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ON A SUBSTRUCTURAL LOGIC WITH MINIMAL NEGATION*

Abstract

In [3] and [1], Adillon and Verdú studied the *intuitionistic contraction-less logic* $\mathbf{IPC}^* \backslash \{c\}$, formulated on the language $\mathcal{L} = \{\wedge, \vee, *, \rightarrow, \top, \bot\}$ of type $(2, 2, 2, 2, 0, 0)$ of intuitionistic logic with two constants $\top$ and $\bot$ and with a binary connective of *fusion* $*$ representing a "context-sensitive" conjunction. In this paper we study some logics related to the $\{\rightarrow, *\}$-fragment of $\mathbf{IPC}^* \setminus \{c\}$ (or **PO**, or logic of *pocrims*): in particular we study equivalent Hilbert style, Gentzen style and algebraic formulations of the expansion **mPO** of the logic **PO**, formulated on a language $\{\rightarrow, *, \neg\}$ with a connective of negation satisfying the properties of Johansson minimal negation.

1. Introduction

"*In the community of researchers in logic is widely accepted the opinion that, among our simpler and deeper intuitions, there is the notion of absurdum, and that among our basic attitudes there is the rational reject of absurdum. This belief finds interesting developments if we relate it to the idea that denying a proposition p is equivalent to the assertion that the proposition p implies an absurdum* ([6], pag. 96)". Following this idea, we will take a fixed formula λ of our language (the *absurdum*) in such a way

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that, for every formula φ, its negation $\neg\varphi$ will be equivalent (modulo some congruence relation on the algebra of formulas) to the formula $\varphi \to \lambda$. Clearly, the properties of the *absurdum* will depend essentially on two conditions: (1) on the hypotheses made on the implication connective$\to$; (2) on the hypotheses made on the logical behavior of the connective $\neg$, by axioms, rules of inference or equations. Regarding point (1), in this paper we will study logics with a **BCK**-type implication; regarding point (2), we must answer the following question: given a logic **S** on (an expansion of) a language $\mathcal{L} = \{\to, \neg\}$, what kind of formula is the *absurdum*? Clearly, the *absurdum* $\bot$ should be such that $\nvdash_{\mathbf{S}} \bot$. Moreover, it seems rather intuitive to consider the *absurdum* as the negation of the *truth*: if $\top$ is a fixed tautology of a logic **S**, the *absurdum* is defined as the formula $\bot =_{def} \neg\top$. Given the *absurdum*, we can proceed to define the difference between *minimal* and *Scotian* negation. Historically, the first appearance of the concept of minimal negation can be found in the works of Kolmorogov ([13], [14]), but it was systematically developed by the Danish logician Ingebrigt Johansson (the same ideas are contained in Jaskowski's article [11], English translation of a paper published in Polish in 1948): adopting the idea that *denying a proposition p is equivalent to the assertion that p implies an absurdum*, Johansson proposed a conception of the absurdum that he called *minimal*, in contraposition to the *intuitionistic* (or *Scotian*) conception of the absurdum. In [12] it is observed that, among the axioms of classical logic (and of intuitionistic logic in Heyting's 1925 axiomatization, as given for instance in [8], pag. 56), there are two which, in a certain sense, could be seen as symmetrical, and which are the formalized expression of the medieval locutions *verum ex quolibet - ex falso quodlibet*, that is $\varphi \to (\psi \to \varphi)$, called *a fortiori* law, or, more commonly, axiom K, and $\varphi \to (\neg\varphi \to \psi)$, called by Jan Łukasiewicz *Duns Scotus* law, since it is stated in the book *In universam logicam quaestiones*, formerly ascribed to Duns Scotus. Johansson explains the meaning of these axioms by interpreting the implication $\to$ in such a way that the formula $\varphi \to \psi$ is true if and only if one of the following three conditions hold: (1) ψ is true; (2) ψ is a logical consequence of φ; (3) φ is false (absurdum).

In particular, following the *intuitionistic* conception, one of the properties of the absurdum is that of it being a *minimum element with respect to the implication*, a property which is formally expressed by the Duns Scotus law. Johansson proposed a calculus, called *minimal logic*, obtained from Heyting's intuitionistic logic by deleting the Duns Scotus law, in which contradictions are not equivalent between themselves, and the absurdum is not the minimum element with respect to implication. We can reformulate Johansson's original definition in terms of abstract algebraic logic. Recall that, if **S** is a deductive system on a propositional language $\mathcal{L}$, and $T \subseteq Fm_{\mathcal{L}}$ is a subset of $\mathcal{L}$-formulas, then $\mathbf{\Omega_{Fm}}(T)$ symbolizes the Leibniz congruence of T, that is the largest congruence of the algebra of formulas **Fm** compatible with T (see [4], pp. 10–11]). More precisely, if **S** is a deductive system on a language expanding $\mathcal{L} = \{\rightarrow, \neg\}$, and if $Th(\mathbf{S})$ stands for the set of **S**-theories, fixing $\bot =_{def} \neg\top$, we shall call **S** a **deductive system with minimal negation** if and only if for every $\varphi, \psi \in Fm_{\mathcal{L}}$ and every $T \in Th(\mathbf{S})$, the following holds:

(m1) $\langle \neg\varphi, \varphi \rightarrow \bot \rangle \in \mathbf{\Omega_{Fm}}(T)$.

