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Goodness of fit test for small diffusions by discrete observations

by

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Running title: Goodness of fit test for diffusions

Abstract

We consider a nonparametric goodness of fit test problem for the drift coefficient of one-dimensional small diffusions. Our test is based on discrete observation of the processes, and the diffusion coefficient is a nuisance function which is estimated in our testing procedure. We prove that the limit distribution of our test is the supremum of the standard Brownian motion, and thus our test is asymptotically distribution free. We also show that our test is consistent under any fixed alternatives.

Keywords. Small diffusion process, discrete time observations, asymptotically distribution free test.

1

1 Introduction

Goodness of fit tests play an important role in theoretical and applied statistics, and the study for them has a long history. Such tests are really useful especially if they are *distribution free*, in the sense that their distributions do not depend on the underlying model. The origin goes back to the Kolmogorov-Smirnov and Crámer-von Mises tests in the i.i.d. case, established early in the 20th century, and they are *asymptotically distribution free*. On the other hand, the diffusion process models have been paid much attention because they are useful in many applications such as Biology, Medicine, Physics and Financial Mathematics. However, the problem of goodness of fit tests for diffusion processes has still been a new issue in recent years. Kutoyants [4] considered this problem in his Section 5.4, but his tests are not asymptotically distribution free. Dachian and Kutoyants [1] and Negri and Nishiyama [6] proposed some asymptotically distribution free tests. However, all their results are based on *continuous time observation* of the diffusion processes. The main contribution of the present paper is that our test is based on *discrete time observation*, which is more realistic in applications.

Consider a one-dimensional stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t S(X_s)ds + \varepsilon \int_0^t \sigma(X_s)dW_s, \quad t \in [0, T], \tag{1}$$

where S and σ are functions which satisfy some properties described in Section 2, and $t \rightsquigarrow W_t$ is a standard Wiener process defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$. Here, $T > 0$ is a fixed time. We consider a case where a unique strong solution X to this SDE exists, and we will consider the asymptotic as $\varepsilon \downarrow 0$. Statistical inference for this model based on continuous observation was studied by Kutoyants [3]. As for discrete observation cases, many researchers have treated the model in some parametric settings; see e.g. Sørensen and Uchida [8] and references therein. In this paper, we are interested in nonparametric goodness of fit test for the drift coefficient S, while the diffusion coefficient σ^2 is an unknown nuisance function which we estimate in our testing procedure. That is, we consider the problem of testing the hypothesis $H_0 : S = S_0$ versus $H_1 : S \neq S_0$ for a given S_0. The meaning of the alternatives "$S \neq S_0$" will be precisely stated in Section 4.

We consider the following situation.

Sampling Scheme. The process $X = \{X_t; t \in [0, T]\}$ is observed at times $0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_{n(\varepsilon)}^\varepsilon = T$, such that $h_\varepsilon = o(\varepsilon^2)$ as $\varepsilon \downarrow 0$, where $h_\varepsilon = \max_{1 \leq i \leq n(\varepsilon)} |t_i^\varepsilon - t_{i-1}^\varepsilon|$. $\diamondsuit$

We may assume $\varepsilon \leq 1$ and $h_\varepsilon \leq 1$ without loss of generality. We will propose an asymptotically distribution free test based on this sampling scheme.

The organization of the article is as follows. In Section 2, we state some conditions for (S, σ) which are assumed throughout this work. Section 3 gives the main result under the null hypothesis, assuming the existence of a consistent estimator for the limit variance. In Section 4, we prove that our test is consistent under any fixed alternatives, assuming the existence of a consistent estimator for

the limit variance again. A consistent estimator for the limit variance is explicitly constructed in Section 5. The proofs for lemmas and a theorem in Section 5 will be given in Section 6, with help from the Appendix.

2 Preliminaries

Let us list some conditions for the pair of functions (S, σ).

