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On Compound Poisson Type Limiting Likelihood Ratio Process Arising in some Change-Point Models

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On Compound Poisson Type Limiting Likelihood Ratio Process Arising in some Change-Point Models*

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Abstract

Different change-point type models encountered in statistical inference for stochastic processes give rise to different limiting likelihood ratio processes. In a previous paper of one of the authors it was established that one of these likelihood ratios, which is an exponential functional of a two-sided Poisson process driven by some parameter, can be approximated (for sufficiently small values of the parameter) by another one, which is an exponential functional of a two-sided Brownian motion. In this paper we consider yet another likelihood ratio, which is the exponent of a two-sided compound Poisson process driven by some parameter. We establish, that similarly to the Poisson type one, the compound Poisson type likelihood ratio can be approximated by the Brownian type one for sufficiently small values of the parameter. We equally discuss the asymptotics for large values of the parameter.

Keywords: compound Poisson process, non-regularity, change-point, limiting likelihood ratio process, Bayesian estimators, maximum likelihood estimator, limiting distribution, limiting variance, asymptotic efficiency

Mathematics Subject Classification (2000): 62F99, 62M99

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1 Introduction

Different change-point type models encountered in statistical inference for stochastic processes give rise to different limiting likelihood ratio processes. In [3] a relation between two of these likelihood ratios was established by one of the authors. More precisely, it was shown that the first one, which is an exponential functional of a two-sided Poisson process driven by some parameter, can be approximated (for sufficiently small values of the parameter) by the second one, defined by

$$Z_0(x) = \exp\left\{W(x) - \frac{1}{2}\,|x|\right\}, \quad x \in \mathbb{R}, \tag{1}$$

where W is a standard two-sided Brownian motion. In this paper we consider another limiting likelihood ratio process arising in some change-point type models.

We introduce the random process Z_γ on $\mathbb{R}$ as the exponent of a two-sided compound Poisson process given by

$$\ln Z_\gamma(x) = \begin{cases} \gamma \sum_{k=1}^{\Pi_+(x)} \varepsilon_k^+ - \frac{\gamma^2}{2}\Pi_+(x), & \text{if } x \geqslant 0, \\ \gamma \sum_{k=1}^{\Pi_-(-x)} \varepsilon_k^- - \frac{\gamma^2}{2}\Pi_-(-x), & \text{if } x \leqslant 0, \end{cases} \tag{2}$$

where $\gamma > 0$, Π_+ and Π_- are two independent Poisson processes of intensity 1 on $\mathbb{R}_+$, $\varepsilon_k^\pm$ are independent standard Gaussian random variables which are also independent of $\Pi_\pm$, and we use the convention $\sum_{k=1}^{0} \varepsilon_k^\pm = 0$. We equally introduce the random variables

$$\begin{aligned} \zeta_\gamma &= \frac{\int_{\mathbb{R}} x\, Z_\gamma(x)\; dx}{\int_{\mathbb{R}} Z_\gamma(x)\; dx}\,, \\ \xi_\gamma^- &= \inf\Big\{z : Z_\gamma(z) = \sup_{x\in\mathbb{R}} Z_\gamma(x)\Big\}, \\ \xi_\gamma^+ &= \sup\Big\{z : Z_\gamma(z) = \sup_{x\in\mathbb{R}} Z_\gamma(x)\Big\}, \\ \xi_\gamma^\alpha &= \alpha\, \xi_\gamma^- + (1-\alpha)\, \xi_\gamma^+, \quad \alpha \in [0,1], \end{aligned} \tag{3}$$

related to this process, as well as their second moments $B_\gamma = \mathbf{E}\zeta_\gamma^2$ and $M_\gamma^\alpha = \mathbf{E}(\xi_\gamma^\alpha)^2$.

The process Z_γ, up to a linear time change, arises (see, for example, Chan and Kutoyants [2]) in some non-regular, namely change-point type, statistical models as the limiting likelihood ratio process, and the variables

ζ_γ and ξ_γ^α as the limiting distributions of the Bayesian estimators and of the appropriately chosen maximum likelihood estimator respectively. Here the maximum likelihood estimator is not unique, and the appropriate choice is a linear combination with weights α and $1-\alpha$ of its minimal and maximal values. In particular, B_γ and M_γ^α are the limiting variances of these estimators, and the Bayesian estimators being asymptotically efficient, the ratio $E_\gamma^\alpha = B_\gamma/M_\gamma^\alpha$ is the asymptotic efficiency of this maximum likelihood estimator.

