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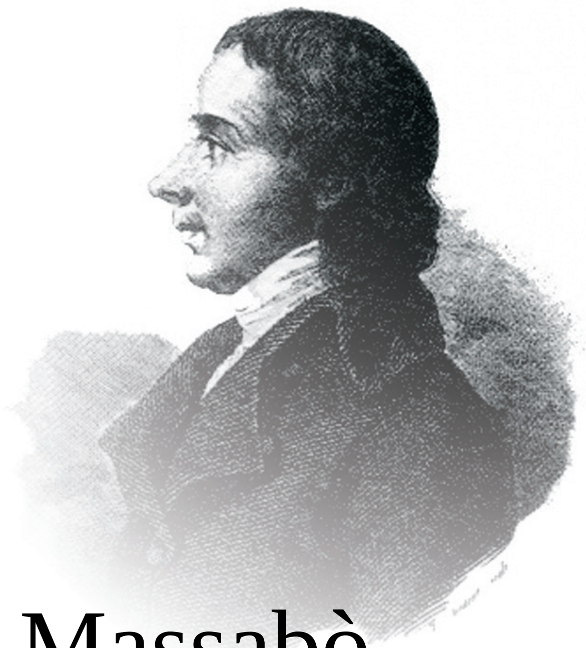

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A binomial model for pricing American-style average options with reset features

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Abstract

We propose a model for pricing American-style period-average reset options. Our approach relies on a binomial tree describing the underlying asset evolution on the reset period. At each node of the tree, we associate a set of representative averages chosen among all the effective averages realized at that node. At the terminal nodes, we associate an option value to each representative average by using the Barone-Adesi and Whaley [1] analytic approximation since, at the end of the reset period, an American period-average reset option becomes a standard American option. Then, we use backward recursion and linear interpolation to compute the option prices.

1 Introduction

In this article, we propose a binomial model for pricing American-style period-average reset options in a Cox, Ross and Rubinstein (CRR) [6] framework. A reset option is a path-dependent contingent claim whose strike price can be adjusted in favor of its holder at predetermined reset dates. The reset criterion concerns the arithmetic average of the underlying asset prices in the sense that, when the average hits the reset barrier in a pre-specified period, the strike price is reset to a new more attractive value. This feature may make the option more valuable and avoids to undergo a large loss when the underlying asset price follows a disadvantageous path. Furthermore, the arithmetic averaging feature reduces the possibilities of price manipulations and mitigates the hedging problems due to sudden changes in option delta near the reset dates. Consequently, these instruments have the desirable feature to be less sensitive to any one day's underlying price.

The payoff of a period-average reset option depends upon the arithmetic average of the underlying asset prices over several time intervals. This results in a huge complexity of the pricing problem in a CRR framework. Despite they are difficult to price, the reason why these options are a valuable resource is that many traded options and warrants are characterized by a strike price that can be reset to a new value if a particular condition is satisfied during a pre-specified period or during the option lifetime.

The pricing problem related to these options was faced, among others, by using tree methodologies. Many authors focus their attention on binomial trees. Among them, it is worth mentioning the valuation method proposed by Gray and Whaley

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[7] who, furthermore, provided a closed form solution for reset options with a single reset date [8].

Later, Kwok and Lau [11] introduced a lattice method for valuing European standard reset options under a risk neutral measure. In this same framework, Kim, Chang and Byun [10] priced arithmetic average reset options both of European and American-style. They investigated the Hull and White (HW) [9] forward shooting grid method generalized by Barraquand and Pudet [2] and applied this method for pricing reset options by using two augmented state variables at each node of the tree, one representing the strike price and the other the arithmetic average.

Other valuable approaches are those ones based on trinomial trees. Among them, an interesting methodology for pricing American-style reset options was proposed by Chang, Chung and Shackleton (CCS) [4] who adapted the HW technique to a Boyle [3] trinomial tree framework investigating the difference between daily reset option contracts and period-average reset ones.