A1. There exists a constant $C > 0$ such that
$$|S(x) - S(y)| \leq C|x - y|, \quad |\sigma(x) - \sigma(y)| \leq C|x - y|.$$

A2. $\sup_{s \in [0,T]} E|X_s|^2 < \infty.$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Diamond$

Under **A1**, the SDE (1) has a unique strong solution X, and notice also that there exists a constant $C' > 0$ such that
$$|S(x)| \leq C'(1 + |x|), \quad |\sigma(x)| \leq C'(1 + |x|).$$

To see this, just put $y = 0$. The constant C' depends on the values $S(0)$ and $\sigma(0)$, however the constant C itself depends on the choice of the functions (S, σ). So it is convenient to introduce the notation
$$K_{S,\sigma} = \max\{C, C'\}.$$

Let us fix some more notations. For given S, let us denote by $x^S = \{x_t^S; t \in [0, T]\}$ the solution to the ordinary differential equation
$$\frac{dx_t^S}{dt} = S(x_t^S) \quad \text{with the initial value } x_0^S = x_0.$$

A3. $\Sigma_{S,\sigma} := \sqrt{\int_0^T \sigma(x_t^S)^2 dt} > 0.$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Diamond$

Let us close this section with making some conventions. We denote by $C[0, T]$ the space of continuous functions on $[0, T]$, and by $\ell^\infty[0, T]$ the space of bounded functions on $[0, T]$. We equip both the spaces with the uniform metric. We denote by "$\to^p$" and "$\to^d$" the convergence in probability and in distribution as $\varepsilon \downarrow 0$, respectively. The notation "$\to$" always means that we take the limit as $\varepsilon \downarrow 0$.

3 Asymptotically distribution free test

Throughout all this section, we shall suppose that **A1** - **A3** are satisfied for some (S_0, σ).

Our test statistics is based on the random field $U^\varepsilon = \{U^\varepsilon(u); u \in [0, T]\}$ defined by
$$U^\varepsilon(u) = \varepsilon^{-1} \sum_{i=1}^{n(\varepsilon)} 1_{[0,u]}(t_i^\varepsilon)[X_{t_i^\varepsilon} - X_{t_{i-1}^\varepsilon} - S_0(X_{t_{i-1}^\varepsilon})|t_i^\varepsilon - t_{i-1}^\varepsilon|].$$

We will approximate U^ε by the following random fields $V^\varepsilon = \{V^\varepsilon(u); u \in [0,T]\}$ and $M^\varepsilon = \{M_u^\varepsilon; u \in [0,T]\}$, defined respectively by:

$$V^\varepsilon(u) = \varepsilon^{-1} \sum_{i=1}^{n(\varepsilon)} 1_{[0,u]}(t_i^\varepsilon) \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} [dX_s - S_0(X_s)ds];$$

$$M_u^\varepsilon = \varepsilon^{-1} \int_0^u [dX_s - S_0(X_s)ds].$$

We present some lemmas which will be proved in Section 6.

Lemma 1 $\sup_{u \in [0,T]} |U^\varepsilon(u) - V^\varepsilon(u)| \to^p 0.$

Lemma 2 $\sup_{u \in [0,T]} |V^\varepsilon(u) - M_u^\varepsilon| \to^p 0.$

Lemma 3 $M^\varepsilon \to^d G$ in $C[0,T]$, where $G = \{G(u); u \in [0,T]\}$ is a Brownian motions with variance
$$EG(u)^2 = \int_{-\infty}^u \sigma(x_t^{S_0})^2 dt.$$

Combining these lemmas, we obtain the following result.

Theorem 4 $U^n \to^d G$ in $\ell^\infty[0,T]$, where G is the process appearing in Lemma 3.

By the continuous mapping theorem, we have the following.

Corollary 5 *It holds that*

$$\sup_{u \in [0,T]} |U^\varepsilon(u)| \to^d \sup_{t \in [0,\Sigma_{S_0,\sigma}^2]} |B_t| =^d \Sigma_{S_0,\sigma} \sup_{t \in [0,1]} |B_t|,$$

where $t \rightsquigarrow B_t$ is a standard Brownian motion, and the notation "$=^d$" means that the distributions are the same.

So we have the main result of the paper.

Theorem 6 *Under $H_0 : S = S_0$, suppose that $\widehat{\Sigma}^\varepsilon$ is a consistent estimator for $\Sigma_{S_0,\sigma}$. Then we have*
$$\frac{\sup_{u \in [0,T]} |U^\varepsilon(u)|}{\widehat{\Sigma}^\varepsilon} \to^d \sup_{t \in [0,1]} |B_t|,$$
where $t \rightsquigarrow B_t$ is a standard Brownian motion.

The construction of a consistent estimator $\widehat{\Sigma}^\varepsilon$ for $\Sigma_{S,\sigma}$ will be discussed in Section 5.

