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## Multi-Dimensional Trigonometric Approximation and Irregularities of Point Distribution

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# Multi-Dimensional Trigonometric 

# Approximation and Irregularities of Point 

Distribution

Leonardo Colzani Giacomo Gigante<br>Giancarlo Travaglini


#### Abstract

Given a positive constant $\alpha$, there exists a constant $c$ such that for every measurable set $\Omega$ in the Euclidean space and $R>0$, there exist entire functions of exponential type $R$ with $A(x) \leq \chi_{\Omega}(x) \leq B(x)$ and $|B(x)-A(x)| \leqslant c(1+R \operatorname{dist}(x, \partial \Omega))^{-\alpha}$. Analogous results hold for the approximation by eigenfunctions of differential operators on manifolds. This leads to Erdös-Turán type estimates for discrepancy on manifolds.


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## 1 Introduction

An infinite sequence of points $\left\{x_{j}\right\}_{j=1}^{+\infty}$ is uniformly distributed in the interval $[0,1]$ if for every $0 \leq a<b \leq 1$,

$$
\lim _{m \rightarrow+\infty}\left\{m^{-1} \sum_{j=1}^{m} \chi_{[a, b]}\left(x_{j}\right)\right\}=b-a .
$$

The criterion of H.Weyl states that a sequence is uniformly distributed if and only if for every $k \neq 0$,

$$
\lim _{m \rightarrow+\infty}\left\{m^{-1} \sum_{j=1}^{m} \exp \left(2 \pi i k x_{j}\right)\right\}=0 .
$$

More precisely, a measure of the irregularities of point distribution is given by the discrepancy

$$
\sup _{0 \leq a<b \leq 1}\left|(b-a)-m^{-1} \sum_{j=1}^{m} \chi_{[a, b]}\left(x_{j}\right)\right| .
$$

The inequality of P.Erdös and P.Turán establishes a quantitative connection between this discrepancy and exponential sums. Given $m$ points $0 \leq x_{1}, x_{2}, \ldots, x_{m} \leq 1$, then for every $n=1,2, \ldots$

$$
\begin{aligned}
& \sup _{0 \leq a<b \leq 1}\left|(b-a)-m^{-1} \sum_{j=1}^{m} \chi_{[a, b]}\left(x_{j}\right)\right| \\
\leq & c n^{-1}+c \sum_{k=1}^{n} k^{-1}\left|m^{-1} \sum_{j=1}^{m} \exp \left(2 \pi i k x_{j}\right)\right| .
\end{aligned}
$$

See [1], [4] and [9]. A proof of the above inequality relies on the ap-
proximation of characteristic functions $\chi_{[a, b]}(x)$ by trigonometric polynomials $P(x)=\sum_{k=-n}^{+n} \widehat{P}(k) \exp (2 \pi i k x)$, so that

$$
\begin{aligned}
& (b-a)-m^{-1} \sum_{j=1}^{m} \chi_{[a, b]}\left(x_{j}\right) \approx(b-a)-m^{-1} \sum_{j=1}^{m} P\left(x_{j}\right) \\
= & (b-a)-\widehat{P}(0)-\sum_{1 \leq|k| \leq n}\left(m^{-1} \sum_{j=1}^{m} \exp \left(2 \pi i k x_{j}\right)\right) \widehat{P}(k) .
\end{aligned}
$$

In particular, a construction of these approximations is due to A.Beurling and A.Selberg, who proved that for every $0 \leq a \leq b \leq 1$ and $n=0,1,2, \ldots$, there exist trigonometric polynomials $P_{ \pm}(x)$ of degree $n$ with

$$
\begin{gathered}
P_{-}(x) \leq \chi_{[a, b]}(x) \leq P_{+}(x) \\
\int_{0}^{1}\left|P_{ \pm}(x)-\chi_{[a, b]}(x)\right| d x=1 /(n+1) .
\end{gathered}
$$

See e.g. [11]. Similar extremal problems have been considered by J.J.Holt and J.D.Vaaler, with precise estimates on the approximation in $-\infty<x<$ $+\infty$ with measure $|x|^{2 \nu+1} d x$ of the function $\operatorname{sgn}(x)$ by functions of finite exponential type. A radialization of these functions then yields an analog of Selberg polynomials for approximation of characteristic functions of multidimensional balls and this has been applied to Erdös-Turán estimates of discrepancy. See [5], [6], [7]. Looking for a generalization of the above results, here we show that analogs of Selberg polynomials exist in several variables and in very general settings and, again, this can be applied to estimates of irregularities of point distribution.

The plan of the paper is the following. In the second section we consider
approximations from above and below of characteristic functions by entire functions of exponential type. We also briefly consider approximations by linear combinations of eigenfunctions of the Laplace-Beltrami operator on a compact Riemannian manifold. In the third section these approximation results are applied to Erdös-Turán estimates of irregularities of point distribution on manifolds. In particular, inspired by A.Lubotzky, R.Phillips and P.Sarnak [10], when the manifold is a compact Lie group or homogeneous space we consider point distributions generated by the action of a free group. Finally, inspired by W.M.Schmidt [12], when the manifold is a torus we obtain explicit estimates for the irregularities of distribution of sequences $\{j p\}_{j=1}^{+\infty}$ with respect to polyhedra and convex domains.