Lattice-based models still play a crucial role in pricing path-dependent derivatives for their simplicity, flexibility and efficiency. We propose a new lattice algorithm for the valuation of American-style period-average reset options. Our model is based on a binomial lattice describing the evolution of the underlying asset on the reset period. At each node of the tree, we associate a set of representative averages whose choice is different from the lattice-based models quoted previously. In fact, we consider only realized averages of the underlying asset prices at each node according to an easy scheme that we already applied for pricing European and American Asian options (see Costabile, Massabó and Russo (CMR) [5] for a detailed description). Since an American period-average reset option becomes a standard American option at the end of the reset period, the CMR methodology is combined with the reset features and with the Barone-Adesi and Whaley (BAW) [1] efficient analytic approximation for American option prices.

The rest of the paper is organized as follows. Section 2 contains a brief description of the CCS tree-based methodology used for pricing reset options. In Section 3, we present a binomial model for pricing American-style period-average options with reset features. Finally, in Section 4, we draw conclusions.

2 Previous tree-based models

In this section, we briefly review the tree-based model proposed by CCS who adapted the HW procedure to a Boyle [3] trinomial tree for pricing American period-average reset options.

The HW model postulates that the underlying asset price evolves in a CRR framework. According to this model, the asset price at each time step increases by the factor $u = \exp(\sigma\sqrt{\Delta t})$ if an up step occurs or decreases by the factor $d = 1/u$ if a down step takes place, where σ is the volatility of the underlying asset price, T is the maturity of the considered contingent claim, $\Delta t = T/n$ and n is the number of time steps. The probability of an up step is the risk-neutral probability $p = (\exp(r\Delta t) - d)/(u - d)$ while the probability of a down step is $(1 - p)$ and r is the risk-free interest rate. Without loss of generality, we assume $t = 0$ as the instant of valuation and denote by $S(i, j)$ the underlying asset price at node (i, j) after j

up steps and $i-j$ down steps, with $S(0,0)=S$.

In a lattice model, the main problem for pricing options with payoff depending on the arithmetic average of the underlying asset prices is the large number of possible payoffs at each node of the tree. It is the case of Asian options. HW solved the problem by considering a set of representative averages at each node of the tree. The representative averages chosen by HW are of the form Se^{mh} where h is a fixed parameter and m is an integer. These averages are fictitious averages and are used to span the range between the minimum and the maximum average at each time step $i\Delta t$, $i=0,\ldots,n$. It is worth mentioning that the set of representative averages is the same for all the nodes lain at time $i\Delta t$.

Once the set of representative averages is associated to each node (i,j), $i=0,\ldots,n$, $j=0,\ldots,i$, HW computed the option price at inception by using a backward induction scheme coupled with linear interpolation.

Following this approach, CCS adapted the HW method by including the reset feature in a Boyle [3] trinomial tree framework. They divided the option lifetime and, consequently, the trinomial tree into two parts. On the first part of the tree corresponding to the reset period, they applied the HW procedure in the sense that the choice of the representative averages associated to each node of the trinomial tree follows the HW scheme. Once a set of representative averages is associated at each node of the tree, they considered the reset feature. Since a period-average reset option becomes an American option at the end of the reset period, CCS considered the path-function (i.e., the arithmetic average) and if it hit the reset barrier they constructed a new tree for the remaining time to maturity by using the reset strike price. In the other cases, they used the original strike price to compute the American standard option price. It means that the strike price for a period-average reset put option is given by

$$K(A)=\begin{cases} K^* & \text{if} \quad A\geq H \\ K & \text{if} \quad A<H \end{cases},$$

where H is the level of the reset barrier, K^* is the reset strike price, K is the original strike price while A is the value of the arithmetic average. Following this procedure, they calculated an option value for all the representative averages associated to the ending nodes of the reset period. Then, the option value at inception is computed following a backward induction scheme and a linear interpolation technique similar to those ones proposed by HW and adapted in a trinomial tree framework.

As in the HW model, a crucial point in the implementation of the CCS model relies on the choice of the value for the parameter h. This choice influences strongly the evaluation process since it determines the number of representative averages to be calculated at each node. To overcome this problem and obtain better convergence, CCS choose h following the approach suggested by Ritchken [12].

Furthermore, it is worth noting that CCS calculated all the American period-average put option prices through an approximation. They observed that the percentages of the difference between the values of American and European period-average puts are very close to their counterpart daily reset puts. Consequently, CCS calculated only approximated values for American period-average reset puts by

$$A(Av) \simeq E(Av)\frac{A(d)}{E(d)},$$

where $A(Av)$ is the value of an American period-average reset put, $E(Av)$ is the value of a European period-average reset put, $A(d)$ is the value of an American daily reset put and $E(d)$ is the value of a European period-average reset put. CCS justified this choice with the fact that it saves a lot of efforts since the calculation of $A(Av)$ is very time consuming. Furthermore, they asserted that a theoretical justification for such a simplification is based upon the Richardson's extrapolation technique used in many other American price approximations.

