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Doctor of Philosophy Dissertation

Pricing and managing life insurance risks

by

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to my family

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Thesis Outline

The aim of this thesis is to investigate about the quantitative models used for pricing and managing life insurance risks. It was done analyzing the existing literature about methods and models used in the insurance field in order to developing (1) new stochastic models for longevity and mortality risks and (2) new pricing functions for life insurance policies and options embedded in such contracts. The motivations for this research are to be searched essentially in:

- a new IAS/IFRS fair value-based accounting for insurance contracts (to be approval, probably in 2013/2014),
- a new risk-based solvency framework for the insurance industry, so-called Solvency II, that will becomes effective in 2013/2014,
- more rigorous quantitative analysis required by the industry in pricing and risk management of insurance risks.

The first part of the thesis (first and second chapters) contains a review of the quantitative models used for interest rates and longevity and mortality modeling. The second part (remaining chapters) describes new methods and quantitative models that it thinks could be useful in the context of pricing and insurance risk management. The organization of the thesis is detailed in the following.

Chapter 1: Interest rates modeling: a review. The chapter reviews some basic concepts and definitions related to interest rates and, briefly, the standard market models for pricing bond and derivatives are explained. Moreover, a review of the main short-rate models (both one- and two-dimensional) is provided.

Chapter 2: Longevity and mortality modeling: a review. The chapter contains a review of the literature related to longevity and mortality models. The chapter is devoted to review traditional and well-established models in longevity and mortality analysis. However, recent developments in the longevity and mortality modeling are considered taking into account contributes coming from the

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industry.

Chapter 3: A new stochastic model for estimating longevity and mortality risks. A new stochastic model for estimating longevity and mortality risks is presented. The model is able to derive the dynamic of the entire term structure of mortality rates by means of a closed-formula. It is consistent with the Gompertz law and is characterized by a stochastic dynamic where two state variables follow an autoregressive stochastic process. The model can be employed to generate stochastic scenarios taking into account the long-term mortality trend observed in historical data and to evaluate longevity and mortality risks when an internal model is used as prescribed by the risk-based Solvency II requirements for European insurance companies. To assess the forecasting capability of the model, Italian population mortality data are used. A comparison with the Solvency II standard formula is performed in order to quantify the Solvency Capital Requirement for mortality and longevity risks.

Chapter 4: Intensity-based framework for longevity and mortality modeling. An intensity-based framework for longevity and mortality modeling is implemented where the use of affine stochastic models is emphasized.

Chapter 5: A new approach for pricing of life insurance policies. A new approach for pricing of insurance contract is presented. According to this approach, insurance contracts are view as a swap in which policyholders exchange cash flows (premiums vs. benefits) with an insurer as in an interest rate swap or credit default swap.

Chapter 6: Calibrating affine stochastic mortality models using term assurance premiums. The chapter is focused on the calibration of affine stochastic mortality models using term assurance premiums. Using a simple bootstrapping procedure, the term structure of mortality rates is derived from a stream of contract quotes with different maturities. This term structure is used to calibrate the parameters of affine stochastic mortality models where the survival probability is expressed in closed-form. The Vasicek, Cox-Ingersoll-Ross, and jump-extended Vasicek models are considered for fitting the survival probabilities term structure. An evaluation of the performance of these models is provided with respect to premiums of three Italian insurance companies.

Chapter 7: Pricing of extended coverage options embedded in life insurance policies. A new model to evaluate the extended coverage option embedded in life insurance contracts is developed. The extended coverage option gives to the policyholder the right to extend the term of the policy after the original maturity maintaining the contractual conditions as valid. This type of option is common in the European insurance market but literature's references related to the evaluation of such type of option are very poor. In this chapter, a pricing model for the extended coverage option embedded in endowment life policies is provided. We take into account interest rates and mortality rates as the main risk factors of the option. We provide an evaluation method in closed-form in which the well-known Black (1976) option pricing formula is used with the assumption that the premiums (single or periodic) of the endowment life insurance contract are lognormal martingales under an appropriate probability's measure. The proposed model could be useful under the new IAS/IFRS market-consistent accounting for insurance contracts and the risk-based Solvency II requirements for the European insurance market. In fact, with the new accounting and solvency regimes, insurers will have to identify all material contractual options embedded in life insurance policies.

Chapter 8: Market-consistent approach for with-profit life insurance contracts and embedded options: a closed formula for the Italian policies. In this chapter, we present a simplified approach in closed-form to provide the market-consistent value for the Italian with-profit life insurance policies. Furthermore, we are able to compute the value of the minimum guaranteed option and the future discretionary benefits as defined by the Solvency II requirements. Our approach could be used in order to quantify the technical provisions for Solvency II purposes. In addition, it could be useful under the new IAS/IFRS principles for insurance contracts. Assuming a specific stochastic dynamic for the segregated fund, we derive a closed formula, under the well-known Black and Scholes framework, such that the value of the contract and the related embedded options can be computed as a function of the effective asset allocation of the segregated fund. We describe the model calibration procedure and provide some numerical results for the Italian with-profit life policies.

Chapter 1

Interest rates modeling: a review

1.1 Cash account

We define C(t) to be the value of a cash account at time $t \ge 0$. We assume the cash account evolves according to the following differential equation

$$d\mathcal{C}(t) = r(t)\mathcal{C}(t)dt, \qquad \mathcal{C}(0) = 1,$$

where r(t) is the instantaneous risk-free interest rate. Consequently,

$$C(t) = \exp\left\{\int_0^t r(u)du\right\}.$$

The quantity,

$$\mathcal{D}(t,T) = \frac{\mathcal{C}(t)}{\mathcal{C}(T)} = \exp\bigg\{-\int_t^T r(u)du\bigg\},$$

is defined as *stochastic discount factor*. It is the value at time t of one unit of cash payable at time T > t.

1.2 Year fraction

We denote by $\tau(t, T)$ the time measure between t and T, which is usually referred to as year fraction between the dates t and T. The particular choice that is made to measure the time between two dates is known as *day-count convention*.

1.3 Day-count convention

In the interest rate market, the appropriate day-count convention is used in order to fix the time measure between two dates. The number of days between two dates includes the first date but not the second. In order to indicate a date d_i , we use the following notation

$$d_i = [day_i, month_i, year_i]_i$$

where day_i , $month_i$ and $year_i$ represent the day, the month and the year of the date d_i , respectively.

In order to quantify the year fraction, different conventions are used as specified in the following.

ACT/365

With this convention a year is 365 days long and the year fraction between two dates is the actual number of days between them divided by 365. Denoting by $d_2 - d_1$ the actual number of days between the two dates, the year fraction is

$$\tau(d_1, d_2) = \frac{d_2 - d_1}{365}.$$

ACT/360

In this case, a year is assumed to be 360 days long. The corresponding year fraction is

$$\tau(d_1, d_2) = \frac{d_2 - d_1}{360}.$$

30/360

With this convention, months are assumed 30 days long and years are assumed 360 days long. The corresponding year fraction is

$$\tau(d_1, d_2) = \frac{day_2 - day_1 + 30(month_2 - month_1) + 360(year_2 - year_1)}{360}$$

ACT/ACT

The convention ACT/ACT means that the accrued interest between two given dates is calculated using the exact number of calendar days between the two dates divided by the exact number of calendar days of the ongoing year.

Under this conventions, it counts the number of whole calendar years between two dates and adds the fractions of the year at the start and end of the period

$$\tau(d_1, d_2) = \frac{y_2 - d_1}{y_2 - y_1} + (n - 3) + \frac{d_2 - y_{n-1}}{y_n - y_{n-1}},$$

where y_i are years end dates such that

$$y_i = \begin{bmatrix} 31, 12, year_i \end{bmatrix},$$

and

$$y_1 \le d_1 \le y_2 < \dots < y_{n-1} \le d_2 \le y_n.$$

If each year is assumed to have 365 days, the expression simplifies to the case ACT/365.

1.4 Zero-coupon bond

A zero-coupon bond with start date in t and maturity in T is a financial security paying one unit of cash at a specified date T without intermediate payments. The price at time t < T is denoted by P(t,T).

There is a close relationship between the zero-coupon bond price P(t,T) and the stochastic discount factor $\mathcal{D}(t,T)$. If the rates are deterministic then \mathcal{D} is deterministic as well and necessarily $\mathcal{D}(t,T) = P(t,T)$. Under stochastic interest rates, P(t,T) corresponds to the expectation of $\mathcal{D}(t,T)$ under the risk-neutral probability measure.

Being \mathscr{R}_t the filtration generated by the term structure of interest rates up to time t, it follows that

$$P(t,T) = \mathbb{E}^{\mathcal{Q}}\left[\frac{\mathcal{C}(t)}{\mathcal{C}(T)}\middle|\mathscr{R}_t\right],$$

where $\mathbb{E}^{\mathcal{Q}}$ is the expectation under the risk-neutral measure denoted by $\mathcal{M}^{\mathcal{Q}}$.

1.5 Spot interest rates

1.5.1 Simply-compounded spot interest rate

The simply compounded spot interest rate prevailing at time t for the maturity T is denoted by $R_s(t,T)$. It is the constant rate at which an investment has to be made to produce an amount of one unit of cash at maturity, starting from P(t,T) units of cash at time t, when accruing occurs proportionally to the investment time. In formulas

$$R_s(t,T) = \frac{1}{\tau(t,T)} \frac{1 - P(t,T)}{P(t,T)} = \frac{1}{\tau(t,T)} \left[\frac{1}{P(t,T)} - 1 \right].$$

The zero-coupon bond price can be expressed in terms of $R_s(t,T)$ as

$$P(t,T) = \frac{1}{1 + \tau(t,T)R_s(t,T)}.$$

XIBOR rates

We refer to the interbank rates quoted in different currencies. Such rates are known as LIBOR rates (London InterBank Offer Rates). We indicate with XI-BOR the generic interbank rate refers to the currency X. For example the XIBOR

quoted in Euro is the EURIBOR. The market XIBOR rate for the period [t, T], that we denote by XIBOR(t, T), is a simply-compounded rate. Consequently,

$$R_s(t,T) = XIBOR(t,T).$$

1.5.2 Annually-compounded spot interest rate

The annually-compounded spot interest rate prevailing at time t for the maturity T is denoted by $R_a(t,T)$. It is the constant rate at which an investment has to be made to produce an amount of one unit of cash at maturity, starting from P(t,T) units of cash at time t, when reinvesting the obtained amounts once a year. In formulas

$$R_a(t,T) = \left(\frac{1}{P(t,T)}\right)^{\frac{1}{\tau(t,T)}} - 1.$$

The zero-coupon bond prices expressed in terms of annually-compounded rates is

$$P(t,T) = (1 + R_a(t,T))^{-\tau(t,T)}.$$

1.5.3 *k*-times-per-year compounded spot interest rate

The k-times-per-year compounded spot interest rate prevailing at time t for the maturity T is denoted by $R_k(t,T)$. It is the constant rate (referred to a one-year period) at which an investment has to be made to produce an amount of one unit of cash at maturity, starting from P(t,T) units of cash at time t, when reinvesting the obtained amounts k times a year. In formulas

$$R_k(t,T) = \left(\frac{k}{P(t,T)}\right)^{\frac{1}{k\tau(t,T)}} - k.$$

The bond prices expressed in terms of k-times-per-year compounded rate is

$$P(t,T) = \left(1 + \frac{R_k(t,T)}{k}\right)^{-k\tau(t,T)}.$$

1.5.4 Continuously-compounded spot interest rate

The continuously-compounded spot interest rate can be obtained as the limit of k-times-per-year compounded rates for the number k going to infinity. In fact, since that

$$\lim_{x \to \infty} \left(1 + \frac{a}{x} \right)^{xb} = e^{ab},$$

it follows that

$$\lim_{k \to \infty} \left(1 + \frac{R_k(t,T)}{k} \right)^{-k\tau(t,T)} = e^{-R_k(t,T)\tau(t,T)}$$

The continuously-compounded spot interest rate prevailing at time t for the maturity T is denoted by $R_c(t,T)$. It is the constant rate at which an investment of P(t,T) units of cash at time t accrues continuously to yield a unit amount of cash at maturity T.

Continuously-compounded rates, commonly defined through the limit relations above, are such that

$$R_c(t,T) = -\frac{\log P(t,T)}{\tau(t,T)}.$$

Consequently, the zero-coupon bond price in terms of the continuously compounded rate is,

$$P(t,T) = e^{-R_c(t,T)\tau(t,T)}.$$

1.5.5 The term structure of spot interest rates

A fundamental curve that can be obtained from the market data of interest rates is the *zero-coupon curve* at a given date t.

The zero-coupon curve at time t is the graph of the function

$$T \mapsto R_q(t,T), \qquad T > t.$$

where g = s, a, k, c represents the compounding type.

The zero-coupon curve is also known as term structure of interest rates.

The term *yield curve* is often used to denote several different curves deduced from the interest-rate-market quotes.

The sequence $R_g(t, T_1), R_g(t, T_2), ..., R_g(t, T_n)$, with respect to the vector of maturities, $T_1, T_2, ..., T_n$, represents the term structure of interest rates under the generic compounding g.

The zero-bond curve at time t can be also represented as

$$T \mapsto P(t,T), \qquad T > t.$$

which, because of the positivity of interest rates, is a *T*-decreasing function starting from P(t,t) = 1. Such a curve is also referred to as the *term structure of discount factors*.

The sequence $P(t,T_1), P(t,T_2), ..., P(t,T_n)$, with respect to the vector of maturities, $T_1, T_2, ..., T_n$, represents the term structure of discount factors.

It is worth to note that the term structure of discount factors does not depends by the compounding type.

Instantaneous spot interest rate (or short rate) 1.5.6

All previous definitions of spot interest rates are equivalent in infinitesimal time intervals.

The instantaneous spot interest rate, denoted by r(t), is obtainable as a limit of all the different rates defined above. In fact, for each t, we have that

$$r(t) = \lim_{T \to t^+} R_s(t, T),$$
$$r(t) = \lim_{T \to t^+} R_a(t, T),$$
$$r(t) = \lim_{T \to t^+} R_k(t, T),$$
$$r(t) = \lim_{T \to t^+} R_c(t, T).$$

The instantaneous spot interest rate is also known as *short rate*.

1.6 Forward interest rates

1.6.1 Forward zero-coupon bond price

The zero-coupon bond forward price in t related to an investment that start in T_1 and pays one unit of cash in T_2 , with $t < T_1 < T_2$, is

$$P(t, T_1, T_2) = \frac{P(t, T_2)}{P(t, T_1)},$$

where $t < T_1 < T_2$.

Simple-compounded forward interest rate 1.6.2

The simply-compounded forward interest rate prevailing at time t for the expire T_1 and maturity T_2 . It is denoted by $F_s(t, T_1, T_2)$ and defined by

$$F_s(t, T_1, T_2) = \frac{1}{\tau(T_1, T_2)} \left[\frac{P(t, T_1)}{P(t, T_2)} - 1 \right].$$

XIBOR rate

Since that the market XIBOR rate is a simply-compounded rates, the simple forward rate $F_s(t, T_1, T_2)$ may be viewed as an estimate of the future spot XIBOR rate. It is denoted by $XIBOR(T_1, T_2)$. Consequently, we assume that

$$F_s(t, T_1, T_2) = \mathbb{E}^{T_2} \bigg[XIBOR(T_1, T_2) \bigg| \mathscr{R}_t \bigg].$$

The expectation \mathbb{E}^{T_2} refers to the T_2 -forward risk-adjusted measure denoted by \mathcal{M}^{T_2} . In the T_2 -forward risk-adjusted measure, or more briefly T_2 -forward measure, the numeraire¹ is represented by the price of a zero-coupon bond maturing at time T_2 .

1.6.3 Instantaneous forward interest rate

The instantaneous forward interest rate prevailing at time t for the maturity $T_1 > t$ is denoted by $f(t, T_1)$ and is defined as

$$f(t, T_1) = \lim_{T_2 \to T_1^+} F(t, T_1, T_2) = -\frac{\delta \log P(t, T_1)}{\delta T_1}$$

We have also that,

$$P(t,T_1) = \exp\bigg\{-\int_t^t f(t,u)du\bigg\}.$$

1.7 Coupon Bond

A *coupon bond*, is a financial claim by which the issuer, or the borrower, is committed to paying back to the bondholder, or the lender, the cash amount borrowed, called principal, plus periodic interests calculated on this amount during a given period of time.

In this section, we treat the pricing of bonds without taking into account the credit risk that characterizes such financial instruments.

1.7.1 Fixed-rate bond

A fixed-rate bond is a bond security that bear fixed coupon rates. This contract ensures the payments at future time $T_1, T_2, ..., T_i, ..., T_n$.

In a fixed-rate bond with one unit of notional, interest rates are expressed as a percentage $K(t, T_n)$ of the notional. The cash flow payed at time T_i is $\tau(T_{i-1}, T_i)K(t, T_n)$ while the last one, payed at time T_n , includes the reimbursement of the notional value of the bond.

The present value of the bond at time t, with $t < T_1$, is

$$B_{fx}(t,T_n) = \sum_{i=1}^n P(t,T_i)\tau(T_{i-1},T_i)K(t,T_n) + P(t,T_n).$$

¹The numeraire represents the market price of a traded asset.

1.7.2 Floating-rate bond

A *floating-rate bond* is a bond security that bear floating coupon rates where the interest rate is a linear function of the *Libor* rate. Usually, the maturity of the rate is equal to the time between two generic interest payment dates.

In a floating-rate bond with one unit of notional, the cash flow payed at time T_i is equal to $\tau(T_{i-1}, T_i) XIBOR(T_{i-1}, T_i)$.

As in the case of the fixed-rate bond, even in the floating-rate bond the last cash flow includes the reimbursement of the notional value.

The present value of the floating-rate bond at time t, with $t < T_1$, is

$$B_{fl}(t,T_n) = \sum_{i=1}^n P(t,T_i)\tau(T_{i-1},T_i)F_s(t,T_{i-1},T_i) + P(t,T_n).$$

1.8 Interest rate swap

An interest rate swap (IRS) is a contract that exchanges interest rate payments between two differently indexed legs, of which one is usually fixed whereas the other one is floating. Interest rates payments are exchanged at times $T_1, T_2, ..., T_i, ..., T_n$. When the fixed leg is paid and the floating leg is received the interest rate swap is termed *payer* IRS and in the other case *receiver* IRS.

The fixed rate of an IRS is also called *swap rate* and, with respect to the period [t, T], it is denoted by $R_{sw}(t, T)$.

The present value of the fixed leg at time $t < T_1$ is

$$Leg_{fx}(t, T_n) = \sum_{i=1}^{n} P(t, T_i) \tau(T_{i-1}, T_i) R_{sw}(t, T_n),$$

while the present value of the floating leg is

$$Leg_{fl}(t, T_n) = \sum_{i=1}^n P(t, T_i) \tau(T_{i-1}, T_i) F_s(t, T_{i-1}, T_i).$$

Consequently, the present value of the payer IRS at time $tt < T_1$ is

$$IRS^{(p)}(t,T_n) = Leg_{fl}(t,T_n) - Leg_{fx}(t,T_n),$$

while the present value of a receiver IRS is

$$IRS^{(r)}(t,T_n) = Leg_{fx}(t,T_n) - Leg_{fl}(t,T_n).$$

If fixed-rate payments and floating-rate payments occur at the same dates and with the same year fractions, we can simplify the pricing formula for the payer IRS as follows

$$IRS^{(p)}(t,T_n) = \sum_{i=1}^n P(t,T_i)\tau(T_{i-1},T_i) \big[F_s(t,T_{i-1},T_i) - R_{sw}(t,T_n) \big].$$

In a similar manner, for the receiver IRS we have

$$IRS^{(r)}(t,T_n) = \sum_{i=1}^n P(t,T_i)\tau(T_{i-1},T_i) \left[R_{sw}(t,T_n) - F_s(t,T_{i-1},T_i) \right]$$

Being the interest rate swaps quoted by the market in term of the swap rate with the fair value of the contract assumed equal to zero, we set $R_{sw}(t, T_n) = K$. Consequently,

$$IRS^{(p)}(t,T_n) = \sum_{i=1}^n P(t,T_i)\tau(T_{i-1},T_i) \big[F_s(t,T_{i-1},T_i) - K \big].$$

and

$$IRS^{(r)}(t,T_n) = \sum_{i=1}^n P(t,T_i)\tau(T_{i-1},T_i) \left[K - F_s(t,T_{i-1},T_i) \right].$$

Moreover, a payer IRS can be thought also as the difference between a floatingrate bond and a fixed-rate one. Namely, an IRS can then be viewed as a contract for exchanging a fixed-rate bond for a floating-rate bond. Consequently, the payer IRS and the receiver IRS, respectively, can be defined also as follows

$$IRS^{(p)}(t,T_n) = B_{fl}(t,T_n) - B_{fx}(t,T_n),$$

and

$$IRS^{(r)}(t, T_n) = B_{fx}(t, T_n) - B_{fl}(t, T_n).$$

1.8.1 Spot swap rate

Interest rate swap are quoted by the market in terms of the spot swap rate $R_{sw}(t,T_n)$. Quotations at time t, with $t = T_0$, refer to IRS with present value equal to zero. Consequently, we have that

$$R_{sw}(t,T_n) = \frac{\sum_{i=1}^n P(t,T_i)\tau(T_{i-1},T_i)F_s(t,T_{i-1},T_i)}{\sum_{i=1}^n P(t,T_i)\tau(T_{i-1},T_i)}, \quad with \ t = T_0.$$

1.8.2 Forward swap rate

We may require the above IRS to be fair at time $t < T_0$, where T_0 is the issue date of the IRS and we look for the particular fixed rate such that the above contract value is zero. This defines a *forward swap rate*. Considering an IRS with present value at time $t < T_0$, issue date of the IRS at time $T_0 < T_1$, and cash flows payments at future times $T_1, T_2, ..., T_i, ..., T_n$. The forward swap rate is

$$F_{sw}(t, T_0, T_n) = \frac{\sum_{i=1}^n P(t, T_i) \tau(T_{i-1}, T_i) F_s(t, T_{i-1}, T_i)}{\sum_{i=1}^n P(t, T_i) \tau(T_{i-1}, T_i)}, \quad with \ t < T_0$$

1.9 Interest rate options

An *option* is a contract in which the seller of the option grants the buyer of the option the right to purchase from the seller the designated underlying at a specified price within a specified period of time. The seller grants this right to the buyer in exchange for a certain sum of money called the option price or option premium.