S will be called **a deductive system with scotian negation** if and only if for every $\varphi, \psi \in Fm_{\mathcal{L}}$ and every $T \in Th(\mathbf{S})$:

(m2) $\vdash_{\mathcal{S}} \varphi \rightarrow (\neg\varphi \rightarrow \psi)$.

The logics **IPC** and $\mathbf{IPC}^* \setminus \{c\}$ are deductive systems with a negation which is *both minimal and scotian* (note that from now on we will take *logic* as a synonymous of *deductive system on a propositional language*). In the last section of this paper we shall see that **mPO** is a deductive system with a *minimal negation* that *is not a scotian one*. Moreover, it has to be remarked that the properties (m1) and (m2) are independent, as we will see in the last section.

2. Hilbert calculi: PO, mPO, BPO

Our starting point will be the logic **PO** of *pocrims* (in [1] called $\{\rightarrow, *\}$-fragment of $\mathbf{IPC}^* \setminus \{c\}$). Let $\mathcal{L}_0 = \{\rightarrow, *\}$ be a propositional language of type (2,2). The logic **PO** is the deductive system $\langle Fm_{\mathcal{L}}, \vdash_{\mathbf{PO}} \rangle$ defined by the following axioms:

(B) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \xi) \rightarrow (\varphi \rightarrow \xi))$

(C) $(\varphi \rightarrow (\psi \rightarrow \xi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \xi))$

(K) $\varphi \rightarrow (\psi \rightarrow \varphi)$

(Adj) $\varphi \rightarrow (\psi \rightarrow (\varphi * \psi))$ *Adjunction law*

(Imp) $(\varphi \rightarrow (\psi \rightarrow \xi)) \rightarrow (\varphi * \psi \rightarrow \xi)$ *Import law*

and the Modus Ponens MP (i.e., $\{\varphi, \varphi \rightarrow \psi\} \vdash \psi$) as rule of inference.

PROPOSITION 2.1 *The following formulas are tautologies of* **PO***:*

(t1) $\varphi \to \varphi$
(t2) $(\varphi \to \varphi) \to (\psi \to \psi)$
(t3) $\varphi * (\varphi \to \psi) \to \psi$
(t4) $(\varphi \to \psi) \to ((\gamma \to \varphi) \to (\gamma \to \psi))$
(t5) $((\varphi \to \varphi) \to \psi) \to \psi$.
(t6) $\varphi * \psi \to \varphi$
(t7) $\varphi * \psi \to \psi * \varphi$
(t8) $(\varphi * \psi \to \xi) \to (\varphi \to (\psi \to \xi))$

Let now $\mathcal{L} = \{\to, *, \neg\}$ be a propositional language of type (2,2,1). The logic **mPO** is the deductive system on $Fm_{\mathcal{L}}$, obtained from **PO** by adding the following axiom:

(mCTR) $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$ *law of minimal contraposition.*

PROPOSITION 2.2. *The following formulas are theorems of the deductive system* ***mPO****:*

(t9) $\varphi \to \neg\neg\varphi$
(t10) $\neg\neg\top \to \top$
(t11) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ *law of weak contraposition*
(t12) $\bot \to \neg\psi$
(t13) $\neg\neg\neg\varphi \to \neg\varphi$

PROPOSITION 2.3. *For every $\varphi \in Fm_{\mathcal{L}}$, minimality holds in* ***mPO****, i.e.:* $\vdash_{\mathbf{mPO}} \neg\varphi \leftrightarrow (\varphi \to \bot)$*, where $\varphi \leftrightarrow \psi$ is defined by $(\varphi \to \psi) * (\psi \to \varphi)$.*

The logic **BPO** (or $\{\to, *, \neg\}$-fragment of $\mathbf{IPC}^* \setminus \{c\}$) is an axiomatic extension of **mPO**, obtained by adjoining the axiom:

(DS) $\varphi \to (\neg\varphi \to \psi)$. *Duns Scotus law*

PROPOSITION 2.4. *The following formulas are theorems of the deductive system* **BPO** *:*

t14 $\varphi * \neg\varphi \to \neg(\varphi \to \varphi)$.
t15 $\neg(\varphi \to \varphi) \to \varphi * \neg\varphi$.
t16 $\varphi * \neg\varphi \to \psi * \neg\psi$.