3 A lattice model for pricing period-average reset options

In this section, we propose a binomial model for pricing American-style period-average reset options. In a period-average reset call option the strike price is reset to a new level, called the reset strike price, only on the pre-specified reset dates if the arithmetic average (i.e., the path-function) of the asset prices during the considered period is lower (greater for a reset put option) than the reset strike price, called the reset barrier.

For pricing these path-dependent derivatives, we divide the lifetime of the option into two periods. In the first period, called reset period, the reset feature operates. At the end of the reset period, a period-average reset option becomes a standard American option even if its exercise price depends on the value of the arithmetic average. If, at the end of the reset period, the arithmetic average of the underlying asset prices is greater than or equal to the value of the reset barrier for put options, we consider the reset strike price for pricing the American option on the second period. Otherwise, we consider the original strike price. The update of the strike price works on the contrary for a call option.

We adapt the CMR adjusted binomial model proposed for pricing Asian options by including the BAW analytic approximation to calculate the prices of the same American-style period-average reset put options considered by CCS.

In order to clarify how the model works, for a period-average reset option characterized by a lifetime T, we divide this lifetime into two subperiods, T_1 and T_2, so that $T = T_1 + T_2$. The first subperiod T_1 identifies the reset period while T_2 identifies the remaining period of time where an American period-average option becomes a standard American option. On the first subperiod of length T_1, we consider a binomial tree based on n time steps with length $\Delta t = T_1/n$. Then, on the second subperiod, we use the BAW analytic approximation for pricing American standard options at the end of the reset period.

We start by operating on the binomial lattice characterized by n steps. At each node, we associate a set of representative averages. The total number of these averages in a binomial tree based on n time steps grows as $n^4/24$. The essence of this approach is to choose a set of effective averages thus, at each node (i, j), we associate a set of $1 + j(i - j)$ averages.

In order to give a clear description of how the model works, we consider a generic node (i, j) of the tree reached after j up steps and $i - j$ down steps. The set of representative averages is obtained as follows. At first, we compute the maximum average associated to the node (i,j), $A_{\max}(i,j)$, that is produced by the trajectory $\tau_{\max}(i,j)$ represented by the path with j up steps followed by $i-j$ down steps (Figure 1 shows the trajectory $\tau_{\max}$ for the node $(4,2)$) and denote it by $A(i,j;1)$ being the first element in the set,

$$A(i,j;1) := A_{\max}(i,j) = \frac{1}{i+1}\left(\sum_{h=0}^{j} Su^h + \sum_{h=0}^{i-j-1} Su^{h+2j-i}\right).$$

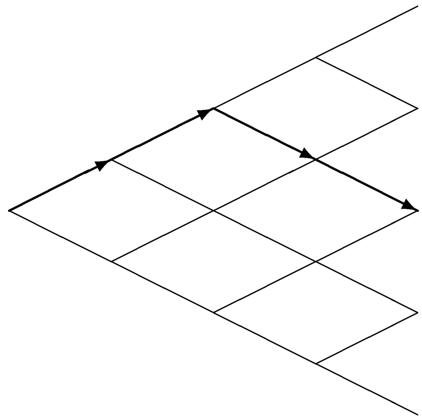

Figure 1: Trajectory $\tau_{\max}(4,2)$.

The minimum average associated to the node (i,j), $A_{\min}(i,j)$, is produced by the trajectory $\tau_{\min}(i,j)$ represented by the path with $i-j$ down steps followed by j up steps (Figure 2 shows the trajectory $\tau_{\min}$ for the node $(4,2)$) and it is denoted by $A(i,j;1+j(i-j))$ being the last element in the set,

$$A(i,j;1+j(i-j)) := A_{\min}(i,j) = \frac{1}{i+1}\left(\sum_{h=0}^{i-j} Sd^h + \sum_{h=0}^{j-1} Sd^{i-2j+h}\right).$$