4 Consistency of the test

Let S_0 be that in Section 3. We denote by $\mathcal{S}$ the class of functions S satisfying **A1** - **A3** and

$$\int_0^{u_S} (S(x_t^S) - S_0(x_t^S))dt \neq 0 \quad \text{for some } u_S \in [0, T]. \tag{2}$$

The precise description of our problem is testing the null hypothesis $H_0 : S = S_0$ versus the alternatives $H_1 : S \in \mathcal{S}$.

We will prove that our test is consistent. Fix $S \in \mathcal{S}$. We can write $U^\varepsilon = U_S^\varepsilon + U_\Delta^\varepsilon$ where

$$U_S^\varepsilon(u) = \varepsilon^{-1} \sum_{i=1}^{n(\varepsilon)} 1_{[0,u]}(t_i)[X_{t_i^\varepsilon} - X_{t_{i-1}^\varepsilon} - S(X_{t_{i-1}^\varepsilon})|t_i^\varepsilon - t_{i-1}^\varepsilon|]$$

and

$$U_\Delta^\varepsilon(u) = \varepsilon^{-1} \sum_{i=1}^{n(\varepsilon)} 1_{[0,u]}(t_i^\varepsilon)(S(X_{t_{i-1}^\varepsilon}) - S_0(X_{t_{i-1}^\varepsilon}))|t_i^\varepsilon - t_{i-1}^\varepsilon|.$$

Now we have

$$\sup_{u \in [0,T]} |U^\varepsilon(u)| \geq \sup_{u \in [0,T]} |U_\Delta^\varepsilon(u)| - \sup_{u \in [0,T]} |U_S^\varepsilon(u)|.$$

Since S satisfies **A1** - **A3**, by the same argument as in Section 3, the random field U_S^ε converges to the corresponding Gaussian random field with S_0 replaced by S. So the second term of the right hand side is $O_P(1)$. As for the first term of the right hand side, we have the following claim.

Lemma 7 *Choose $u_S \in [0, T]$ as in (2). Then it holds that $|U_\Delta^\varepsilon(u_S)| \neq O_P(1)$.*

We therefore obtain the consistency of the test.

Theorem 8 *Suppose that $\widehat{\Sigma}^\varepsilon$ is a consistent estimator for $\Sigma_{S,\sigma}$. Under $H_1 : S \in \mathcal{S}$, it holds that*

$$\frac{\sup_{u \in [0,T]} |U^\varepsilon(u)|}{\widehat{\Sigma}^\varepsilon} \neq O_P(1).$$

5 Consistent estimator for $\Sigma_{S,\sigma}$

In order to construct an asymptotically distribution free test, we need a consistent estimator for $\Sigma_{S,\sigma}$. The following result gives us an answer.

Theorem 9 *For any (S, σ) which satisfies **A1** and $\sup_{s \in [0,T]} E|X_s|^4 < \infty$ (which is stronger than **A2**),*

$$\widehat{\Sigma}^\varepsilon = \sqrt{\varepsilon^{-2} \sum_{i=1}^{n(\varepsilon)} |X_{t_i^\varepsilon} - X_{t_{i-1}^\varepsilon}|^2}$$

is a consistent estimator for $\Sigma_{S,\sigma}$.

6 Proofs

Proof of Lemma 1. Without loss of generality, we may assume that $\varepsilon \leq 1$ and $h_\varepsilon \leq 1$. It follows from Lemma 12 that

$$
\begin{aligned}
E &\left(\sup_{u\in[0,T]} |U^\varepsilon(u) - V^\varepsilon(u)| \right) \\
&\leq \quad \varepsilon^{-1} E \sum_{i=1}^{n(\varepsilon)} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} |S_0(X_{t_{i-1}^\varepsilon}) - S_0(X_s)| ds \\
&\leq \quad \varepsilon^{-1} \sum_{i=1}^{n(\varepsilon)} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} K_{S_0,\sigma} E|X_{t_{i-1}} - X_s| ds \\
&\leq \quad \varepsilon^{-1} T K_{S_0,\sigma} C_1 h_\varepsilon^{1/2} \\
&\to \quad 0.
\end{aligned}
$$

So we have the assertion of the lemma.