On the other hand, many change-point type statistical models encountered in various fields of statistical inference for stochastic processes rather have as limiting likelihood ratio process, up to a linear time change, the process Z_0. In this case, the limiting distributions of the Bayesian estimators and of the maximum likelihood estimator are given by

$$\zeta_0 = \frac{\int_{\mathbb{R}} x\, Z_0(x)\, dx}{\int_{\mathbb{R}} Z_0(x)\, dx} \quad \text{and} \quad \xi_0 = \operatorname*{argsup}_{x\in\mathbb{R}} Z_0(x) \tag{4}$$

respectively, and the limiting variances of these estimators are $B_0 = \mathbf{E}\zeta_0^2$ and $M_0 = \mathbf{E}\xi_0^2$.

A well-known example is the model of a discontinuous signal in a white Gaussian noise exhaustively studied by Ibragimov and Khasminskii in [10, Chapter 7.2] (see also their previous work [9]), but one can also cite change-point type models of dynamical systems with small noise (see Kutoyants [12] and [13, Chapter 5]), those of ergodic diffusion processes (see Kutoyants [14, Chapter 3]), a change-point type model of delay equations (see Küchler and Kutoyants [11]), an i.i.d. change-point type model (see Deshayes and Picard [4]), a model of a discontinuous periodic signal in a time inhomogeneous diffusion (see Höpfner and Kutoyants [8]), and so on.

Let us also note that Terent'yev in [16] determined explicitly the distribution of ξ_0 and calculated the constant $M_0 = 26$. These results were taken up by Ibragimov and Khasminskii in [10, Chapter 7.3], where by means of numerical simulation they equally showed that $B_0 = 19.5 \pm 0.5$, and so $E_0 = 0.73 \pm 0.03$. Later in [7], Golubev expressed B_0 in terms of the second derivative (with respect to a parameter) of an improper integral of a composite function of modified Hankel and Bessel functions. Finally in [15], Rubin and Song obtained the exact values $B_0 = 16\,\zeta(3)$ and $E_0 = 8\,\zeta(3)/13$, where ζ is Riemann's zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$.

In this paper we establish that the limiting likelihood ratio processes Z_γ and Z_0 are related. More precisely, we show that as $\gamma \to 0$, the process $Z_\gamma(y/\gamma^2)$, $y \in \mathbb{R}$, converges weakly in the space $\mathcal{D}_0(-\infty,+\infty)$ (the Skorohod space of functions on $\mathbb{R}$ without discontinuities of the second kind and vanishing at infinity) to the process Z_0. So, the random variables $\gamma^2\zeta_\gamma$ and $\gamma^2\xi_\gamma^\alpha$ converge weakly to the random variables ζ_0 and ξ_0 respectively. We show equally that the convergence of moments of these random variables holds, that is, $\gamma^4 B_\gamma \to 16\,\zeta(3)$, $\gamma^4 M_\gamma^\alpha \to 26$ and $E_\gamma^\alpha \to 8\,\zeta(3)/13$.

These are the main results of the present paper, and they are presented in Section 2, where we also briefly discuss the second possible asymptotics $\gamma \to +\infty$. The necessary lemmas are proved in Section 3 and, finally, in Section 4 we discuss some directions for future development.

2 Asymptotics of the limiting likelihood ratio

Consider the process $X_\gamma(y) = Z_\gamma(y/\gamma^2)$, $y \in \mathbb{R}$, where $\gamma > 0$ and Z_γ is defined by (2). Note that

$$\frac{\int_{\mathbb{R}} y\, X_\gamma(y)\; dy}{\int_{\mathbb{R}} X_\gamma(y)\; dy} = \gamma^2\zeta_\gamma,$$

$$\inf\Big\{z : X_\gamma(z) = \sup_{y\in\mathbb{R}} X_\gamma(y)\Big\} = \gamma^2\xi_\gamma^-$$

and

$$\sup\Big\{z : X_\gamma(z) = \sup_{y\in\mathbb{R}} X_\gamma(y)\Big\} = \gamma^2\xi_\gamma^+,$$

where the random variables ζ_γ and $\xi_\gamma^\pm$ are defined by (3). Remind also the process Z_0 on $\mathbb{R}$ defined by (1) and the random variables ζ_0 and ξ_0 defined by (4). Recall finally the quantities $B_\gamma = \mathbf{E}\zeta_\gamma^2$, $M_\gamma^\alpha = \mathbf{E}(\xi_\gamma^\alpha)^2$, $E_\gamma^\alpha = B_\gamma/M_\gamma^\alpha$, $B_0 = \mathbf{E}\zeta_0^2 = 16\,\zeta(3)$, $M_0 = \mathbf{E}\xi_0^2 = 26$ and $E_0 = B_0/M_0 = 8\,\zeta(3)/13$. Now we can state the main result of the present paper.