## 2 Approximation by entire functions

The following is the main result in this section.

Theorem 1 Given a positive constant $\alpha$, there exists a constant $c$ such that for every measurable set $\Omega$ in the Euclidean space $\mathbb{R}^{d}$ and $R>0$, there exist entire functions of exponential type $R$ satisfying $A(x) \leq \chi_{\Omega}(x) \leq B(x)$ and $|B(x)-A(x)| \leqslant c(1+R \text { dist }(x, \partial \Omega))^{-\alpha}$. Roughly speaking, the approximation $|B(x)-A(x)|$ is essentially one at points with distances $1 / R$ from the boundary of $\Omega$, while $|B(x)-A(x)|$ is essentially zero at larger distances. This approximation is essentially optimal. Indeed, for every pair of entire functions of exponential type $R$ with $A(x) \leq \chi_{\Omega}(x) \leq B(x)$, if $\operatorname{dist}(x, \partial \Omega) \leq\left(2 R \sup _{z \in \mathbb{R}^{d}}\{|A(z)|,|B(z)|\}\right)^{-1}$ then $B(x)-A(x) \geq 1 / 2$.

We would like to emphasize that the constant $c$ in the theorem is independent of the measurable set $\Omega$ and that there are no regularity assumptions on this set. When a set is regular, then there are few points at small distance from the boundary and the approximation is bad only on a small set, but when a set is fractal, then there are many points at small distance from the boundary and the approximation is bad on a large set.

One can turn the above pointwise estimates into integral ones. Denoting by $\mu$ the Lebesgue measure, the boundary of $\Omega$ has finite $\delta$ dimensional Minkowski measure if

$$
\limsup _{\varepsilon \rightarrow 0+} \varepsilon^{\delta-d} \mu(\{\operatorname{dist}(x, \partial \Omega)<\varepsilon\})<+\infty
$$

Corollary 2 Assume that the boundary of $\Omega$ has finite $\delta$ dimensional Minkowski measure. Then for every $R>0$ there exist entire functions of exponential type $R$ with $A(x) \leq \chi_{\Omega}(x) \leq B(x)$ and

$$
\int_{\mathbb{R}^{d}}|B(x)-A(x)| d x \leq c R^{\delta-d}
$$

For measurable sets periodic with respect to the integer lattice $\mathbb{Z}^{d}$, the above approximating entire functions are periodic too, hence they are trigonometric polynomials which may be seen as generalizations of Beurling Selberg polynomials.

Corollary 3 Given a positive constant $\alpha$ there exists a constant $c$ such that for every domain $\Omega$ in the torus $\mathbb{T}^{d}=\mathbb{R}^{d} / \mathbb{Z}^{d}$ and $R=0,1,2, \ldots$, there exist trigonometric polynomials of degree $R$ with $A(x) \leq \chi_{\Omega}(x) \leq B(x)$ and
$|B(x)-A(x)| \leqslant c(1+R \operatorname{dist}(x, \partial \Omega))^{-\alpha}$.

Proof of Theorem 1. Let $m(\xi)$ be a smooth radial function on $\mathbb{R}^{d}$ with $m(\xi)=0$ if $|\xi| \geq 1 / 2$ and $\int_{\mathbb{R}^{d}} m(\xi)^{2} d \xi=1$. Then the convolution $m * m(\xi)$ is a smooth radial function with $m * m(0)=1$ and $m(\xi)=0$ if $|\xi| \geq 1$.

Define

$$
K(x)=\int_{\mathbb{R}^{d}} m * m(\xi) \exp (2 \pi i \xi \cdot x) d \xi
$$

Since $m * m(\xi)$ is smooth, this kernel $K(x)$ and its derivatives have fast decay at infinity, $\left|\partial^{\beta} K(x) / \partial x^{\beta}\right| \leq c(1+|x|)^{-d-\gamma}$ for every $\beta$ and $\gamma$. Moreover, it is non negative, since it is the square of the Fourier transform of $m(\xi)$, and it has mean one, $\int_{\mathbb{R}^{d}} K(x) d x=m * m(0)=1$. Define $K_{R}(x)=R^{d} K(R x)$. The convolutions $K_{R} * \chi_{\Omega}(x)$ are entire functions of exponential type $R$ which approximate $\chi_{\Omega}(x)$ and suitable modifications of these approximations satisfy all the required properties. Let

$$
H_{\alpha, R}(x)=(1+R \operatorname{dist}(x, \partial \Omega))^{-\alpha}
$$

Claim For every $\alpha>0$ there exists $c$ such that for every $x$,

$$
\left|\chi_{\Omega}(x)-K_{R} * \chi_{\Omega}(x)\right| \leq c H_{\alpha, R}(x) .
$$