Proof of Lemma 2. Notice that

$$
\begin{aligned}
M_u^\varepsilon &= \quad \varepsilon^{-1} \sum_{i=1}^n \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} 1_{[0,u]}(s)[dX_s - S_0(X_s)ds] \\
&= \quad V^\varepsilon(u) + \varepsilon^{-1} \int_{t_{i-1}^\varepsilon}^u [dX_s - S_0(X_s)ds] \quad \forall u \in [t_{i-1}^\varepsilon, t_i^\varepsilon) \\
&= \quad V^\varepsilon(u) + \int_{t_{i-1}^\varepsilon}^u \sigma(X_s)dW_s \quad \forall u \in [t_{i-1}^\varepsilon, t_i^\varepsilon)
\end{aligned}
$$

and that $M_T^\varepsilon = V^\varepsilon(T)$. Now we have

$$
\begin{aligned}
E \left| \sup_{u\in[0,T]} |V^\varepsilon(u) - M_u^\varepsilon| \right|^4 &= \quad \sum_{i=1}^{n(\varepsilon)} E \sup_{u\in[t_{i-1}^\varepsilon, t_i^\varepsilon)} |V^\varepsilon(u) - M_u^\varepsilon|^4 \\
&\leq \quad \sum_{i=1}^{n(\varepsilon)} E \sup_{u\in[t_{i-1}^\varepsilon, t_i^\varepsilon]} \left| \int_{t_{i-1}^\varepsilon}^u \sigma(X_s)dW_s \right|^4 .
\end{aligned}
$$

It follows from Burkholder-Davis-Gundy's inequality (see e.g. Theorem 26.12 of Kallenberg [2]) that, for a constant c_k depending only on $k = 4$, the right hand

side is bounded by

$$c_4 \sum_{i=1}^{n(\varepsilon)} E \left| \int_{t_{i-1}^{\varepsilon}}^{t_i^{\varepsilon}} \sigma(X_s)ds \right|^2 \leq c_4 \sum_{i=1}^{n(\varepsilon)} E \left(\int_{t_{i-1}^{\varepsilon}}^{t_i^{\varepsilon}} 1 ds \cdot \int_{t_{i-1}^{\varepsilon}}^{t_i^{\varepsilon}} \sigma(X_s)^2 ds \right)$$

$$\leq c_4 T \max_{1 \leq i \leq n(\varepsilon)} \int_{t_{i-1}^{\varepsilon}}^{t_i^{\varepsilon}} E\sigma(X_s)^2 ds$$

$$\leq c_4 T h_\varepsilon \sup_{s \in [0,T]} E\sigma(X_s)^2$$

$$\rightarrow 0.$$

The proof is finished. $\qquad\square$

Proof of Lemma 3. When $S = S_0$, it holds that

$$M_u^\varepsilon = \int_0^u \sigma(X_s)dW_s$$

We will apply the central limit theorem for continuous martingales.

$$\langle M^\varepsilon \rangle_u = \int_0^u \sigma(X_s)^2 ds$$

$$= \int_0^u (\sigma(X_s)^2 - \sigma(x_s^{S_0})^2) ds + \int_0^u \sigma(x_s^{S_0})^2 ds$$

$$= (I) + (II).$$

Now, using Lemma 10, we have

$$|(I)| \leq \int_0^u |\sigma(X_s)^2 - \sigma(x_s^{S_0})^2| ds$$

$$= \int_0^T |\sigma(X_s) - \sigma(x_s^{S_0})||\sigma(X_s) + \sigma(x_s^{S_0})| ds$$

$$\leq K_{S_0,\sigma} \sup_{t \in [0,T]} |X_t - x_t^{S_0}| \cdot \int_0^T |\sigma(X_s) + \sigma(x_s^{S_0})| ds$$

$$\leq K_{S_0,\sigma} \exp(K_{S_0,\sigma}T) \cdot \varepsilon \sup_{t \in [0,T]} \left| \int_0^t \sigma(X_s)dW_s \right| \cdot \int_0^T |\sigma(X_s) + \sigma(x_s^{S_0})| ds$$

$$= O_P(\varepsilon).$$

So we have $\langle M^\varepsilon \rangle_u \rightarrow^p \int_0^u \sigma(x_s^{S_0})^2 ds$, and the weak convergence of the process $u \rightsquigarrow M_u^\varepsilon$ holds.