The price at which the instrument may be bought or sold is called the exercise or strike price. The date after which an option is void is called the expiration date. The market quotes *American option*, that may be exercised any time up to and including the expiration date, and *European option* that may be exercised only on the expiration date.

When an option seller grants the buyer the right to purchase the designated instrument, it is called a *call option*. When the option buyer has the right to sell the designated instrument to the seller, the option is called a *put option*. The buyer of any option is said to be long the option; the seller is said to be short the option.

In this section, we present the most popular over-the-counter (OTC) interest rate options (Zero-coupon bond option, Caplet/Floorlet, Cap/Floor, Swaption) and explain the standard market model for their valuation.

1.9.1 Black (1976) model

Practitioners use Black (1976) model to price and hedge standard fixed-income derivatives. This model, which is particularly tractable and simple to use, remains currently the reference for the market in terms of pricing and hedging. However, this model is based on a strong simplifying assumption related to a stationary interest rate. Another drawback is the absence of a model for explain the dynamic of the term structure.

1.9.2 Zero-coupon bond option

A zero-coupon bond option is an option with a zero-coupon bond as underlying. In this section, we consider European zero-coupon bond options with strike equal to K and maturity option in T_1 . The maturity of the zero-coupon bond is in $T_2 > T_1$.

The payoff of the zero-coupon bond call option, for one unit of notional, is

$$[P(T_1, T_2) - K]^+$$

Since that, the spot price at time T_1 is equal to the forward price

$$P(T_1, T_2) = P(T_1, T_1, T_2),$$

we can rewrite the option payoff as

$$[P(T_1, T_1, T_2) - K]^+.$$

Considering the price of a zero-coupon bond expiring in T_1 as numeraire, the value of the option is

$$ZCBC(t, T_1, T_2, K) = P(t, T_1)\mathbb{E}^{T_1} \left[\frac{\left(P(T_1, T_1, T_2) - K \right)^+}{P(T_1, T_1)} \right] = P(t, T_1)\mathbb{E}^{T_1} \left[\left(P(T_1, T_1, T_2) - K \right)^+ \right].$$

Moreover, we assume that under the T_1 -forward measure, denoted by \mathcal{M}^{T_1} , the forward price is a lognormal martingale

$$dP(t, T_1, T_2) = \sigma_P P(t, T_1, T_2) dW^{T_1}(t),$$

where σ_P is the volatility of the zero-coupon bond forward price and $dW^{T_1}(t)$ is a standard Brownian motion under \mathcal{M}^{T_1} .

Applying the Black (1976) model, the present value of the zero-coupon bond call option at time $t < T_1$ is

$$ZCBC(t, T_1, T_2, K) = P(t, T_1) \big[P(t, T_1, T_2) N(d_1) - KN(d_2) \big],$$

where

$$d_{1} = \frac{\log\left[\frac{P(t,T_{1},T_{2})}{K}\right] + \frac{1}{2}\sigma_{P}^{2}(T_{1}-t)}{\sigma_{P}\sqrt{(T_{1}-t)}},$$

and

$$d_2 = d_1 - \sigma_P \sqrt{(T_1 - t)}.$$

The term $N(\cdot)$ denotes the standard normal cumulative distribution function. Analogously, the price at time t of a zero-coupon bond put option, with one unit of notional, is

$$ZCBP(t, T_1, T_2, K) = P(t, T_1) [KN(-d_2) - P(t, T_1, T_2)N(-d_1)].$$

Moreover, by the put-call parity, the price at time t of a zero-coupon bond put option can be written as

$$ZCBP(t, T_1, T_2, K) = ZCBC(t, T_1, T_2, K) - P(t, T_2) + KP(t, T_1).$$

1.9.3 Caplet and floorlet

We define a *caplet* as a call option on the XIBOR rate. The most usual ones are the 3-month, 6-month and 1-year XIBOR rates.

Consider the value of a caplet at time t, with strike equal to K. The reference rate of the caplet is $XIBOR(T_1, T_2)$. The rate is fixed at time T_1 and is related to the period $[T_1, T_2]$. The maturity of the Caplet is at time T_2 . The payoff of the caplet, for one unit of notional, is

$$\tau(T_1, T_2) [XIBOR(T_1, T_2) - K]^+.$$

Instead of using the XIBOR rate, let us use as underlying variable the forward XIBOR rate $F_s(t, T_1, T_2)$.

Since that, the spot rate at time T_1 is equal to the forward rate

$$XIBOR(T_1, T_2) = F_s(T_1, T_1, T_2),$$

we can rewrite the option payoff as

$$[F_s(T_1, T_1, T_2) - K]^+.$$

Consequently, a caplet is a call option on the simple forward XIBOR rate. Considering the price of a zero-coupon bond expiring in T_2 as numeraire, the value of the caplet is

$$\begin{split} Caplet(t,T_1,T_2,K) &= \tau(T_1,T_2)P(t,T_2)\mathbb{E}^{T_2}\bigg[\frac{\big(F_s(T_1,T_1,T_2)-K\big)^+}{P(T_2,T_2)}\bigg] = \\ \tau(T_1,T_2)P(t,T_2)\mathbb{E}^{T_2}\bigg[\big(F_s(T_1,T_1,T_2)-K\big)^+\bigg]. \end{split}$$

Moreover, we assume that under the T_2 -forward measure the simple forward rate is a lognormal martingale

$$dF_s(t, T_1, T_2) = \sigma_{F_s} F_s(t, T_1, T_2) dW^{T_2}(t),$$

where σ_{F_s} is the volatility of the simple forward rate and $dW^{T_2}(t)$ is a standard Brownian motion under \mathcal{M}^{T_2} .

Applying the Black (1976) model, the present value of the Caplet at time t is

$$Caplet(t, T_1, T_2, K) = \tau(T_1, T_2) P(t, T_2) \big[F_s(t, T_1, T_2) N(d_1) - K N(d_2) \big],$$

where

$$d_1 = \frac{\log\left[\frac{F_s(t,T_1,T_2)}{K}\right] + \frac{1}{2}\sigma_{F_s}^2(T_1 - t)}{\sigma_{F_s}\sqrt{(T_1 - t)}},$$

and

$$d_2 = d_1 - \sigma_{F_s} \sqrt{(T_1 - t)}.$$

Analogously, the payoff of the floorlet is

$$\tau(T_1, T_2) [K - L(T_1, T_2)]^+,$$

and its present value is

$$Floorlet(t, T_1, T_2, K) = \tau(T_1, T_2) P(t, T_2) [KN(-d_2) - F_s(t, T_1, T_2)N(-d_1)].$$

It is important to note that there is equivalence of a caplet to a zero-coupon bond option. A caplet is like a zero-coupon bond put option with strike

$$K' = \frac{1}{1 + \tau(T_1, T_2)K},$$

and $1 + \tau(T_1, T_2)K$ unit of notional. Consequently, we have that

$$Caplet(t, T_1, T_2, K) = [1 + \tau(T_1, T_2)K]ZCBP(t, T, S, K'),$$

and

$$Floorlet(t, T_1, T_2, K) = [1 + \tau(T_1, T_2)K]ZCBC(t, T, S, K').$$

1.9.4 Cap and floor

A *cap* is a contract that corresponds to a sum of caplet where each exchange payment is executed only if it has positive value.

We consider a cap with strike K and maturity at time T_n where the reference rate reset at future times $T_0, T_1, ..., T_{i-1}, ..., T_{n-1}$. The cap price at time t will be equal to the sum of the prices of the single caplets

$$Cap(t, T_n, K) = \sum_{i=1}^{n} Caplet(t, T_{i-1}, T_i, K) = \sum_{i=1}^{n} \tau(T_{i-1}, T_i) P(t, T_i) [F_s(t, T_{i-1}, T_i) N(d_{1,i}) - KN(d_{2,i})].$$

where

$$d_{1,i} = \frac{\log\left[\frac{F_s(t,T_{i-1},T_i)}{K}\right] + \frac{1}{2}\bar{\sigma}_{F_s}^2(T_{i-1}-t)}{\bar{\sigma}_{F_s}\sqrt{(T_{i-1}-t)}},$$

and

$$d_{2,i} = d_{1,i} - \bar{\sigma}_{F_s} \sqrt{(T_{i-1} - t)}.$$

The quantity $\bar{\sigma}_{F_s}$ is the so-called *flat* or *par* volatility. The flat volatility is a kind of average volatility of the set of individual caplet volatilities. The OTC market quotes the implied volatility of caps and floors for different maturities. Analogously, the price at time t of a floor will be

 $Floor(t, T_n, K) = \sum_{i=1}^{n} Floorlet(t, T_{i-1}, T_i, K) = \sum_{i=1}^{n} \tau(T_{i-1}, T_i) P(t, T_i) [KN(-d_{2,i}) - F_s(t, T_{i-1}, T_i)N(-d_{1,i})].$

Since that the price of a caplet/floorlet can be written in terms of the price of a zero-coupon bond option, it holds that

$$Cap(t, T_n, K) = \sum_{i=1}^{n} \left[1 + \tau(T_{i-1}, T_i) K \right] ZCBP(t, T_{i-1}, T_i, K'_i), \quad (1.-18)$$

and

$$Floor(t, T_n, K) = \sum_{i=1}^{n} \left[1 + \tau(T_{i-1}, T_i) K \right] ZCBC(t, T_{i-1}, T_i, K_i'), \quad (1.-17)$$

where

$$K_{i}^{'} = \frac{1}{1 + \tau(T_{i-1}, T_{i})K}.$$
(1.-16)

1.9.5 Swaption

An european *swaption* is an option allowing the holder to enter some specified underlying interest rate swap contract on a specified date, which is the expiration date of the option.

There are two kinds of european swaption:

- the *receiver swaption* is an option that gives the buyer the right to receive the fixed leg of the swap,
- the *payer swaption* is an option that gives the buyer the right to pay the fixed leg of the swap.

Consider an europen swaption that give the right to enter a payer interest rate swap at time $T_0 > t$ such that fixed-rate payments and floating-rate payments occur at the same dates with the same year fractions. The time T_0 is the maturity of the swaption and it coincides with the issue date of the underlying. The interest rate swap has maturity at time T_n and the stream of cash flows is payed at the future dates $T_1, T_2, ..., T_i, ..., T_n$, with $T_0 < T_1$. In order to define the payoff of the swaption, we have to consider the value in $t < T_0$ of a forwad start interest rate swap

$$IRS^{(p)}(t, T_0, T_n) = \sum_{i=1}^n P(t, T_i) \tau(T_{i-1}, T_i) \big[F_s(t, T_{i-1}, T_i) - F_{Sw}(t, T_0, T_n) \big],$$

where $F_{Sw}(t, T_0, T_n)$ is the forward swap rate such that the value of the forward start interest rate swap is equal to zero at time t.

Since that a payer swaption is in-the-money when $IRS^{(p)}(T_0, T_0, T_n) > 0$, assuming $K = F_{Sw}(t, T_0, T_n)$, the payoff of the payer swaption can be formalized as follows

$$\left[IRS^{(p)}(T_0, T_0, T_n)\right]^+ = \left[\sum_{i=1}^n P(T_0, T_i)\tau(T_{i-1}, T_i)[F_s(T_0, T_{i-1}, T_i) - K]\right]^+.$$

Multiplying and dividing the simple forward rate of the above formula for

$$\sum_{i=1}^{n} P(T_0, T_i) \tau(T_{i-1}, T_i),$$

we have that

$$\left\{\sum_{i=1}^{n} P(T_0, T_i)\tau(T_{i-1}, T_i) \left[\frac{\sum_{i=1}^{n} P(T_0, T_i)\tau(T_{i-1}, T_i)F_s(T_0, T_{i-1}, T_i)}{\sum_{i=1}^{n} P(T_0, T_i)\tau(T_{i-1}, T_i)} - K\right]\right\}^+.$$

Consequently, the payoff of a payer swaption can be written as

$$\left[\sum_{i=1}^{n} P(T_0, T_i) \tau(T_{i-1}, T_i) \left[F_{Sw}(T_0, T_0, T_n) - K \right] \right]^+.$$

Finally, the payoff of a payer swaption with one unit of notional and strike K is

$$\sum_{i=1}^{n} P(T_0, T_i) \tau(T_{i-1}, T_i) \big[F_{Sw}(T_0, T_0, T_n) - K) \big]^+.$$

Following Jamshidian (1997) and considering as numeraire the quantity²

$$\sum_{i=1}^{n} P(t, T_i) \tau(T_{i-1}, T_i),$$

the value of a payer swaption is

$$Swpt^{(p)}(t, T_0, T_n, K) = \sum_{i=1}^{n} P(t, T_i) \tau(T_{i-1}, T_i) \mathbb{E}^{Sw} \bigg[\big(F_{Sw}(T_0, T_0, T_n) - K \big)^+ \bigg],$$

²This quantity is known as *annuity*.

where the expectation is computed under the so-called *swap measure* benoted by \mathcal{M}^{Sw} .

In addition, we assume that under the swap measure the forward swap rate is a lognormal martingale

$$dF_{Sw}(t, T_1, T_2) = \sigma_{F_{Sw}} F_{Sw}(t, T_1, T_2) dW^{Sw}(t),$$

where $\sigma_{F_{Sw}}$ is the instantaneous percentage volatility of the forward swap rate and $dW^{Sw}(t)$ is a standard Brownian motion under \mathcal{M}^{Sw} .

Applying the Black (1976) model, the present value at time t of a payer swaption, is

$$Swpt^{(p)}(t, T_0, T_n, K) = \sum_{i=1}^{n} P(t, T_i) \tau(T_{i-1}, T_i) \big[F_{Sw}(t, T_0, T_n) N(d_1) - K N(d_2) \big],$$

where

$$d_{1} = \frac{\log\left(\frac{F_{Sw}(t,T_{0},T_{n})}{K}\right) + \frac{1}{2}\sigma_{F_{Sw}}^{2}(T_{0}-t)}{\sigma_{F_{Sw}}\sqrt{(T_{0}-t)}},$$

and

$$d_2 = d_1 - \sigma_{F_{Sw}} \sqrt{(T_0 - t)}.$$

Analogously, the present value at time t of a receiver swaption is

$$Swpt^{(r)}(t, T_0, T_n, K) = \sum_{i=1}^{n} P(t, T_i) \tau(T_{i-1}, T_i) \big[KN(-d_2) - F_{Sw}(t, T_0, T_n)N(-d_1) \big].$$

1.10 One-factor affine interest rate models

In one-factor short rate models a single synthetic variable is modelled. This variable is the instantaneous short rate denoted by r(t). Such models are based on the assumption that by modelling the one dimensional instantaneous short rate, it is possible to deduce the current yield curve and its evolution.

Under stochastic interest rates, the price P(t,T) at time t of a risk-free zero coupon bond that pays one unit of cash at time T under the risk-neutral measure $\mathcal{M}^{\mathcal{Q}}$ is

$$P(t,T) = \mathbb{E}^{\mathcal{Q}}\left[exp\left\{-\int_{t}^{T} r(u)du\right\} \middle| \mathscr{R}_{t}\right],$$

where $\mathbb{E}^{\mathcal{Q}}$ is the expectation under the risk-neutral measure denoted by $\mathcal{M}^{\mathcal{Q}}$.

1.10.1 Merton model

Merton (1973) assumes the following equation for the short rate under the riskneutral measure

$$dr(t) = \theta dt + \sigma dW(t),$$

where θ defines the average direction that r(t) moves at time t and σ is the instantaneous standard deviation of the short rate.

In the Merton model, zero-coupon bonds can be valued analitically,

$$P(t,T) = G(t,T) \exp\left\{-H(t,T)r(t)\right\},\$$

where

$$G(t,T) = \exp\left\{\frac{\sigma^2}{6}(T-t)^3 - \frac{\theta}{2}(T-t)^2\right\},\,$$

and

$$H(t,T) = T - t.$$

The model is analitically tractable and it is easy to apply. The main disadvantage of the model is that it has not mean reversion.

1.10.2 Vasicek model

The first major development in modelling the interest rate using a one-factor model was done by Vasicek (1977). Vasicek assumed that the instantaneous spot rate evolves as an Ornstein-Uhlenbeck process with constant coefficients where the short rate r(t) is mean reverting, since the expected rate tends, for t going to infinity, to a long term average rate. In addition, it is able to price discount bonds analitically.

The short rate follows the stochastic differential equation

$$dr(t) = k(\theta - r(t))dt + \sigma dW(t), \qquad r(0) = r_0,$$

where r_0 , k, θ , and σ are positive constants and dW(t) represents a standard brownian motion.

In the Vasicek model, the short rate is normally distributed. The model implies that, for each time t, the rate r(t) can be negative with positive probability and the possibility of negative rates is indeed a major drawback of the Vasicek model. Using the Vasicek model, the price of a zero-coupon bond can be obtained by

$$P(t,T) = G(t,T) \exp\bigg\{-H(t,T)r(t)\bigg\},\$$

where

$$G(t,T) = \exp\left\{\left(\theta - \frac{\sigma^2}{2k^2}\right) \left[H(t,T) - T + t\right] - \frac{\sigma^2}{4k}H(t,T)^2\right\},\$$

and

$$H(t,T) = \frac{1}{k} \left[1 - e^{-k(T-t)} \right].$$

1.10.3 Cox-Ingersoll-Ross model

Another approach for modeling the short rate was developed by Cox, Ingersoll and Ross (1985). Assuming the dynamic of the Cox-Ingersoll-Ross model (CIR model hereafter), the short rate r(t) satisfies

$$dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t), \qquad r(0) = r_0,$$

where r_0 , k, θ , and σ are positive constants. The principal advantage of the CIR model over the Vasicek model is that the short rate is guaranteed to remain non-negative. The condition $2k\theta > \sigma^2$ has to be imposed to ensure that r(t) remains positive.

The process r(t) follows a non-central chi-squared distribution. We denote by $\chi^2(\cdot, v, \zeta)$ the non-central chi-squared distribution function with v degree of freedom and non-centrality parameter ζ . Consequently, the distribution of the short rate has tails that are fatter than in the gaussian case.

Under the CIR model, the price of a zero-coupon bond can be computed analytically and is given by

$$P(t,T) = G(t,T) \exp\bigg\{-H(t,T)r(t)\bigg\},\$$

where

$$G(t,T) = \left[\frac{2\gamma \exp\left[\frac{k+\gamma}{2}(T-t)\right]}{2\gamma + (k+\gamma)\{\exp\left[\gamma(T-t)\right] - 1\}}\right]^{\frac{2k\theta}{\sigma^2}},$$

$$H(t,T) = \frac{2\{\exp\left[\gamma(T-t)\right] - 1\}}{2\gamma + (k+\gamma)\{\exp\left[\gamma(T-t)\right] - 1\}},$$

and

$$\gamma = \sqrt{k^2 + 2\sigma^2}.$$

1.10.4 Jump-extended Vasicek model

The Vasicek model can be generalized adding a jump shock component to the standard diffusion shock. The model is rich enough to capture possible asymmetries in the size and the probability of positive and negative shocks. This makes it particularly suitable to fit higher-order aspects of the distribution of interest rates, such as skewness and kurtosis.

The dynamic of the short rate follows the stochastic differential equation

$$dr(t) = k(\theta - r(t))dt + \sigma dW(t) + dJ(t),$$

where J is a pure jump process such that

$$J(t) = \sum_{M(t)}^{i=1} Y_i,$$

with M that represents a time-homogeneous Poisson process with intensity $\lambda > 0$ and Y_i being exponentially distributed with parameter $\eta > 0$.

We present explicit closed formula for the affine jump diffusion model along with the solutions found in Duffie and Garleanu (2001) and Christensen (2002) whose results are summarized in Lando (2004).

The price of a zero-coupon bond can be computed analytically by the following formula

$$P(t,T) = J(t,T)G(t,T)\exp\bigg\{-H(t,T)r(t)\bigg\},\$$

where

$$J(t,T) = \exp\left\{-\lambda(T-t)\right\} \left[1 + \frac{1}{k}(k+\eta)\left(e^{k(T-t)} - 1\right)\right]^{\frac{\lambda}{k+\eta}},$$
$$G(t,T) = \exp\left\{\left(\theta - \frac{\sigma^2}{2k^2}\right)\left[H(t,T) - T + t\right] - \frac{\sigma^2}{4k}H(t,T)^2\right\},$$

and

$$H(t,T) = \frac{1}{k} \left[1 - e^{-k(T-t)} \right].$$

1.10.5 Jump-extended Cox-Ingersoll-Ross model

As in the case of Vasicek model, the Cox-Ingersoll-Ross model can be generalized adding a jump shock component to the standard diffusion shock. The dynamics of r(t) satisfy,

$$dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t) + dJ(t),$$

where J is a pure jump process. As in the Jump-extended Vasicek, one assume that

$$J(t) = \sum_{M(t)}^{i=1} Y_i,$$

where M is a time-homogeneous Poisson process with intensity $\lambda > 0$ and Y_i being exponentially distributed with parameter $\eta > 0$.

