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by

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Abstract. This is the text of a talk given in Dalmine on May 9, 2007, during one of the "scientific meetings" of the Faculty of Engineering of the Bergamo University. Since the aim of these meetings is to bring together scientists working in different fields, this presentation was intended to be understandable also for non-mathematicians. Using the example of the open Toda lattice, we give a very brief account of the history of integrable systems and we present the main ideas of a geometric approach based on the notion of bi-Hamiltonian system.

1   Integrable systems

Around 1855, right after the birth of Hamiltonian mechanics, an important result was found by Jacobi and Liouville. It states that under suitable assumptions the motion a given (mechanical) system can be (almost) explicitly described. In order to understand the nature of these assumptions, let us consider a system coming from the modern theory of integrable systems, namely, the open Toda lattice. It appeared in the study of nonlinear crystals and, in its simplest form, consists in n particles (with masses equal to 1) moving on the line under a nearest-neighbor interaction of exponential type. If we denote with q_i the positions of the particles and with p_i their momenta (coinciding with their velocities in this case), then we can write the kinetic energy and the potential energy of the system as

$$T = \frac{1}{2}\sum_{i=1}^{n} p_i{}^2 \ , \qquad V = \sum_{i=1}^{n-1} \mathrm{e}^{\,q_i - q_{i+1}} \ ,$$

so that the Hamiltonian is given by

$$H = T + V = \frac{1}{2}\sum_{i=1}^{n} p_i{}^2 + \sum_{i=1}^{n-1} \mathrm{e}^{\,q_i - q_{i+1}} \ .$$

The equations governing the motion of the system are the Hamilton equations

$$\dot{q}_i = H_{p_i} \ , \qquad \dot{p}_i = -H_{q_i} \ , \qquad i = 1, \ldots, n \ , \tag{1}$$

where the dot denotes a derivative with respect to time and expressions like f_x stand for the partial derivative of f with respect to the variable x. In matrix form the Hamilton equation can be written as

$$\begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \\ \dot{p}_1 \\ \vdots \\ \dot{p}_n \end{pmatrix} = P \begin{pmatrix} H_{q_1} \\ \vdots \\ H_{q_n} \\ H_{p_1} \\ \vdots \\ H_{p_n} \end{pmatrix} \ ,$$

where P is the $2n \times 2n$ matrix

$$P = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \ .$$

A function $I = I(q_1, \cdots, q_n, p_1, \cdots, p_n)$ is said to be a conserved quantity if it remains constant during the motion of the system. It is not difficult to check that it happens if and only if $\{I, H\} = 0$, where

$$\{I, H\} = \sum_{i=1}^{n} (I_{q_i} H_{p_i} - I_{p_i} H_{q_i}) \tag{2}$$

$$= \begin{pmatrix} I_{q_1}, \cdots, I_{q_n}, I_{p_1}, \cdots, I_{p_n} \end{pmatrix} P \begin{pmatrix} H_{q_1} \\ \vdots \\ H_{q_n} \\ H_{p_1} \\ \vdots \\ H_{p_n} \end{pmatrix}$$

is the so-called Poisson bracket between I and H. For example, the total momentum $\sum_{i=1}^{n} p_i$ and the energy H are clearly conserved quantity for the Toda lattice. Moreover, it can be shown that the Toda lattice is an integrable system, that is, there exist n conserved quantities $I_1, \ldots, I_n$ which are (independent and) in involution. This means that their Poisson brackets are all zero: $\{I_j, I_k\} = 0$ for all j, k.

After the discovery, made in 1890 by Poincaré, that one of the most important systems in celestial mechanics — the three-body problem — is not integrable, the field of integrable systems was gradually abandoned. Nowadays it occupies again a central place in mathematics and in physics thanks to the

invention (made in 1967) of the inverse scattering method for the solution of the Korteweg-deVries equation

$$u_t + uu_x + u_{xxx} = 0$$

and for other so-called soliton equations. This led to the notion of infinite-dimensional integrable system and to new methods for studying finite and infinite-dimensional integrable systems, such as the Lax representation and the bi-Hamiltonian formulation (to be discussed in the following sections). In this "new era" a lot of new examples of finite-dimensional integrable systems were found. The Toda lattice is one of these.

2 Lax representation

In 1968 Peter Lax found what is now called the Lax representation of the Korteweg-deVries equation, and he explained in this way some features of this equation. Some years later, Flaschka and Manakov found such a representation for the Toda lattice. Let us see what this means in the case of 3 particles. We introduce the *Lax matrix*

$$L = \begin{pmatrix} p_1 & \mathrm{e}^{\frac{1}{2}(q_1-q_2)} & 0 \\ \mathrm{e}^{\frac{1}{2}(q_1-q_2)} & p_2 & \mathrm{e}^{\frac{1}{2}(q_2-q_3)} \\ 0 & \mathrm{e}^{\frac{1}{2}(q_2-q_3)} & p_3 \end{pmatrix}$$

of the system, and the matrix

$$B = \frac{1}{2} \begin{pmatrix} 0 & \mathrm{e}^{\frac{1}{2}(q_1-q_2)} & 0 \\ -\mathrm{e}^{\frac{1}{2}(q_1-q_2)} & 0 & \mathrm{e}^{\frac{1}{2}(q_2-q_3)} \\ 0 & -\mathrm{e}^{\frac{1}{2}(q_2-q_3)} & 0 \end{pmatrix} \; .$$

It can be easily checked that the Hamilton equations (1) imply that $\dot{L} = [L, B]$, where $[L, B] = LB - BL$ is the matrix commutator. A very important consequence of this fact is that the functions $I_k = \frac{1}{k}\mathrm{tr}L^k$ are conserved quantities. Indeed, $\dot{I}_k = \frac{1}{k}\mathrm{tr}[L^k, B] = 0$. It is not difficult to show that I_1, I_2, and I_3 are independent, whereas the proof of the involutivity is more complicated. We notice that $I_1 = \sum_i p_i$ is the total momentum, while $I_2 = H$. The physical meaning of I_3 is still unclear.