The other representative averages for the node (i,j), $A(i,j;k)$, $k=2,\ldots,j(i-j)$, are computed recursively as follows. If $S_{\max}(i,j;k)$ is the greatest value of the underlying asset prices on a given trajectory, not belonging to $\tau_{\min}(i,j)$, that produces the average $A(i,j;k)$, then

$$A(i,j;k+1) = A(i,j;k) - \frac{1}{i+1}\left[S_{\max}(i,j;k) - S_{\max}(i,j;k)d^2\right].$$

In other words, the $(k+1)$-th representative path is obtained from the previous one by simply substituting $S_{\max}(i,j;k)$ with $S_{\max}(i,j;k)d^2$. This procedure continues as long as the last trajectory, $\tau_{\min}(i,j)$, is reached. Clearly, starting from $\tau_{\max}(i,j)$, we reach $\tau_{\min}(i,j)$ after $j(i-j)$ substitutions so that, at the (i,j)-th node, we associate a set made up of $1+j(i-j)$ representative averages.

The following example further clarifies how the algorithm works. Figure 3 illus-

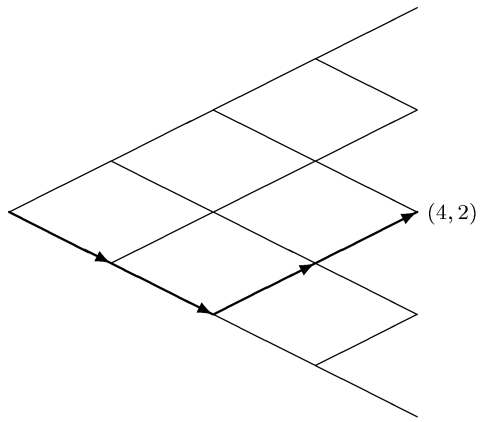

Figure 2: Trajectory $\tau_{\min}(4,2)$.

trates a binomial tree that describes the evolution of the underlying asset price on $n = 4$ steps. Consider all the trajectories reaching the node $(4,4)$. Since there is only one trajectory, there is only one average computed using the values $(S, Su, Su^2, Su^3, Su^4)$. For the node $(4,2)$, the first average of the set, $A_{\max}(4,2) = A(4,2;1)$, is computed by using the values (S, Su, Su^2, Su, S). The highest value used to compute $A(4,2;1)$ is $S_{\max}(4,2;1) = Su^2$. The second average is computed by considering the path obtained by substituting $S_{\max}(4,2;1)$ with $S_{\max}(4,2;1)d^2 = S$. Hence, $A(4,2;2)$ is computed using the vector (S, Su, S, Su, S). The remaining averages associated to the node $(4,2)$ are computed using the vectors (S, Sd, S, Su, S), (S, Sd, S, Sd, S) and (S, Sd, Sd^2, Sd, S). Following this procedure, the set of representative averages associated to the node $(4,2)$ contains all the effective averages but the one generated by the vector (S, Su, S, Sd, S) (the corresponding path is depicted in Figure 3 with thick lines marked with terminal arrows).

It is worth mentioning that there may exist some cases such that the average $A(i,j;k)$ is produced by a path in which $S_{\max}(i,j;k)$ is reached more than one time. In such cases, the next representative average $A(i,j;k+1)$ is computed by substituting only the first value $S_{\max}(i,j;k)$, reached by the trajectory starting from inception, with the value $S_{\max}(i,j;k)d^2$. In the same way, the set of representative averages is constructed for all the nodes of the tree[3].

Consider now all the averages associated to the node $(n,j), j = 0, ..., n$, with $n\Delta t = T_1$, and denote these averages $A(n,j;k), j = 0, ..., n, k = 1, ..., 1 + j(n-j)$. The n-th step of the binomial tree coincides with the date T_1 and it remains a period of length T_2 before that the American period-average reset option expires. On this time period, the American period-average reset option becomes an American standard option. Consequently, for each determination of the average $A(n,j;k)$, we have to calculate the American option price. It means that each average is considered as the underlying asset for the American option valuation.

[3]The sets of representative averages that we propose are generated starting from the greatest average, $A_{\max}(i,j)$, and ending to the smallest one, $A_{\min}(i,j)$. The same sets of representative averages could be generated in a symmetrical way simply starting from the average associated to the minimum path, $\tau_{\min}(i,j)$, and ending to that one associated to the highest path, $\tau_{\max}(i,j)$, but by substituting in each iteration only the last value $S_{\min}(i,j;k)$ starting from inception.