Observe that the logic **BPO** could have equivalently been defined (as in [1], where this logic is called $\{*, \to, \bot\}$-fragment of $\mathbf{IPC}^* \setminus \{c\}$, here $\mathbf{BPO}^{\bot}$) on a language $\mathcal{L}_1 = \{*, \to, \bot\}$ of type (2,2,0) with the constant $\bot$ as primitive, with axioms (B), (C), (K), (Adj), (Imp), the MP rule and axiom

$$(\mathrm{DS}^{\bot}) \qquad \bot \to \psi$$

instead of axioms (mCTR) and (DS): it is indifferent, in the case of **BPO**, if we assume $\bot$ or $\neg$ as primitive symbol in the language, since, if we assume $\bot$ (respectively, $\neg$) as primitive, we can define negation $\neg$ as $\neg\varphi =_{df} \varphi \to \bot$ (respectively, by Proposition 2.4, we can define the absurdum $\bot$ as $\bot =_{df} p_0 * \neg p_0$). However, it is not so if we are interested in questions about the existence of negations which are weaker than the Scotian one, or in obtaining the independence of properties (m1) and (m2): in this case the logic $\mathbf{BPO}^{\{\bot\}}$ has not enough expressive means, and the standard treatments of negation with $\bot$ as primitive (as the one given, for instance, in [15]) fail in the purpose: therefore we are constrained to diverge from Adillon's and Verdú's treatment, and take the negation connective $\neg$ as a primitive symbol in the language.

In the following section we shall present two Gentzen calculi $L_{\mathbf{mPO}}$ and $L_{\mathbf{BPO}}$ on the language $\mathcal{L} = \{\to, *, \neg\}$, and we will prove that these two Gentzen calculi are equivalent (in the sense defined in sections 2.6 and 2.7 of [16], [17]) modulo two inverse translations, to the deductive system **mPO** and **BPO** respectively. Equivalent Gentzen-style formulations for the logics **BCK**, **PO** and $\mathbf{IPC}^* \setminus \{c\}$, formulated on a language $\mathcal{L}' = \{\to, *, \top, \bot\}$ with two propositional constants and without the negation connective, have been given in [2], [3] and [1].

3. Gentzen calculi and equivalence proofs

The Gentzen calculus $L_{\mathbf{mPO}}$ is a calculus of type $(\omega, \{0,1\})$ on the language $\mathcal{L} = \{\to, *, \neg\}$, defined by the following rules and axiom:

$$\varphi \Rightarrow \varphi \ (Axiom)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ (w \Rightarrow) \qquad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Rightarrow \Delta}\ (e \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi \qquad \varphi, \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}\ (Cut)$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \varphi * \psi}\ (\Rightarrow *) \qquad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi * \psi \Rightarrow \Delta}\ (* \Rightarrow)$$

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ (\Rightarrow\to) \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Pi, \psi \Rightarrow \Delta}{\Gamma, \Pi, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma, \neg\varphi \Rightarrow \emptyset}\ (\neg \Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \emptyset}{\Gamma \Rightarrow \neg\varphi}\ (\Rightarrow \neg)$$

As it is easily seen by inspecting the rules of the calculus $L_{\mathbf{mPO}}$, this calculus not only lacks the *contraction* rule, but also the *right weakening* rule.

We shall now prove the equivalence between the Gentzen calculus $L_{\mathbf{mPO}}$ and the deductive system **mPO**. Given the calculus $L_{\mathbf{mPO}}$, define the *Gentzen system* $\mathcal{G}_{\mathbf{mPO}} = \langle Seq^{(\omega,\{0,1\})}_{\mathcal{L}_{\mathbf{mPO}}}, \vdash_{L_{\mathbf{mPO}}}\rangle$ and the deductive system $\mathbf{S}_{\mathcal{G}_{\mathbf{mPO}}} = \langle Fm_{\mathcal{L}}, \vdash_{\mathcal{S}_{\mathcal{G}_{\mathbf{mPO}}}}\rangle$ associated with $\mathcal{G}_{\mathbf{mPO}}$ in the following way:

$$\Gamma \vdash_{\mathcal{S}_{\mathcal{G}_{\mathbf{mPO}}}} \varphi \text{ iff } \{\emptyset \Rightarrow \gamma : \gamma \in \Gamma\} \vdash_{L_{\mathbf{mPO}}} \emptyset \Rightarrow \varphi.$$

The following Theorem (given without proof) asserts the equivalence between the Gentzen system $\mathcal{G}_{\mathbf{mPO}}$ and the deductive system $\mathcal{S}_{\mathcal{G}_{\mathbf{mPO}}}$:

THEOREM 3.1. *Let τ and ρ the translations of $\mathcal{G}_{\mathbf{mPO}}$ in $\mathcal{S}_{\mathcal{G}_{\mathbf{mPO}}}$ and of $\mathcal{S}_{\mathcal{G}_{\mathbf{mPO}}}$ in $\mathcal{G}_{\mathbf{mPO}}$ respectively, defined by:* $\tau_{(m,1)}(p_0, \ldots, p_{m-1} \Rightarrow q_0) = \begin{cases} (p_0 * p_1 * \ldots * p_{m-1}) \to q_0 & \text{if } m \neq 0 \\ q_0 & \text{if } m = 0 \end{cases}$

$\tau_{(m,0)}(p_0, \ldots, p_{m-1} \Rightarrow \emptyset) = \neg(p_0 * p_1 * \ldots * p_{m-1})$
$\tau_{(0,0)}(\emptyset \Rightarrow \emptyset) = \neg\top = \bot.$

$\rho(p_0) = \{\emptyset \Rightarrow p_0\}.$
Then the following hold:

(i) $\{\Gamma_i \Rightarrow \Delta_i : i \in I\} \vdash_{L_{\mathbf{mPO}}} \Gamma \Rightarrow \Delta$ *if and only if* $\{\tau(\Gamma_i \Rightarrow \Delta_i) : i \in I\} \vdash_{\mathcal{S}_{\mathcal{G}_{\mathbf{mPO}}}} \tau(\Gamma \Rightarrow \gamma)$

(ii) $\varphi \dashv\vdash_{\mathcal{S}_{\mathcal{G}_{\mathbf{mPO}}}} \tau\rho(\varphi)$.