Theorem 1 *The process X_γ converges weakly in the space $\mathcal{D}_0(-\infty,+\infty)$ to the process Z_0 as $\gamma \to 0$. In particular, the random variable $\gamma^2\zeta_\gamma$ converge weakly to the random variable ζ_0 and, for any $\alpha \in [0,1]$, the random variable $\gamma^2\xi_\gamma^\alpha$ converge weakly to the random variable ξ_0. Moreover, for any $k > 0$ we have*

$$\gamma^{2k}\,\mathbf{E}\zeta_\gamma^k \to \mathbf{E}\zeta_0^k \quad \textit{and} \quad \gamma^{2k}\,\mathbf{E}(\xi_\gamma^\alpha)^k \to \mathbf{E}\xi_0^k,$$

and in particular $\gamma^4 B_\gamma \to 16\,\zeta(3)$, $\gamma^4 M_\gamma^\alpha \to 26$ and $E_\gamma^\alpha \to 8\,\zeta(3)/13$.

The results concerning the random variable ζ_γ are direct consequence of Ibragimov and Khasminskii [10, Theorem 1.10.2] and the following three lemmas.

Lemma 2 *The finite-dimensional distributions of the process X_γ converge to those of Z_0 as $\gamma \to 0$.*

Lemma 3 *For all $\gamma > 0$ and all $y_1, y_2 \in \mathbb{R}$ we have*

$$\mathbf{E}\left|X_\gamma^{1/2}(y_1) - X_\gamma^{1/2}(y_2)\right|^2 \leqslant \frac{1}{4}\left|y_1 - y_2\right|.$$

Lemma 4 *For any $c \in \,]\,0\,,\,1/8\,[$ we have*

$$\mathbf{E}X_\gamma^{1/2}(y) \leqslant \exp\bigl(-c\,|y|\bigr)$$

for all sufficiently small γ and all $y \in \mathbb{R}$.

Note that these lemmas are not sufficient to establish the weak convergence of the process X_γ in the space $\mathcal{D}_0(-\infty, +\infty)$ and the results concerning the random variable ξ_γ^α. However, the increments of the process $\ln X_\gamma$ being independent, the convergence of its restrictions (and hence of those of X_γ) on finite intervals $[A, B] \subset \mathbb{R}$ $\bigl($that is, convergence in the Skorohod space $\mathcal{D}[A, B]$ of functions on $[A, B]$ without discontinuities of the second kind$\bigr)$ follows from Gihman and Skorohod [6, Theorem 6.5.5], Lemma 2 and the following lemma.

Lemma 5 *For any $\varepsilon > 0$ we have*

$$\lim_{h\to 0}\ \lim_{\gamma\to 0}\ \sup_{|y_1 - y_2|<h} \mathbf{P}\Bigl\{\bigl|\ln X_\gamma(y_1) - \ln X_\gamma(y_2)\bigr| > \varepsilon\Bigr\} = 0.$$

Now, Theorem 1 follows from the following estimate on the tails of the process X_γ by standard argument $\bigl($see, for example, Ibragimov and Khasminskii [10]$\bigr)$.

Lemma 6 *For any $b \in \,]\,0\,,\,1/12\,[$ we have*

$$\mathbf{P}\Biggl\{\sup_{|y|>A} X_\gamma(y) > e^{-bA}\Biggr\} \leqslant 4\,e^{-bA}$$

for all sufficiently small γ and all $A > 0$.