A jump extensions of the CIR model with analytical closed-form solution for the price of the zero-coupon bond was proposed Duffie and Garleanu (2001) and Christensen (2002). The main results are summarized in Lando (2004).

The price of a zero-coupon bond can be computed analytically by the following formula

$$P(t,T) = J(t,T)G(t,T)\exp\bigg\{-H(t,T)r(t)\bigg\},\$$

where

$$J(t,T) = \exp\left\{\frac{2\lambda\eta}{\gamma - k - 2\eta}(T-t)\right\} \left[1 + \frac{(\gamma + k + 2\eta)\left(e^{\gamma(T-t)} - 1\right)}{2\gamma}\right]^{\frac{-2\lambda\eta}{\sigma^2 - 2\eta k - 2\eta^2}},$$
$$G(t,T) = \left[\frac{2\gamma \exp\left[\frac{k+\gamma}{2}(T-t)\right]}{2\gamma + (k+\gamma)\{\exp\left[\gamma(T-t)\right] - 1\}}\right]^{\frac{2k\theta}{\sigma^2}},$$
$$H(t,T) = \frac{2\{\exp\left[\gamma(T-t)\right] - 1\}}{2\gamma + (k+\gamma)\{\exp\left[\gamma(T-t)\right] - 1\}},$$

and

$$\gamma = \sqrt{k^2 + 2\sigma^2}.$$

Double-jump-extended Vasicek model 1.10.6

The jump-extended Vasicek model, proposed by Chacko and Das (2002) allows separate distributions for the upward jumps and downward jumps. This can significantly reduce the probability of negative interest rates by allowing more flexibility in estimating parameters for different interest rate regimes.

It implies the following process to model the short rate

$$dr(t) = k(\theta - r(t))dt + \sigma dW(t) + J_u dN_u(\lambda_u) - J_d dN_d(\lambda_d), \qquad r(0) = r_0,$$

where r_0 , k, θ , and σ are positive constants. The up-jump variable J_u and the down-jump variable J_d are exponentially distributed random variables with parameters η_u and η_d , respectively. The two Poisson parameters $dN_u(\lambda_u)$ and
$dN_d(\lambda_d)$ are distributed, independently, with intensities λ_u and λ_d . Also for this model, the price of a zero-coupon bond can be computed analytically

$$P(t,T) = G(t,T) \exp\bigg\{-H(t,T)[r(t)-\theta]\bigg\},\$$

where

$$G(t,T) = \exp\left\{\left[(T-t) - H(t,T)\right]\left(\frac{\sigma^2}{2k^2}\right) - \frac{\sigma^2 H(t,T)^2}{4k} + \left.-(\lambda_u + \lambda_d)(T-t) + \frac{\lambda_u \eta_u}{k\eta_u + 1}\log\left|\left(1 + \frac{1}{k\eta_u}\right)e^{k(T-t)} - \frac{1}{k\eta_u}\right| + \frac{\lambda_d \eta_d}{k\eta_d - 1}\log\left|\left(1 - \frac{1}{k\eta_d}\right)e^{k(T-t)} + \frac{1}{k\eta_d}\right| - \theta(T-t)\right\},$$

and

$$H(t,T) = \frac{1}{k} \left[1 - e^{-k(T-t)} \right].$$

The above solution is identical to that given by Chacko and Das (2002), though expressed in a slightly different form.³

1.10.7 Ho-Lee model

Ho and Lee (1986) proposed the first no-arbitrage model of the term structure of interest rates. They presented the model in the form of a binomial tree of bond prices. However, it is possible to show that the continuous-time limit of the model is

$$dr(t) = \theta(t)dt + \sigma dW(t),$$

where σ (positive constant) is the instantaneous standard deviation of the short rate and $\theta(t)$ is a function of time chosen so as to exactly fit the term structure of interest rates being currently observed in the market. The variable $\theta(t)$ can be evaluated analitically

$$\theta(t) = f^M(0,t) + \sigma^2 t,$$

where $f^M(0,t)$ is the market instantaneous forward rate. We can define the short rate also as

$$r(t) = \varphi(t) + x(t),$$

where x(t) follows a stochastic differential equation as

$$dx(t) = \sigma dW(t), \qquad x(0) = 0,$$

³See Nawalkha, Beliaeva, and Soto (2007).

and the deterministic function $\varphi(t)$ is such that

$$\varphi(t) = f^M(0,t) + \frac{\sigma^2 t^2}{2}.$$

In the Ho-Lee model, zero-coupon bonds and European option on zero-coupon bonds can be avalued analitically.

The expression for the price of a zero-coupon bond at time t with payoff in T is

$$P(t,T) = G(t,T) \exp\left\{-H(t,T)r(t)\right\},\,$$

where

$$G(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\left\{f^M(0,t)(T-t) - \frac{\sigma^2 t}{2}(T-t)^2\right\},\$$

and

$$H(t,T) = T - t.$$

The quantity $P(0,T)^M$ is the market price in t of a zero-coupon bond with maturity in T.

The price at time t of a European call option, with maturity T_1 written on a zero-coupon bond maturing at time T_2 with strike K is

$$ZCBC(t, T_1, T_2, K) = P(t, T_1) [P(t, T_1, T_2)N(d_1) - KN(d_2)],$$

where

$$d_1 = \frac{\log\left[\frac{P(t,T_1,T_2)}{K}\right] + \frac{1}{2}\Sigma_P^2(t,T_1,T_2)}{\Sigma_P(t,T_1,T_2)},$$

and

$$d_2 = d_1 - \Sigma_P(t, T_1, T_2).$$

The quantity $\Sigma_P(t, T_1, T_2)$ is such that

$$\Sigma_P(t, T_1, T_2) = \sigma(T_2 - T_1)\sqrt{T_1 - t}.$$

Similarly, the price at time t of a European put option is

$$ZCBP(t, T_1, T_2, K) = P(t, T_1) [KN(-d_2) - P(t, T_1, T_2)N(-d_1)].$$

The Ho-Lee model is an analitically tractable no-arbitrage model. It is easy to apply and provides an exact fit to the current term structure of interest rates. One disadvantage of the model is that it has no mean reversion.

1.10.8 Hull-White model

A drawback of the Vasicek model is that it assumes that the dynamics of the short rate depend on constant, unobservable parameters. This model produces an endogenous term structure of interest rates that will not necessarily match the current term structure.

Hull and White (1990) have proposed an extension of the Vasicek model in which, by introduction of a time-varying parameter, the model is able to provide an exact fit to the currently-observed yield curve. This model is also known as the Hull-White extended Vasicek model. In the original version of the Hull-White model, also the term structure of spot or forward-rate volatilities can be fitted exactly but the perfect fitting to a volatility term structure can be rather dangerous and must be carefully dealt with. For this reason, we concentrate on the extension of the Vasicek model where k and σ are positive constants and θ is chosen so as to exactly fit the term structure of interest rates.

The Hull-White model assumes that the instantaneous short-rate process evolves according to the following stochastic differential equation,

$$dr(t) = [\theta(t) - kr(t)]dt + \sigma dW(t),$$

where k and σ are positive constants and $\theta(t)$ is a function of time chosen so as to exactly fit the term structure of interest rates being currently observed in the market. The variable $\theta(t)$ can be evaluated analitically

$$\theta(t) = \frac{\delta f^M(0,t)}{\delta T} + k f^M(0,t) + \frac{\sigma^2}{2k} \left(1 - e^{-2kt}\right). \label{eq:theta}$$

We can define the short rate also as

$$r(t) = \varphi(t) + x(t),$$

where x(t) follows a stochastic differential equation as

$$dx(t) = -kx(t)dt + \sigma dW(t), \qquad x(0) = 0,$$

and the deterministic function $\varphi(t)$ is such that

$$\varphi(t) = f^M(0,t) + \frac{\sigma^2}{2k^2} (1 - e^{-kt})^2.$$

In Hull-White model, zero-coupon bonds can be evalued analitically. The price at time t of a zero-coupon bond with payoff in T is given by

$$P(t,T) = G(t,T) \exp\bigg\{-H(t,T)r(t)\bigg\},\$$

where

$$G(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\left\{H(t,T)f^M(0,t) - \frac{\sigma^2}{4k} \left(1 - e^{-2kt}\right)H(t,T)^2\right\},\$$

and

$$H(t,T) = \frac{1}{k} \left[1 - e^{-k(T-t)} \right].$$

The price at time t of a European call option with strike K, maturity at time T_1 , written on a pure discount bond maturing at time T_2 is

$$ZCBC(t, T_1, T_2, K) = P(t, T_1) \big[P(t, T_1, T_2) N(d_1) - KN(d_2) \big],$$

where

$$d_1 = \frac{\log\left[\frac{P(t,T_1,T_2)}{K}\right] + \frac{1}{2}\Sigma_P^2(t,T_1,T_2)}{\Sigma_P(t,T_1,T_2)},$$

and

$$d_2 = d_1 - \Sigma_P(t, T_1, T_2).$$

The quantity $\Sigma_P^2(t, T_1, T_2)$ is such that

$$\Sigma_P^2(t, T_1, T_2) = \sigma^2 \frac{1 - e^{-2k(T_1 - t)}}{2k} \frac{1}{k^2} \left[1 - e^{-k(T_2 - T_1)} \right]^2,$$

Analogously, the price at time t of a European put option is

$$ZCBP(t, T_1, T_2, K) = P(t, T_1) [KN(-d_2) - P(t, T_1, T_2)N(-d_1)].$$

Using the pricing formula of the European zero-coupon bond option it is possible to derive the pricing formula for caplets. The value at time t of a caplet that resets at time T_{i-1} , pays at time T_i , and has strike K is,

$$Caplet(t, T_{i-1}, T_i, K) = [1 + K\tau(T_{i-1}, T_i)]ZCBP(t, T_{i-1}, T_i, K'_i),$$

with

$$K_i' = \frac{1}{1 + K\tau(T_{i-1}, T_i)}.$$

Analogously, the value at time t of a floorlet that resets at time T_{i-1} , pays at time T_i and has strike K is

 $Floorlet(t, T_{i-1}, T_i, K) = [1 + K\tau(T_{i-1}, T_i)]ZCBC(t, T_{i-1}, T_i, K'_i).$

Consequently, the cap value is

$$Cap(t, T_n, K) = \sum_{i=1}^{n} \left[1 + K\tau(T_{i-1}, T_i) \right] ZCBP(t, T_{i-1}, T_i, K'_i),$$

and the price of the corresponding floor is

$$Floor(t, T_n, K) = \sum_{i=1}^{n} \left[1 + K\tau(T_{i-1}, T_i) \right] ZCBC(t, T_{i-1}, T_i, K'_i).$$

1.10.9 Shift-extended Cox-Ingersoll-Ross model (CIR++)

In this subsection, we present an extension of the Cox-Ingersoll-Ross (1985) model referred to as CIR++. Brigo and Mercurio (2006) have proposed a simple method to extend a time-homogeneous short-rate model, so as to exactly reproduce any observed term structure of interest rates while preserving the possible analytical tractability of the original model. In the case of the Vasicek (1977) model, the extension is perfectly equivalent to that of Hull and White (1990).

The model assumes the short rate as a sum of a state variable x(t) and another variable $\varphi(t)$ which is a deterministic function of time. The short rate is

$$r(t) = \varphi(t) + x(t),$$

where x(t) follows a stochastic differential equation as

$$dx(t) = k(\theta - x(t))dt + \sigma\sqrt{(x(t))}dW_t, \qquad x(0) = x_0,$$

with x_0 , k, θ , and σ positive constants such that $2k\theta > \sigma^2$. The deterministic function $\varphi(t)$ is such that

$$\varphi(t) = f^M(0,t) - \frac{2k\theta(e^{\gamma t} - 1)}{2\gamma + (k+\gamma)(e^{\gamma t} - 1)} - x_0 \frac{4\gamma^2 e^{\gamma t}}{[2\gamma + (k+\gamma)(e^{\gamma t} - 1)]^2}$$

The price at time t of a zero-coupon bond with payoff in T is given by

$$P(t,T) = \frac{P^M(0,T)G(0,t)\exp\left[-H(0,t)x_0\right]}{P^M(0,t)G(0,T)\exp\left[-H(0,T)x_0\right]}G(t,T)e^{-x(t)}.$$

where

$$G(t,T) = \left[\frac{2\gamma \exp\left[\frac{k+\gamma}{2}(T-t)\right]}{2\gamma + (k+\gamma)\{\exp\left[\gamma(T-t)\right] - 1\}}\right]^{\frac{2k\theta}{\sigma^2}},$$

$$H(t,T) = \frac{2\{\exp\left[\gamma(T-t)\right] - 1\}}{2\gamma + (k+\gamma)\{\exp\left[\gamma(T-t)\right] - 1\}}$$

and

$$\gamma = \sqrt{k^2 + 2\sigma^2}.$$

The price at time t of a European call option with maturity $T_1 > t$ and strike price K on a zero-coupon bond maturing at $T_2 > T_1$ is

$$ZCBC(t, T_1, T_2, K) = P(t, T_1) \left[P(t, T_1, T_2) \chi^2(z_1, v, \zeta_1) - K \chi^2(z_2, v, \zeta_2) \right],$$

where $\chi^2(z_i, v, \zeta_i)$ denotes the noncentral chi-squared distribution function with v degrees of freedom and non-centrality parameter ζ_i , with i = 1, 2. In the case of CIR++ model, z_1, z_2, v, ζ_1 , and ζ_2 are such that,

$$\begin{split} z_1 &= \frac{2}{H(T_1, T_2)} \\ \times \bigg[\log \frac{G(T_1, T_2)}{K} - \log \frac{P^M(0, T_1) G(0, T_2) \exp \big[-H(0, T_2) x_0 \big]}{P^M(0, T_2) G(0, T_1) \exp \big[-H(0, T_1) x_0 \big]} \bigg] \big[\varrho + \varsigma + H(T_1, T_2) \big], \end{split}$$

$$\begin{split} z_2 &= \frac{2}{H(T_1, T_2)} \\ \times \bigg[\log \frac{G(T_1, T_2)}{K} - \log \frac{P^M(0, T_1) G(0, T_2) \exp \big[-H(0, T_2) x_0 \big]}{P^M(0, T_2) G(0, T_1) \exp \big[-H(0, T_1) x_0 \big]} \bigg] \big[\varrho + \varsigma \big], \end{split}$$

$$v = \frac{4k\theta}{\sigma^2},$$

$$\zeta_1 = \frac{2\varrho^2 x(t) e^{\gamma(T_1 - t)}}{\varrho + \varsigma + H(T_1, T_2)},$$

$$\zeta_2 = \frac{2\varrho^2 x(t) e^{\gamma(T_1 - t)}}{\varrho + \varsigma},$$

where

$$\varrho = \frac{2\gamma}{\sigma^2 \{ \exp\left[\gamma(T-t)\right] - 1 \}},$$

and

$$\varsigma = \frac{k + \gamma}{\sigma^2}.$$

By put call parity, the price at time t of a European put option is

$$ZCBP(t, T_1, T_2, K) = ZCBC(t, T_1, T_2, K) - P(t, T_2) + KP(t, T_1).$$

The value at time t of a caplet that resets at time T_1 , pays at time T_2 , and has strike K is

$$Caplet(t, T_1, T_2, K) = [1 + K\tau(T_1, T_2)]ZCBP(t, T_1, T_2, K'),$$

with

$$K' = \frac{1}{1 + K\tau(T_1, T_2)},$$

Analogously, the value at time t of a floorlet that resets at time T_1 , pays at time T_2 and has strike K is

$$Floorlet(t, T_1, T_2, K) = [1 + K\tau(T_1, T_2)]ZCBC(t, T_1, T_2, K').$$

Consequently, the value of the cap is,

$$Cap(t, T_n, K) = \sum_{i=1}^{n} [1 + K\tau(T_{i-1}, T_i)] ZCBP(t, T_{i-1}, T_i, K'_i),$$

where

$$K_i' = \frac{1}{1 + K\tau(T_{i-1}, T_i)},$$

and the price of the corresponding floor is

$$Floor(t, T_n, K) = \sum_{i=1}^{n} [1 + K\tau(t_{i-1}, t_i)] ZCBC(t, t_{i-1}, t_i, K'_i).$$

1.11 Two-Factor affine interest rate models

Using one-factor short rate models, the evolution of the whole curve is characterized by the evolution of the single quantity r(t) and the rates for all maturities in the curve are perfectly correlated.

Truly, interest rates are known to exhibit some decorrelation (i.e. non-perfect correlation), so that a more satisfactory model of curve evolution has to be found. Whenever the correlation plays a more relevant role, or when a higher precision is needed anyway, we need to move to a model allowing for more realistic correlation patterns. This can be achieved with multifactor models, and in particular with two-factor models.

If the short rate is obtained as a function of two driving diffusion components (typically a summation, leading to an additive model) the model is said to be two-factor. Two-factor models provide a more realistic correlation and volatility structures in the evolution of the interest-rate curve.

We move to analyze the major two-factor short-rate models.

1.11.1 Two-factor Gaussian model (G2)

The Two-factor Gaussian model has the following dynamic,

$$\begin{aligned} r(t) &= x(t) + y(t) \\ dx(t) &= k(\theta - x(t))dt + \sigma dW_x(t), \\ dy(t) &= h(\vartheta - y(t))dt + v dW_y(t), \\ dW_x(t)dW_x(t) &= \rho, \end{aligned}$$

where ρdt is the instantaneously-correlated sources of randomness.

Given the analytical tractability of the two-factor Gaussian model, it is possible to derive the closed formula for the price of a zero coupon bond also when $\rho \neq 0$. Let $X = m_X + \sigma_X N_X$ and $Y = m_Y + \sigma_Y N_Y$ be two random variables such that N_X and N_Y are two correlated standard Gaussian random variables with $[N_X, N_Y]$ jointly Gaussian vector with correlation ρ . Then,

$$\mathbb{E}\big[\exp(-X)\big] = \exp\bigg[-m_X + \frac{1}{2}\sigma_X^2\bigg].$$

Consequently, the quantity

$$X = \int_t^T x(u) + y(u)du,$$

is a Gaussian random variable with mean

$$m_X = (\theta + \vartheta)(T - t) - \left[\theta - x_0\right] \frac{1}{k} \left[1 - \exp\left[-k(T - t)\right]\right]$$
$$-\left[\vartheta - y_0\right] \frac{1}{h} \left[1 - \exp\left[-h(T - t)\right]\right],$$

and variance

$$\begin{split} \sigma_X^2 &= \left(\frac{\sigma}{k}\right)^2 \bigg\{ (T-t) - \frac{2}{k} \bigg[1 - \exp\left[-k(T-t)\right] \bigg] + \frac{1}{2k} \bigg[1 - \exp\left[-2k(T-t)\right] \bigg] \bigg\} \\ &+ \left(\frac{v}{h}\right)^2 \bigg\{ (T-t) - \frac{2}{h} \bigg[1 - \exp\left[-h(T-t)\right] \bigg] + \frac{1}{2h} \bigg[1 - \exp\left[-2h(T-t)\right] \bigg] \bigg\} \\ &+ \frac{2\rho\sigma v}{kh} \bigg\{ (T-t) - \frac{1}{k} \bigg[1 - \exp\left[-k(T-t)\right] \bigg] - \frac{1}{h} \bigg[1 - \exp\left[-h(T-t)\right] \bigg] \bigg\} \\ &+ \frac{1}{k+h} \bigg[1 - \exp\left[-(k+h)(T-t)\right] \bigg] \bigg\}. \end{split}$$

Consequently, adopting a two-factor Gaussian model, the price of a zero-coupon bond is such that

$$P(t,T) = \mathbb{E}\left[\exp\left(-\int_{t}^{T} x(u) + y(u)du\right)\right] = \exp\left[-m_{X} + \frac{1}{2}\sigma_{X}^{2}\middle|\mathscr{R}_{t}\right],$$

where $\mathbb{E}^{\mathcal{Q}}$ is the expectation under the risk-neutral measure.

1.11.2 Shift-extended two-factor Gaussian model (G2++)

Brigo and Mercurio (2006) explain the two additive factor Gaussian model, denoted by G2++, as a model of the instantaneous short rate process given by the sum of two correlated gaussian factors. The drawback of negative interest rates as a consequence of the normal distributional assumption still exists. However, the additional factor in the model explains the actual variability of market rates more precisely than models that only look at one source of randomness. As a result of the gaussian assumption, this model remains analytically tractable, therefore there exists a formula that one can use to price pure discount bonds. This characteristic of the model, allows one to use this model, together with a numerical procedure, to price options with any payoff.

