
3	$\ln L_T(\phi)$	-2819.432	-2677.750	-2672.033	-2661.983	-2654.901
	AIC	-2831.432	-2692.750	-2690.033	-2682.983	-2678.901
	BIC	-2861.211	-2729.967	-2734.684	-2735.066	-2738.414
	MSE	13.645	13.557	13.492	13.338	13.304

References

Basawa I. V. and Prakasa Rao B. L. S. (1980). *Statistical Inference for Stochastic Processes.* Academic Press, London.

Dempster A. P., Laird N. M., Rubin D. B. (1977). Maximum likelihood from incomplete data via the EM algorithm (with Discussion). *Journal of the Royal Statistical Society, Series B*, **39**, 1-38.

Douc R., Moulines E., Rydén T. (2001). *Asymptotic properties of the maximum-likelihood estimator in autoregressive models with Markov regime.* `http//www.tsi.enst.fr/~douc`.

Franses P. H. and van Dijk D. (2000). *Nonlinear Time Series Models in Empirical Finance.* Cambridge University Press, Cambridge.

Frühwirth-Schnatter S. (1994). Data Augmentation and Dynamic Linear Models. *Journal of Time Series Analysis*, **15**, 183-202.

Hamilton J. D. (1994). *Time Series Analysis.* Princeton University Press, Princeton.

Kim C. J. (1993). Dynamic linear models with Markow-switching. *Journal of Econometrics*, **60**, 1-22.

Kim C. J. and Nelson C. R. (1999). *State-Space Models with Regime Switching: Classical and Gibbs-Sampling Approaches with Applications.* The MIT Press, Cambridge.

Krozlig H.-M. (1997). *Markov-Switching Vector Autoregression: Modelling, Statistical Inference and Applications to Business Cycle Analysis.* Springer, Berlin.

MacDonald I. L. and Zucchini W. (1997). *Hidden Markov and Other Models for Discrete-valued Time Series.* Chapman & Hall, London.

Wu C. F. J. (1983). On the Convergence Properties of the EM Algorithm. *The Annals of Statistics*, **11**, 95-103.