LEMMA 3.2. *Let τ be the translation of $\mathcal{L}$-sequents in $\mathcal{L}$-formulas defined in Theorem 3.1. Then for every rule $\frac{\{\Gamma_i \Rightarrow \gamma_i : i \leq 2\}}{\Gamma \Rightarrow \gamma}$ of $L_{\mathbf{mPO}}$, we have*

$$\{\tau(\Gamma_i \Rightarrow \Delta_i) : i \leq 2\} \vdash_{\mathbf{mPO}} \tau(\Gamma \Rightarrow \Delta).$$

PROOF. For the cases $(w \Rightarrow)$, $(e \Rightarrow)$, (Cut), $(\rightarrow\Rightarrow)$, $(\Rightarrow\rightarrow)$, $(* \Rightarrow)$ and $(\Rightarrow *)$, the reader is sent to [2], Lemma 4 and [3], Lemma 9. The case of $\neg$ is straightforward.

LEMMA 3.3. *Let τ be the translation of $\mathcal{L}$-sequences in $\mathcal{L}$-formulas defined in Theorem 3.1. Then*

$$\{\Gamma_i \Rightarrow \gamma_i : i \in I\} \vdash_{L_{\mathbf{mPO}}} \Gamma \Rightarrow \gamma \textit{ implies}$$
$$\{\tau(\Gamma_i \Rightarrow \Delta_i) : i \in I\} \vdash_{\mathbf{mPO}} \tau(\Gamma \Rightarrow \Delta).$$

PROOF. By induction on the length of the proof (see [2], Lemma 4).

We proceed to prove the equivalence between the deductive systems $\mathcal{G}_{\mathbf{mPO}}$ and **mPO**.

THEOREM 3.4. $\vdash_{\mathcal{S}_{\mathcal{G}_{\mathbf{mPO}}}} = \vdash_{\mathbf{mPO}}$

PROOF. $\supseteq$): Suppose that $\Gamma \vdash_{\mathbf{mPO}} \varphi$. We shall show that $\Gamma \vdash_{\mathcal{S}_{\mathcal{G}_{\mathbf{mPO}}}} \varphi$ by induction on the length of the proof.

- $n = 1$
  (i). If $\varphi \in \Gamma$ then it is trivial.
  (ii). If φ is an instance of axiom, we shall show that $\emptyset \vdash_{L_{\mathbf{mPO}}} \emptyset \Rightarrow \varphi$. We limit ourselves to axiom (mCTR).

$$(mCTR)\quad \dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi \qquad \dfrac{\psi \Rightarrow \psi}{\psi, \neg\psi \Rightarrow \emptyset}\,(\neg \Rightarrow)}{\varphi, \psi, \varphi \rightarrow \neg\psi \Rightarrow \emptyset}\,(\rightarrow\Rightarrow)}{\varphi \rightarrow \neg\psi, \psi, \varphi \Rightarrow \emptyset}\,(e \Rightarrow)}{\varphi \rightarrow \neg\psi, \psi \Rightarrow \neg\varphi}\,(\Rightarrow \neg)}{\emptyset \Rightarrow (\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi)}\,(\Rightarrow\rightarrow)$$

- $n > 1$ If φ is obtained by Modus Ponens from the formulas ξ and $\xi \rightarrow \varphi$, then by induction hypothesis we have $\Gamma \vdash_{\mathcal{S}_{\mathcal{G}_{\mathbf{mPO}}}} \xi$ and $\Gamma \vdash_{\mathcal{S}_{\mathcal{G}_{\mathbf{mPO}}}} \xi \rightarrow \varphi$, and from $\{\xi, \xi \rightarrow \varphi\} \vdash_{\mathcal{S}_{\mathcal{G}_{\mathbf{mPO}}}} \varphi$ we finally obtain $\Gamma \vdash_{\mathcal{S}_{\mathcal{G}_{\mathbf{mPO}}}} \varphi$.

$\subseteq$) : Suppose that $\Gamma \vdash_{\mathcal{S}_{\mathcal{G}_{\mathbf{mPO}}}} \varphi$, that is, $\{\emptyset \Rightarrow \gamma : \gamma \in \Gamma\} \vdash_{L_{\mathbf{mPO}}} \emptyset \Rightarrow \varphi$, by the previous Lemma one obtains $\{\tau(\emptyset \Rightarrow \gamma) : \gamma \in \Gamma\} \vdash_{\mathbf{mPO}} \tau(\emptyset \Rightarrow \varphi)$, that is $\Gamma \vdash_{\mathbf{mPO}} \varphi$.