All the above lemmas will be proved in the next section, but before let us discuss the second possible asymptotics $\gamma \to +\infty$. One can show that in this case, the process Z_γ converges weakly in the space $\mathcal{D}_0(-\infty,+\infty)$ to the process $Z_\infty(x) = \mathbb{1}_{\{-\eta<x<\tau\}}$, $x \in \mathbb{R}$, where η and τ are two independent exponential random variables with parameter 1. So, the random variables ζ_γ, ξ_γ^-, ξ_γ^+ and ξ_γ^α converge weakly to the random variables

$$\begin{aligned}
\zeta_\infty &= \frac{\int_{\mathbb{R}} x\, Z_\infty(x)\; dx}{\int_{\mathbb{R}} Z_\infty(x)\; dx} = \frac{\tau-\eta}{2}\,,\\
\xi_\infty^- &= \inf\Big\{z : Z_\infty(z) = \sup_{x\in\mathbb{R}} Z_\infty(x)\Big\} = -\eta,\\
\xi_\infty^+ &= \sup\Big\{z : Z_\infty(z) = \sup_{x\in\mathbb{R}} Z_\infty(x)\Big\} = \tau
\end{aligned}$$

and

$$\xi_\infty^\alpha = \alpha\,\xi_\infty^- + (1-\alpha)\,\xi_\infty^+ = (1-\alpha)\,\tau - \alpha\,\eta$$

respectively. One can equally show that, moreover, for any $k>0$ we have

$$\mathbf{E}\zeta_\gamma^k \to \mathbf{E}\zeta_\infty^k \quad\text{and}\quad \mathbf{E}(\xi_\gamma^\alpha)^k \to \mathbf{E}(\xi_\infty^\alpha)^k,$$

and in particular, denoting $B_\infty = \mathbf{E}\zeta_\infty^2$, $M_\infty^\alpha = \mathbf{E}(\xi_\infty^\alpha)^2$ and $E_\infty^\alpha = B_\infty/M_\infty^\alpha$, we finally have

$$\begin{aligned}
B_\gamma \to B_\infty &= \mathbf{E}\Big(\frac{\tau-\eta}{2}\Big)^2 = \frac{1}{2}\,,\\
M_\gamma^\alpha \to M_\infty^\alpha &= \mathbf{E}\big((1-\alpha)\,\tau - \alpha\,\eta\big)^2 = 6\left(\alpha-\frac{1}{2}\right)^2 + \frac{1}{2}
\end{aligned}\tag{5}$$

and

$$E_\gamma \to E_\infty^\alpha = \frac{1}{12\left(\alpha-\frac{1}{2}\right)^2+1}\,.\tag{6}$$

Let us note that these convergences are natural, since the process Z_∞ can be considered as a particular case of the process Z_γ with $\gamma = +\infty$ if one admits the convention $+\infty\cdot 0 = 0$.

Note also, that the process Z_∞ is the limiting likelihood ratio process in the problem of estimating the parameter θ by i.i.d. uniform observations on $[\theta,\theta+1]$. So, in this problem, the variables ζ_∞ and ξ_∞^α are the limiting distributions of the Bayesian estimators and of the maximum likelihood estimator respectively, B_∞ and M_∞^α are the limiting variances of these estimators and, the Bayesian estimators being asymptotically efficient, E_∞^α is the asymptotic efficiency of the maximum likelihood estimator.

Finally observe, that the formulae (5) and (6) clearly imply that in the latter problem (as well as in any problem having Z_∞ as limiting likelihood ratio) the best choice of the maximum likelihood estimator is $\alpha = 1/2$, and that the so chosen maximum likelihood estimator is asymptotically efficient. This choice was also suggested in Kutoyants [2] for problems having Z_γ as limiting likelihood ratio. For large values of γ this suggestion is confirmed by our asymptotic results. However, we see that for small values of γ the choice of α will not be so important, since all the limits in Theorem 1 do not depend on α.

3 Proofs of the lemmas

First we prove Lemma 2. Note that the restrictions of the process $\ln X_\gamma$ (as well as those of the process $\ln Z_0$) on $\mathbb{R}_+$ and on $\mathbb{R}_-$ are mutually independent processes with stationary and independent increments. So, to obtain the convergence of all the finite-dimensional distributions, it is sufficient to show the convergence of one-dimensional distributions only, that is,

$$\ln X_\gamma(y) \Rightarrow \ln Z_0(y) = W(y) - \frac{|y|}{2} = \mathcal{N}\Bigl(-\frac{|y|}{2}, |y|\Bigr)$$

for all $y \in \mathbb{R}$. Moreover, these processes being symmetric, it is sufficient to consider $y \in \mathbb{R}_+$ only. Here and in the sequel "$\Rightarrow$" denotes the weak convergence of the random variables, and $\mathcal{N}(m, V)$ denotes a "generic" random variable distributed according to the normal law with mean m and variance V.