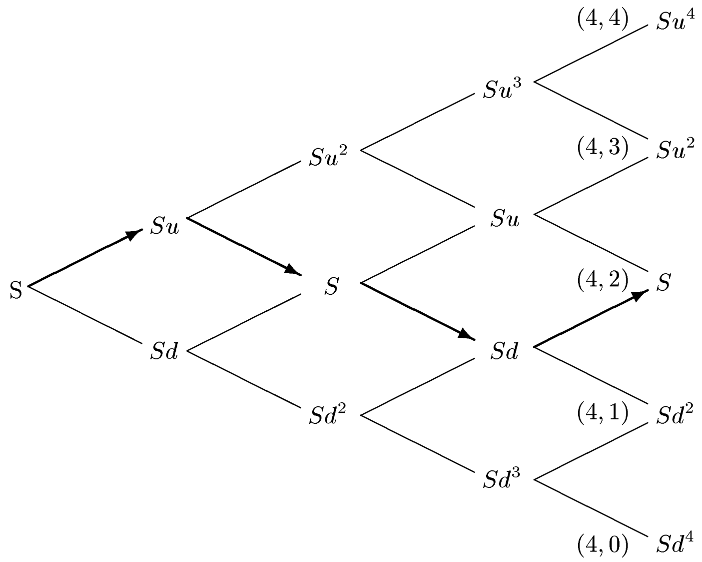

Figure 3: Path not considered for the average calculation.

The second step of our algorithm is to associate an American option price at each one of the values $A(n,j;k)$. The pricing of American options has usually resorted to finite difference or binomial approximation methods but, while these approximation methods yield accurate American option values, they are computationally expensive to use in our framework. This is the reason why we adopt the analytic approximation for American option values proposed by BAW.

For adapting the BAW model to our framework, on the remaining period of length T_2, we define the new strike price associated to each determination of $A(n,j;k)$ as

$$K(A(n,j;k)) = \begin{cases} K^* & \text{if} \quad A(n,j;k) \geq H \\ K & \text{if} \quad A(n,j;k) < H \end{cases}, \tag{1}$$

where K^* is the reset strike price, K is the original strike price while H is the level of the reset barrier. Hence, using the strike price defined in (1), the BAW model (see Appendix for further details) supplies us the American put price, $P(n,j;k)$, associated to $A(n,j;k)$, $j = 0, \ldots, n$, $k = 1, \ldots, 1 + j(n-j)$.

To compute the other option prices, we follow a backward induction scheme similar to that one proposed by HW but we must also consider the reset feature. In this case, the price of a period-average American reset put option in each state of nature $(i,j;k)$ is given by

$$\begin{aligned} P(i,j;k) = \max\left\{e^{-r\Delta t}[pP(i+1,j+1;k_u) + (1-p)P(i+1,j;k_d)]\right., \\ \left. K(A(i,j;k)) - Su^j d^{i-j}\right\}, \end{aligned} \tag{2}$$

where

$$K(A(i,j;k)) = \begin{cases} K^* & \text{if} \quad A(i,j;k) \geq H \\ K & \text{if} \quad A(i,j;k) < H \end{cases},$$

and $P(i+1,j;k_d)$ and $P(i+1,j+1;k_u)$ are the put option prices associated respectively to $[(i+1)A(i,j;k) + dS(i,j)]/(i+2)$ and $[(i+1)A(i,j;k) + uS(i,j)]/(i+2)$.

As a matter of fact, we suppose that the k-th value of the average at node (i, j), $A(i, j; k)$, leads to the k_u-th value of the average at node $(i+1, j+1)$, $A(i+1, j+1; k_u)$, when there is an upward movement in the stock price and to the k_d-th value of the average at node $(i + 1, j)$, $A(i + 1, j; k_d)$, when there is a downward movement in the stock price. Since we consider only a selected subset of effective averages, $A(i + 1, j + 1; k_u)$ and $A(i + 1, j; k_d)$ could appear in the sets of representative averages. In the other cases, $P(i + 1, j + 1; k_d)$ and $P(i + 1, j + 1; k_u)$ are computed using linear interpolation as follows. $P(i + 1, j + 1; k_u)$ is computed using linear interpolation between $P(i + 1, j + 1; k_1)$ and $P(i + 1, j + 1; k_2)$, where k_1 and k_2 are chosen so that $A(i + 1, j + 1; k_1)$ and $A(i + 1, j + 1; k_2)$ are the values of the representative averages at level $(i + 1)\Delta t$ closest to $A(i + 1, j + 1; k_u)$ such that $A(i + 1, j + 1; k_1) \leq A(i + 1, j + 1; k_u) \leq A(i + 1, j + 1; k_2)$. The interpolation technique starts by calculating the quantity