Finally, from the previous results we easily get the following Equivalence Theorem:

THEOREM 3.6. *$L_{\mathbf{mPO}}$ and* **mPO** *are equivalent, that is, the following hold:*

(i) $\{\Gamma_i \Rightarrow \Delta_i : i \in I\} \vdash_{L_{\mathbf{mPO}}} \Gamma \Rightarrow \Delta$ iff $\{\tau(\Gamma_i \Rightarrow \Delta_i) : i \in I\} \vdash_{\mathbf{mPO}} \tau(\Gamma \Rightarrow \Delta)$,

(ii) $\varphi \dashv\vdash_{\mathbf{mPO}} \tau\rho(\varphi)$.

The Gentzen calculus $L_{\mathbf{BPO}}$ is the calculus of type $(\omega, \{0,1\})$, on the language $\mathcal{L}$, obtained from the calculus $L_{\mathbf{mPO}}$ by adjoining the following *right weakening* rule:

$$\frac{\Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow \psi}\ (\Rightarrow w)$$

Let τ and ρ the translations of $\mathcal{L}$-sequences in $\mathcal{L}$-formulas defined in Theorem 3.1, and define $\mathcal{S}_{\mathcal{G}_{\mathbf{BPO}}}$ as we did in the case of **mPO**. Observing that (DS) is derivable in $L_{\mathbf{BPO}}$, and that the translation of the rule of right-weakening holds in **BPO**, we get the following:

THEOREM 3.6. $\vdash_{\mathcal{S}_{\mathcal{G}_{\mathbf{BPO}}}} = \vdash_{\mathbf{BPO}}$*, and this implies that $L_{\mathbf{BPO}}$ and* **BPO** *are equivalent.*

4. Algebraic Semantics: Minimal and Bounded Pocrims

In this section we shall analyze the algebraic structures which correspond to the deductive systems **PO**, **mPO** and **BPO**.

Let $\mathcal{L}_0 = \{\rightarrow, *, \top\}$ be a propositional language of type (2,2,0)[2]. The structure $\mathbf{A} = \langle \mathbf{A}, \rightarrow, *, \top, \leq \rangle$ is a *partially ordered commutative residuated*

[2]Note that the quasivariety PO is defined on a language $\{\rightarrow, *, \top\}$ with a propositional constant $\top$, differing from the logic **PO** defined on a language $\{\rightarrow, *\}$. When proving the algebraizability of this logic, this could be incoherent with the definitions of algebraizability given in [4], which requires the languages to be the same. But since in every pocrim **A** the top element $\top$ can be defined, for every $a \in A$, as $a \rightarrow a$, and this implies that the class PO could be formulated on language $\{\rightarrow, *\}$ without any essential change, it is completely harmless to assume the constant $\top$ in the language, and in what follows we will do so.

integral monoid, briefly a **pocrim**, in symbols $\mathbf{A} \in \mathsf{PO}$) if, for every $a, b, c \in A$, the following hold (see [5]):

- $\langle A, *, \top \rangle$ is a commutative monoid with neutral element $\top$;
- $\langle A, \leq \rangle$ is a partially ordered set such that $\leq$ is compatible with $*$ (i.e., $a \leq b$ implies $a * c \leq b * c$) and $\top$ is the maximum of $\langle A, \leq \rangle$;
- $\langle \mathbf{A}, \leq \rangle$ has the *residuum property*, that is

$$a * c \leq b \text{ if and only if } c \leq a \to b.$$

It has been proved (see e.g. [1]) that every pocrim is the $\{\to, *, \top\}$-*subreduct* of a *residuated lattice*. Moreover, it is known that every $\{\to, \top\}$-reduct of a pocrim is a BCK algebra.

We recall some basic definitions and results about pocrims. If $\mathbf{A} \in \mathsf{PO}$ and $F \subseteq A$, F is said to be an *implicative filter* of $\mathbf{A}$ if it satisfies $[F1]$: $\top \in F$; and $[F2]$: if $a, a \to b \in F$ then $b \in F$ (or equivalently, if it satisfies $[F1]$; $[F3]$: if $a \leq b$ and $a \in F$ then $b \in F$; and $[F4]$: if $a, b \in F$ then $a * b \in F$). If $\mathbf{A} \in \mathsf{PO}$ and $X \subseteq A$, the *filter generated by X in* $\mathbf{A}$ ($\mathcal{F}i_{\mathbf{A}}(X)$) is defined as $\mathcal{F}i_{\mathbf{A}}(X) = \bigcap \{F \in \mathcal{F}i(\mathbf{A}) | \mathbf{F} \supseteq \mathbf{X}\}$. Defining recursively $\varphi \to_n \psi$ and φ^n by $\varphi \to_0 \psi =_{df} \psi$ and $\varphi \to_{n+1} \psi =_{df} \varphi \to (\varphi \to_n \psi)$, and respectively by $\varphi^0 =_{df} \top$ and $\varphi^{n+1} =_{df} \varphi * \varphi^n$, we get the following Lemma about implicative filters and pocrims.