The characteristic function $\varphi_\gamma(t)$ of $\ln X_\gamma(y)$ is

$$\begin{aligned}
\varphi_\gamma(t) &= \mathbf{E}\, e^{it\ln X_\gamma(y)} = \mathbf{E}\, e^{it\gamma\sum_{k=1}^{\Pi_+(y/\gamma^2)} \varepsilon_k^+ - it\frac{\gamma^2}{2}\Pi_+(y/\gamma^2)} \\
&= \mathbf{E}\,\mathbf{E}\Bigl(e^{it\gamma\sum_{k=1}^{\Pi_+(y/\gamma^2)} \varepsilon_k^+ - it\frac{\gamma^2}{2}\Pi_+(y/\gamma^2)} \Bigm| \mathscr{F}_{\Pi_+}\Bigr) \\
&= \mathbf{E}\Biggl(e^{-it\frac{\gamma^2}{2}\Pi_+(y/\gamma^2)} \prod_{k=1}^{\Pi_+(y/\gamma^2)} \mathbf{E}\, e^{it\gamma\varepsilon_k^+}\Biggr) \\
&= \mathbf{E}\, e^{-it\frac{\gamma^2}{2}\Pi_+(y/\gamma^2) - \frac{t^2\gamma^2}{2}\Pi_+(y/\gamma^2)} = \mathbf{E}\, e^{-\frac{\gamma^2}{2}(it+t^2)\Pi_+(y/\gamma^2)}
\end{aligned}$$

where we have denoted $\mathscr{F}_{\Pi_+}$ the σ-algebra related to the Poisson process Π_+, used the independence of ε_k^+ and Π_+ and recalled that $\mathbf{E}\, e^{it\varepsilon_k^+} = e^{-t^2/2}$.

Then, noting that $\Pi_+(y/\gamma^2)$ is a Poisson random variable of parameter y/γ^2 with moment generating function $\mathbf{E}\, e^{t\Pi_+(y/\gamma^2)} = \exp\bigl(\frac{y}{\gamma^2}(e^t - 1)\bigr)$, we get

$$\begin{aligned} \ln \varphi_\gamma(t) &= \frac{y}{\gamma^2}\Bigl(e^{-\frac{\gamma^2}{2}(it+t^2)} - 1\Bigr) = \frac{y}{\gamma^2}\Bigl(-\frac{\gamma^2}{2}(it+t^2) + o(\gamma^2)\Bigr) \\ &= -\frac{y}{2}(it+t^2) + o(1) \to -\frac{y}{2}(it+t^2) = \ln \mathbf{E}\, e^{it\mathcal{N}(-y/2,\, y)} \end{aligned}$$

as $\gamma \to 0$, and so Lemma 2 is proved.

Now we turn to the proof of Lemma 4 (we will prove Lemma 3 just after). For $y > 0$ we have

$$\begin{aligned} \mathbf{E} X_\gamma^{1/2}(y) &= \mathbf{E}\,\mathbf{E}\Bigl(e^{\frac{\gamma}{2}\sum_{k=1}^{\Pi_+(y/\gamma^2)} \varepsilon_k^+ - \frac{\gamma^2}{4}\Pi_+(y/\gamma^2)} \Bigm| \mathscr{F}_{\Pi_+}\Bigr) \\ &= \mathbf{E}\, e^{-\frac{\gamma^2}{4}\Pi_+(y/\gamma^2) + \frac{\gamma^2}{8}\Pi_+(y/\gamma^2)} = \mathbf{E}\, e^{-\frac{\gamma^2}{8}\Pi_+(y/\gamma^2)} \\ &= \exp\biggl(\frac{y}{\gamma^2}\Bigl(e^{-\frac{\gamma^2}{8}} - 1\Bigr)\biggr). \end{aligned}$$

The process X_γ being symmetric, we have

$$\mathbf{E} X_\gamma^{1/2}(y) = \exp\biggl(\frac{|y|}{\gamma^2}\Bigl(e^{-\frac{\gamma^2}{8}} - 1\Bigr)\biggr) \tag{7}$$

for all $y \in \mathbb{R}$ and, since

$$\frac{1}{\gamma^2}\Bigl(e^{-\frac{\gamma^2}{8}} - 1\Bigr) = \frac{1}{\gamma^2}\Bigl(-\frac{\gamma^2}{8} + o(\gamma^2)\Bigr) \to -\frac{1}{8}$$

as $\gamma \to 0$, for any $c \in \,]\,0\,,\,1/8\,[$ we have $\mathbf{E} X_\gamma^{1/2}(y) \leqslant \exp\bigl(-c\,|y|\bigr)$ for all sufficiently small γ and all $y \in \mathbb{R}$. Lemma 4 is proved.