To prove this estimate, recall that $K_{R}(y)$ is positive with mean one and fast decay at infinity. Hence

$$
\begin{aligned}
& \left|\chi_{\Omega}(x)-K_{R} * \chi_{\Omega}(x)\right|=\left|\int_{\mathbb{R}^{d}} K_{R}(y)\left(\chi_{\Omega}(x)-\chi_{\Omega}(x-y)\right) d y\right| \\
& \quad \leq \int_{\{|y| \geq \operatorname{dist}(x, \partial \Omega)\}} K_{R}(y) d y \leq c(1+R \operatorname{dist}(x, \partial \Omega))^{-\alpha} .
\end{aligned}
$$

Claim For every $\alpha>0$ there exist positive constants $c$ and $C$ such that for every $x$,

$$
c H_{\alpha, R}(x) \leq K_{R} * H_{\alpha, R}(x) \leq C H_{\alpha, R}(x)
$$

To prove the estimate from above, observe that $H_{\alpha, R}(x-y) \approx H_{\alpha, R}(x)$ if $|y| \leq 2^{-1} \operatorname{dist}(x, \partial \Omega)$. Hence,

$$
\begin{aligned}
& K_{R} * H_{\alpha, R}(x)=\int_{\mathbb{R}^{d}} K_{R}(y) H_{\alpha, R}(x-y) d y \\
& \leq\left\{\max _{|y| \leq 2^{-1} \operatorname{dist}(x, \partial \Omega)} H_{\alpha, R}(x-y)\right\} \int_{\mathbb{R}^{d}} K_{R}(y) d y \\
&+\left\{\max _{y \in \mathbb{R}^{d}} H_{\alpha, R}(x-y)\right\}_{\left\{|y| \geq 2^{-1} \operatorname{dist}(x, \partial \Omega)\right\}} K_{R}(y) d y \\
& \leq c H_{\alpha, R}(x) .
\end{aligned}
$$

To prove the estimate from below, again observe that $H_{\alpha, R}(x-y) \approx$ $H_{\alpha, R}(x)$ if $|y| \leq 1 / R$. Hence,

$$
\begin{gathered}
K_{R} * H_{\alpha, R}(x)=\int_{\mathbb{R}^{d}} K_{R}(y) H_{\alpha, R}(x-y) d y \\
\geq\left\{\min _{|y| \leq 1 / R} H_{\alpha, R}(x-y)\right\} \int_{\{|y| \leq 1 / R\}} K_{R}(y) d y \\
\geq c H_{\alpha, R}(x) .
\end{gathered}
$$

The constants $c$ and $C$ in both claims may be chosen independent of $\Omega$. To conclude the proof of the theorem, define

$$
\begin{aligned}
& A(x)=K_{R} * \chi_{\Omega}(x)-\gamma K_{R} * H_{\alpha, R}(x), \\
& B(x)=K_{R} * \chi_{\Omega}(x)+\gamma K_{R} * H_{\alpha, R}(x) .
\end{aligned}
$$

By the above claims, if $\gamma$ is large enough, then

$$
\begin{gathered}
\chi_{\Omega}(x)-A(x)=\gamma K_{R} * H_{\alpha, R}(x)-\left(K_{R} * \chi_{\Omega}(x)-\chi_{\Omega}(x)\right) \\
\geq \gamma c_{1} H_{\alpha, R}(x)-c_{2} H_{\alpha, R}(x) \geq 0, \\
B(x)-\chi_{\Omega}(x)=\gamma K_{R} * H_{\alpha, R}(x)-\left(\chi_{\Omega}(x)-K_{R} * \chi_{\Omega}(x)\right) \\
\geq \\
\geq \gamma c_{1} H_{\alpha, R}(x)-c_{2} H_{\alpha, R}(x) \geq 0 .
\end{gathered}
$$

Finally,

$$
B(x)-A(x)=2 \gamma K_{R} * H_{\alpha, R}(x) \leq c H_{\alpha, R}(x) .
$$

The fact that this order of approximation is optimal is a consequence of the inequality of S.Bernstein between the maxima of an entire function and its derivatives. If $F(z)$ is an entire function of exponential type $R$, then

$$
|F(x)-F(y)| \leq|x-y| \sup _{z \in \mathbb{R}^{d}}|\nabla F(z)| \leq|x-y| R \sup _{z \in \mathbb{R}^{d}}|F(z)| .
$$

Hence, if $A(x) \leq \chi_{\Omega}(x) \leq B(x)$ are entire functions of exponential type $R$, if $x$ is in $\Omega$ and $y$ is outside $\Omega$ with $|x-y| \leq\left(2 R \sup _{z \in \mathbb{R}^{d}}\{|A(z)|,|B(z)|\}\right)^{-1}$, then $B(x)-A(y) \geq 1$ and

$$
\begin{aligned}
& B(x)-A(x)=(B(x)-A(y))-(A(x)-A(y)) \geq 1-1 / 2 \\
& B(y)-A(y)=(B(x)-A(y))-(B(x)-B(y)) \geq 1-1 / 2
\end{aligned}
$$