Lemma 4.1. *If* $\mathbf{A} \in \mathsf{PO}$, *then*

(i) for every $a \in A$, $a \in \mathcal{F}i_{\mathbf{A}}(X)$ *iff*
there exist $n < \omega$ *and* $b_1 \dots b_n \in X$ *s.t.* $(b_1 * \dots * b_n) \to a = \top$*;*

(ii) for every $X \subseteq A$,
$\mathcal{F}i_{\mathbf{A}}(X) = \{a \in A | \exists n < \omega, \exists b_1 \dots b_n \in X : b_1 * \dots * b_n \leq a\}$*;*

(iii) $\mathcal{F}i_{\mathbf{A}}(\emptyset) = \{\top\}$*;*

(iv) $\mathcal{F}i_{\mathbf{A}}(\{a\}) = \{b \mid \exists n < \omega : a^n \leq b\} = \{b \mid \exists n < \omega : a \to_n b = \top\}$.

Let $\mathcal{L} = \{\to, *, \neg, \top\}$ be a language of type (2,2,1,0). The structure $\langle A, \to, *, \top, \neg, \leq \rangle$ is a **minimal pocrim** (briefly: $\mathbf{A} \in \mathsf{MPO}$) if:

- the $\{\to, *, \top, \leq\}$-reduct of $\mathbf{A}$ is a pocrim;
- the following equation holds: $(x \to \neg y) \to (y \to \neg x) \approx \top$.

The structure $\mathbf{A} = \langle \mathbf{A}, \to, *, \top, \neg, \leq \rangle$ is a **bounded pocrim** (briefly: $\mathbf{A} \in \mathsf{BPO}$), if the following hold:

- $\mathbf{A}$ is a minimal pocrim,
- $\bot =_{def} \neg \top$ is the *minimum element* of $\langle A, \leq \rangle$.

We have the following characterization of the classes MPO and BPO:

THEOREM 4.2. *An algebra* $\mathbf{A} = \langle \mathbf{A}, \rightarrow, *, \neg, \top \rangle$ *of type (2,2,1,0) satisfies, for every* $x, y, z \in A$*, the following equations and quasiequations:*

a1 $x \rightarrow \top \approx \top$

a2 $(y \rightarrow z) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) \approx \top$

a3 $\top \rightarrow x \approx x$

a4 $(x * y) \rightarrow z \approx y \rightarrow (x \rightarrow z)$

a5 if $x \rightarrow y \approx \top$ *and* $y \rightarrow x \approx \top$ *then* $x \approx y$.

a6 $(x \rightarrow \neg y) \rightarrow (y \rightarrow \neg x) \approx \top$

iff the relation $\leq = \{\langle a, b \rangle \ : \ a \rightarrow b = \top\}$ *is a partial order on* A *and the structure* $\mathbf{A} = \langle \mathbf{A}, \rightarrow, *, \neg, \top, \leq \rangle$ *is a minimal pocrim.*

An algebra $\mathbf{A} = \langle \mathbf{A}, \rightarrow, *, \neg, \top \rangle$ *of type (2,2,1,0) satisfies, for every* $x, y, z \in A$*, equations and quasiequations [a1]-[a6] and the following equation:*

a7 $x \rightarrow (\neg x \rightarrow y) \approx \top$

iff the relation $\leq = \{\langle a, b \rangle \in A \times A \ : \ a \rightarrow b = \top\}$ *is a partial order on* A *and the structure* $\mathbf{A} = \langle \mathbf{A}, \rightarrow, *, \neg, \top, \leq \rangle$ *is a bounded pocrim.*

PROOF. (Sketch) In [10] it is proved that an algebra $\mathbf{A} = \langle \mathbf{A}, \rightarrow, *, \top \rangle$ is a pocrim iff it satisfies equations [a1]-[a4] and the quasiequation [a5]: the generalization to minimal and bounded pocrims is then an easy matter of calculation.

Modifying the proof of Higgs (in [9]) for the class PO of pocrims, it can be proved that the classes MPO and BPO are quasivarieties but are not varieties, that is, are not equational classes. In [1], Teor. 2.62 is proved that the class PO is the equivalent quasivariety semantics of the logic **PO**. Analogous results can be proved about the logics **mPO** and **BPO**.

THEOREM 4.3. *The class* MPO *(*BPO*) is the equivalent quasivariety semantics of the logic* **mPO** *(respectively* **BPO***).*