Further we verify Lemma 3. We first consider the case $y_1, y_2 \in \mathbb{R}_+$ (say $y_1 \geqslant y_2$). Using (7) and taking into account the stationarity and the independence of the increments of the process $\ln X_\gamma$ on $\mathbb{R}_+$, we can write

$$\begin{aligned} \mathbf{E}\,\bigl|X_\gamma^{1/2}(y_1) - X_\gamma^{1/2}(y_2)\bigr|^2 &= \mathbf{E} X_\gamma(y_1) + \mathbf{E} X_\gamma(y_2) - 2\,\mathbf{E} X_\gamma^{1/2}(y_1) X_\gamma^{1/2}(y_2) \\ &= 2 - 2\,\mathbf{E} X_\gamma(y_2)\,\mathbf{E}\frac{X_\gamma^{1/2}(y_1)}{X_\gamma^{1/2}(y_2)} \\ &= 2 - 2\,\mathbf{E} X_\gamma^{1/2}\bigl(|y_1 - y_2|\bigr) \\ &= 2 - 2\exp\biggl(\frac{|y_1 - y_2|}{\gamma^2}\Bigl(e^{-\frac{\gamma^2}{8}} - 1\Bigr)\biggr) \\ &\leqslant -2\,\frac{|y_1 - y_2|}{\gamma^2}\Bigl(e^{-\frac{\gamma^2}{8}} - 1\Bigr) \leqslant \frac{1}{4}\,|y_1 - y_2|\,. \end{aligned}$$

The process X_γ being symmetric, we have the same result for the case $y_1, y_2 \in \mathbb{R}_-$.

Finally, if $y_1 y_2 \leqslant 0$ (say $y_2 \leqslant 0 \leqslant y_1$), we have

$$\begin{aligned}
\mathbf{E}\left|X_\gamma^{1/2}(y_1) - X_\gamma^{1/2}(y_2)\right|^2 &= 2 - 2\,\mathbf{E}X_\gamma^{1/2}(y_1)\,\mathbf{E}X_\gamma^{1/2}(y_2) \\
&= 2 - 2\exp\biggl(\frac{|y_1|}{\gamma^2}\Bigl(e^{-\frac{\gamma^2}{8}} - 1\Bigr) + \frac{|y_2|}{\gamma^2}\Bigl(e^{-\frac{\gamma^2}{8}} - 1\Bigr)\biggr) \\
&= 2 - 2\exp\biggl(\frac{|y_1 - y_2|}{\gamma^2}\Bigl(e^{-\frac{\gamma^2}{8}} - 1\Bigr)\biggr) \\
&\leqslant \frac{1}{4}\,|y_1 - y_2|\,,
\end{aligned}$$

and so, Lemma 3 is proved.

Now let us check Lemma 5. First let $y_1, y_2 \in \mathbb{R}_+$ (say $y_1 \geqslant y_2$) such that $\Delta = |y_1 - y_2| < h$. Then, noting that conditionally to $\mathscr{F}_{\Pi_+}$ the random variable

$$\ln X_\gamma(\Delta) = \gamma \sum_{k=1}^{\Pi_+(\Delta/\gamma^2)} \varepsilon_k^+ - \frac{\gamma^2}{2}\Pi_+(\Delta/\gamma^2)$$

is Gaussian with mean $-\frac{\gamma^2}{2}\Pi_+(\Delta/\gamma^2)$ and variance $\gamma^2\Pi_+(\Delta/\gamma^2)$, we get