The above results for the Euclidean spaces $\mathbb{R}^{d}$ and $\mathbb{T}^{d}$ easily extend to non euclidean settings, such as expansions in eigenfunctions of differential operators on manifolds. Let $\mathcal{M}$ be a smooth $d$ dimensional compact manifold without boundary, with Riemannian distance $\operatorname{dist}(x, y)$ and measure $\mu$ normalized so that $\mu(\mathcal{M})=1$. Let $\Delta$ be the Laplace-Beltrami operator, with eigenvalues $\left\{\lambda^{2}\right\}$ and let $\left\{\varphi_{\lambda}(x)\right\}$ be a complete orthonormal system of eigenfunctions. To every function in $\mathbb{L}^{2}(\mathcal{M}, d \mu)$ one can associate a Fourier transform and a Fourier series,

$$
\widehat{f}(\lambda)=\int_{\mathcal{M}} f(y) \overline{\varphi_{\lambda}(y)} d \mu(y), \quad f(x)=\sum_{\lambda} \widehat{f}(\lambda) \varphi_{\lambda}(x) .
$$

Fourier series on compact Lie groups and symmetric spaces are examples. In particular, the Laplace operator $-\sum_{j=1}^{d} \partial^{2} / \partial x_{j}^{2}$ on the torus $\mathbb{T}^{d}=\mathbb{R}^{d} / \mathbb{Z}^{d}$ has eigenfunctions $\{\exp (2 \pi i k \cdot x)\}_{k \in \mathbb{Z}^{d}}$ with eigenvalues $\left\{4 \pi^{2}|k|^{2}\right\}_{k \in \mathbb{Z}^{d}}$ and the eigenfunction expansions are classical trigonometric series. Similarly, the eigenfunctions of the Laplace operator on the surface of a sphere are homogeneous harmonic polynomials and the eigenfunction expansions are spherical harmonic expansions. In the setting of manifolds an analog of the trigonometric polynomials is given by finite linear combination of eigenfunctions $\sum_{\lambda} c_{\lambda} \varphi_{\lambda}(x)$. Indeed it can be shown that there is a close relation between approximation by functions of exponential type and by eigenfunctions. See for example [3]. The following is a generalization of the above theorem and corollary.

Theorem 4 Given a positive constant $\alpha$ there exists a constant $c$ with the following property: given a domain $\Omega$ in $\mathcal{M}$ and $R>0$, there exist linear combinations of eigenfunctions with eigenvalues at most $R^{2}$ such that $A(x) \leq$ $\chi_{\Omega}(x) \leq B(x)$ and $|B(x)-A(x)| \leqslant c(1+R \text { dist }(x, \partial \Omega))^{-\alpha}$.

Proof. The proof of this theorem is similar to the proof of Theorem 1. One only needs a suitable family of approximations of the identity adapted to the manifold, analogous to the convolution kernels $R^{d} K(R x)$.

Claim Given $\beta$ and $R>0$, there exist kernels with the following properties:

$$
\begin{gathered}
K_{R}(x, y)=\sum_{\lambda<R} c_{\lambda} \varphi_{\lambda}(x) \overline{\varphi_{\lambda}(y)}, \\
\left|K_{R}(x, y)\right| \leq c R^{d}(1+R \operatorname{dist}(x, y))^{-d-\beta}, \\
\int_{\mathcal{M}} K_{R}(x, y) d \mu(y)=1, \\
\int_{\mathcal{M}}\left|K_{R}(x, y)\right| d \mu(y) \leq 1+c / R .
\end{gathered}
$$

The last two conditions mean that, up to a negligible error, these kernels are essentially positive. The construction of such kernels on Lie groups and symmetric spaces is well known and in these cases it is possible to obtain positivity. We do dot know whether positivity can be achieved in our general setting, however in the sequel this essential positivity will suffice. Given $m(\xi)$ as in the proof of Theorem 1 and $h(|\xi|)=m * m(\xi)$, define

$$
K_{R}(x, y)=\sum_{\lambda} h\left(R^{-1} \lambda\right) \varphi_{\lambda}(x) \overline{\varphi_{\lambda}(y)} .
$$

It is possible to prove that these kernels have asymptotic expansions with Euclidean main terms $R^{d} K(R \operatorname{dist}(x, y))$ and remainders controlled by $c / R$. Although the details are not completely trivial, the techniques can be found in Chapter XII of [14], or in [2]. Finally, define

$$
\begin{aligned}
& A(x)=\int_{\mathcal{M}} K_{R}(x, y)\left(\chi_{\Omega}(y)-\gamma H_{\alpha, R}(y)\right) d \mu(y) \\
& B(x)=\int_{\mathcal{M}} K_{R}(x, y)\left(\chi_{\Omega}(y)+\gamma H_{\alpha, R}(y)\right) d \mu(y) .
\end{aligned}
$$

Then, as in the proof of Theorem 1, it is possible to show that these functions satisfy the required properties.