The model is equivalent to the two-factor Hull-White model. However, the formulation described by Brigo and Mercurio is easier to implement in practice. The dynamic of the instantaneous process is,

$$r(t) = \varphi(t) + x(t) + y(t), \qquad r(0) = r_0,$$

$$dx(t) = -kx(t)dt + \sigma dW_x(t),$$

$$dy(t) = -hy(t)dt + vdW_y(t),$$

$$dW_x(t)dW_x(t) = \rho.$$

where r_0, k, h, σ , and v are positive constants and $-1 \le \rho \le 1$. The deterministic function $\varphi(t)$ is such that

$$\varphi(t) = f^{M}(0,t) + \frac{\sigma^{2}}{2k^{2}} (1 - e^{-kt})^{2} + \frac{v^{2}}{2h^{2}} (1 - e^{-ht})^{2} + \rho \frac{\sigma v}{kh} (1 - e^{-kt}) (1 - e^{-ht}).$$

The price at time t of a zero-coupon bond with payoff in T is given by

 $P_x(t,T) = G(t,T) \exp\left[-H_x(t,T)x(t) - H_y(t,T)y(t)\right],$

where

$$H_x(t,T) = \frac{1}{k} \left[1 - e^{-k(T-t)} \right],$$
$$H_y(t,T) = \frac{1}{h} \left[1 - e^{-h(T-t)} \right],$$

and

$$G(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\left\{\frac{1}{2} \left[V(t,T) - V(0,T) + V(0,t)\right]\right\}.$$

The quantity V(t,T) is such that

$$\begin{split} V(t,T) &= \frac{\sigma^2}{k^2} \bigg[T - t + \frac{2}{k} e^{-k(T-t)} - \frac{1}{2k} e^{-2k(T-t)} - \frac{3}{2k} \bigg] + \\ \frac{v^2}{h^2} \bigg[T - t + \frac{2}{h} e^{-h(T-t)} - \frac{1}{2h} e^{-2h(T-t)} - \frac{3}{2h} \bigg] + \\ 2\rho \frac{\sigma v}{kh} \bigg[T - t + \frac{e^{-k(T-t)} - 1}{k} + \frac{e^{-h(T-t)} - 1}{h} - \frac{e^{-(k+h)(T-t)} - 1}{k+h} \bigg]. \end{split}$$

The price at time t of a European call option with maturity T_1 and strike K, written on a zero-coupon bond with maturity T_2 is given by

$$ZCBC(t, T_1, T_2, K) = P(t, T_1) \big[P(t, T_1, T_2) N(d_1) - KN(d_2) \big],$$

where

$$d_1 = \frac{\log\left(\frac{P(t,T_2)}{KP(t,T_1)}\right) + \frac{1}{2}\Sigma_P^2(t,T_1,T_2)}{\Sigma_P(t,T_1,T_2)}$$

and

$$d_2 = d_1 - \Sigma_P(t, T_1, T_2).$$

The quantity $\Sigma_P^2(t, T_1, T_2)$ is such that

$$\begin{split} \Sigma_P^2(t, T_1, T_2) &= \frac{\sigma^2}{2k^3} \bigg[1 - e^{-k(T_2 - T_1)} \bigg]^2 \bigg[1 - e^{-2k(T_1 - t)} \bigg] + \\ \frac{v^2}{2h^3} \bigg[1 - e^{-h(T_2 - T_1)} \bigg]^2 \bigg[1 - e^{-2h(T_1 - t)} \bigg] + \\ 2\rho \frac{\sigma v}{kh(k+h)} \bigg[1 - e^{-k(T_2 - T_1)} \bigg] \bigg[1 - e^{-h(T_2 - T_1)} \bigg] \bigg[1 - e^{-(k+h)(T_1 - t)} \bigg] \end{split}$$

Similarly, the price at time t of a European put option is

$$ZCBP(t, T_1, T_2, K) = P(t, T_1) \big[KN(-d_2) - P(t, T_1, T_2)N(-d_1) \big].$$

Starting from the pricing formula of the European zero-coupon bond option it is possible to derive the pricing formula for caplets. The value at time t of a caplet that resets at time t_{i-1} , pays at time t_i , and has strike K is,

$$Caplet(t, t_{i-1}, t_i, K) = [1 + K\tau(t_{i-1}, t_i)]ZCBP(t, t_{i-1}, t_i, K'_i),$$

with

$$K_i' = \frac{1}{1 + K\tau(t_{i-1}, t_i)}.$$

Analogously, the value at time t of a floorlet that resets at time T_{i-1} , pays at time T_i and has strike K is

$$Floorlet(t, T_{i-1}, T_i, K) = [1 + K\tau(T_{i-1}, T_i)]ZCBC(t, T_{i-1}, T_i, K'_i).$$

Consequently, the value of the cap is

$$Cap(t, T_n, K) = \sum_{i=1}^{n} [1 + K\tau(t_{i-1}, t_i)] ZCBP(t, t_{i-1}, t_i, K'_i),$$

and the price of the corresponding floor is

$$Floor(t, T_n, K) = \sum_{i=1}^{n} [1 + K\tau(t_{i-1}, t_i)] ZCBC(t, t_{i-1}, t_i, K'_i).$$

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Chapter 2

Longevity and mortality modeling: a review

2.1 Introduction

Recently, stochastic mortality models have received increased attention among practitioners and academic researchers. The introduction of International Financial Reporting Standards (IFRS) market-consistent accounting and risk-based Solvency II requirements for the European insurance market has called for the integration of mortality risk analysis into stochastic valuation models.¹ Furthermore, the issuance of mortality/longevity-linked securities requires stochastic models to price financial instruments related to demographic risks.

Several proposals for modeling and forecasting mortality rates have been proffered. The leading statistical model of mortality forecasting in the literature is the one proposed by Lee and Carter (1992). The use of the Lee-Carter model or one similar to it was recommended by two U.S. Social Security Technical Advisory Panels. There is further support for this model in other countries.

However, an important body of literature regarding models that describe death arrival as the first jump time of a Poisson process with stochastic force of mortality have appeared since the turn of the century.² In these models, the same mathematical tools used in interest rate and credit risk modeling are applied. Milevsky and Promislow (2001) were the first to propose a stochastic force of mortality model. Since then, several other stochastic models have been pro-

¹See European Community (2009), European Commission (2010), and IFRS Foundation (2010) for further information.

²In actuarial science, force of mortality represents the instantaneous rate of mortality at a certain age measured on an annualized basis. It is identical in concept to the failure rate or hazard function.

posed.

In this chaper, a review of the most significant mortality models existing in literature is provided.

2.2 Mortality data: source and structure

2.2.1 Mortality data source

In order to perform mortality research one needs access to accurate and reliable data that cover a long enough period.

To analyze mortality trends on a national level is needed the number of individuals alive and deceased at all ages over a range of years. This information differs between several countries. Moreover, mortality differs between males and females.

To facilitate the investigation of human mortality, an international project, the *Human Mortality Database* (HMD), was initiated by the Department of Demography at the University of California Berkeley, USA. This project provides detailed mortality and population data which can be accessed online and may be used freely for research purposes. Currently the HMD provides information on 34 countries. For each country the HMD offers basic quantities in mortality studies: the deceased and survivors by sex, age, year of death, and birth cohort. Though the age range covered is the same in all countries (from age 0 to 110+), the range of years covered differs from country to country. The longest series is provided for Sweden (1751-2006), whereas other countries have data from the nineteenth century (Scandinavian countries, Belgium, England, France, Italy, Netherlands, New Zealand and Switzerland). For some European countries, Japan, Australia, Canada, Taiwan and the USA, the series of data first start in the twentieth century.³

2.2.2 Mortality data structure

Typically, mortality data are presented in the so-called *life tables* that include the following information:

- t : reference year,
- x : age,
- $l_x(t)$: number of survivors at exact age x with respect to the reference year t, assuming $l_0(t) = 100,000$,
- $d_x(t)$: number of deaths with respect to the reference year t between ages x and x + 1.

³Human Mortality Database (2008).

National mortality data are generally published on an annual basis and by individual year of age.

Data used for calculating population mortality can be presented in the form of *central rate of mortality*

$$m_x(t) = \frac{\# \text{ deaths during year t aged x last birthday}}{average \text{ population during year t aged x last birthday}}$$

The central rate of mortality reflects deaths per unit of exposure over an entire year, assuming that the population changes uniformly over the year.

Some authors choose to model m_x directly, while others choose to model the mortality rate as the underlying probability that an individual aged exactly x at time t will survive until time t + 1.

2.3 Relevant quantities

2.3.1 Probability of death

With respect to a reference year t and an individual aged x, a standard measure of mortality is the probability in t that an individual aged x dies before one year. We denote this probability by $D_x(t)$ and it is such that

$$D_x(t) = \frac{d_x(t)}{l_x(t)}.$$

A simple approximation for $D_x(t)$, assuming a uniform distribution of deaths over the year, is

$$D_x(t) \approx \frac{m_x(t)}{1 + \frac{1}{2}m_x(t)}.$$

2.3.2 Survival probability

Given the rate of mortality $D_x(t)$, it is possible to define the survival probability which reflects the probability that an individual aged x survives over one year. This probability is denoted by $S_x(t)$ and it is such that

$$S_x(t) = 1 - \frac{d_x(t)}{l_x(t)}.$$

Survival and death probability are such that

$$S_x(t) = 1 - D_x(t).$$

2.3.3 Survival function

We define X as the non negative and continuous random variable describing time from birth of an individual until death. The basic quantity to describe time-todeath distribution is the *survival function*, which is defined as the probability of an individual surviving beyond age x,

$$S_x = Prob(X > x).$$

The survival function is the complement of the cumulative distribution function, that is

$$D_x = Prob(X \le x).$$

Consequently,

$$S_x = 1 - D_x.$$

Moreover, the survival function is the integral of the probability density function, denoted by f(x), from x to infinity,

$$S_x = Prob(X > x) = \int_x^\infty f(t)dt.$$

2.3.4 Force of mortality

A fundamental quantity is the *force of mortality* denoted by μ_x (with respect to an individual aged x). It describes the instantaneous rate of death of an individual aged x that is alive until x. In formula

$$\mu_x = \lim_{\Delta x \to 0} = \frac{\operatorname{Prob}(x < X \le x + \Delta x | X > x)}{\Delta x} = \frac{f(x)}{S_x} = -\frac{d \log S_x}{dx}$$

In actuarial science, force of mortality represents the instantaneous rate of mortality at a certain age measured on an annualized basis. The concept is identical to the failure rate or hazard function.

The survival function is related to the force of mortality according to the following expression

$$S_x = \exp\bigg\{-\int_0^x f(u)du\bigg\}.$$

2.4 Mortality models over age

2.4.1 Gompertz law (1825)

The Gompertz law (1825) is commonly known as the most successful law to model the dying out process of living organisms. It is based on the biological concept of organism senescence, in which mortality rates increase exponentially with age. Gompertz observed that death rates increase exponentially with age. He suggested representing the hazard rate as

$$\mu_x = ae^{bx},$$

with parameters a > 0 and b > 0. Commonly, a represents the mortality at time zero and b is the rate of increase of mortality and is frequently used as a measure of the rate of aging. The probability density function for the Gompertz distribution is,

$$f(x) = ae^{bx} \exp\left[\frac{a}{b}\left(1 - e^{bx}\right)\right].$$

2.4.2 Makeman law (1860)

Makeham (1860) extended Gompertz equation by adding an age-independent term, c > 0, to account for risks of death that do not depend on age. This model is known also as Gompertz-Makeham model. In formula

$$\mu_x = c + ae^{bx}$$

In the Makeham model, the probability density function is equal to

$$f(x) = ae^{bx} \exp\left[-cx + \frac{a}{b}\left(1 - e^{bx}\right)\right].$$

2.4.3 Perks law (1932)

Perks (1932) was the first to proposed a logistic modification of the Gompertz-Makeham models. The logistic function proposed to model the late-life mortality deceleration is

$$\mu_x = c + \frac{ae^{bx}}{1 + \alpha e^{bx}}.$$

We can see that this includes Makeham law as the special case when $\alpha = 0$.

2.5 Mortality models over age and over time

2.5.1 Lee-Carter model (1992)

One of the seminal works on mortality modeling is the Lee-Carter model introduced by Lee and Carter (1992). The Lee-Carter model is the leading statistical model of mortality forecasting in the demographic literature.

The model emerged as the benchmark for forecasting mortality rates and its use

or one similar to it was recommended by two U.S. Social Security Technical Advisory Panels. Further support for this model has been proposed in other countries. In the Lee-Carter approach, the central rate of mortality is modelled as a two variable function. It is a one factor stochastic model where the mortality rate is a function of three parameters expressed in the form

$$\log[m_x(t)] = \beta_x^{(1)} + \beta_x^{(2)} k_t^{(2)}.$$

The state variable $k_t^{(2)}$ follows a one-dimensional random walk with drift

$$k_t^{(2)} = \mu + k_{t-1}^{(2)} + \sigma z_t^{(2)},$$

in which μ is a constant drift term, σ is a constant volatility and $z_t^{(2)}$ is a onedimensional i.i.d. standard gaussian error.

The coefficient $\beta_x^{(1)}$ is the drift term expressed as a function of a particular age group. This term describes the age-specific pattern of mortality. The coefficient $\beta_x^{(2)}$ is a function of the age group and describes the sensitivity of mortality rate, specified by $k_t^{(2)}$, to changes through time. The state variable $k_t^{(2)}$ describes the change in mortality rates over time without any differentiation between age groups.

Since that for this model there is an identifiability problem in parameter estimation, Cairns et al. (2007) suggested to impose two constraints to circumvent this problem,

$$\sum_{t} k_t^{(2)} = 0,$$
$$\sum_{x} \beta_x^{(2)} = 1.$$

The model is calibrated on historical data, namely population and number of deaths. The model is extremely easy to calibrate, given the limited number of parameters and their intuitive meaning.

2.5.2 Brouhns-Denuit-Vermunt model (2002)

Brouhns, Denuit and Vermunt (2002) improve the Lee-Carter approach embedding in the original method a Poisson regression model which is perfectly suited for age-sex-specific mortality rates.

They consider that the number of deaths recorded at age x during the year t, denoted by D_{xt} , has a Poisson distribution with parameter $E_{xt}\mu_x(t)$, where E_{xt} represents the exposure-to-risk (i.e., E_{xt} is the number of person years from which D_{xt} occurred) and $\mu_x(t)$ is the force of mortality. The force of mortality is assumed to have the log-bilinear form

$$\log[\mu_x(t)] = \beta_x^{(1)} + \beta_x^{(2)} k_t^{(2)},$$

where the parameters are essentially the same as in the classical Lee-Carter model.

2.5.3 Renshaw-Haberman model (2006)

The Lee-Carter extension model designed by Renshaw and Haberman (2006) is a generalized version of the Lee-Carter model. It allows for the modeling and extrapolation of age-specific cohort effects

$$\log[m_x(t)] = \beta_x^{(1)} + \beta_x^{(2)} k_t^{(2)} + \beta_x^{(3)} \gamma_{t-x}^{(3)}.$$

The state variable $k_t^{(2)}$ follows a one-dimensional random walk with drift

$$k_t^{(2)} = \mu + k_{t-1}^{(2)} + \sigma z_t^{(2)},$$

in which μ is a constant drift term, σ is a constant volatility and $z_t^{(2)}$ is a onedimensional i.i.d. standard gaussian error.

Following Dowd et al. (2008), the cohort effect $\gamma_{t-x}^{(3)}$ is modelled as an ARIMA(1,1,0) process indipendent of $k_t^{(2)}$

$$\Delta \gamma_{t-x}^{(3)} = \mu_{\gamma} + \alpha_{\gamma} (\Delta \gamma_{t-x-1}^{(3)} - \mu_{\gamma}) + \sigma_{\gamma} z_{t-x}^{(\gamma)}.$$

The quantity $\gamma_{t-x}^{(3)}$ is a random cohort effect expressed as a function of the year of birth (t-x) and $k_t^{(3)}$. The impact of this cohort effect can be varied by age through $\beta_x^{(3)}$.

This model has similar identifiability problems to the previous one. Also in this case Cairns et al. (2007) suggested to impose the following constraints

$$\sum_{t} k_{t}^{(2)} = 0,$$
$$\sum_{x} \beta_{x}^{(2)} = 1,$$
$$\sum_{x,t} \gamma_{t-x}^{(3)} = 0,$$
$$\sum_{x} \beta_{x}^{(3)} = 1.$$

2.5.4 Currie model (2006)

Currie (2006) proposed a simplified version and a special case of the Renshaw-Haberman model (2006) with $\beta_x^{(2)} = 1$ and $\beta_x^{(3)} = 1$.

In the Currie model, the age period and cohort effects influence mortality rates independently. The model can be expressed in the form

$$\log[m(t,x)] = \beta_x^{(1)} + k_t^{(2)} + \gamma_{t-x}^{(3)},$$

where the variables $k_t^{(2)}$ and $\gamma_{t-x}^{(3)}$ are defined as in the previous model. Currie (2006) uses P-splines to fit $\beta_x^{(1)}$, $k_t^{(2)}$ and $\gamma_{t-x}^{(3)}$ to ensure smoothness.

For this model, Cairns et al. (2007) have suggested the following constraints

$$\sum_{t} k_t^{(2)} = 0,$$
$$\sum_{x,t} \gamma_{t-x}^{(3)} = 0.$$

2.5.5 Cairns-Blake-Dowd model (CBD)

The Cairns, Blake, and Dowd model (2006a) differs from the previous stochastic models and assume a functional relationship between mortality rates across ages. It is fitted, directly, to initial mortality rates instead of central mortality rates. This model can be expressed as

logit
$$q(t,x) = k_t^{(1)} + k_t^{(2)}(x - \overline{x}),$$

where \overline{x} is the mean age in the sample range of ages with lenght n_a such that

$$\overline{x} = \frac{\sum_{i=1}^{n_a} x_i}{n_a}.$$

The state variables follow a two-dimensional random walk with drift

$$\begin{split} k_t^{(1)} &= \mu^{(1)} + k_{t-1}^{(1)} + \sigma^{(1)} z_t^{(1)}, \\ k_t^{(2)} &= \mu^{(2)} + k_{t-1}^{(2)} + \sigma^{(2)} z_t^{(2)}, \end{split}$$

where the parameters $\mu^{(1)}$ and $\mu^{(2)}$ are constant drift terms, $\sigma^{(1)}$ and $\sigma^{(2)}$ are constant volatilities while $z_t^{(1)}$ and $z_t^{(2)}$ are indipendent and i.i.d. standard gaussian errors.

It is important to note that this model has no identifiability problems.

2.5.6 A first generalisation of the Cairns-Blake-Dowd model (CBD1)

In Cairns et al. (2007) a first generalisation of the Cairns-Blake-Dowd model (CBD) including a cohort effect is presented. The functional form of the model is

$$logit \ q(t,x) = k_t^{(1)} + k_t^{(2)}(x - \overline{x}) + \gamma_{t-x}^{(3)},$$

where the variables \overline{x} , $k_t^{(1)}$, $k_t^{(2)}$ and $\gamma_{t-x}^{(3)}$ are defined as in the previous case. In this case there is a identifiability problems that can be solve according to the suggestion contains in Cairns et al. (2007).

2.5.7 A second generalisation of the Cairns-Blake-Dowd model (CBD2)

In Cairns et al. (2007) a second generalisation of the Cairns-Blake-Dowd model (CBD) adding a quadratic term into the age effect is showed. This model is able to take into account some curvature identified in the *logit* q(t, x) plots in the US data

$$logit \ q(t,x) = k_t^{(1)} + k_t^{(2)}(x - \overline{x}) + k_t^{(3)} \left[(x - \overline{x})^2 - \sigma_x^{(2)} \right] + \gamma_{t-x}^{(4)}.$$

The state variables $k_t^{(1)}$, $k_t^{(2)}$ and $k_t^{(3)}$ follow a three-dimensional random walk with drift, and $\gamma_{t-x}^{(4)}$ is a cohort effect that is modelled as an AR(1) process. The constant

$$\sigma_x^{(2)} = \frac{\sum_{i=1}^{n_a} (x - \overline{x})^2}{n_a},$$

is the mean of $(x - \overline{x})^2$.

2.5.8 A third generalisation of the Cairns-Blake-Dowd model (CBD3)

A further generalization of the Cairns-Blake-Dowd model (CBD) is reported in Cairns et al. (2007) and it is such that

logit
$$q(t,x) = k_t^{(1)} + k_t^{(2)}(x-\overline{x}) + \gamma_{t-x}^{(3)}(x_c-x),$$

for some constant parameter x_c to be estimated.

To avoid identifiability problems one constraint is introduced

$$\sum_{x,t} \gamma_{t-x}^{(3)} = 0$$

Also in this case, the variables $k_t^{(1)}$, $k_t^{(2)}$ and $\gamma_{t-x}^{(3)}$ are defined as in the previous models.

2.6 Discrete-time models

2.6.1 Lee-Young model

Lee (2000) and Yang (2001) proposed the following model for stochastic mortality in which the actual mortality experience is modelled as

$$D_x(t) = \hat{D}_x(t) \exp\left[X_t - \frac{1}{2}\sigma_Y^2 + \sigma_Y, Z_Y(t)\right],$$

where

• $D_x(t)$ is the actual probability of a life aged x at time t dying in year t+1,

• $\hat{D}_x(t)$ represents the probability (estimated) that an individual aged x at time t will die before time t + 1 for each integer x and t.

The quantity X_t is such that

$$X_{t} = X_{t-1} - \frac{1}{2}\sigma_{X}^{2} + \sigma_{X}Z_{X}(t),$$

where $Z_X(t)$ and $Z_Y(t)$ are i.i.d. standard normal variates.

2.6.2 Smith-Oliver model

The Smith (2005) and Oliver (2004) proposed a stochastic model for the long term trends in mortality. This model assumes that the estimates of future survival probabilities change on an annual basis where the change in estimates of survival probabilities is driven by a random variation factor which follows a Gamma distribution.

The model produces stochastic variation around the deterministic best estimates of mortality and can be formalized as follows

$$S_x(t+1,T,T+1) = S_x(t,T,T+1)^{det_x(t+1,T,T+1)},$$

where $S_x(t, T, T + 1)$ is the probability, based on information available at time t+1, that if the individual survives to time T he will then survive to time T+1. The model is driven by the deterioration factor, denoted by $det_x(t+1, T, T+1)$, such that

$$det_x(t+1, T, T+1) = b_x(t+1, T, T+1)G(t+1).$$

The quantity $b_x(t+1,T,T+1)$ is a bias correction factor and G(1), G(2), ... is a series of i.i.d. gamma random variables with both shape and scaling parameters equal to some constant α . Following Cairns (2007), it is assumed that there exists a probability measure $\mathcal{M}^{\mathcal{Q}}$ under which the prices of all assets discounted by the cash account are martingales. Hence $E^Q[G(t)] = 1$ and variance $Var^Q[G(t)] = \alpha^{-1}$.