$$\begin{aligned}
\mathbf{P}\Bigl\{\bigl|\ln X_\gamma(y_1) - \ln X_\gamma(y_2)\bigr| > \varepsilon\Bigr\} &\leqslant \frac{1}{\varepsilon^2}\,\mathbf{E}\bigl|\ln X_\gamma(y_1) - \ln X_\gamma(y_2)\bigr|^2 \\
&= \frac{1}{\varepsilon^2}\,\mathbf{E}\bigl|\ln X_\gamma(\Delta)\bigr|^2 \\
&= \frac{1}{\varepsilon^2}\,\mathbf{E}\,\mathbf{E}\left(\bigl(\ln X_\gamma(\Delta)\bigr)^2 \,\middle|\, \mathscr{F}_{\Pi_+}\right) \\
&= \frac{1}{\varepsilon^2}\,\mathbf{E}\left(\gamma^2\Pi_+(\Delta/\gamma^2) + \frac{\gamma^4}{4}\bigl(\Pi_+(\Delta/\gamma^2)\bigr)^2\right) \\
&= \frac{1}{\varepsilon^2}\left(\Delta + \frac{\gamma^4}{4}\left(\frac{\Delta}{\gamma^2} + \frac{\Delta^2}{\gamma^4}\right)\right) \\
&= \frac{1}{\varepsilon^2}\bigl((1 + \gamma^2/4)\Delta + \Delta^2/4\bigr) \\
&< \frac{1}{\varepsilon^2}\bigl(\beta(\gamma)\,h + h^2/4\bigr)
\end{aligned}$$

where $\beta(\gamma) = 1 + \gamma^2/4 \to 1$ as $\gamma \to 0$. So, we have

$$\begin{aligned}
\lim_{\gamma\to 0}\ \sup_{|y_1 - y_2| < h} \mathbf{P}\Bigl\{\bigl|\ln X_\gamma(y_1) - \ln X_\gamma(y_2)\bigr| > \varepsilon\Bigr\} &\leqslant \lim_{\gamma\to 0}\frac{1}{\varepsilon^2}\bigl(\beta(\gamma)\,h + h^2/4\bigr) \\
&= \frac{1}{\varepsilon^2}\left(h + \frac{h^2}{4}\right),
\end{aligned}$$

and hence

$$\lim_{h\to 0}\ \lim_{\gamma\to 0}\ \sup_{|y_1-y_2|<h} \mathbf{P}\Big\{\big|\ln X_\gamma(y_1)-\ln X_\gamma(y_2)\big|>\varepsilon\Big\}=0,$$

where the supremum is taken only over $y_1, y_2 \in \mathbb{R}_+$.

The process X_γ being symmetric, we have the same conclusion with the supremum taken over $y_1, y_2 \in \mathbb{R}_-$.

Finally, for $y_1 y_2 \leqslant 0$ (say $y_2 \leqslant 0 \leqslant y_1$) such that $|y_1-y_2|<h$, using the elementary inequality $(a-b)^2 \leqslant 2(a^2+b^2)$ we get

$$\begin{aligned}
\mathbf{P}\Big\{\big|\ln X_\gamma(y_1)-\ln X_\gamma(y_2)\big|>\varepsilon\Big\} &\leqslant \frac{1}{\varepsilon^2}\,\mathbf{E}\big|\ln X_\gamma(y_1)-\ln X_\gamma(y_2)\big|^2\\
&\leqslant \frac{2}{\varepsilon^2}\left(\mathbf{E}\big|\ln X_\gamma(y_1)\big|^2+\mathbf{E}\Big|\ln X_\gamma\big(|y_2|\big)\Big|^2\right)\\
&= \frac{2}{\varepsilon^2}\big(\beta(\gamma)y_1+y_1^2/4+\beta(\gamma)\,|y_2|+|y_2|^2/4\big)\\
&< \frac{2}{\varepsilon^2}\Big(\beta(\gamma)h+h^2/4\Big),
\end{aligned}$$

which again yields the desired conclusion. Lemma 5 is proved.