## 3 An Erdös-Turán type inequality

As advertised in the Introduction, the approximation results in the previous section have simple and straightforward applications to multi-dimensional versions of the classical Erdös-Turán inequality.

Theorem 5 Let $\Omega$ be a domain in the manifold $\mathcal{M}$ and let $\left\{x_{j}\right\}_{j=1}^{m}$ be a sequence of $m$ points. Also let $H_{\alpha, R}(x)=(1+R \text { dist }(x, \partial \Omega))^{-\alpha}$. Then, for some constant $c$ independent of $\Omega$ and of $\left\{x_{j}\right\}_{j=1}^{m}$ and for every $R>0$,

$$
\begin{gathered}
\left|\mu(\Omega)-m^{-1} \sum_{j=1}^{m} \chi_{\Omega}\left(x_{j}\right)\right| \leq \\
c \int_{\mathcal{M}} H_{\alpha, R}(x) d \mu(x)+c \sum_{0<\lambda<R}\left(\left|\widehat{\chi}_{\Omega}(\lambda)\right|+\left|\widehat{H}_{\alpha, R}(\lambda)\right|\right)\left|m^{-1} \sum_{j=1}^{m} \varphi_{\lambda}\left(x_{j}\right)\right| .
\end{gathered}
$$

Proof. If $A(x) \leq \chi_{\Omega}(x) \leq B(x)$ are defined as above, then

$$
\begin{aligned}
A(x), B(x)= & \int_{\mathcal{M}}\left(\sum_{\lambda} h\left(R^{-1} \lambda\right) \varphi_{\lambda}(x) \overline{\varphi_{\lambda}(y)}\right)\left(\chi_{\Omega}(y) \pm \gamma H_{\alpha, R}(y)\right) d \mu(y) \\
& =\sum_{\lambda} h\left(R^{-1} \lambda\right)\left(\widehat{\chi}_{\Omega}(\lambda) \pm \gamma \widehat{H}_{\alpha, R}(\lambda)\right) \varphi_{\lambda}(x) .
\end{aligned}
$$

Recalling that 0 is an eigenvalue with eigenfunction $\varphi_{0}(x)=1$ and also that $h(t)$ is bounded, with $h(0)=1$ and $h(t)=0$ if $t \geq 1$, one obtains

$$
\begin{gathered}
\mu(\Omega)-m^{-1} \sum_{j=1}^{m} \chi_{\Omega}\left(x_{j}\right) \leq \mu(\Omega)-m^{-1} \sum_{j=1}^{m} A\left(x_{j}\right) \\
=\gamma \widehat{H}_{\alpha, R}(0)-\sum_{\lambda>0} h\left(R^{-1} \lambda\right)\left(\widehat{\chi}_{\Omega}(\lambda)-\gamma \widehat{H}_{\alpha, R}(\lambda)\right)\left(m^{-1} \sum_{j=1}^{m} \varphi_{\lambda}\left(x_{j}\right)\right) \\
\leq c \int_{\mathcal{M}} H_{\alpha, R}(x) d \mu(x)+c \sum_{0<\lambda<R}\left(\left|\widehat{\chi}_{\Omega}(\lambda)\right|+\left|\widehat{H}_{\alpha, R}(\lambda)\right|\right)\left|m^{-1} \sum_{j=1}^{m} \varphi_{\lambda}\left(x_{j}\right)\right| .
\end{gathered}
$$

Similarly

$$
\begin{gathered}
-\mu(\Omega)+m^{-1} \sum_{j=1}^{m} \chi_{\Omega}\left(x_{j}\right) \leq-\mu(\Omega)+m^{-1} \sum_{j=1}^{m} B\left(x_{j}\right) \\
\leq c \int_{\mathcal{M}} H_{\alpha, R}(x) d \mu(x)+c \sum_{0<\lambda<R}\left(\left|\widehat{\chi}_{\Omega}(\lambda)\right|+\left|\widehat{H}_{\alpha, R}(\lambda)\right|\right)\left|m^{-1} \sum_{j=1}^{m} \varphi_{\lambda}\left(x_{j}\right)\right| .
\end{gathered}
$$

One might ask for a neater statement of the above theorem, without explicit references to the functions $H_{\alpha, R}(x)$. When the boundary of $\Omega$ has finite $\delta$ dimensional Minkowski measure, then $\int_{\mathcal{M}} H_{\alpha, R}(x) d \mu(x)$ is dominated by $c R^{\delta-d}$. However, it is not so clear how to eliminate $\left\{\widehat{H}_{\alpha, R}(\lambda)\right\}$. Indeed, even if in some average sense this Fourier transform is smaller than $\left\{\widehat{\chi}_{\Omega}(\lambda)\right\}$, the pointwise inequality $\left|\widehat{H}_{\alpha, R}(\lambda)\right| \leq c\left|\widehat{\chi}_{\Omega}(\lambda)\right|$ can be false, for example when $\widehat{\chi}_{\Omega}(\lambda)=0$. On the other hand the function $H_{\alpha, R}(x)$ can be replaced by any function $G_{\alpha, R}(x)$ with similar behavior, $0<c<G_{\alpha, R}(x) / H_{\alpha, R}(x)<C<$ $+\infty$. For example, when the domain $\Omega$ is defined by an inequality $\varphi(x)<0$, with $\nabla \varphi(x) \neq 0$ if $\varphi(x)=0$, one can take $G_{\alpha, R}(x)=\left(1+(R \varphi(x))^{2}\right)^{-\alpha / 2}$.