The Smith-Oliver model provides us with an elegant approach to simulating stochastic mortality where no approximations are required.

There are, however, two potential drawbacks to the model. First, the model only accommodates a single source of randomness through G(t). In contrast, historical data suggests that more than one factor may be appropriate: specifically, changes in mortality rates at different ages are not perfectly correlated. Second, there is no flexibility in the way in which the volatility term structure is specified.

2.6.3 Generalisation of the Smith-Oliver model

Cairns (2007) proposes a generalisation of the Smith-Olivier model that moves away from dependence on a single source of risk and allows for full control over the variances and correlations

$$S_x(t+1,t,T) = S_x(t,t,T)^{det_x(t+1,T)}.$$

The deterioration factor $det_x(t+1,T)$ is calculated as

 $det_x(t+1,T) = g_x(t+1,T)G_x(t+1,T).$

For each x and for each T > t + 1, we have that

$$G_x(t+1,T) \sim Gamma[\alpha_x(t+1,T), \alpha_x(t+1,T)],$$

and

$$g_x(t+1,T) = -\frac{\alpha_x(t+1,T)[S_x(t,t,T)^{-1/\alpha_x(t+1,T)} - 1]}{\log[S_x(t,t,T)]}.$$

2.7 Continuous-time models

A fairly recent stream of academic literature models the force of mortality as a stochastic process.

In these models, the death arrival is modelled as the first jump time of a Poisson process with stochastic force of mortality where the same mathematical tools used in interest rate and credit risk modeling are applied.

Cairns, Blake, and Dowd (2006b) suggest that affine stochastic models need to incorporate non-mean reverting elements; Luciano and Vigna (2005) propose non-mean reverting affine processes for modeling the force of mortality. In the non-mean reverting models, the deterministic part of the mortality rate process increases exponentially in a manner that is consistent with the exponential growth that is the main feature of the Gompertz model.⁴

2.7.1 Milevsky-Promislow model

Milevsky and Promislow (2001) have used a stochastic force of mortality, whose expectation at any future date, under an appropriate choice of the parameters, has a Gompertz specification.

They investigate a so-called mean reverting Brownian Gompertz specification with the force of mortality $\mu_x(t)$ that is modelled by the following process

$$\begin{split} \mu_x(t) &= \mu_x(0) \exp(gt + \sigma Y_t), \\ dY_t &= -bY_t dt + bW_t, \end{split}$$

⁴The model is based on the Gompertz law (1825) founded on the biological concept of organism senescence, in which mortality rates increase exponentially with age.

where $g, \sigma, \mu_x(0)$ are positive constants, $Y_0 = 0$ and $b \ge 0$. Essentially, the model is equivalent to a Gompertz model with a time-varying scaling factor.

2.7.2 Dahl model

Dahl (2004) models the process for $\mu_x(t)$ as follows

$$d\mu_x(t) = \left[\delta^{\alpha}(t,x)\mu_x(t) + \zeta^{\alpha}(t,x)\right]dt + \sqrt{\delta^{\sigma}(t,x)\mu_x(t)\zeta^{\sigma}(t,x)}dW(t).$$

In Dahl model, the survival probability can be written as

$$S_x(t,T) = G(t,T) \exp\left[-H(t,T)\mu_x(t)\right],$$

where the deterministic functions G(t, T) and H(t, T) are derived from differential equations involving $\delta^{\alpha}(t, x)$, $\delta^{\sigma}(t, x)$, $\zeta^{\alpha}(t, x)$, and $\zeta^{\sigma}(t, x)$.

2.7.3 Biffis model

Biffis (2005) has extended the Dahl model adding a jump process into the SDE.

2.7.4 Luciano-Vigna model

Luciano and Vigna (2005) has proposed non-mean reverting affine stochastic mortality models.

They observes that the force of mortality extrapolated from the mortality tables does not seem to present a mean reverting behaviour, but rather an exponential one. This fact leads to the simple idea of dropping the mean reverting term in the classical affine processes used in finance and choosing processes whose deterministic part increases exponentially. Consequently, non-mean reverting processes are consistent with all the deterministic exponential models presented in the actuarial literature.

Four affine models with these characteristics are presented in the paper

- the Ornstein Uhlenbeck process without jumps,
- the Ornstein Uhlenbeck process with jumps,
- the Feller process without jumps,
- the Feller process with jumps.

2.7.5 Schrager model

Schrager (2004) has proposed the following general form for the force of mortality

$$\mu_x(t) = g_0(x) + \sum_{i=1}^M Y_i(t)g_i(x),$$

where g_i is a function with the positive half line as its range and Y(t) are factors driving the uncertainty in the mortality intensity.

The M-dimensional factor dynamics are given by the following diffusion

$$dY(t) = A(\theta - Y(t))dt + \Sigma \sqrt{V_t} dW_t,$$

$$Y(0) = \bar{Y}.$$

The matrices A and Σ are $M \times M$ matrices and θ is an M vector. The matrix V_t is a diagonal matrix holding the diffusion coefficients of the factors on the diagonal such that

$$V_t = diag[\alpha + \beta Y(t)],$$

where α and β are such that

$$\alpha = \left[\alpha_1, \alpha_2, ..., \alpha_M\right]',$$
$$\beta = \left[\beta_1, \beta_2, ..., \beta_M\right]'.$$

Considering the special case of the Gaussian Thiele Model, by choosing

$$g_0(x) = 0,$$

$$g_1(x) = \exp \left[-\tau_1(x) \right],$$

$$g_2(x) = \exp \left[-\tau_2(x-\eta)^2 \right],$$

$$g_3(x) = \exp \left[-\tau_3(x) \right],$$

it follows that

$$\mu_x(t) = Y_1(t) \exp\left[-\tau_1(x)\right] + Y_2(t) \exp\left[-\tau_2(x-\eta)^2\right] + Y_3(t) \exp\left[-\tau_3(x)\right].$$

2.8 Models from the industry

2.8.1 Barrie and Hibbert & Heriot-Watt University model

Barrie and Hibbert⁵ use a simple model for mortality uncertainty developed with researchers at Heriot-Watt University.⁶ The model is characterized by an adjustment applied to best-estimate mortality rates where

 $^{^5\}mathrm{Barrie}$ and Hibbert supplies software to insurance and financial companies to enable them to understand and estimate their risks.

 $^{^{6}}$ We refer to McCullock et al. (2005).

- the adjustment factor follows a random walk without drift,
- the model implies two sources of mortality uncertainty.

The model has the following dynamic

$$D_x(t) = \hat{D}_x(t) \exp\left[Y_t - \frac{\sigma_Y^2}{2} - t\frac{\sigma_X^2}{2}\right],$$

$$Y_t = X_t + \sigma_Y Z_{Y_t},$$

$$X_t = X_{t-1} + \sigma_X Z_{X_t},$$

where,

- $D_x(t)$ is the actual probability of a life aged x at time t dying in year t+1,
- $\hat{D}_x(t)$ is the best estimate at time 0 of the probability of a life aged x at time t dying in year t + 1,
- Z_{Y_t} and Z_{X_t} are i.i.d. standard normal variates.

It is assumed that $Y_0 = 0$. This ensured that the ratio of actual to estimated mortality rates is a martingale. Consequently, the current value of this ratio is the best estimate of its value at any time in the future.

2.8.2 Extended Barrie and Hibbert & Heriot-Watt University model

The model represents an extended version of the previous one where a stochastic trend factor is added.⁷

In this model, the actual probability $D_x(t)$ of a life aged x dying in year t + 1 is represented by

$$D_x(t) = \hat{D}_x(t) \left[\exp\left(\delta(s) - \frac{1}{2}s\sigma_{\delta}^2\right) \right]^{t-s}$$

$$\times \exp\left[Y(t) - Y(s) - \frac{1}{2}(t-s)\sigma_X^2 - (t-s)\sigma_{\delta}^2 - \frac{1}{2}\sigma_Y^2 \right],$$

where

$$\begin{aligned} Y_t &= X_t + Z_{Y_t}, \\ X_t &= X_{t-1} + \delta_t + Z_{X_t}, \\ \delta_t &= \delta_{t-1} + Z_{\delta_t}. \end{aligned}$$

The stochastic process Y(t) is observable and models the year-by-year random transient change. It relates to another two non-observable stochastic processes

 $^{^7\}mathrm{We}$ refer to a the 3-month Internship project carried out at Barrie and Hibbert in conjunction with Heriot-Watt University.

X(t) and $\delta(t)$, which model the long-term true trend in mortality. The quantities Z_{X_t} , Z_{Y_t} and Z_{δ_t} are stochastic drivers following independent $N(0, \sigma_X^2)$, $N(0, \sigma_Y^2)$ and $N(0, \sigma_{\delta}^2)$ distributions, respectively. The exponential term, that represent the trend factor is

$$\left[\exp\left(\delta(s) - \frac{1}{2}s\sigma_{\delta}^2\right)\right]^{t-s}$$

2.8.3 Munich Re model

Edwalds et al. (2008), in name of Munich Re,⁸ propose the use of a predictive modeling technique called projection pursuit regression (PPR) to model mortality data. PPR is a form of Generalized Additive Models (GAM) models, but PPR has extra flexibility allowing one to model interactions between various predictors without requiring additional effort.

2.8.4 ING model

Van Broekhoven (2002), in name of ING,⁹ proposed a model by means of that it is possible deriving the best estimate and the price of mortality risk.

In order to fit the expected mortality as well as possible for the applicable group of insured persons the model take into account (1) the current mortality for a specified group of (insured) persons and (2) the expected changes in the level of this mortality in the future. The first part is the level of the mortality, the second part is the trend.

2.8.5 Partner Re model

Duchassing and Suter (2009), in name of Partner Re,¹⁰ has developed a simple stochastic model which could be an alternative to some of the well known models. The model can be applied to company specific best estimates for future mortality rates. The following stochastic model is applied for the mortality process

$$D_x(t) = D_x(t)C_t + \epsilon_t,$$

where

- $\hat{D}_x(t)$ is the expected mortality,
- $D_x(t)$ the real mortality,
- $C_0 = 1$,

⁸Munich Re is a reinsurance company.

⁹ING is an insurance company.

¹⁰Partner Reinsurance Europe Limited is a reinsurance company.

- $C_t = \exp(X_t)C_{t-1}$,
- X_t are i.i.d. $N(\mu, \sigma)$ variates,
- C and ϵ are indipendent.

Based on the England and Wales data, the authors estimated the underlying parameters μ and σ using various smoothing methods for particular age and calendar year ranges and tested their validity.

2.8.6 Towers Perrin model for the Solvency II longevity shock

The proposal of Towers Perrin¹¹ for the Solvency II longevity shock consists in projecting the evolution of the base population taking into account the number of people alive from one year to another, including a random variable within the annual mortality improvement factor.¹² The functional form of the model is the following

$$l'_{x+1} = l_x \times \left[1 - D_x \times \left(1 - \mu_x + \sigma_x \times K_{aleatory}\right)\right],$$

where,

- l_x : population alive in year 0, given an x age,
- D_x : probability of death within a year under the standard table, given an age of x years old at the start of the year,
- l'_{x+1} : population alive in year x + 1, given an age of x years old at the beginning,
- μ_x : mean of the mortality improvement factor for the analysed age range,
- σ_x : standard deviation of the mortality improvement factor for the analysed age range,
- $K_{aleatory}$: is a standard normal random variable.

Setting equal to n the number of years in the projection, the population alive at the end of the projection would be

$$l'_{x+1} = l_x \times \left[1 - D_x \times \prod_{i=1}^n \left(1 - \mu_{x+i} + \sigma_{x+i} \times K_{aleatory} \right) \right].$$

¹¹Towers Perrin is an independent consulting company.

 $^{^{12}}$ See Unespa-Towers Perrin (2009).

2.8.7 Milliman model for longevity risk under Solvency II

Silverman and Simpson (2011), in name of Milliman,¹³ proposed a stochastic model for longevity risk where mortality projections reflects three sources af volatility:

- randomized dates of death,
- future mortality improvements trends volatility,
- potential extreme longevity occurrences.

In the Milliman model, the annual rate of mortality improvement, at attained age x and duration t, reflects stochastic volatility in the following way

$$\Delta \hat{D}_x(t) = 1 - [1 - \Delta D_x(t) + \delta_x(t)],$$

where

- $\Delta D_x(t)$ is the expected annual rate of mortality improvement at attained age x and duration t = 0, 1, 2, ...,
- $\delta_x(t)$ is the stochastic adjustment to mortality improvement of attained age x and duration t.

The stochastic adjustment is such that

$$\delta_x(t) = \hat{M}_x(t) - M_x,$$

with

$$\hat{M}_x(t) = W_x^T(t) \frac{M_x^*(t)}{\left[\prod_{s=1}^T M_x^*(s)\right]^{-T}},$$

and M_x is the average of 10-year improvement factors over the entire period for each age x. In particular,

$$M_x^*(t) = W_x^T(t) + \sigma_x^1 \epsilon_t^x,$$

and

$$W_x^T(t) = M_x + \sigma_x \epsilon_t^x,$$

where

• σ_x is the standard deviation of average annualized mortality improvement factors over each of the consecutive *T*-year periods within the *N* years of historical experience,

¹³Milliman is among the world's largest independent actuarial and consulting firms.

- σ_x^1 is the average of standard deviations of annual mortality improvement rates within each of the consecutive *T*-year periods contained in the *N* years of historical data,
- ϵ_t^x are correlated standard normal random variables.

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Chapter 3

A new stochastic model for estimating longevity and mortality risks

3.1 Introduction

Longevity risk and mortality risk are critical components of a life insurance company's risk profile because these risks impact life products where benefits are paid upon the insured's survival and life products based on the insured's death. In order to quantify a life insurer's longevity and mortality risks, several proposals for modeling and forecasting mortality rates have been suggested. The leading statistical model of mortality forecasting is the one proposed by Lee and Carter (1992) and that has been recommended by two U.S. Social Security Technical Advisory Panels.¹ The Lee-Carter model (and variants of it) is a discrete model where the parameters can be calibrated to historical mortality experience.

A different approach in mortality modeling involves the use of continuoustime stochastic models of the force of mortality.² Milevsky and Promislow (2001) — the first to propose a stochastic force of mortality model — and others focused on the use of affine stochastic models.³ Other authors have proposed the use of

¹Other models have been proposed by Lee (2000), Yang (2001), Cairns et al. (2006a), Currie et al. (2004), Lin and Cox (2005), Renshaw and Haberman (2003), Brouhns et al. (2002), and Giacometti et al. (2009).

²In actuarial science, force of mortality represents the instantaneous rate of mortality at a certain age measured on an annualized basis. It is identical in concept to failure rate or hazard function.

³For other force of mortality models see Dahl (2004), Biffis (2005), Denuit and Devolder (2006), Schrager (2006), Luciano and Vigna (2005), and Giacometti et al. (2011).

CHAPTER 3. A NEW STOCHASTIC MODEL FOR ESTIMATING LONGEVITY 62 AND MORTALITY RISKS

stochastic models to quantity the amount of capital that insurance companies need to reserve in order to deal with their exposure to longevity and mortality risks. Several studies have analyzed stochastic models for longevity and mortality risks with respect to Solvency II capital requirements that will become effective January 2013 for European insurers.⁴

In this chapter, we propose a new stochastic model for the estimation of longevity and mortality risks. We use the mortality rates closed-formula implied in affine stochastic mortality models in order to extract two time-varying parameters that will be used as a proxy for factors affecting the shape of the mortality rates curve across time. The parameters of the closed-formula are estimated yearly by means of an optimization procedure. To explain the dynamic of the parameters, we apply a two-dimensional autoregressive process. Our approach is similar to that used for modeling the term structure of interest rates where a time-varying functional form for the term structure is assumed and the dynamic of the parameters is analyzed using a stochastic process.⁵ In summary, our model has the following attributes: (1) the dynamic of the mortality rates is explained by two state variables that follow a two-dimensional autoregressive process of order $1;^{6}$ (2) the mortality rate increases exponentially in a consistent manner with the Gompertz law; 7 (3) the model incorporates the decreasing trend observable in historical mortality data, and; (4) the term structure of mortality rates can be obtained with a closed-formula for each age and for each point in time.

Our model could be useful for the modeling of mortality/longevity risks under insurance solvency regimes mandated by Solvency II. It could potentially offer an appropriate tool for the valuation of longevity and mortality risks where an internal assessment of the insurance business must be provided according to a solvency investigation based on internal models.

Using Italian population data, we provide empirical support for our proposed model. Moreover, we analyze the consistency of the shocks proposed in the Solvency II standard formula by assessing the impact of comparable shocks using the stochastic model we propose.

There are four sections that follow. In the next, we present model, followed in Section 3.3 by a description of the estimation methodology. Empirical results of

⁴See Olivieri and Pitacco (2008), Olivieri (2009), Unespa-Tower Perrin (2009), Borger (2010), Stevens et al. (2010), Borger et al. (2011), Silverman and Simpson (2011), Plat (2011).

 $^{^{5}}$ See McCulloch (1971), Nelson and Siegel (1987), Bliss (1989), Bliss (1996) and Fabozzi et al. (2005).

⁶See Rachev et al. (2007) for details about the autoregressive process.

 $^{^{7}}$ The Gompertz law (1825) is founded on the biological concept of organism senescence, in which mortality rates increase exponentially with age.
the model are provided in Section 3.4. Some empirical results related to Solvency II are analyzed in Section 3.5 while our conclusions are summarized in the last section.

3.2 The proposed model

Our proposed stochastic model is a closed-formula for the term structure of mortality rates where the rates with different maturities can be computed explicitly. The closed-formula is a function of two state variables describing the dynamic of mortality rates along time and age dimensions. We assume a two-dimensional autoregression process of order one to explain the dynamic of the state variables. Consequently, the closed-formula is time-varying in the sense that the entire term structure of the mortality rates changes stochastically over time following the dynamic of the autoregression process. The model is consistent with the Gompertz law and it is able to take into account the long-term mortality trend observed in historical data. In this section, we present the model's functional form and describe how it can be calibrated to historical data.

3.2.1 Notation

In order to describe the model, we introduce and define the following quantities:

- x = reference age with x = 1, 2, ..., X;
- t = reference year with t = 1, 2, ..., T;
- m = reference maturity of the mortality rate (i.e., the number of years ofter the reference year t);
- $D_x(t, t+m) = \text{death probability (i.e., the probability in t that an individual aged x dies within the period <math>[t, t+m]$;
- $S_x(t, t + m) =$ survival probability (i.e., the probability in t that an individual aged x dies after t + m) and is such that,

$$S_x(t, t+m) = 1 - D_x(t, t+m).$$

Modeling the death event according to the Poisson distribution and denoting the mortality rate by $\mu_x(t, t+m)$,⁸ the survival probability at time t of an individual aged x can be computed as,

 $S_x(t, t+m) = \exp[-\mu_x(t, t+m)m].$

⁸For further details see Cairns, Blake, and Dowd (2006b).

Consequently, the mortality rate is equal to:

$$\mu_x(t,t+m) = -\frac{\log[S_x(t,t+m)]}{m}$$

All of the previous quantities have to be considered theoretical quantities. In order to distinguish the values derived by historical data from the values implied by the theoretical model, we denote by $\hat{\mu}_x(t,t+m)$ the historical mortality rate. In the same way, we denote by $\hat{D}_x(t,t+m)$ and $\hat{S}_x(t,t+m)$ the historical death probability and the historical survival probability, respectively.

3.2.2 Model specification

Instead of defining directly a parametric functional form for the term structure of the mortality rates, we assume that the force of mortality, for a fixed age x, increases exponentially over time and satisfies the following differential equation consistent with its empirical observed behavior,

$$d\mu_x(t) = k\mu_x(t)dt, \quad \mu_x(0) = h_x,$$

where (1) $\mu_x(t)$ is the instantaneous force of mortality assumed as deterministic such that,

$$\mu_x(t) = \lim_{m \to 0} \mu_x(t, t+m),$$

(2) the parameter k is a positive constant, and (3) the parameter h_x is age dependent.

The differential equation that we propose is derived from the dynamic of an affine stochastic mortality model where only the deterministic component of the equation, without considering the mean-reversion, is taken into account. In this way, our model takes into account the so-called non-mean reverting effect that assures consistency with the Gompertz law. Cairns et al. (2006b) suggest that affine stochastic models need to incorporate non-mean reverting elements, while Luciano and Vigna (2005) and Russo et al.(2011) propose non-mean reverting affine processes for modeling the force of mortality. In these models, the deterministic part of the mortality rate process increases exponentially as in the model that we propose.

Then, in order to introduce a parametric function, we observe that affine stochastic mortality models imply a closed-form expression for the survival probabilities,⁹ with the consequence that, for t = 0, we have

$$S_x(0,m) = \exp\left[\frac{h_x}{k} \left[1 - \exp(km)\right]\right],$$

 $^{^{9}}$ We apply the same mathematical tools used in interest rate and credit risk modeling. See Duffie, Pan and Singleton (2000) and Brigo and Mercurio (2006) for further details.

where h_x and k are the model's parameters.

Due to the term structure closed-formula implied in affine stochastic models, we are able to define the entire term structure of mortality rates with respect to the rate's maturity m.

In order to explain the dynamic of the parameters across time, we estimate them yearly by means of an optimization procedure. Consequently, the closedformula that we have defined previously becomes time-varying. We use the two time-varying parameters as a proxy for factors affecting the term structure of mortality rates across time. We denote by $h_x(t)$ and k(t) the two time-varying parameters. Consequently, although the dynamic of our model across the maturity m and ages x is deterministic, the dynamic of the model across time t is stochastic.