$$\omega(i+1, j+1; k_u) = \frac{A(i+1, j+1; k_u) - A(i+1, j+1; k_1)}{A(i+1, j+1; k_2) - A(i+1, j+1; k_1)},$$

so that the option price, $P(i + 1, j + 1; k_u)$, is then given by

$$\begin{aligned} P(i+1, j+1; k_u) &= P(i+1, j+1; k_1) + \\ \omega(i+1, j+1; k_u)\,[P(i+1, j+1; k_2) &- P(i+1, j+1; k_1)]\,. \end{aligned}$$

The same interpolation technique is used to estimate the option price $P(i+1, j; k_d)$.

The quantity $K(A(i, j; k)) - Su^j d^{i-j}$ in (2) represents the early exercise value for an American-style period-average reset put option. If we opportunely change the reset criterion in the following way

$$K(A(i, j; k)) = \begin{cases} K^* & \text{if} \quad A(i, j; k) < H \\ K & \text{if} \quad A(i, j; k) \geq H \end{cases},$$

our algorithm may be easily extended to American-style period-average reset call options whose price is given by

$$\begin{aligned} C(i, j; k) = \max \{ & e^{-r\Delta t}[pC(i+1, j+1; k_u) + (1-p)C(i+1, j; k_d)]\,, \\ & Su^j d^{i-j} - K(A(i, j; k)) \}\,, \end{aligned}$$

where $C(i+1, j; k_d)$ and $C(i+1, j+1; k_u)$ are computed similarly as $P(i+1, j; k_d)$ and $P(i + 1, j + 1; k_u)$. Clearly, we have to consider the BAW algorithm for call options that is different from that one presented for put options (see Barone-Adesi and Whaley [1] for further details).

Table 1 illustrates the numerical results supplied by our model for American-style period-average reset put options. The underlying asset at inception is $S = 100$, the original strike price is $K = 100$, the reset strike price is $K^* = 120$, the reset criterion is set to $H = 120$ while the time to maturity of the option is $T = 1$ year. The reset period is considered three months away from inception. We report the results for four different levels of the risk-free interest rate r (continuously compounded) and for three different levels of volatility σ.

It is worth noting that our model allow us to compute the option prices straightforwardly. On the contrary, as mentioned in the previous section, CCS calculated

n	$r = 0.1$ $\sigma = 0.7$ CCS=24.40 $LSM = 24.37$	$r = 0.1$ $\sigma = 0.5$ CCS=16.33 $LSM = 16.28$	$r = 0.1$ $\sigma = 0.3$ CCS=8.43 $LSM = 8.42$	$r = 0.08$ $\sigma = 0.7$ CCS=25.21 $LSM = 25.17$
10	23.9214	16.3694	8.6606	24.5682
20	24.1907	16.5430	8.7520	24.8493
30	24.2945	16.6035	8.7933	24.9522
40	24.3458	16.6311	8.8123	25.0036
50	24.3750	16.6447	8.8245	25.0332
60	24.3880	16.6584	8.8324	25.0464
70	24.4024	16.6670	8.8381	25.0610
80	24.4150	16.6750	8.8455	25.0739
90	24.4201	16.6560	8.8455	25.0790
n	$r = 0.08$ $\sigma = 0.5$ CCS= 17.04 $LSM = 17.00$	$r = 0.08$ $\sigma = 0.3$ CCS=9.00 $LSM = 8.96$	$r = 0.06$ $\sigma = 0.7$ CCS=26.07 $LSM = 26.02$	$r = 0.06$ $\sigma = 0.5$ CCS=17.80 $LSM = 17.78$
10	16.9407	9.1382	25.2563	17.5520
20	17.1292	9.2372	25.5448	17.7534
30	17.1897	9.2817	25.6469	17.8136
40	17.2171	9.3012	25.6985	17.8409
50	17.2309	9.3136	25.7287	17.8551
60	17.2448	9.3218	25.7423	17.8691
70	17.2535	9.3277	25.7571	17.8778
80	17.2617	9.3326	25.7702	17.8862
90	17.2422	9.3354	25.7751	17.8663
n	$r = 0.06$ $\sigma = 0.3$ CCS= 9.63 $LSM = 9.59$	$r = 0.04$ $\sigma = 0.7$ CCS= 26.99 $LSM = 26.95$	$r = 0.04$ $\sigma = 0.5$ CCS=18.63 $LSM = 18.59$	$r = 0.04$ $\sigma = 0.3$ CCS=10.33 $LSM = 10.26$
10	9.6522	25.9921	18.2129	10.2145
20	9.7634	26.2845	18.4232	10.3393
30	9.8099	26.3860	18.4833	10.3867
40	9.8298	26.4384	18.5105	10.4069
50	9.8425	26.4692	18.5249	10.4201
60	9.8509	26.4830	18.5393	10.4289
70	9.8571	26.4980	18.5480	10.4353
80	9.8621	26.5112	18.5565	10.4405
90	9.8650	26.5159	18.5361	10.4436