Inspired by the work of A.Lubotzky, R.Phillips, P.Sarnak on the problem of distributing points on a sphere, we now consider irregularities of point distributions generated by the action of a free group on a homogeneous space.

Corollary 6 Let $\mathcal{G}$ be a compact Lie group, $\mathcal{K}$ a closed subgroup, $\mathcal{M}=\mathcal{G} / \mathcal{K}$ a homogeneous space of dimension $d$ with normalized invariant measure $\mu$. Let $\mathcal{H}$ be a finitely generated free subgroup in $\mathcal{G}$ and assume that the action of $\mathcal{H}$ on $\mathcal{M}$ is free. Given a positive integer $k$, let $\left\{\sigma_{j}\right\}_{j=1}^{m}$ be an ordering of the elements in $\mathcal{H}$ with length at most $k$. For every function $f(x)$ in $\mathbb{L}^{2}(\mathcal{M}, d \mu)$ define

$$
T f(x)=m^{-1} \sum_{j=1}^{m} f\left(\sigma_{j} x\right) .
$$

This operator is self adjoint and has an orthonormal complete system of eigenfunctions. Assume that all the eigenfunctions with eigenvalue 1 are constant, while the other eigenvalues $T(\lambda)$ satisfy the bound $|T(\lambda)| \leq$ $\mathrm{cm}^{-1 / 2} \log (m)$ with some constant $c$ independent of $m$. Finally, let $\Omega$ be a domain in $\mathcal{M}$ with boundary of finite $\delta$ dimensional Minkowski measure. Then, for every point $p$ in $\mathcal{M}$, one has

$$
\left|\mu(\Omega)-m^{-1} \sum_{j=1}^{m} \chi_{\Omega}\left(\sigma_{j} p\right)\right| \leq c m^{(\delta-d) /(2 d-\delta)} \log ^{(2 d-2 \delta) /(2 d-\delta)}(m) .
$$

The above constant $c$ may be chosen independent of $m$ and $p$ and of the positioning of $\Omega$ in $\mathcal{M}$.
A.Lubotzky, R.Phillips and P.Sarnak have shown that the required Ramanujan bounds on the eigenvalues hold when the homogeneous space is the two dimensional sphere $\mathbb{S O}(3) / \mathbb{S O}(2)$ and the free group is generated by rota-
tions of angles $\arccos (-3 / 5)$ around orthogonal axes. In particular, when $\mathcal{M}$ is a sphere and $\Omega$ is a spherical cap, the above corollary is already contained in [10]. However, while their proof relies on explicit estimates for Fourier coefficients of spherical caps, here we avoid explicit computations and obtain results which apply to more general domains.

Proof. Since the operators $T$ and $\Delta$ commute, they have a common orthonormal system of eigenfunctions, $\Delta \varphi_{\lambda}(x)=\lambda^{2} \varphi_{\lambda}(x)$ and $T \varphi_{\lambda}(x)=T(\lambda) \varphi_{\lambda}(x)$. Our assumption is precisely that $|T(\lambda)| \leq c m^{-1 / 2} \log (m)$ if $\lambda \neq 0$. Hence, by Theorem 5,

$$
\begin{gathered}
\left|\mu(\Omega)-m^{-1} \sum_{j=1}^{m} \chi_{\Omega}\left(\sigma_{j} p\right)\right| \\
\leq c \int_{\mathcal{M}} H_{\alpha, R}(x) d \mu(x)+c \sum_{0<\lambda<R}\left(\left|\widehat{\chi}_{\Omega}(\lambda)\right|+\left|\widehat{H}_{\alpha, R}(\lambda)\right|\right)\left|T(\lambda) \varphi_{\lambda}(p)\right| \\
\leq c \int_{\mathcal{M}} H_{\alpha, R}(x) d \mu(x) \\
+c m^{-1 / 2} \log (m) \sum_{0<\lambda<R}\left(\left|\widehat{\chi}_{\Omega}(\lambda)\right|+\left|\widehat{H}_{\alpha, R}(\lambda)\right|\right)\left|\varphi_{\lambda}(p)\right|
\end{gathered}
$$

If the boundary of $\Omega$ has finite $\delta$ dimensional Minkowski measure and if $\alpha>d-\delta$, then

$$
\int_{\mathcal{M}} H_{\alpha, R}(x) d \mu(x) \leq c R^{\delta-d}
$$

The remaining sums can be estimated as follows.