The dynamic of the mortality rates across time t, across age x, and across maturity m can be explained by the following functional form

$$S_x(t,t+m) = \exp\left\{\frac{h_x(t)}{k(t)}\left[1 - \exp\left[k(t)m\right]\right]\right\}.$$

In terms of mortality rates, our model becames

$$\mu_x(t,t+m) = -\frac{h_x(t)}{k(t)m} \bigg[1 - \exp\left[k(t)m\right] \bigg],$$

where $h_x(t)$ and k(t) represent the two fundamental state variables.

In order to disantangle age and time dependence, we define the state variable $h_x(t)$ as

$$h_x(t) = h(t)g(x),$$

where,

- h(t) is the state variable that explains the dynamic of the force of mortality considered as a latent factor;
- g(x) is a deterministic function of x such that,

$$g(x) = \hat{\mu}_x(\bar{t}, \bar{t}+1),$$

with \bar{t} representing the last available date in mortality rates for the time series.

For t = 1, 2, ..., T, the variables k(t) and h(t) are estimated in order to derive the time series $\{\hat{k}(t)\}_{t=1}^{T}$ and $\{\hat{h}(t)\}_{t=1}^{T}$.

We model the dynamic of the first differences computed on $\{\hat{k}(t)\}_{t=1}^{T}$ and $\{\hat{h}(t)\}_{t=1}^{T}$ with a two-dimensional autoregression process of order one, denoted by AR(1). Setting,

$$\Delta h(t) = h(t) - h(t-1)$$

$$\Delta k(t) = k(t) - k(t-1),$$

we estimate,

$$\Delta h(t) = \alpha_0 + \alpha_1 \Delta h(t-1) + \sigma_h \varepsilon_h(t)$$

$$\Delta k(t) = \beta_0 + \beta_1 \Delta k(t-1) + \sigma_k \varepsilon_k(t).$$

The parameters α_0 and β_0 are constant drift terms while α_1 and β_1 quantify the sensitivities of the state variables with respect to the regressors. The quantities σ_h and σ_k are constant volatility parameters while $\varepsilon_h(t)$ and $\varepsilon_k(t)$ are correlated standard normal errors with correlation coefficient equal to ρ .

The parameters of the AR(1) process are independently estimated using the ordinary least squares (OLS) method.

Basically, our model possesses the following characteristics:

- It describes the dynamic of mortality rates across the maturity m providing a closed-formula for the term structure of mortality rates.
- It describes the dynamic of mortality rates across age x in a manner that is consistent with the Gompert law.
- It describes the dynamic of mortality rates across time t.
- It incorporates the decreasing trend observable in historical mortality data.

3.3 Estimation procedure

In this section, we describe the estimation procedure.

3.3.1 Input data

In order to estimate the model, we use data contained in life tables. Typically, life table includes the following information:

• $l_x(t)$ = number of survivors of age x at the start of the reference year t;

• $d_x(t, t+1) =$ number of deaths with respect to the reference year t between age x and x + 1.

Given the reference year t and age x, we compute the one-year survival probability as,

$$\hat{S}_x(t,t+1) = 1 - \frac{d_x(t,t+1)}{l_x(t)}.$$

Consequently, we derive the one year death rate as

$$\hat{\mu}_x(t,t+1) = -\log[\hat{S}_x(t,t+1)].$$

In order to estimate the proposed model, we use a $T \times X$ matrix of historical data containing the one-year death rate for t = 1, 2, ... T and x = 1, 2, ... X.

3.3.2 Calibrating the vectors $\{h(t)\}$ and $\{k(t)\}$

The estimation procedure of the model involves the calibration of the state variables h(t) and k(t) by means of an optimization procedure. The *t*-th element of the time series $\{\hat{h}(t)\}_{t=1}^{T}$ and $\{\hat{k}(t)\}_{t=1}^{T}$ is calibrated by minimizing the sum of the square difference between the observed mortality rate and the theoretical one implied by our model at time t with respect to all available ages.

Denoting by $e_x(t)$ the square difference between the observed mortality rate and the theoretical one, we have that,

$$e_x(t) = [\hat{\mu}_x(t,t+1) - \mu_x(t,t+1)]^2.$$

The estimates of h(t) and k(t) are the solution of the following optimization problem,

$$\arg\min_{h(t),k(t)}\sum_{x=1}^{X}e_{x}(t).$$

Repeating this procedure for t = 1, 2, ..., T, we obtain the time series $\{\hat{h}(t)\}_{t=1}^{T}$ and $\{\hat{k}(t)\}_{t=1}^{T}$.

3.3.3 Modeling the dynamic of the state parameters

In order to model the dynamic of the state variables, h(t) and k(t), we estimate the parameters of the AR(1) on the time series $\{h(t)\}$ and $\{k(t)\}$.

Using the OLS method, we estimate the parameters of the two processes independently. Applying the OLS method on the time series $\{h(t)\}$, we obtain the estimators of the parameters α_0 and α_1 . In the same way, applying the OLS method on the time series $\{k(t)\}$, we obtain the estimators of the parameters β_0 and β_1 . Consequently, computing the expected values of h(t) and k(t), we obtain the time series of the empirical residuals $\hat{\varepsilon}_h(t)$ and $\hat{\varepsilon}_k(t)$.

Denoting the covariance by $\sigma_{h,k}(t) = \rho \sigma_h \sigma_k$, the two vectors of residuals are then used to estimate the variance-covariance matrix (Σ) of the model,

$$\Sigma = \left[\begin{array}{cc} \sigma_h^2 & \sigma_{h,k} \\ \sigma_{h,k} & \sigma_k^2 \end{array} \right].$$

3.4 Empirical results

3.4.1 Data

In order to provide an empirical application of the model, we use Italian population annual data from 1950 to 2008. We did not consider pre-1950 data in order to avoid the impact of the two world wars on the volatility of mortality rates. We consider all the available ages (up to 110). The source of data is *The Human Mortality Database* of the University of California, Berkeley (www.mortality.org).

The graphical presentation of the one-year death rates from 1950 to 2008 for all the available ages is shown in Figure 3.1.

Figure 3.1: Historical one-year death rate from 1950 to 2008



3.4.2 Estimation results

To provide the time series of the two state variables k(t) and h(t), the optimization procedures for each reference year t as described in Section 3.2 are used. Figures 3.2-3.5 show the vectors $\{k(t)\}$, $\{h(t)\}$, $\{\Delta h(t)\}$ and $\{\Delta k(t)\}$. In Table 1, we report some basic descriptive statistics.

Figure 3.2: Time series of $\{h(t)\}$



Figure 3.3: Time series of $\{k(t)\}$





Figure 3.4: Time series of $\{\Delta h(t)\}$

Figure 3.5: Time series of $\{\Delta k(t)\}$



Table 3.1: Descriptive statistics on $\{\Delta h(t)\}\$ and $\{\Delta k(t)\}\$

state variable	mean	standard deviation	skewness	kurtosis
$\Delta h(t)$	-0.002750	0.051399	0.148946	3.528705
$\Delta k(t)$	0.000025	0.004237	0.446829	2.928592

Note that the time series of h(t) (Figure 3.2) show a decreasing trend, while the series of the first differences (Figure 3.4) appear to be stationary. This observation led us to employ the stochastic processes of $\Delta h(t)$ and $\Delta k(t)$ in place of h(t) and k(t).

To confirm our assumption, we performed a formal Augmented Dickey-Fuller test for unit root. The null hypothesis of a unit root in the dynamics of the h(t)and k(t) was not rejected. We did find that after differencing once, the time series of $\Delta h(t)$ and $\Delta k(t)$ appear to be stationary.

In order to model the dynamic of $\{\Delta h(t)\}\$ and $\{\Delta k(t)\}\$ we evaluated which type of autoregressive process was more appropriate. We did so by first estimating a vector autoregression process of order one — so-called VAR(1) — to model the two time series. Unfortunately, we obtained low values for the significant of the parameters. Consequently, we decided to verify the estimation results using a pure autoregressive process (AR). Because by doing so we obtained significant estimates for the parameters of the model, we adopted a two-dimensional AR(1) process in place of a VAR(1) process.

To verify our assumption that the residuals of the AR(1) process for $\{\Delta h(t)\}$ and $\{\Delta k(t)\}$ follow a normal distribution, we performed a Kolmogorov-Smirnov test to compare the values in the vectors $\{\varepsilon_h(t)\}$ and $\{\varepsilon_k(t)\}$ with the standard normal distribution values. The null hypothesis that the residuals of the regression on $\{\Delta h(t)\}$ and $\{\Delta k(t)\}$ have a standard normal distribution is accepted at the 5% significance level. Our findings are consistent with the results reported in Unespa-Tower Perrin (2009), where using the same statistical test the normal distribution hypothesis was not rejected at the 5% significance level.

Consequently, we have estimated the parameters of the AR(1) process starting from the time series $\{\Delta h(t)\}$ and $\{\Delta k(t)\}$ as described in Section 3.3. The parameters estimation, the values of standard deviations and *t*-statistics are reported in Table 3.2. The estimation results show that the regression coefficients α_1 and β_1 are significant. We also estimated the variance-covariance matrix Σ with the following results,

$$\hat{\Sigma} = \left| \begin{array}{ccc} 0.002021 & 0.000057 \\ 0.000057 & 0.000013 \end{array} \right|.$$

Consequently, we find that $\hat{\rho} = 0.351761$.

parameter	coefficient	standard error	t-statistic
$lpha_0$	-0.005803	0.006018	-0.964165
α_1	-0.420959	0.116953	-3.599397
β_0	0.000114	0.000480	0.238224
β_1	-0.529630	0.113183	-4.679427

Table 3.2: Estimation results for the AR(1) process

3.4.3 Simulation results

The simulation procedure involves generating a four-dimensional ipercube of mortality rates, where the four dimensions are represented by:

- x = the reference age with x = 1, 2, ..., X,
- t =the reference year with t = T + 1, T + 2, ..., T + N,
- m = the maturity of the mortality rate with m = 1, 2, ..., M,
- s = the number of simulations with s = 1, 2, ..., S.

Here we report the simulation results for S = 1,000.¹⁰ The ipercube consists of a set of mortality rates, one for each reference year, for each age, for each maturity, and for each simulation step.

In Figure 3.6, the projection of the one-year mortality rate for an individual aged 50 is provided for a period of 20 years. From the projection, one can appreciate how our model explains the mortality trend over a long-time horizon. Moreover, the level of the mortality rate decreases in time which is consistent with the decreasing trend that is observed for the historical data as shown in Figure 3.1.

In Figure 3.7, we provide the same projection as Figure 3.6 but this time plotting a confidence level of 90%.

3.4.4 Backtesting

Our focus in this section is on backtesting techniques for verifying the accuracy of our proposed model. Backtesting is a statistical testing framework that consists of checking whether actual mortality rates are in line with a model's forecast.

$$\varepsilon_h(t) = z_h(t)$$

$$\varepsilon_k(t) = z_h(t)\rho + z_k(t)\sqrt{1-\rho^2}$$

¹⁰We have simulated correlated values of the state variables starting from independent standard normal random numbers, denoted by $z_h(t)$ and $z_k(t)$. Consequently, $\varepsilon_h(t)$ and $\varepsilon_k(t)$ are such that,



Figure 3.6: Projection of the one-year mortality rate (x = 50): simulation results

Figure 3.7: Projection of the one year-mortality rate (x = 50): confidence interval



We divided the available time series of one-year mortality rates (from 1950 to 2008) into two parts: in-sample and out-of-sample. The in-sample part is from 1950 to 1980; the out-of-sample part is from 1981 to 2008. We compared the historical data from 1981 to 2008 with the related estimation. In addition, we provided an upper and lower bound with respect to the 0.5% and 99.5% confidence level. We can see from backtest results shown in Figure 3.8 that our model is able to correctly predict the trend and the volatility of the one-year mortality rate.

We also provided a comparison between historical and expected value across ages. For the reference year 2008, we computed the expected value and the confidence interval for each age. The results are shown in Figure 3.9. Note the consistency between the observed and expected values.



Figure 3.8: Backtesting (x = 50)

3.5 The Solvency II European project

The so-called Solvency II rules for European insurance companies that will become effective January 1, 2013 sets forth two capital requirements representing different levels of supervisory intervention: the Solvency Capital Requirement



Figure 3.9: Backtesting for different ages

(SCR) and the Minimum Capital Requirement (MCR).¹¹ According to the Solvency II directive, insurers should hold an amount of capital that enables them to absorb unexpected losses and meet their obligations to policyholders. The calculation of this requirement must be made on the basis of the value at risk (VaR) with a confidence level of 99.5% over a time horizon of one year.

The capital requirements must be computed using a standard formula or by means of an internal model. If the standard formula is adopted, the overall risk can be split into several modules. Separate capital requirements are computed for each risk and then aggregated with linear correlation matrices to allow for the benefit of diversification. The capital requirement for each risk is determined as the 99.5% VaR of the available capital over a one-year time horizon. In lieu of the standard formula, with the approval of the insurance supervisor, internal models can be used to compute the capital requirement. On the one hand, internal models should provide a more accurate quantification of the capital requirements. On the other hand, internal models are often more complex than the standard formula and generally are based on stochastic models.

As an alternative to the standard formula, stochastic mortality models can be used as internal models for evaluating the impact of longevity and mortality risk within the overall risk framework. The typical reasons for an insurer's adoption of

 $^{^{11}\}mathrm{See}$ European Commission (2009) and European Community (2009) for further details about Solvency II.

an internal model are threefold: (1) more accurate measure of the risks, (2) moral suasion from the capital market and rating agencies, and (3) encouragement by regulators.

3.5.1 Using the proposed model under Solvency II regime

In the Solvency II standard model, the capital charges for longevity and mortality risks are computed as the change in liabilities with respect to a percentage shock applied to the current level of the mortality rates. In particular, for longevity risk, a reduction of the mortality rates is taken into account while for mortality risk an increase of the mortality rates is considered. The percentage shock is used for all ages is the same.

The percentage shocks as a part of the standard formula of Solvency II are currently being established by a series of Quantitative Impact Studies (QIS) in which the effects of the new capital requirements are analyzed. According to the latest Quantitative Impact Study as of this writing, the so-called QIS5,¹² the capital charge for longevity risk is captured by a permanent 20% decrease in the mortality rates, while the capital charge for mortality risk is captured by a permanent 15% increase.

We compared the percentage shocks proposed in the QIS5 with the analogous results of our stochastic model with respect to Italian population data. In order to obtain comparable results with the Solvency II longevity and mortality shocks, we have considered the empirical distribution of $\mu_x(t, t + m)$ projected over a time horizon of one year. Using the empirical distribution, we calculated the percentile at the 0.5% and 99.5% confidence levels over the time horizon of one-year where the percentile at 0.5% represents the VaR for longevity risk and the percentile at 99.5% the VaR for mortality risk. The VaR for both are computed over a one-year time horizon as prescribed by the Solvency II directive.

Transforming the value of the percentile into a percentage shock, we find that our model implies a 12.7% decrease in mortality rates for the longevity risk capital charge and a 12.8% increase in mortality rates for the mortality risk capital charge. The percentage shocks are the same for all the ages. This finding suggests that for computing the capital charge for longevity risk, a reduction of 20% in mortality probabilities as mandated by Solvency II seems unrealistically high. However, the percentage shock for mortality risk from our model appears to be consistent with Solvency II. That is, if the shock value proposed by Solvency II for mortality risk can be considered realistic with respect to the Italian population data, the shock value for longevity risk appears to be too conservative.

¹²See European Commission (2010) for further details.

3.6 Conclusions

In this chapter, we propose a stochastic model for the estimation of longevity and mortality risks that can be employed as an internal model for risk evaluation in determining risk-based Solvency II requirements for European insurance companies. Our model provides a closed-formula for computing the mortality rates at different maturities for different ages and for each reference year. Because the model has two stochastic drivers that follow an autoregressive stochastic process, it is capable of accounting for the observed long-term mortality trend and it is consistent with the Gompertz law.

Calibrating our stochastic model to historical data for the Italian population, we find that the estimated values for the model's parameters are statistically significant.

Moreover, we performed a backtesting analysis where we found that our model produced highly accurate forecasts of mortality rates.

We also analyzed the shock values specified in the Solvency II standard formula for longevity and mortality risks. Applying our model, we calculated the percentage shocks for the expected longevity and mortality risks in a manner consistent with the VaR at 0.5% and 99.5% confidence levels over a one-year time horizon. For the Italian population data, we found that the shock values computed with our model are consistent with the assumption of the standard formula in which an increase of 15% is mandating for the purpose of computing the mortality risk. In contrast, our results suggest that the standard formula of Solvency II could lead to an over-estimation of the capital requirements for longevity risks when a decrease of 20% of the mortality rates is required to quantify the capital charge for longevity risks.

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Chapter 4

Intensity-based framework for longevity and mortality modeling

4.1 Introduction

The use of intensity-based models has seen a remarkable surge during the last decade in the modeling of credit risk.¹

In this chapter, following Cairns et al. (2006) and Luciano and Vigna (2005), an intensity-based framework for mortality modeling is presented. In fact, the modeling of mortality in life insurance became very similar to that of default in the credit risk literature. Consequently, the mortality intensity can be thought of as a hazard rate in the context of the Poisson process approach.

4.2 Quantitative measures of mortality and longevity

As a standard measure of mortality, we consider the probability in t that an individual aged x dies within the period [t,T] with t < T. This probability is denoted by $D_x(t,T)$. Given the probability in t that an individual aged x dies within the period $[T_{i-1}, T_i]$ it holds that,

$$D_x(T_{i-1}, T_i) = D_x(t, T_i) - D_x(t, T_{i-1}).$$

We consider also the survival probability which reflects the probability in t that an individual aged x survives over T. We denote this probability by $S_x(t,T)$. Clearly, it holds the following relation

$$S_x(t,T) = 1 - D_x(t,T).$$

¹See Lando (2004), Schonbucher (2000), Duffie, Pan and Singleton (2000), and Brigo and Mercurio (2006).

4.3 Mortality reduced form models

Reduced form models can be considered to model the death event describing death arrival as the first jump time of a Poisson process with deterministic or stochastic force of mortality.

4.3.1 Time-homogeneous Poisson process: constant force of mortality

Time-homogeneous Poisson process is the standard Poisson process.

Considering an individual aged x at time t, it is assumed that the death arrival is the first jump-time of a time-homogeneous Poisson process indicated by $\{A_{\tau}, \tau \geq 0\}$. Denoting by $\mu_x(t,T)$ the mortality rate related to the period [t,T] and assuming the death arrival as the first jump time of a Poisson Process, it follows that

$$Prob[A_{\tau} = a] = \frac{e^{-\mu_x(t,T)\tau(t,T)}[\mu_x(t,T)\tau(t,T)]^a}{a!}.$$

Setting a = 0, the survival probability at time t that an individual aged x survives over T is

$$S_x(t,T) = \exp\left\{-\mu_x(t,T)\tau(t,T)\right\}.$$

The term structure of mortality rates

A stream of mortality rates related to the respective maturities represents the term structure of mortality rates. Setting a vector of maturities, $T_1, T_2, ..., T_n$, the term structure of mortality rates is represented by the sequence

$$\mu_x(t, T_1), \mu_x(t, T_2), \dots, \mu_x(t, T_n).$$

4.3.2 Time-inhomogeneous Poisson process: time-varying deterministic force of mortality

Time-inhomogeneous Poisson process can be built on the based of time-homogeneous Poisson process with *deterministic and time-varying force of mortality*. We denote by $\mu_x(t)$ the force of mortality in t related to an individual aged x to take into account that it is time-varying. The force of mortality is such that

$$\mu_x(t) = \lim_{T \to t} \mu_x(t, T).$$

We assume that the death event can be considered as the first jump time of a Poisson process where

$$\Lambda_{\mu_x}(t) = \exp\bigg\{\int_0^t \mu_x(u) du\bigg\},\,$$

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represents the *cumulative force of mortality* or *cumulative hazard rate* or *hazard function*.

Under the assumption of time-inhomogeneous Poisson process, we have that

$$Prob[A_{\tau} = a] = \frac{e^{-\int_t^T \mu_x(u)du} \left[\int_t^T \mu_x(u)du\right]^a}{a!}$$

Consequently,

$$\mathcal{S}_x(t,T) = \frac{\Lambda_{\mu_x}(t)}{\Lambda_{\mu_x}(T)} = \exp\bigg\{-\int_t^T \mu_x(u)du\bigg\},\,$$

where $S_x(t,T)$ is the so-called *survivor index*. It is equal to the probability at time t that an individual aged x survives over T.

4.3.3 Double stochastic Poisson process (Cox process): stochastic intensity

The Poisson process can have deterministic or stochastic intensity.

In the case of double stochastic Poisson process (Cox process) the force of mortality is stochastic.

Indicating by \mathcal{M}_t the filtration generated by the evolution of the term structure of the mortality rates, we have that

$$S_x(t,T) = \mathbb{E}^{\mathcal{Q}}\left[\frac{\Lambda_{\mu_x}(t)}{\Lambda_{\mu_x}(T)}\middle| \mathcal{M}_t\right].$$

4.4 Affine processes as stochastic mortality models

Affine models are very popular in quantitative finance. They have been employed in finance since decades, and they have found growing interest due to their computational tractability. The main property of affine models is that the conditional cumulant function, defined as the logarithmic of the conditional characteristic function, is affine in the state variable.

Affine models are very often used to model the short term of interest rates because they lead to closed-form of the bond prices and yields whatever the maturity. This approach has been introduced in continuous time by Vasicek (1977) where the short term interest rate is assumed to follow a gaussian Ornstein-Uhlenbeck process. Another important affine model used in financial modeling was proposed by Cox, Ingersoll, and Ross (1985), the so-called CIR model.

Besides these applications in risk-free rate modeling, affine processes are also used in credit risk modeling. In this case, affine models lead to a closed-form solution for the survival probability.²

²See Duffie, Pan and Singleton (2000).