PROOF. (Sketch) The deductive system **mPO** (and therefore its axiomatic extension **BPO**) are, following the definition of Blok and Pigozzi ([4], p. 46), Rasiowa's *standard systems of implicative extensional propositional calculi*, and therefore are algebraizable. The Leibniz congruence of these logics is defined by $\mathbf{\Omega_{Fm}}(\mathbf{T}) = \{\langle \varphi, \psi \rangle : \varphi \rightarrow \psi, \psi \rightarrow \varphi \in \mathbf{T}\}$, or equivalently by $\mathbf{\Omega_{Fm}}(\mathbf{T}) = \{\langle\varphi,\psi\rangle : \varphi \leftrightarrow \psi \in \mathbf{T}\}$, where, again, we write $\varphi \leftrightarrow \psi$ as an abbreviation of the formula $(\varphi \to \psi) * (\psi \to \varphi)$. By [4], Thm. 2.17, the equivalent quasivariety semantics of **mPO** is the quasivariety K axiomatised by the equations $\varphi \approx \top$, for every axiom φ of **mPO**, and the two quasiequations: $[\mathcal{MP}]$ *if* $x \to y \approx \top$ *and* $x \approx \top$ *then* $y \approx \top$; and $[\mathcal{Q}]$: *if* $x \to y \approx \top$ *and* $y \to x \approx \top$ *then* $x \approx y$. Using Theorem 4.2, it is then easy to show that, given an arbitrary algebra $\mathbf{A} = \langle \mathbf{A}, \to, *, \neg, \top\rangle$ of type (2,2,1,0), $\mathbf{A} \in \mathsf{K}$ if and only if $\mathbf{A} \in \mathsf{MPO}$. The same holds for the logic **BPO** and the quasivariety BPO.

5. Properties of the minimal contraction-less logic mPO

In this section we give an example of a five-element minimal pocrim, which is useful in suggesting a simple intuitive visualization of various (logically) interesting properties of the quasivariety MPO (and hence of the logic **mPO**), and in characterizing the differences between *minimal* and *scotian* negations (a more general treatment of negations on BCK-related structures, with a particular attention to independence results, is given in [7]). But before doing this, we will show that properties (m1) and (m2) are mutually independent. Consider the three-element **BCK** algebra with a negation operation, where the connectives $\to$ and $\neg$ are defined by the following tables (see [7], pag. 341):

$\to$	1	a	0
1	1	a	0
a	1	1	a
0	1	1	1

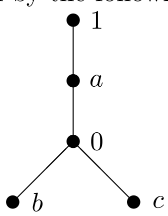
	$\neg$
1	0
a	a
0	0

we have that $\mathbf{A} \models \mathbf{x} \to (\neg\mathbf{x} \to \mathbf{y}) \approx \top$, i.e. the algebraic counterpart of (m2) holds, but in $\mathbf{A}$ we do not have $\langle \neg x, x \to \bot\rangle \in \mathbf{\Omega_A}(\emptyset)$, since $0 \to \bot = 0 \to \neg 1 = 0 \to 0 = 1$, but $\neg 0 = 0$ therefore $\neg 0 \neq 0 \to \bot$, and since the matrix $\langle \mathbf{A}, \top\rangle$ is reduced (i.e. $\Omega_{\mathbf{A}}(\top) = Id_{\mathbf{A}}$, the identity on $\mathbf{A}$), we have that $\langle \neg 0, 0 \to \bot\rangle \notin \mathbf{\Omega_A}(\emptyset)$, i.e. (m1) does not hold. The symmetric case of an example satisfyng (m1) but not (m2) is given in what follows.

Coming back to the properties of the logic **mPO**, the main fact to be noticed is that, in its equivalent quasivariety MPO, the absurdum $\bot =_{def}$

$\neg\top$ is not, in general, the minimum element, and moreover, for every a, b, it is not always the case that $a * \neg a \leq b$: these facts imply that in the logic **mPO** the *absurdum* $\perp$ and the *contradictions* $x * \neg x$ do not need to be equivalent, as it happens in logics with intuitionistic negation (such that **BPO**). At a logical level, this fact tells us that the logic **mPO** is *paraconsistent*: the assumption of the *absurdum* or of a contradiction in a set of premises, does not imply the inconsistency of the set (i.e., does not imply that the consequences of the set of premises are the set of all formulas). Recall that, on the contrary, in intuitionistic logic we have $Th_{\mathbf{IPC}}(\perp) = Fm_{\mathcal{L}}$, where $Th_{\mathbf{IPC}}(\perp)$ is the theory generated by $\perp$: we shall see that this property does not hold in the logic **mPO**.

Let $A = \{1, a, 0, b, c\}$, and let $\mathbf{A_0} = \langle \mathbf{A}, \to, *, \top, \leq \rangle$ be the five-element pocrim whose ordering is given by the following graph:



where the operations $\to$ and $*$ are defined by the following tables:

$\to$	1	a	0	b	c
1	1	a	0	b	c
a	1	1	a	b	c
0	1	1	1	b	c
b	1	1	1	1	c
c	1	1	1	b	1

$*$	1	a	0	b	c
1	1	a	0	b	c
a	a	a	0	b	c
0	0	0	0	b	c
b	b	b	b	b	c
c	c	c	c	c	c

and define on A the following negation:

	$\neg$
1	0
a	a
0	1
b	1
c	1

Consider the structure $\mathbf{A} = \langle \mathbf{A}, \to, *, \neg, \top, \leq \rangle$. As a first fact we have that $\mathbf{A}$ is a *minimal* pocrim, but not a *bounded* pocrim: that is $\mathbf{A} \in \mathsf{MPO}$

but $\mathbf{A} \notin \mathsf{BPO}$, from which it follows that $\mathsf{BPO} \not\subseteq \mathsf{MPO}$, that is, **BPO** is a proper extension of **mPO** (and both are proper expansions of **PO**).