$$
\begin{gathered}
\sum_{0<\lambda<R}\left|\widehat{\chi}_{\Omega}(\lambda)\right|\left|\varphi_{\lambda}(p)\right| \\
\leq\left\{\sum_{0<\lambda<1}\left|\widehat{\chi}_{\Omega}(\lambda)\right|^{2}\right\}^{1 / 2}\left\{\sum_{0<\lambda<1}\left|\varphi_{\lambda}(p)\right|^{2}\right\}^{1 / 2} \\
+\sum_{k=0}^{\left[\log _{2}(R)\right]}\left\{\sum_{\lambda \geq 2^{k}}\left|\widehat{\chi}_{\Omega}(\lambda)\right|^{2}\right\}^{1 / 2}\left\{\sum_{\lambda<2^{k+1}}\left|\varphi_{\lambda}(p)\right|^{2}\right\}^{1 / 2} .
\end{gathered}
$$

If the boundary of $\Omega$ has finite $\delta$ dimensional Minkowski measure, then the modulus of continuity in $\mathbb{L}^{2}(\mathcal{M}, d \mu)$ of the characteristic function $\chi_{\Omega}(x)$ has an order $(d-\delta) / 2$ and, by an analog for eigenfunction expansions of the approximation theorem of D.Jackson,

$$
\sum_{\lambda \geq 2^{k}}\left|\widehat{\chi}_{\Omega}(\lambda)\right|^{2} \leq c 2^{(\delta-d) k} .
$$

See e.g. [3]. Moreover, by the estimates of H.Weyl on the spectral function,

$$
\sum_{\lambda<2^{k+1}}\left|\varphi_{\lambda}(p)\right|^{2} \leq c 2^{d k}
$$

See e.g. Theorem 17.5.3 in [8]. Collecting these estimates one obtains

$$
\sum_{0<\lambda<R}\left|\widehat{\chi}_{\Omega}(\lambda)\right|\left|\varphi_{\lambda}(p)\right| \leq c R^{\delta / 2}
$$

Similarly,

$$
\sum_{0<\lambda<R}\left|\widehat{H}_{\alpha, R}(\lambda)\right|\left|\varphi_{\lambda}(p)\right| \leq c R^{\delta / 2}
$$

Hence,

$$
\left|\mu(\Omega)-m^{-1} \sum_{j=1}^{m} \chi_{\Omega}\left(\sigma_{j} p\right)\right| \leq c R^{\delta-d}+c m^{-1 / 2} \log (m) R^{\delta / 2} .
$$

Choosing $R=m^{1 /(2 d-\delta)} \log ^{2 /(\delta-2 d)}(m)$, one concludes that

$$
\left|\mu(\Omega)-m^{-1} \sum_{j=1}^{m} \chi_{\Omega}\left(\sigma_{j} p\right)\right| \leq c m^{(\delta-d) /(2 d-\delta)} \log ^{(2 d-2 \delta) /(2 d-\delta)}(m) .
$$

Finally, observe that all these estimates are invariant under the group action.

As we said, the above corollary applies to distributions of points on a sphere generated by suitable free groups of rotations. As a final example, we consider distributions of points in a torus. Since $\mathbb{T}^{d}=\mathbb{R}^{d} / \mathbb{Z}^{d}$ is commutative, a free subgroup has only one generator and it is isomorphic to the integers, hence it has the form $\{j p\}_{j=-\infty}^{+\infty}$, where the multiples $j p$ are modulo $\mathbb{Z}^{d}$ and at least one coordinate of $p$ is irrational. A common orthonormal system of eigenfunctions for the operators

$$
\Delta f(x)=-\sum_{j=1}^{d} \frac{\partial^{2}}{\partial x_{j}^{2}} f(x), \quad T f(x)=m^{-1} \sum_{j=1}^{m} f(x+j p),
$$

are the exponentials $\{\exp (2 \pi i k \cdot x)\}_{k \in \mathbb{Z}^{d}}$ and the eigenfunction expansions are classical trigonometric Fourier series. The eigenvalues of these operators can be computed explicitly, in particular $\Delta$ has eigenvalues $\left\{4 \pi^{2}|k|^{2}\right\}_{k \in \mathbb{Z}^{d}}$,
while

$$
\begin{aligned}
& T(\exp (2 \pi i k \cdot x))=m^{-1} \sum_{j=1}^{m} \exp (2 \pi i k \cdot(x+j p)) \\
= & \left(\exp (\pi i(m+1) k \cdot p) \frac{\sin (\pi m k \cdot p)}{m \sin (\pi k \cdot p)}\right) \exp (2 \pi i k \cdot x) .
\end{aligned}
$$

Denoting by $\|t\|$ the distance of $t$ to the nearest integer,

$$
\left|\exp (\pi i(m+1) k \cdot p) \frac{\sin (\pi m k \cdot p)}{m \sin (\pi k \cdot p)}\right| \leq \min \{1,1 /(2 m\|k \cdot p\|)\}
$$

If $p$ has a rational coordinate, then $\|k \cdot p\|$ can be zero, if $p$ has an irrational coordinate, then $\|k \cdot p\|$ can be arbitrarily close to zero. In both cases the Ramanujan estimates required by the above corollary are not satisfied. Nevertheless, for the distribution of points $\{j p\}_{j=-\infty}^{+\infty}$ the following results hold.