Proof of Lemma 7. We simply denote $u = u_S$. We consider the following random

7

variables:

$$
\begin{aligned}
A_1^\varepsilon &= \varepsilon U_\Delta^\varepsilon(u) \\
&= \sum_{i=1}^{n(\varepsilon)} 1_{[0,u]}(t_i^\varepsilon) \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} (S(X_{t_{i-1}^\varepsilon}) - S_0(X_{t_{i-1}^\varepsilon}))ds; \\
A_2^\varepsilon &= \sum_{i=1}^{n(\varepsilon)} 1_{[0,u]}(t_i^\varepsilon) \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} (S(X_s) - S_0(X_s))ds; \\
A_3^\varepsilon &= \sum_{i=1}^{n(\varepsilon)} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} 1_{[0,u]}(s)(S(X_s) - S_0(X_s))ds; \\
A_4 &= \int_0^u (S(x_s^S) - S_0(x_s^S))ds.
\end{aligned}
$$

First, it holds that

$$
\begin{aligned}
E|A_1^\varepsilon - A_2^\varepsilon| &\le \sum_{i=1}^{n(\varepsilon)} E \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \left\{ |S(X_{t_{i-1}^\varepsilon}) - S(X_s)| + |S_0(X_{t_{i-1}^\varepsilon}) - S_0(X_s)| \right\} ds \\
&\le (K_{S,\sigma} + K_{S_0,\sigma}) \sum_{i=1}^{n(\varepsilon)} E \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} |X_{t_{i-1}^\varepsilon} - X_s| ds \\
&\le (K_{S,\sigma} + K_{S_0,\sigma}) T C_1 h_\varepsilon^{1/2} \\
&\to 0,
\end{aligned}
$$

where C_1 is a constant appearing in Lemma 12. So we have $|A_1^\varepsilon - A_2^\varepsilon| \to^p 0$.

Next,

$$
\begin{aligned}
A_2^\varepsilon - A_3^\varepsilon &= \sum_{i=1}^{n(\varepsilon)} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} (1_{[0,u]}(t_i^\varepsilon) - 1_{[0,u]}(s))\{S(X_s) - S_0(X_s)\}ds \\
&= \int_{t_{i-1}^\varepsilon}^u \{S(X_s) - S_0(X_s)\}ds \quad \forall u \in [t_{i-1}^\varepsilon, t_i^\varepsilon).
\end{aligned}
$$

If $u = u_S = T$, then $A_2^\varepsilon = A_3^\varepsilon$. Since

$$
\begin{aligned}
\max_{1 \le i \le n(\varepsilon)} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \{|S(X_s)| + |S_0(X_s)|\}ds \\
\le h_\varepsilon \sup_{s \in [0,T]} \{|S(X_s)| + |S_0(X_s)|\} = O_P(h_\varepsilon),
\end{aligned}
$$

we have $|A_2^\varepsilon - A_3^\varepsilon| \to^p 0$.

Finally, notice that

$$
A_3^\varepsilon - A_4 = \int_0^u (S(X_s) - S_0(X_s))ds - \int_0^u (S(x_s^S) - S_0(x_s^S))ds.
$$

8

It follows from Lemma 10 that

$$
\begin{aligned}
\int_0^u |S(X_s) - S(x_s^S)|ds &\leq \int_0^T K_{S,\sigma}|X_s - x_s^S|ds \\
&\leq TK_{S,\sigma} \sup_{t\in[0,T]} |X_t - x_t^S| \\
&\leq TK_{S,\sigma} \cdot \exp(K_{S,\sigma}T) \cdot \varepsilon \sup_{t\in[0,T]} \left| \int_0^t \sigma(X_s)dW_s \right| \\
&= O_P(\varepsilon).
\end{aligned}
$$

By the same way, it holds that

$$
\int_0^u |S_0(X_s) - S_0(x_s^S)|ds \leq TK_{S_0,\sigma} \cdot \exp(K_{S,\sigma}T) \cdot \varepsilon \sup_{t\in[0,T]} \left| \int_0^t \sigma(X_s)dW_s \right| = O_P(\varepsilon).
$$

Thus we have $|A_3^\varepsilon - A_4| \to 0$.