PROPOSITION 5.1. *The following hold in* $\mathbf{A}$*:*

(1) $\mathbf{A} \not\models \mathbf{x} \to (\neg\mathbf{x} \to \mathbf{y}) \approx \top$*;*

(2) $\mathbf{A} \not\models \mathbf{x} * \neg\mathbf{x} \approx \mathbf{y} * \neg\mathbf{y}$*;*

(3) $\mathcal{F}i_{\mathbf{A}}(\bot) = \{x \mid \exists n < \omega : \bot^n \to x = \bot \to_n x = \top\} \neq A$.

PROOF. (1) $0 \to (\neg 0 \to b) = 0 \to (1 \to b) = 0 \to b = b \neq 1$. We then have $\mathsf{MPO} \not\models x \to (\neg x \to y) \approx \top$ and $\mathsf{MPO} \not\models (x * \neg x) \to y \approx \top$.
(2) $b = b * \neg b \neq c * \neg c = c$. Observe that $b * \neg b \not\leq 0 * \neg 0 = \bot$.
(3) We have $\mathcal{F}i_{\mathbf{A}}(\{\bot\}) = \{0, a, 1\} \neq A$, $\mathcal{F}i_{\mathbf{A}}(\{b\}) = A \backslash \{c\}$ and $\mathcal{F}i_{\mathbf{A}}(\{c\}) = A \setminus \{b\}$.

From Proposition 2.3, Theorem 4.3 and Proposition 5.1, we finally get:

THEOREM 5.2. ***mPO** is a logic with minimal negation, but it is not a logic with scotian negation, i.e.*
(1) $\vdash_{\mathbf{mPO}} \neg\varphi \leftrightarrow \varphi \to \bot$*;*
(2) $\not\vdash_{\mathbf{mPO}} \varphi \to (\neg\varphi \to \psi)$*.*
*Moreover, **mPO** is paraconsistent in the following sense:*
(3) $Th_{\mathbf{mPO}}(\bot) \neq Fm_{\mathcal{L}}$*;*
(4) there exist $\varphi, \psi \in Fm_{\mathcal{L}}$ *s.t.* $\not\vdash_{\mathbf{mPO}} \varphi * \neg\varphi \leftrightarrow \psi * \neg\psi$*.*

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References

[1] R. J. Adillon, *Contribució a l'estudi de les lógiques proposicionals intuïcionistes i de Gödel sense contracció, i dels seus fragments*, Ph.D. Thesis, University of Barcelona, Barcelona, 2001.

[2] R. Adillon and V. Verdú, *A Gentzen system equivalent to BCK-logic*, **Bullettin of the Section of Logic** 25 (1996), pp. 73–79.

[3] R. Adillon and V. Verdú, *On a contraction-less intuitionistic propositional logic with conjunction and fusion*, **Studia Logica** 65 (2000), pp. 11–30.

[4] W. Blok and D. Pigozzi, *Algebraizable logics*, **Memoirs of the AMS** 396 (1989), Providence (U.S.A.).

[5] W. Blok and J. Raftery, *Varieties of commutative residuated integral pomonoids and their residuation subreducts*, **Journal of Algebra** 190 (1997), pp. 280–328.

[6] Casari, Ettore, **Introduzione alla logica**, UTET, Torino, 1997.

[7] F. García Olmedo and A. J. Rodriguez, *Negation and BCK-algebras*, **Math. Log. Quart.** 49 (2003), pp. 336–346.

[8] G. Gentzen, *The collected papers of Gerhard Gentzen*(ed. by M.E. Szabo), **Studies in Logic and the Foundations of Mathematics**, North Holland, Amsterdam, 1969.

[9] D. Higgs, *Dually residuated commutative monoids with identity element as least element do not form an equational class*, **Mathematica Japonica** 29 (1984), pp. 69–75.

[10] K. Iseki, *On BCK-algebras with condition (S)*, **Mathematica Japonica** 24 (1980), pp. 625–626.

[11] S. Jaskowski, *Propositional calculus for contradictory deductive systems*, **Studia Logica** 24 (1969), pp. 143–157.

[12] I. Johansson, *Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus*, **Compositio Mathematica** 4 (1936), pp. 119–136.

[13] A. Kolmogorov, *O principe tertium non datur*, **Matematiceskii Sbornik** 32 (1925), pp. 646–667.

[14] A. Kolmogorov, *Zur Deutung der intuitionistischen Logik*, **Matthematische Zeitschrift** 35 (1932), pp. 58–65.

[15] A. S. Troelstra and H. Schwichtenberg, **Basic Proof Theory**, Cambridge University Press, 1996.

[16] J. Rebagliato and V. Verdú, *On the algebraization of some Gentzen systems*, **Fundamenta Informaticae** 18 (1993), pp. 319–338.

[17] J. Rebagliato and V. Verdú, *Algebraizable Gentzen systems and the deduction theorem for Gentzen systems*, **Mathematics Preprints Series** 175 (1995), University of Barcelona, Barcelona (Spain).

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