Corollary 7 If $\Omega$ is a polyhedron in the torus $\mathbb{T}^{d}$, then for every $\varepsilon>0$ and almost every $p$ in $\mathbb{T}^{d}$ there exists a constant $c$ such that for every $m$,

$$
\left|\mu(\Omega)-m^{-1} \sum_{j=1}^{m} \chi_{\Omega}(j p)\right| \leq c m^{-1} \log ^{d+1+\varepsilon}(m) .
$$

The above constant c may be chosen independent of the positioning of the polyhedron in the torus.

Proof. To be explicit, let $\Omega$ be a parallelepiped defined by $\left\{\gamma_{j} \leq x_{j} \leq \delta_{j}\right\}$. In this case

$$
\begin{gathered}
\hat{\chi}_{\Omega}(k)=\int_{\Omega} \exp (-2 \pi i k \cdot x) d x \\
=\exp \left(-\pi i \sum_{j=1}^{d}\left(\delta_{j}+\gamma_{j}\right) k_{j}\right) \prod_{j=1}^{d} \frac{\sin \left(\pi\left(\delta_{j}-\gamma_{j}\right) k_{j}\right)}{\pi k_{j}} .
\end{gathered}
$$

Hence

$$
\left|\widehat{\chi}_{\Omega}(k)\right| \leq \prod_{j=1}^{d} \min \left\{\left|\delta_{j}-\gamma_{j}\right|, \pi^{-1}\left|k_{j}\right|^{-1}\right\}
$$

The behavior of the Fourier transform of a polyhedron is similar, there is a decay of order $-d$ along directions not normal to the faces or edges and a decay of order $h-d$ along directions normal to $h$ dimensional faces. Moreover, the Fourier transform of $H_{\alpha, R}(x)$ satisfies similar estimates. Also, $\int_{\mathbb{T}^{d}} H_{\alpha, R}(x) d x \leq c / R$. Hence, by Theorem 5,

$$
\begin{gathered}
\left|\mu(\Omega)-m^{-1} \sum_{j=1}^{m} \chi_{\Omega}(j p)\right| \\
\leq c R^{-1}+c \sum_{0<|k|<R}\left(\min \left\{1, m^{-1}\|k \cdot p\|^{-1}\right\} \prod_{j=1}^{d} \min \left\{1,\left|k_{j}\right|^{-1}\right\}\right) .
\end{gathered}
$$

Finally, by a result in [12], for almost every $p$ the last sum is dominated by $c R^{-1}+c m^{-1} \log ^{d+1+\varepsilon}(R)$ and the desired result follows by choosing $R=m$.

Corollary 8 If $\Omega$ is a convex body in the torus $\mathbb{T}^{d}$ with smooth boundary with positive Gaussian curvature, then for every $\varepsilon>0$ and almost every $p$ in
$\mathbb{T}^{d}$ there exists a constant $c$ such that for every $m$,

$$
\left|\mu(\Omega)-m^{-1} \sum_{j=1}^{m} \chi_{\Omega}(j p)\right| \leq c m^{-2 /(d+1)} \log ^{2+\varepsilon}(m)
$$

The above constant $c$ does not depend on the positioning of the convex body in the torus.

Proof. By classical estimates on the decay of oscillatory integrals with non degenerate critical points, $\left|\widehat{\chi}_{\Omega}(k)\right| \leq c(1+|k|)^{-(d+1) / 2}$ and $\left|\widehat{H}_{\alpha, R}(k)\right| \leq$ $c R^{-1}(1+|k|)^{-(d-1) / 2}$. See e.g. Chapter VIII in [13]. Hence, as in the previous corollary, for almost every $p$,

$$
\begin{gathered}
\left.\mid \mu(\Omega)-m^{-1} \sum_{j=1}^{m} \chi_{\Omega}(j p)\right) \mid \\
\leq c R^{-1}+c \sum_{0<|k|<R}|k|^{-(d+1) / 2} \min \left\{1, m^{-1}\|k \cdot p\|^{-1}\right\} \\
\leq c R^{-1}+c m^{-1} R^{(d-1) / 2} \sum_{0<|k|<R}|k|^{-d}\|k \cdot p\|^{-1} \\
\leq c R^{-1}+c m^{-1} R^{(d-1) / 2} \log ^{d+1+\varepsilon}(R) .
\end{gathered}
$$

The desired result follows by choosing $R=m^{2 /(d+1)} \log ^{-2-\varepsilon}(m)$.

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