Consequently, we obtain $A_1^\varepsilon \to^p A_4 \neq 0$, which implies that $|U_\Delta^\varepsilon(u_S)| \neq O_P(1)$.
□

Proof of Theorem 9. By Itô's formula, we have

$$
|X_{t_i^\varepsilon}|^2 - |X_{t_{i-1}^\varepsilon}|^2 = 2\int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} X_s dX_s + \varepsilon^2 \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \sigma(X_s)^2 ds.
$$

Since

$$
|\widehat{\Sigma}^\varepsilon|^2 = \varepsilon^{-2} \sum_{i=1}^{n(\varepsilon)} \left\{ |X_{t_i^\varepsilon}|^2 - |X_{t_{i-1}^\varepsilon}|^2 - 2X_{t_{i-1}^\varepsilon}(X_{t_i^\varepsilon} - X_{t_{i-1}^\varepsilon}) \right\},
$$

it is enough to show that

$$
\varepsilon^{-2} \sum_{i=1}^{n(\varepsilon)} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} (X_s - X_{t_{i-1}^\varepsilon})dX_s \to^p 0
$$

and

$$
\int_0^T \sigma(X_s)^2 ds \to^p \Sigma_{S,\sigma}^2.
$$

The latter is proved by the same argument as that in the proof of Lemma 3. As for the former, observe that

$$
\varepsilon^{-2} \left| \sum_{i=1}^{n(\varepsilon)} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} (X_s - X_{t_{i-1}^\varepsilon})dX_s \right|
$$

$$
\leq \varepsilon^{-2} \sum_{i=1}^{n(\varepsilon)} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} |X_s - X_{t_{i-1}^\varepsilon}||S(X_s)|ds + \varepsilon^{-1} \left| \sum_{i=1}^{n(\varepsilon)} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} (X_s - X_{t_{i-1}^\varepsilon})\sigma(X_s)dW_s \right|
$$

9

By Lemma 12, the expectation of the first term on the right hand side is

$$\varepsilon^{-2}\sum_{i=1}^{n(\varepsilon)}\int_{t_{i-1}^{\varepsilon}}^{t_i^{\varepsilon}}E(|X_s-X_{t_{i-1}^{\varepsilon}}||S(X_s)|)ds$$

$$\leq \varepsilon^{-2}\sum_{i=1}^{n(\varepsilon)}\int_{t_{i-1}^{\varepsilon}}^{t_i^{\varepsilon}}\sqrt{E|X_s-X_{t_{i-1}^{\varepsilon}}|^2}\sqrt{E|S(X_s)|^2}ds$$

$$\leq \varepsilon^{-2}\sum_{i=1}^{n(\varepsilon)}\int_{t_{i-1}^{\varepsilon}}^{t_i^{\varepsilon}}\sqrt{C_2\max\{h_\varepsilon^2,\varepsilon^2 h_\varepsilon\}}\sqrt{E|S(X_s)|^2}ds$$

$$\leq \varepsilon^{-2}T\sqrt{C_2}\max\{h_\varepsilon,\varepsilon h_\varepsilon^{1/2}\}\cdot\sup_{s\in[0,T]}\sqrt{E|S(X_s)|^2}$$

$$\to 0.$$

On the other hand, the expectation of the square of the second term on the right hand side is

$$\varepsilon^{-2}\sum_{i=1}^{n(\varepsilon)}E\int_{t_{i-1}^{\varepsilon}}^{t_i^{\varepsilon}}|X_s-X_{t_{i-1}^{\varepsilon}}|^2\sigma(X_s)^2 ds$$

$$\leq \varepsilon^{-2}\sum_{i=1}^{n(\varepsilon)}\int_{t_{i-1}^{\varepsilon}}^{t_i^{\varepsilon}}\sqrt{E|X_s-X_{t_{i-1}^{\varepsilon}}|^4}\sqrt{E\sigma(X_s)^4}ds$$

$$\leq \varepsilon^{-2}T\sqrt{C_4 h_\varepsilon^2}\sup_{s\in[0,T]}\sqrt{E\sigma(X_s)^4}$$

$$\to 0.$$

This proves the consistency of our estimator. $\square$

Appendix

In the main part of this article, we use the following inequality which is well known.

Lemma 10 *For any solution $X=\{X_t; t\in[0,T]\}$ to the SDE (1), it holds that*

$$\sup_{t\in[0,T]}|X_t-x_t|\leq\exp(K_{S,\sigma}T)\cdot\varepsilon\sup_{t\in[0,T]}\left|\int_0^t\sigma(X_s)dW_s\right|.$$

Proof. The proof is a simple application of Gronwall's inequality: apply Lemma 4.13 of Liptser and Shiryaev [5] for $c_0=\varepsilon\sup_{t\in[0,T]}|\int_0^t\sigma(X_s)dW_s|$, $c_1=K_{S,\sigma}$, $c_2=0$, $u(t)=|X_t-x_t|$, and $v(t)=1$. $\square$

The following fact is used in the article many times, so we state it as a lemma here.