Table 1: American-style period-average reset option prices.

all the American period-average put option values through an approximation.

There is another remarkable difference between our model and the CCS one. The CCS approach calculates option prices based on daily observations of the average by considering 20 steps of the trinomial tree between two adjacent observations of the average. The reason why they used this approach is that, in a trinomial tree, the number of averages at each time step grows up very fast making the pricing problem computationally unmanageable. Instead, our approach is based on observations of the averages at each time step. Consequently, the option prices supplied by our model are a bit different than those ones calculated by CCS and the least squares Monte Carlo method even if a larger number of time steps might assure the full convergence to the values used for comparison.

4 Conclusions

We proposed a new algorithm for pricing American-style period-average reset options. The model is based on a binomial tree describing the evolution of the underlying asset in a CRR framework. These options are characterized by a reset barrier and when the path-function (i.e., the arithmetic average) hits the barrier the strike price is set to a new level. We use the CMR adjusted binomial model on the reset period. The main feature of this method is to choose sets of representative averages made up of effective averages calculated on real paths reaching each node of the tree. At the end of the reset period, an American period-average reset option becomes a standard American option so that we adopt the BAW analytic approximation to associate an option value to each representative average lain at the terminal nodes. Then, we apply backward recursion and linear interpolation to compute the option prices. The empirical results supplied by our approach are accurate with respect to other existing pricing models even if a larger number of time steps is required to achieve the full convergence to the benchmark values.

Appendix

The BAW analytic approximation for American options

In our framework, the BAW model works as follows. If we consider the put option case, the American put price $P(n,j;k)$ associated to each determination of the average $A(n,j;k)$ is calculated as

$$P(n,j;k) = \begin{cases} p(n,j;k) + A_1 \left(A(n,j;k)/A^*(n,j;k)\right)^{q_1} & \text{if } \; A(n,j;k) > A^*(n,j;k) \\ K(A(n,j;k)) - A(n,j;k) & \text{if } \; A(n,j;k) \leq A^*(n,j;k) \end{cases}, \tag{A.1}$$

where $p(n,j;k)$ is the European put option price calculated on the subperiod T_2 with strike $K(A(n,j;k))$,

$$A_1 = -\frac{A^*(n,j;k)}{q_1} \left\{1 - N\left[-d_1(A^*(n,j;k))\right]\right\},$$

($N\left[-d_1(A^*(n,j;k))\right]$ is the cumulative distribution function of a standard normal density function)

$$d_1(A^*(n,j;k)) = \frac{1}{\sigma\sqrt{T_2}}\left[\log\left(\frac{A^*(n,j;k)}{K(A(n,j;k))}\right) + \left(r + \frac{1}{2}\sigma^2\right)T_2\right],$$

$$q_1 = \frac{1}{2}\left[-(M-1) - \sqrt{(M-1)^2 + 4\frac{M}{1-e^{-rT_2}}}\right],$$

$$M = \frac{2r}{\sigma^2},$$

and $A^*(n,j;k)$ is defined as critical value. For each $A(n,j;k)$, the critical value is determined by solving the following equation