We apply reduced-form models into mortality risk modeling where the force of mortality is driven by affine diffusions processes. In particular, the force of mortality is an affine function of one latent factor and the survival probabilities are known in closed-form. It is important to know that thank to the affine formulation it is possible to obtain analytical formulas for the entire term structure of mortality rates. Different types of affine models used in interest rate modeling can be taken into account to model stochastic force of mortality.

Below, we provide a brief description of Vasicek and CIR model affine models with the related closed-formula for survival probability.

4.4.1 Vasicek model

For the Vasicek (1977) model, we assume that the force of mortality follows the stochastic differential equation

$$d\mu_x(t) = k(\theta - \mu_x(t))dt + \sigma dW(t), \qquad \mu_x(0) = \mu_x$$

with μ_x and σ positive constants, θ constrained to be positive, and k constrained to be strictly negative. The main drawback of this process is that the force of mortality can be negative with positive probability. For this model, the survival probability can be obtained by

$$\begin{split} S_x(t,T) &= G(t,T)e^{-H(t,T)\mu_x(t)},\\ G(t,T) &= exp\bigg\{\bigg(\theta - \frac{\sigma^2}{2k^2}\bigg)\bigg[H(t,T) - \tau(t,T)\bigg] - \frac{\sigma^2}{4k}H(t,T)^2\bigg\},\\ H(t,T) &= \frac{1}{k}\bigg[1 - e^{-k\tau(t,T)}\bigg]. \end{split}$$

4.4.2 Cox-Ingersoll-Ross model

Assuming the dynamic of the Cox, Ingersoll, and Ross (1985) model (CIR model hereafter), the force of mortality $\mu_x(t)$ satisfies

$$d\mu_x(t) = k(\theta - \mu_x(t))dt + \sigma\sqrt{\mu_x(t)}dW(t), \qquad \mu_x(0) = \mu_x,$$

with μ_x and σ positive constants, θ constrained to be positive, and k constrained to be strictly negative. The principal advantage of the CIR model over the Vasicek model is that the hazard rate is guaranteed to remain non-negative. However, the condition $2k\theta > \sigma^2$ is not applicable and the hazard rates can be equal to zero with positive probability. Survival probabilities can still be computed analytically

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and are given by

$$S_x(t,T) = G(t,T)e^{-H(t,T)\mu_x(t)},$$

$$G(t,T) = \left[\frac{2\gamma e^{\frac{1}{2}(k+\gamma)\tau(t,T)}}{2\gamma + (k+\gamma)(e^{\gamma\tau(t,T)} - 1)}\right]^{\frac{2k\theta}{\sigma^2}},$$

$$H(t,T) = \frac{2(e^{\gamma\tau(t,T)} - 1)}{2\gamma + (k+\gamma)(e^{\gamma\tau(t,T)} - 1)},$$

$$\gamma = \sqrt{k^2 + 2\sigma^2}.$$

4.5 How correlating interest rates and mortality rates

Usual assumption is to assume mortality and interest rates as independent. However, in extreme scenarios there must be some dependence (e.g. war and inflation, pandemic and economic growth,...).

A two-factor Vasicek process can be used to take into account correlation between interet rate and mortality. The two-factor gaussian process has the following dynamic,

$$dr(t) = k(\theta - r(t))dt + \sigma dW_r(t), \quad r(0) = r_0,$$

$$d\mu_x(t) = h(\vartheta - \mu_x(t))dt + v dW_\mu(t), \quad \mu_x(0) = \mu_x,$$

$$dW_r(t)dW_\mu(t) = \rho.$$

Given the analytical tractability of the two-factor gaussian model, it is possible to derive the closed formula for the price of the so-called zero coupon longevity bond.

Let $X = m_X + \sigma_X N_X$ and $Y = m_Y + \sigma_Y N_Y$ be two random variables such that N_X and N_Y are two correlated standard gaussian random variables with $[N_X, N_Y]$ jointly gaussian vector with correlation ρ . Then,

$$\mathbb{E}\big[\exp(-X)\big] = \exp\bigg[-m_X + \frac{1}{2}\sigma_X^2\bigg].$$

Consequently, the quantity

$$X = \int_t^T r(u) + \mu_x(u) du,$$

is a gaussian random variable with mean

$$m_X = (\theta + \vartheta)(T - t) - \left[\theta - r_0\right] \frac{1}{k} \left[1 - \exp\left[-k(T - t)\right]\right]$$
$$-\left[\vartheta - \mu_x\right] \frac{1}{h} \left[1 - \exp\left[-h(T - t)\right]\right],$$

and variance

$$\begin{aligned} \sigma_X^2 &= \left(\frac{\sigma}{k}\right)^2 \bigg\{ (T-t) - \frac{2}{k} \bigg[1 - \exp\left[-k(T-t)\right] \bigg] + \frac{1}{2k} \bigg[1 - \exp\left[-2k(T-t)\right] \bigg] \bigg\} \\ &+ \left(\frac{v}{h}\right)^2 \bigg\{ (T-t) - \frac{2}{h} \bigg[1 - \exp\left[-h(T-t)\right] \bigg] + \frac{1}{2h} \bigg[1 - \exp\left[-2h(T-t)\right] \bigg] \bigg\} \\ &+ \frac{2\rho\sigma v}{kh} \bigg\{ (T-t) - \frac{1}{k} \bigg[1 - \exp\left[-k(T-t)\right] \bigg] - \frac{1}{h} \bigg[1 - \exp\left[-h(T-t)\right] \bigg] \bigg\} \\ &+ \frac{1}{k+h} \bigg[1 - \exp\left[-(k+h)(T-t)\right] \bigg] \bigg\}. \end{aligned}$$

Consequently, the price of a zero-coupon longevity bond that pays one unit of cash in case of life and 0 in case of death of the individual aged x is

$$\bar{P}(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r(u) + \mu_x(u)du\right)\right] = \exp\left[-m_X + \frac{1}{2}\sigma_X^2\right].$$

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Chapter 5

A new approach for pricing of life insurance policies

5.1 Introduction

The present chapter is focus on a new approach for pricing of life insurance policies.

We consider life insurance contracts as a swap where the pricing function is similar to the pricing function of an interest rate swap (IRS) or credit default swap (CDS). According to our pricing approach, policyholders exchange cash flows (premiums vs. benefits) with an insurer as with an IRS or CDS. We present our pricing approach with respect to term assurance, pure endowment and endowment policies.

5.2 The basic building block

Interest rates and the mortality rates are the most important risk factors that affect the valuation of insurance contracts. In the following, we introduce some basic standard concepts in interest rates and mortality rates modeling.

5.2.1 Interest rates modeling

We define C(t) to be the value of a bank account at time $t \ge 0$. We assume the bank account evolves according to the following differential equation

$$d\mathcal{C}(t) = r(t)\mathcal{C}(t)dt, \qquad \mathcal{C}(0) = 1,$$

where r(t) is the instantaneous risk-free interest rate. As a consequence,

$$C(t) = \exp\left(\int_0^t r(u)du\right).$$

Being \mathscr{R}_t the filtration generated by the term structure of interest rates up to time t, it follows that

$$P(t,T) = \mathbb{E}^{\mathcal{Q}}\left[\frac{\mathcal{C}(t)}{\mathcal{C}(T)}\middle|\mathscr{R}_t\right],$$

where P(t,T) is the price at time t of a risk-free zero coupon bond that pays one unit of cash at time T and $\mathbb{E}^{\mathcal{Q}}$ is the expectation under the risk-neutral measure $\mathcal{M}^{\mathcal{Q}}$.

5.2.2 Mortality modeling

We consider the force of mortality related to an individual aged x, denoted by $\mu_x(t)$.¹ We assume that the death event can be considered as the first jump time of a Poisson process where the quantity,

$$\Lambda_{\mu_x}(t) = \exp\bigg(\int_0^t \mu_x(u) du\bigg),$$

represents the cumulative force of mortality, also known as *hazard function*. Indicating by \mathcal{M}_t the filtration generated by the evolution of the term structure of the mortality rates, we have that

$$S_x(t,T) = \mathbb{E}^{\mathcal{Q}}\left[\frac{\Lambda_{\mu_x}(t)}{\Lambda_{\mu_x}(T)}\middle|\mathscr{M}_t\right] = \mathbb{E}^{\mathcal{Q}}\left[\mathcal{S}_x(t,T)\middle|\mathscr{M}_t\right],$$

where $S_x(t,T)$ is the survival probability, namely the probability in t that an individual aged x dies after T, and $S_x(t,T)$ is the so-called *survivor index*. Starting from $S_x(t,T)$ we define also the death probability, indicated by $D_x(t,T)$, as the probability that an individual aged x dies within the period [t,T]. Consequently, we have that

$$D_x(t,T) = 1 - S_x(t,T),$$

and

$$D_x(T_{i-1}, T_i) = D_x(t, T_i) - D_x(t, T_{i-1}) = S_x(t, T_{i-1}) - S_x(t, T_i).$$

¹In actuarial science, force of mortality represents the instantaneous rate of mortality at a certain age measured on an annualized basis. It is identical in concept to the failure rate or hazard function.

5.2.3 Zero-coupon longevity bond

We use the convenient assumption that, under the risk neutral measure $\mathcal{M}^{\mathcal{Q}}$, mortality rates are independent by interest rates. Under this assumption, the price of a *zero-coupon longevity bond* that pays one unit of cash in case of life and zero in case of death of an individual aged x is

$$\bar{P}(t,T) = \mathbb{E}^{\mathcal{Q}} \left[\frac{\mathcal{C}(t)}{\mathcal{C}(T)} \middle| \mathscr{R}_{t} \right] \mathbb{E}^{\mathcal{Q}} \left[\frac{\Lambda_{\mu_{x}}(t)}{\Lambda_{\mu_{x}}(T)} \middle| \mathscr{M}_{t} \right] = \mathbb{E}^{\mathcal{Q}} \left[\frac{\mathcal{C}(t)}{\mathcal{C}(T)} \frac{\Lambda_{\mu_{x}}(t)}{\Lambda_{\mu_{x}}(T)} \middle| \mathscr{C}_{t} \right] = \mathbb{E}^{\mathcal{Q}} \left[\mathcal{D}(t,T) \mathcal{S}_{x}(t,T) \middle| \mathscr{C}_{t} \right] = P(t,T) S_{x}(t,T),$$

where \mathscr{C}_t is the combined filtration for both the term structure of mortality rates and the term structure of interest rates.

5.2.4 Temporary life annuity

A *life annuity* is an insurance contract where an insurer makes a series of future payments to a policyholder in exchange for an immediate payment of a lump sum (single-payment annuity) or a series of regular payments (regular-payment annuity). The value of the annuity depends by the survival probability of the insured.

We consider a temporary life annuity for n periods (at the beginning of the period), with respect to an insured aged x and a fixed benefit equal to one unit of cash. The insurer makes regular payments starting from the issue date of the contract. Assuming that the issue date T_0 of the annuity coincide with the valuation date t, the expected present value of the temporary life annuity, denoted by $A(t, T_n, x)$, with maturity at time T_n is

$$A(t, T_n, x) = \tau(T_0, T_1) + \sum_{i=1}^{n-1} \tau(T_i, T_{i+1}) P(t, T_i) S_x(t, T_i)$$
$$= \tau(T_0, T_1) + \sum_{i=1}^{n-1} \tau(T_i, T_{i+1}) \bar{P}(t, T_i),$$

where $\tau(T_i, T_{i+1})$ is the time measure as a fraction of the year between the dates T_i and T_{i+1} according to some convention.

5.2.5 Forward start temporary life annuity

We denote by $A(t, T_0, T_n, x)$ the value in t of a forward start temporary life annuity with start date in $T_0 > t$ and maturity in T_n . The present value of a forward start temporary life annuity is

$$A(t, T_0, T_n, x) = \sum_{i=0}^{n-1} \tau(T_i, T_{i+1}) P(t, T_i) S_x(t, T_i) = \sum_{i=0}^{n-1} \tau(T_i, T_{i+1}) \bar{P}(t, T_i).$$

5.3 Pricing life insurance contracts as a swap

5.3.1 Term assurance as a swap: pricing function

A term life insurance or term assurance is a life insurance contract which provides coverage for a limited period of time in exchange for premium payments. Although this form of life insurance can have a fixed or variable payment over time, here we only consider the fixed payment case. If the insured dies during the term, the death benefit will be paid to the beneficiary; no benefit is provided by the policy should the insured survive to the end of the policy period.

As explained above, a term assurance can be considered a swap in which policyholders exchange cash flows (premiums vs. benefits) with an insurer just as with a generic interest rate swap or credit default swap. The policyholder pays to an insurer a constant periodic premium Q (or a single premium U) to insure the life of an individual aged x (insured) against the death event during a certain number of years. We consider the case where the beneficiary of the contract receives a fixed amount C in the case of the insured's death. We assume that the payment related to the effective death time is postponed to the first discrete time T_i .

Consider a term assurance related to an individual aged x. Given a set of n annual payments at discrete times $T_1, T_2, ..., T_i, ..., T_n$, the expected present value of the term assurance at time $t = T_0 < T_1$ is the difference between the expected present value of the premium leg, denoted by $Leg_{pm}(t, T_n, x)$, and the expected present value of the protection leg, denoted by $Leg_{pr}(t, T_n, x)$.

Denoting by $Q_{Ta}(t, T_n, x)$ the premium of the term assurance, the expected present value of the premium leg is

$$Leg_{pm}(t, T_n, x) = Q_{Ta}(t, T_n, x)\tau(T_0, T_1)$$
$$+Q_{Ta}(t, T_n, x)\sum_{i=1}^{n-1}\tau(T_i, T_{i+1})P(t, T_i)S_x(t, T_i),$$

and the expected present value of the protection leg is

$$Leg_{pr}(t, T_n, x) = C \sum_{i=1}^{n} P(t, T_i) D_x(T_{i-1}, T_i).$$

At the valuation date t, the present value of the term assurance from the prospective of the insurance company, denoted by $TA^{(c)}(t, T_n, x)$, is

$$TA^{(c)}(t, T_n, x) = Leg_{pm}(t, T_n, x) - Leg_{pr}(t, T_n, x) = 0.$$

Analogously, the present value of the term assurance from the prospective of the policyholder, denoted by $TA^{(p)}(t, T_n, x)$, is

$$TA^{(p)}(t, T_n, x) = Leg_{pr}(t, T_n, x) - Leg_{pm}(t, T_n, x) = 0.$$

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Consequently, the value of the periodic premium is

$$Q_{Ta}(t, T_n, x) = \frac{C \sum_{i=1}^{n} P(t, T_i) D_x(T_{i-1}, T_i)}{\tau(T_0, T_1) + \sum_{i=1}^{n-1} \tau(T_i, T_{i+1}) P(t, T_i) S_x(t, T_i)}$$

Considering a contract with a single premium, it follows that

$$U_{Ta}(t, T_n, x) = Leg_{pr}(t, T_n, x).$$

For the policyholder (or the investor), a term assurance contract can be viewed as a long position on the death rate to n years: if the n year's death rate increases, the fair value of the contract increases from the prospective of the policyholder. From the prospective of the insurer (or another counterparty), the contract can be viewed as a short position on the n years death rate. Consequently, if the death rate decreases, the fair value of the contract increases from the prospective of the insurer.

5.3.2 Pure endowment as a swap: pricing function

A pure endowment is a life insurance contract which provides coverage with respect to payments for a limited period of time. In this case, if the insured does not dies during the term, a benefit C is payed at the end of the period. Also in this case, although this form of insurance can have a fixed payment or one that changes over time, here we only consider the fixed payment case.

A pure endowment can be considered as a swap where the policyholder, during a period of n years, pays to an insurer a constant periodic premium Q (or a single premium U) to ensure the payment of the fixed amount C if the insured of age x survives up to the year n.

Given a set of n annual payments at discrete times $T_1, T_2, ..., T_i, ..., T_n$, the present value of the premium leg and the present value of the protection leg, at time $t = T_0 < T_1$, are

$$Leg_{pm}(t, T_n, x) = Q_{Pe}(t, T_n, x)\tau(T_0, T_1) + Q_{Pe}(t, T_n, x) \sum_{i=1}^{n-1} P(t, T_i)\tau(T_i, T_{i+1})S_x(t, T_i),$$

and

$$Leg_{pr}(t, T_n, x) = P(t, T_n)S_x(t, T_n)C.$$

The present value of the pure endowment at time t, denoted by $PE(t, T_n, x)$, is the difference between the present value of the premium leg and the present value of the protection leg. At the valuation date t, the present value of the pure endowment from the prospective of the insurance company, denoted by $PE^{(c)}(t, T_n, x)$, is

$$PE^{(c)}(t, T_n, x) = Leg_{pm}(t, T_n, x) - Leg_{pr}(t, T_n, x) = 0.$$

Analogously, the present value of the pure endowment form the prospective of the policyholder, denoted by $PE^{(p)}(t, T_n, x)$, is

$$PE^{(p)}(t, T_n, x) = Leg_{pr}(t, T_n, x) - Leg_{pm}(t, T_n, x) = 0.$$

Consequently, the value of the periodic premium is

$$Q_{Pe}(t,T_n,x) = \frac{P(t,T_n)S_x(t,T_n)C}{\tau(T_0,T_1) + \sum_{i=1}^{n-1} P(t,T_i)\tau(T_i,T_{i+1})S_x(t,T_i)}.$$

Considering a contract with a single premium U, it follows that

$$U_{Pe}(t, T_n, x) = Leg_{pr}(t, T_n, x).$$

5.3.3 Endowment as a swap: pricing function

An *endowment* is a life insurance contract which provides coverage for a limited period of time in exchange for premium payments. If the insured dies during the term, the death benefit C will be paid to the beneficiary; if the insured does not dies during the term, the benefit C is payed at the end of the period. Consequently, an endowment can be considered as a combination of a term assurance and a pure endowment.

Also in this case, although this form of insurance can have a fixed payment or one that changes over time, here we only consider the fixed payment case.

An endowment can be considered as a swap where the policyholder, during a period of n years, pays to an insurer a constant periodic premium Q (or a single premium U) to ensure the payment of the fixed amount C.

Given a set of n annual payments at discrete times $T_1, T_2, ..., T_i, ..., T_n$, the present value of the premium leg and the present value of the protection leg, at time $t = T_0 < T_1$, are

$$Leg_{pm}(t, T_n, x) = Q_E(t, T_n, x)\tau(T_0, T_1) + Q_E(t, T_n, x) \sum_{i=1}^{n-1} P(t, T_i)\tau(T_i, T_{i+1})S_x(t, T_i),$$

and

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$$Leg_{pr}(t, T_n, x) = C \sum_{i=1}^{n} P(t, T_i) D_x(T_{i-1}, T_i) + P(t, T_n) S_x(t, T_n) C.$$

At the valuation date t, the present value of the endowment from the prospective of the insurance company, denoted by $E^{(c)}(t, T_n, x)$, is

$$E^{(c)}(t, T_n, x) = Leg_{pm}(t, T_n, x) - Leg_{pr}(t, T_n, x) = 0.$$

Analogously, the present value of the endowment form the prospective of the policyholder, denoted by $E^{(p)}(t, T_n, x)$, is

$$E^{(p)}(t, T_n, x) = Leg_{pr}(t, T_n, x) - Leg_{pm}(t, T_n, x) = 0.$$

Consequently, the value of the periodic premium will be such that,

$$Q_E(t,T_n,x) = \frac{C\sum_{i=1}^n P(t,T_i) D_x(T_{i-1},T_i) + P(t,T_n) S_x(t,T_n) C}{\tau(T_0,T_1) + \sum_{i=1}^{n-1} P(t,T_i) \tau(T_i,T_{i+1}) S_x(t,T_i)}.$$

Considering a contract with a single premium U, it follows that

$$U_E(t, T_n, x) = Leg_{pr}(t, T_n, x).$$

5.4 References

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Chapter 6

Calibrating affine stochastic mortality models using term assurance premiums

6.1 Introduction

A fundamental issue in the use of any stochastic mortality model involves the quantification of the parameters. Parameter estimation can be based on historical data employing various statistical procedures. This approach is appropriate for risk management purposes where parameter estimation is provided under real-world measure. However, for the purpose of pricing, a risk-neutral measure should be considered and risk-neutral parameter calibration should be provided using stochastic mortality models. Under a risk-neutral measure, the main issue involves the mortality risk premium estimation¹ deriving risk-neutral survival probabilities leading to an arbitrage-free price for mortality/longevity-linked insurance or financial products.

Several instruments such as so-called longevity or survivor bonds or survivor swaps have been proposed. In December 2003, Swiss Re issued a three-year life catastrophe bond. In November 2004, BNP Paribas announced the issuance of the first longevity bond, the so-called EIB/BNP bond,² a bond that has attracted considerable attention among practitioners and researchers. Cairns, Blake, and Dowd (2006a) used the published pricing value of the EIB/BNP bond to calibrate the parameters of a 2-factor stochastic mortality model.

¹See Milevsky, Promislow, and Young (2005).

²See Blake, Cairns, and Dowd (2006) and Cairns, Blake, Dawson, and Dowd (2005).

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We believe that the market for life insurance policies can be utilized for pricing the mortality risk premium. In this chapter, we develop a new procedure to calibrate the parameters of affine stochastic mortality models using insurance contract premiums. The fundamental idea is to utilize real quotes from simple insurance products such as term assurance contracts to calibrate the parameters of affine stochastic mortality models. We consider these life insurance contracts as a "swap" where the pricing function is similar to the pricing function of an interest rate swap or credit default swap.

An important step in the proposed model involves deriving the term structure of mortality rates by means of a bootstrapping technique, a procedure similar to bootstrapping of the default rates using credit default swaps. Then, the term structure of mortality rates is used to calibrate the parameters of affine stochastic mortality models by means of an optimization procedure.