Lemma 11 *Let $f:\mathbb{R}\to\mathbb{R}$ be a measurable function such that $|f(x)|\leq H(1+|x|)$ for some $H>0$. Let $X=\{X_t; t\in[0,T]\}$ be any stochastic process. Let $k>0$ and assume $\sup_{t\in[0,T]}E|X_t|^k<\infty$. Then, it holds that $\sup_{t\in[0,T]}E|f(X_t)|^k<\infty$.*

Proof. Since

$$|x + y|^k \leq ||x| + |y||^k \leq |2 \max\{|x|, |y|\}|^k \leq 2^k \{|x|^k + |y|^k\},$$

the lemma is trivial. □

The following lemma is rather well known, but we give a full proof for references.

Lemma 12 *Let $X = \{X_t; t \in [0, T]\}$ be a solution to the SDE (1) for (S, σ) which satisfies **A1**. Let $k > 0$ and assume $\sup_{t \in [0,T]} E|X_t|^{k \vee 2} < \infty$. Then, there exists a constant $C_k > 0$, such that for any $0 \leq t \leq t' \leq T$ and any $\varepsilon > 0$*

$$E|X_{t'} - X_t|^k \leq C_k \max\{|t' - t|^k, \varepsilon^k |t' - t|^{k/2}\}.$$

In particular, if $|t' - t| \leq 1$ and $\varepsilon \leq 1$, then

$$E|X_{t'} - X_t|^k \leq C_k |t' - t|^{k/2}.$$

Remark. The constant C_k is *not* a universal constant depending only on k. It actually depends on S, σ, T. However, it does not depend on t, t', ε.

Proof. First we consider the case $k \geq 2$. Notice that

$$X_{t'} - X_t = \int_t^{t'} S(X_s)ds + \varepsilon \int_t^{t'} \sigma(X_s)dW_s.$$

It follows from Hölder's inequality that

$$\left| \int_t^{t'} S(X_s)ds \right|^k \leq |t' - t|^{k-1} \int_t^{t'} |S(X_s)|^k ds.$$

Taking the expectation, it holds that

$$E \left| \int_t^{t'} S(X_s)ds \right|^k \leq |t' - t|^k \sup_{s \in [0,T]} E|S(X_s)|^k.$$

On the other hand, it follows from Burkholder-Davis-Gundy's inequality that there exists a constant $c_k > 0$, depending only on k, such that

$$E \left| \varepsilon \int_t^{t'} \sigma(X_s)dW_s \right|^k \leq c_k \varepsilon^k E \left| \int_t^{t'} \sigma(X_s)^2 ds \right|^{k/2}.$$

When $k > 2$, it follows from Hölder's inequality that

$$\varepsilon^k \left| \int_t^{t'} \sigma(X_s)^2 ds \right|^{k/2} \leq \varepsilon^k |t' - t|^{(k/2)-1} \int_t^{t'} |\sigma(X_s)|^k ds.$$

11

Taking the expectation, we have

$$\varepsilon^k E \left| \int_t^{t'} \sigma(X_s)^2 ds \right|^{k/2} \leq \varepsilon^k |t' - t|^{k/2} \sup_{s \in [0,T]} E|\sigma(X_s)|^k.$$

When $k = 2$, we actually have

$$\begin{aligned} \varepsilon^2 E \int_t^{t'} \sigma(X_s)^2 ds &= \varepsilon^2 \int_t^{t'} E\sigma(X_s)^2 ds \\ &\leq \varepsilon^2 |t' - t| \sup_{s \in [0,T]} E\sigma(X_s)^2. \end{aligned}$$

Thus the proof for the case $k \geq 2$ is finished.

For $k \in (0, 2)$, by Jensen's inequality, it holds that

$$(E|X_{t'} - X_t|^k)^{2/k} \leq E|X_{t'} - X_t|^2 \leq C_2 \max\{|t' - t|^2, \varepsilon^2|t' - t|\},$$

thus we obtain the desired inequality. □

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