$$K(A(n,j;k)) - A^*(n,j;k) = p^*(n,j;k) - \left\{1 - N\left[-d_1(A^*(n,j;k))\right]\right\}\frac{A^*(n,j;k)}{q_1}, \tag{A.2}$$

where $p^*(n,j;k)$ is the European put option price whose underlying is $A^*(n,j;k)$. Since (A.2) cannot be solved straightforwardly, BAW developed an iterative procedure. To begin, evaluate both sides of (A.2) at some seed value, $A_i(n,j;k)$, that is

$$LHS(A_i(n,j;k)) = K(A(n,j;k)) - A_i(n,j;k), \tag{A.3}$$

and

$$RHS(A_i(n,j;k)) = p_i(n,j;k) - \left\{1 - N\left[-d_1(A_i(n,j;k))\right]\right\}\frac{A_i(n,j;k)}{q_1}, \tag{A.4}$$

where $p_i(n,j;k)$ is the European put option price whose underlying is $A_i(n,j;k)$ and $i=1$.

It would be unlikely that $LHS(A_i(n,j;k)) = RHS(A_i(n,j;k))$ on the initial guess $A_1(n,j;k)$. Consequently, a second guess must be calculated. To develop the next guess $A_{i+1}(n,j;k)$, first find the slope b_i of RHS at $A_i(n,j;k)$ that is

$$b_i = N[d_1(A_i(n,j;k))]\left(1 - \frac{1}{q_1}\right) - \frac{1}{q_1\sigma\sqrt{T_2}}n[-d_1(A_i(n,j;k))] - 1, \tag{A.5}$$

where $n[-d_1(A_i(n,j;k))]$ is the univariate normal density function.

The next step is to find where the line tangent to the curve RHS at $A_i(n,j;k)$ intersects the exercisable proceeds of the American put, $K(A(n,j;k)) - A(n,j;k)$, that is

$$RHS(A_i(n,j;k)) + b_i[A(n,j;k) - A_i(n,j;k)] = K(A(n,j;k)) - A(n,j;k),$$

and then isolate $A(n,j;k)$ to find $A_{i+1}(n,j;k)$,

$$A_{i+1}(n,j;k) = \frac{K(A(n,j;k)) + b_iA_i(n,j;k) - RHS(A_i(n,j;k))}{1-b_i}. \tag{A.6}$$

Equation (A.6) will provide the second and subsequent guesses of $A(n,j;k)$, updating (A.3), (A.4), (A.5), and (A.6) at each new iteration. The iterative procedure will continue until the relative absolute error falls within an acceptable tolerance level as it is

$$\frac{\mid LHS(A_i(n,j;k)) - RHS(A_i(n,j;k)) \mid}{K(A(n,j;k))} < 0.00001. \tag{A.7}$$

This iterative technique converges reasonable quickly by setting the seed value $A_1(n,j;k)$ equal to the option's exercise price $K(A(n,j;k))$ and by imposing the tolerance criterion (A.7).

To arrive to an approximate value of $A^*(n,j;k)$, consider the information contained in (A.2). If the time to expiration of the put option is equal to 0, the critical value above which the option will be exercised is the exercise price $K(A(n,j;k))$. At the other extreme, if the time remaining to expiration is infinite, the critical value may be calculated exactly by substituting $T_2 = +\infty$ into (A.2), that is

$$A^*(n,j;k)(\infty) = \frac{K(A(n,j;k))}{1 - \frac{1}{q_1(\infty)}},$$

where $q_1(\infty) = -M$. Finally, in (A.2), the critical value is a decreasing function of time to expiration, and an approximate analytic expression for the critical value is

$$A^*(n,j;k) = A^*(n,j;k)(\infty) + [K(A(n,j;k)) - A^*(n,j;k)]e^{h_1}, \tag{A.8}$$

where

$$h_1 = \left(rT_2 - 2\sigma\sqrt{T_2}\right)\left[\frac{K(A(n,j;k)}{K(A(n,j;k)) - A^*(n,j;k)(\infty)}\right].$$

Equation (A.8) provides the seed values for the iterative procedures that determine the critical values. In our case, their use ensures convergence in three iterations or less. After that we have calculated each critical value $A^*(n,j;k)$, we can use (A.1) to calculate the American put option price associated to each average $A(n,j;k)$ at the level n of the binomial tree.

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