It remains to verify Lemma 6. Taking into account the symmetry of the process $\ln X_\gamma$, as well as the stationarity and the independence of its increments on $\mathbb{R}_+$, we obtain

$$\begin{aligned}
\mathbf{P}\Big\{\sup_{|y|>A} X_\gamma(y)>e^{-bA}\Big\} &\leqslant 2\,\mathbf{P}\Big\{\sup_{y>A} X_\gamma(y)>e^{-bA}\Big\}\\
&\leqslant 2\,e^{bA/2}\ \mathbf{E}\sup_{y>A} X_\gamma^{1/2}(y)\\
&= 2\,e^{bA/2}\ \mathbf{E}X_\gamma^{1/2}(A)\ \mathbf{E}\sup_{y>A}\frac{X_\gamma^{1/2}(y)}{X_\gamma^{1/2}(A)}\\
&= 2\,e^{bA/2}\ \mathbf{E}X_\gamma^{1/2}(A)\ \mathbf{E}\sup_{z>0} X_\gamma^{1/2}(z).
\end{aligned} \tag{8}$$

In order to estimate the last factor we write

$$\begin{aligned}
\mathbf{E}\sup_{z>0} X_\gamma^{1/2}(z) &= \mathbf{E}\ \exp\left(\frac{1}{2}\ \sup_{z>0}\left(\gamma\sum_{k=1}^{\Pi_+(z/\gamma^2)}\varepsilon_k^+-\frac{\gamma^2}{2}\Pi_+(z/\gamma^2)\right)\right)\\
&= \mathbf{E}\ \exp\left(\frac{1}{2}\ \sup_{n>0}\left(\gamma\sum_{k=1}^{n}\varepsilon_k^+-\frac{n\gamma^2}{2}\right)\right).
\end{aligned}$$

Now, let us observe that the random process $S_n = \sum_{k=1}^{n} \varepsilon_k^+$, $n \in \mathbb{N}$, has the same law as the restriction on $\mathbb{N}$ of a standard Brownian motion W. So,

$$\begin{aligned}
\mathbf{E} \sup_{z>0} X_\gamma^{1/2}(z) &= \mathbf{E} \exp\left(\frac{1}{2} \sup_{n>0} \bigl(\gamma W(n) - n\gamma^2/2\bigr)\right) \\
&= \mathbf{E} \exp\left(\frac{1}{2} \sup_{n>0} \bigl(W(n\gamma^2) - n\gamma^2/2\bigr)\right) \\
&\leqslant \mathbf{E} \exp\left(\frac{1}{2} \sup_{t>0} \bigl(W(t) - t/2\bigr)\right) = \mathbf{E} \exp\left(\frac{1}{2}\, S_0\right)
\end{aligned}$$

with an evident notation. It is known that the random variable S_0 is exponential of parameter 1 (see, for example, Borodin and Salminen [1]) and hence, using its moment generating function $\mathbf{E}\, e^{tS_0} = (1-t)^{-1}$, we get

$$\mathbf{E} \sup_{z>0} X_\gamma^{1/2}(z) \leqslant 2. \tag{9}$$

Finally, taking $b \in \left]\,0\,,\,1/12\,\right[$ we have $3b/2 \in \left]\,0\,,\,1/8\,\right[$ and, combining (8), (9) and using Lemma 4, we finally obtain

$$\mathbf{P}\left\{ \sup_{|y|>A} X_\gamma(y) > e^{-bA} \right\} \leqslant 4\, e^{bA/2} \exp\Bigl(-\frac{3b}{2} A\Bigr) = 4\, e^{-bA}$$

for all sufficiently small γ and all $A > 0$, which concludes the proof.

4 Final remarks

In conclusion let us mention, that a more general compound Poisson type limiting likelihood ratio process $Z_{\gamma,f}$ appears in many change-point type statistical models (see, for example, Fujii [5]). It is still the exponent of a two-sided compound Poisson process, but the jumps of the latter are not necessarily Gaussian. More precisely, it is given by

$$\ln Z_{\gamma,f}(x) = \begin{cases} \sum_{k=1}^{\Pi_+(x)} \ln \frac{f(\varepsilon_k^+ + \gamma)}{f(\varepsilon_k^+)}\,, & \text{if } x \geqslant 0, \\ \sum_{k=1}^{\Pi_-(-x)} \ln \frac{f(\varepsilon_k^- - \gamma)}{f(\varepsilon_k^-)}\,, & \text{if } x \leqslant 0, \end{cases}$$

where $\gamma > 0$, Π_+ and Π_- are two independent Poisson processes of intensity 1 on $\mathbb{R}_+$, and $\varepsilon_k^\pm$ are independent random variables with density f, mean 0 and variance 1 which are also independent of $\Pi_\pm$. Our guess is that the results of the present paper hold in this general situation under some regularity conditions on f. All the proofs, except the one of Lemma 6, can be easily adapted and we are currently working on the proof of that lemma.

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