3 Bi-Hamiltonian systems

We have just seen that the Lax representation allows one to construct conserved quantities for a given system, but it does not explain why these quantities are in involution. Now we will show that the bi-Hamiltonian formulation clarifies this point. This idea is mainly due to Magri.

Starting from the end of the 1970s, a particular attention in the theory of integrability has been payed to systems admitting more than one Hamiltonian representation. The first examples belonged to the class of infinite-dimensional systems, like the Korteweg-deVries equation, but it was realized soon that also finite-dimensional integrable systems are likely to possess a bi-Hamiltonian representation.

Let us consider again the three-particle open Toda lattice. We have seen that the equations of motion are

$$\begin{cases} \dot{q}_1 = p_1 \\ \dot{q}_2 = p_2 \\ \dot{q}_3 = p_3 \\ \dot{p}_1 = -e^{\,q_1-q_2} \\ \dot{p}_2 = e^{\,q_1-q_2} - e^{\,q_2-q_3} \\ \dot{p}_3 = e^{\,q_2-q_3} \end{cases} \tag{3}$$

Let us introduce the vector field

$$X = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ -e^{\,q_1-q_2} \\ e^{\,q_1-q_2} - e^{\,q_2-q_3} \\ e^{\,q_2-q_3} \end{pmatrix}$$

associated with the system. We know that it admits a canonical Hamiltonian formulation

$$X = P \begin{pmatrix} H_{q_1} \\ H_{q_2} \\ H_{q_3} \\ H_{p_1} \\ H_{p_2} \\ H_{p_3} \end{pmatrix}$$

where

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

But Damianou showed that the Toda vector field X can also be written as

$$P' \begin{pmatrix} K_{q_1} \\ K_{q_2} \\ K_{q_3} \\ K_{p_1} \\ K_{p_2} \\ K_{p_3} \end{pmatrix} \ ,$$

where $K = I_1 = p_1 + p_2 + p_3$ is the total momentum and

$$P' = \begin{pmatrix} 0 & -1 & -1 & p_1 & 0 & 0 \\ 1 & 0 & -1 & 0 & p_2 & 0 \\ 1 & 1 & 0 & 0 & 0 & p_3 \\ -p_1 & 0 & 0 & 0 & -e^{q_1-q_2} & 0 \\ 0 & -p_2 & 0 & e^{q_1-q_2} & 0 & -e^{q_2-q_3} \\ 0 & 0 & -p_3 & 0 & e^{q_2-q_3} & 0 \end{pmatrix} \ .$$

There are two important points. First of all, the matrix P' defines a Poisson tensor, meaning that

$$\{F,G\}' = \left(F_{q_1}, F_{q_2}, F_{q_3}, F_{p_1}, F_{p_2}, F_{p_3}\right) P' \begin{pmatrix} G_{q_1} \\ G_{q_2} \\ G_{q_3} \\ G_{p_1} \\ G_{p_2} \\ G_{p_3} \end{pmatrix}$$

has the same formal properties of the canonical Poisson bracket (2) associated with P. Secondly, the Poisson tensors P and P' are compatible, i.e., their linear combinations are still Poisson tensors. For these reasons, the Toda vector field X is said to be a bi-Hamiltonian vector field.

Finally, we will show how this property can be used to construct a maximal set of integrals of motion for the Toda lattice, that are automatically in involution. For any function $F = F(q_1, q_2, q_3, p_1, p_2, p_3)$, let us denote with $dF = \left(F_{q_1}, F_{q_2}, F_{q_3}, F_{p_1}, F_{p_2}, F_{p_3}\right)$ its differential. We have that

$$X = P(dH)^T = P'(dK)^T \ ,$$

where $(\cdot)^T$ stands for the transpose of a matrix (in particular, of a vector). One can consider the vector field $X_2 = P'(dH)^T$ and show that it is bi-Hamiltonian too. This means that there exists a function I such that $X_2 = P(dI)^T$. It turns out that I coincides with the conserved quantity I_3 constructed with the Lax matrix. Using the fact that the matrices P and P' are skewsymmetric, it is very easy to prove that, e.g., K and I are in involution. Indeed,

$$\{K, I\} = dK\, P(dI)^T = dK\, P'(dH)^T = -dH\, P'(dK)^T = -dH\, P(dH)^T = 0 \ .$$

The generalization to an arbitrary number n of particles is straightforward. The Toda system is still bi-Hamiltonian, and one can use the bi-Hamiltonian formulation to find n conserved quantities and to show that they are in involution.

Further reading

F. Magri, M. Pedroni, *Multi-Hamiltonian systems*, Encyclopedia of Mathematical Physics, Volume 3, pp. 459–464, Elsevier, 2006.

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