Our model can be used for pricing mortality/longevity-linked securities and derivatives. Furthermore, it can be applied to the calculation of the technical provisions of insurance contracts under a market-consistent accounting regime. In fact, the introduction of the IFRS market-consistent accounting for insurance contracts (enforcement expected to begin in 2013) and the risk-based Solvency II requirements for the European insurance market (enforcement scheduled to begin in 2013) will involve taking into account the market-consistent value of the technical provisions related to insurance contracts. Under these regulations, insurance companies will have to identify all material contractual options embedded in the life insurance policies that they issue. Our model could represent a useful framework when a calibrated stochastic mortality model is needed in the evaluation of mortality/longevity-linked options.

We provide an empirical application of the model using premiums of contracts with different maturities issued by three Italian insurance companies. The performance of Vasicek, Cox-Ingersoll-Ross, and jump-extended Vasicek models are analyzed for individuals at different ages.

The organization of the chapter is as follows. In Section 6.2, the proposed model is described. The empirical results are presented in Section 6.3 and the conclusions are provided in Section 6.4.

6.2 Proposed model for calibrating affine stochastic mortality models on term assurance premiums

As discussed in the previous section, a fundamental issue in the use of any stochastic mortality model is the quantification of the parameters. Although historical parameter estimation under real-world measure is appropriate for risk management purposes, for pricing purposes a risk-neutral measure is needed.

A different approach to calibrating stochastic mortality models could be based on transactions in the life settlement market. In this market, the contract's policyholder can sell the policy to a third party (an investor). Transactions of this type, referred to as viatical settlements, have been available in the United States since 1911; the volume of these transactions was roughly \$18 to \$19 billion in 2009. Unfortunately, this form of investment is still underdeveloped in Europe and it is accessible only through hedge funds, structured products, and funds of funds for qualified investors.³

In order to provide an alternative approach for risk-neutral calibration of stochastic mortality models, the approach we propose involves estimating the parameters of affine stochastic mortality models using the quotes of life insurance contracts.⁴ More specifically, using insurance contracts such as term assurance, we infer the risk-neutral survival probability implied in the quotes. For this purpose, term assurance is treated as a "swap" (any insurance contract can be viewed as a swap) in which the policyholder (or the investor) exchanges with the insurer (or a new counterparty) the premium payments against the contingent benefit payment. Viewing these contracts as a swap, we propose a bootstrapping procedure to derive the term structure of mortality rates implied by the contracts. The term structure of mortality rates obtained by the bootstrapping procedure is used as an input to calibrate the parameters of affine stochastic mortality models by means of an optimization procedure.

It is important to note that our model requires estimating a different model

³See The Economist (2009) and United States Senate (2009) for further information.

⁴We specify the dynamics under a risk-neutral pricing measure Q. Unfortunately, at the present time the market for life insurance contracts is currently far from being liquid. Consequently, from a theoretical point of view the life insurance market is incomplete and the risk-neutral measure Q is not unique. A different way of generating risk-neutral measures involves using the Wang transform (Wang, 2000, 2002, 2003). Lin and Cox (2004), Denuit, Devolder and Goderniaux (2004), and Dowd, Blake, Cairns, and Dawson (2005) apply the Wang transform to mortality/longevity-linked securities. An alternative approach was adopted by the Solvency II requirements. Under Solvency II, the market-consistent value of the technical provisions for non-hedgeable risks (indicating market incompleteness) is given by the sum of a "best estimate" and a "risk margin" (see European Community (2009) and European Commission (2010) for details).

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for each cohort. Consequently, when several term structures of mortality rates have to be considered to manage different cohorts, the correlation between cohorts must be taken into account.

In summary, our proposed model involves the following four tasks: (1) obtaining real quotes of term assurance contracts; (2) recasting the pricing function of these contracts in terms of a swap; (3) employing a bootstrapping procedure to construct the mortality rates term structure, and; (4) calibrating the parameters of stochastic mortality models based on the bootstrapped term structure.

6.2.1 Term assurance as a swap: pricing function

Term life insurance or term assurance is a life insurance contract which provides coverage for a limited period of time in exchange for premium payments. Although this form of life insurance can have a fixed or variable payment over time, here we only consider the fixed payment case. If the insured dies during the term, the death benefit will be paid to the beneficiary; no benefit is provided by the policy should the insured survive to the end of the policy period.

As explained above, a term assurance can be considered a swap in which policyholders exchange cash flows (premiums vs. benefits) with an insurer just as with a generic interest rate swap or credit default swap. The policyholder pays to an insurer a constant annual premium Q (or a single premium U) to insure the life of an individual aged x (insured) against the death event during a certain number of years. We consider the case where the beneficiary of the contract receives a fixed amount C in the case of the insured's death. We assume that the payment related to the effective death time is postponed to the first discrete time T_i .

Consider a term assurance related to an individual aged x. Given a set of n annual payments at discrete time $T_1, T_2, ..., T_i, ..., T_n$, the expected present value of the term assurance at time $t < T_1$ is the difference between the expected present value of the premium leg, denoted by $TA_{pml}(t, T_n, x)$, and the expected present value of the protection leg, denoted by $TA_{prl}(t, T_n, x)$. Denoting by $Q(t, T_n, x)$ the premium of the term assurance, the expected present value of the premium leg is

$$TA_{pml}(t, T_n, x) = Q(t, T_n, x) + Q(t, T_n, x) \sum_{i=1}^{n-1} \tau(T_{i-1}, T_i) P(t, T_i) S_x(t, T_i),$$

and the expected present value of the protection leg is

$$TA_{prl}(t, T_n, x) = C \sum_{i=1}^n P(t, T_i) D_x(T_{i-1}, T_i),$$

where

 $P(t, T_i)$ = the price of a risk-free zero-coupon bond evaluated at time t with maturity at time $[T_i]$;

 $\tau(T_{i-1}, T_i)$ = the time measure as a fraction of the year between the dates T_{i-1} and T_i according to some convention;

 $D_x(T_{i-1}, T_i)$ = the death probability; it is the probability at time t that an individual aged x dies within the period $[T_{i-1}, T_i]$ with

$$D_x(T_{i-1}, T_i) = D_x(t, T_i) - D_x(t, T_{i-1});$$

 $S_x(t,T_i)$ = the survival probability; it is the probability at time t that an individual aged x dies after time T_i with

$$S_x(t,T_i) = 1 - D_x(t,T_i).$$

Consequently, the value of the premium is

$$Q(t, T_n, x) = \frac{C \sum_{i=1}^{n} P(t, T_i) D_x(T_{i-1}, T_i)}{1 + \sum_{i=1}^{n-1} \tau(T_{i-1}, T_i) P(t, T_i) S_x(t, T_i)}.$$

At the valuation date, the value of the swap, denoted by $TA(t, T_n, x)$, is equal to zero,

$$TA(t, T_n, x) = TA_{pml}(t, T_n, x) - TA_{prl}(t, T_n, x) = 0.$$

Considering a contract with a single premium, it follows that

$$U(t, T_n, x) = TA_{prl}(t, T_n, x).$$

Then, for the policyholder (or the investor), a term assurance contract can be viewed as a long position on the death rate to n years: if the n year's death rate increases, the fair value of the contract increases from the prospective of the policyholder. From the prospective of the insurer (or another counterparty), the contract can be viewed as a short position on the n years death rate. Consequently, if the death rate decreases, the fair value of the contract increases from the prospective of the insurer.

6.2.2 Bootstrapping the term structure of mortality rates from term assurance contracts

In this section, we derive a term structure of mortality rates from a vector of premiums related to term assurance quotes for different discrete maturities and

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for each x. The procedure is very similar to the bootstrapping of default rates related to a reference obligation/reference entity using quoted premiums for credit default swaps (CDS).⁵ CDS contracts with different maturities are used to extract the piecewise constant default rates using an iterative procedure. Our approach is similar where term assurance contracts with different maturities are used.

In the bootstrapping procedure, we assume deterministic interest rates and mortality rates. In addition, we assume independence between interest rates and mortality rates, an assumption that we restrict to the bootstrapping procedure. In implementing pricing functions for mortality/longevity-linked securities, stochastic interest rates and mortality rates can be assumed and some dependence can be considered between interest rates and mortality (e.g., war and inflation, pandemic and economic growth, and the like).

By modeling the death event according to the Poisson distribution and denoting the mortality rate by $\bar{\mu}_x(t, T_i)$, it is possible to compute the survival probability of an individual aged x by means of the following relation

$$S_x(t,T_i) = e^{-\bar{\mu}_x(t,T_i)\tau(t,T_i)}.$$

Consequently, it is possible to express the mortality rate as

$$\bar{\mu}_x(t,T_i) = -\frac{\log[S_x(t,T_i)]}{\tau(t,T_i)}.$$

The vector of mortality rates related to the respective maturities represents the term structure of mortality rates. It also can be expressed in terms of the survival probabilities computed according to the relation (8).

From a series of maturities, $T_1, T_2, ..., T_i, ..., T_n$, we develop the bootstrapping procedure to obtain a vector of mortality rates that represents the term structure of mortality rates, $\bar{\mu}_x(t, T_1), \bar{\mu}_x(t, T_2), ..., \bar{\mu}_x(t, T_i), ..., \bar{\mu}_x(t, T_n)$. Suppose that a set of *n* term assurance contracts is quoted in terms of their annual premiums $Q(t, T_1, x), Q(t, T_2, x), ..., Q(t, T_i, x), ..., Q(t, T_n, x)$ with respect to maturities $T_1, T_2, ..., T_i, ..., T_n$.⁶ Starting from a term assurance contract with maturities T_1 and setting $T_0 = t$, the pricing formula is

$$Q(t, T_1, x) - P(t, T_1)D_x(t, T_1)C = 0.$$

After setting

$$D_x(t,T_1) = 1 - S_x(t,T_1) = 1 - e^{-\bar{\mu}_x(t,T_1)\tau(t,T_1)},$$

⁵See Schonbucher (2000).

⁶Pure premiums are considered in order to obtain the term structure of mortality rates. In practice, only the part of the premium which is sufficient to pay losses and loss adjustment expenses is considered, but not other expenses. The various types of loading (commission, expenses, taxes, and so on) are ignored.

it follows that

$$Q(t, T_1, x) - P(t, T_1) \left[1 - e^{-\bar{\mu}_x(t, T_1)\tau(t, T_1)} \right] C = 0.$$

Solving with respect to $\bar{\mu}_x(t, T_1)$ we obtain

$$\bar{\mu}_x(t,T_1) = -\frac{1}{\tau(t,T_1)} \ln \left[1 - \frac{Q(t,T_1,x)}{P(t,T_1)C} \right].$$

So, considering a contract with maturity T_2 , it is possible to compute $\bar{\mu}_x(t, T_2)$ given $\bar{\mu}_x(t, T_1)$ as an input to the following pricing function

$$Q(t, T_2, x) + Q(t, T_2, x)\tau(t, T_1)P(t, T_1)S_x(t, T_1) - C\sum_{i=1}^2 P(t, T_i)D_x(T_{i-1}, T_i) = 0$$

Solving with respect to $\bar{\mu}_x(t, T_2)$ we obtain

$$\begin{split} \bar{\mu}_x(t,T_2) &= -\frac{1}{\tau(t,T_2)} \ln \left[1 - \frac{Q(t,T_2,x) \left[1 + \tau(t,T_1) P(t,T_1) S_x(t,T_1) \right]}{P(t,T_2) C} \right] \\ &- \frac{D_x(t,T_1) \left[P(t,T_2) - P(t,T_1) \right]}{P(t,T_2)} \end{split}$$

Iterating the procedure described above up to n, the term structure of mortality rates, $\bar{\mu}_x(t, T_1), \bar{\mu}_x(t, T_2), ..., \bar{\mu}_x(t, T_i), ..., \bar{\mu}_x(t, T_n)$, is obtained with respect to the maturities $T_1, T_2, ..., T_i, ..., T_n$.

6.2.3 Affine stochastic models as mortality models

To model the force of mortality, we employ the same affine stochastic models utilized in interest rate and credit risk modeling.⁷ Affine stochastic models such as the Vasicek, Cox-Ingersoll-Ross, and jump-extended Vasicek models are analyzed in the calibration procedure. It is important to note that modeling the force of mortality as an affine function leads to the analytical representations of survival probabilities with closed-form solution.

Denoting by $\mu_x(t)$ the stochastic force of mortality, below we provide a brief description of the three affine models showing the related closed formula for survival probability.

Vasicek model

For the Vasicek (1977) model, we assume that the force of mortality follows the stochastic differential equation

$$d\mu_x(t) = k(\theta - \mu_x(t))dt + \sigma dW(t), \qquad \mu_x(0) = \mu_x$$

⁷See Brigo and Mercurio (2006) for further details.

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with μ_x and σ positive constants, θ constrained to be positive, and k constrained to be strictly negative. The main drawback of this process is that the force of mortality can be negative with positive probability. For this model, the survival probability can be obtained by

$$\begin{split} S_x(t,T) &= G_\mu(t,T) e^{-H_\mu(t,T)\mu_x(t)}, \\ G_\mu(t,T) &= exp \bigg\{ \bigg(\theta - \frac{\sigma^2}{2k^2} \bigg) \bigg[H_\mu(t,T) - \tau(t,T) \bigg] - \frac{\sigma^2}{4k} H_\mu(t,T)^2 \bigg\}, \\ H_\mu(t,T) &= \frac{1}{k} \bigg[1 - e^{-k\tau(t,T)} \bigg]. \end{split}$$

Cox-Ingersoll-Ross model

Assuming the dynamic of the Cox, Ingersoll, and Ross (1985) model (CIR model hereafter), the force of mortality $\mu_x(t)$ satisfies

$$d\mu_x(t) = k(\theta - \mu_x(t))dt + \sigma \sqrt{\mu_x(t)}dW(t), \qquad \mu_x(0) = \mu_x,$$

with μ_x and σ positive constants, θ constrained to be positive, and k constrained to be strictly negative. The principal advantage of the CIR model over the Vasicek model is that the hazard rate is guaranteed to remain non-negative. However, the condition $2k\theta > \sigma^2$ is not applicable and the hazard rates can be equal to zero with positive probability. Survival probabilities can still be computed analytically and are given by

$$\begin{split} S_x(t,T) &= G_{\mu}(t,T)e^{-H_{\mu}(t,T)\mu_x(t)},\\ \gamma &= \sqrt{k^2 + 2\sigma^2},\\ G_{\mu}(t,T) &= \left[\frac{2\gamma e^{\frac{1}{2}(k+\gamma)\tau(t,T)}}{2\gamma + (k+\gamma)(e^{\gamma\tau(t,T)} - 1)}\right]^{\frac{2k\theta}{\sigma^2}},\\ H_{\mu}(t,T) &= \frac{2(e^{\gamma\tau(t,T)} - 1)}{2\gamma + (k+\gamma)(e^{\gamma\tau(t,T)} - 1)}. \end{split}$$

Jump-extended Vasicek model

The jump-extended Vasicek model, proposed by Chacko and Das (2002), implies the following process to model the force of mortality

$$d\mu_x(t) = k(\theta - \mu_x(t))dt + \sigma dW(t) + J_u dN_u(\lambda_u) - J_d dN_d(\lambda_d)$$

6.2. PROPOSED MODEL FOR CALIBRATING AFFINE STOCHASTIC MORTALITY MODELS ON TERM ASSURANCE PREMIUMS

where J_u and J_d are exponentially distributed random variables with parameters η_u and η_d , respectively. In this model, $\mu_x(0)$, θ , and σ , are positive constants, and k is constrained to be strictly negative.⁸ Also for this model, survival probabilities can be computed analytically as

$$\begin{aligned} S_x(t,T) &= G_\mu(t,T)e^{-H_\mu(t,T)[\mu_x(t)-\theta]},\\ G_\mu(t,T) &= exp\bigg\{\bigg[\tau(t,T) - H_\mu(t,T)\bigg]\bigg(\frac{\sigma^2}{2k^2}\bigg) - \frac{\sigma^2 H_\mu(t,T)^2}{4k} + \\ &-(\lambda_u + \lambda_d)\tau(t,T) + \frac{\lambda_u \eta_u}{k\eta_u + 1}\log\bigg|\bigg(1 + \frac{1}{k\eta_u}\bigg)e^{k\tau(t,T)} - \frac{1}{k\eta_u}\bigg| + \\ &+ \frac{\lambda_d \eta_d}{k\eta_d - 1}\log\bigg|\bigg(1 - \frac{1}{k\eta_d}\bigg)e^{k\tau(t,T)} + \frac{1}{k\eta_d}\bigg| - \theta\tau(t,T)\bigg\},\end{aligned}$$

$$H_{\mu}(t,T) = \frac{1}{k} \left[1 - e^{-k\tau(t,T)} \right]$$

The above solution is identical to that given by Chacko and Das (2002), though expressed in a slightly different form.⁹

6.2.4 Model calibration

We calibrate each model's parameters by minimizing the sum of squares relative differences between mortality rates implied in the quotes and mortality rates implied by a specific affine model. This calibration technique is analogous to the calibration of affine stochastic interest rates models with respect to the term structure of interest rates.

The relative error is defined as

$$\varepsilon_i(\beta) = \frac{\bar{\mu}_x^{mkt}(t, T_i) - \bar{\mu}_x(t, T_i)}{\bar{\mu}_x^{mkt}(t, T_i)},$$

where $\bar{\mu}_x^{mkt}(t, T_i)$ is the mortality rate implied in the contracts and $\bar{\mu}_x(t, T_i)$ is the mortality rate computed using the survival probability closed-formula related to the considered affine model.

Denoting by β the set of the parameters of the affine model, the calibration procedure is such that

$$\widehat{\beta} = \arg\min_{\beta} \varepsilon'(\beta) \varepsilon(\beta).$$

⁸The model allows jumps with a positive size, in which case the mortality increases (in the case of wars, for instance), or jumps with a negative size, in which case mortality decreases (in the case of medical advancements, for instance).

⁹See Nawalkha, Beliaeva, and Soto (2007).

6.3 Empirical results

In this section, some numerical results related to premiums of Italian insurance companies are presented. We applied bootstrapping and calibration procedures to term assurance pure premiums of three Italian insurance companies in force during 2010:¹⁰

- 1. AXA MPS ASSICURAZIONI VITA S.p.A. AXA Group
- 2. CATTOLICA, Società Cattolica di Assicurazione S.C. CATTOLICA Group
- 3. GENERTEL LIFE S.p.A. GENERALI Group

More specifically, we used premiums with respect to males aged 30, 40, and 50. The premiums are denominated in euros and related to an insured amount of euro 1,000. For each age, only contracts with a maturity of 5, 10, 15, 20, and 25 years are available. Consequently, premiums related to intermediate maturities are obtained by applying linear interpolation.¹¹ The data are reported in Table 6.1-6.3. The pure premiums of AXA MPS and GENERTEL LIFE appear very similar. There are some differences with respect to the premiums of CATTOLICA.¹²

For the five contracts available for each company with different discrete maturities (5, 10, 15, 20, and 25 years), the term structure of mortality rates is derived from the vector of premiums related to term assurance with different time horizons and for each x. In order to evaluate the pricing function of the term assurance contracts and to implement the bootstrapping procedure, the term structure of risk-free interest rates denominated in euros as of December 31, 2009 was used.¹³

$$Q(t, T_i, x) = \frac{Q(t, T_{j-1}, x)(T_j - T_i) + Q(t, T_j, x)(T_i - T_{j-1})}{T_j - T_{j-1}}.$$

 $^{12}\mathrm{A}$ further analysis could be provided using re-insurance rates but it was difficult to find such data.

¹⁰Data used for the analysis are public. Premium data are reported in the informative-sheet available on the web-site of the companies:

AXA MPS: http://www.axa-mpsvita.it/

CATTOLICA: http://www.cattolica.it/

GENERTEL LIFE: http://www.genertellife.it/

¹¹Suppose that a series of maturities, $(T_1, T_2, ..., T_j, ..., T_m)$ and the related vector of premiums $(Q(t, T_1, x), Q(t, T_2, x), ..., Q(t, T_j, x), ..., Q(t, T_m, x))$ are available in the market. If the consecutive maturities of two contracts are T_{j-1} and T_j and the premiums related to these maturities are $Q(t, T_{j-1}, x)$ and $Q(t, T_j, x)$, then the interpolated premium $Q(t, T_i, x)$ related to the maturity T_i , such that $T_{j-1} < T_i < T_j$, is given by:

¹³This is the official EUR-Swap yield curve, without illiquidity premium, adopted in the 5th Quantitative Impact Study of Solvency II (QIS5). Yield curve data are available on the web site of EIOPA, the European Insurance and Occupational Pensions Authority (http://www.eiopa.europa.eu/). Since annual interest rates are published, we computed the risk-free zero-coupon

	AXA MPS	CATTOLICA	GENERTEL LIFE
Year	$Q(t, T_j, x)$	$Q(t, T_j, x)$	$Q(t, T_j, x)$
5	1,017391	0,918919	0,990000
10	1,121739	0,891892	1,090000
15	1,286957	0,954955	1,260000
20	1,565217	1,144144	1,530000
25	1,965217	1,423423	1,920000

Table 6.1: Premiums in euros for an insured amount of euro 1,000. Age = 30

Table 6.2: Premiums in euros for an insured amount of euro 1,000. Age = 40

	AXA MPS	CATTOLICA	GENERTEL LIFE
Year	$Q(t, T_j, x)$	$Q(t, T_j, x)$	$Q(t, T_j, x)$
5	1,704348	1,126126	1,670000
10	2,182609	1,522523	2,130000
15	2,800000	2,000000	2,740000
20	$3,\!686957$	2,675676	$3,\!610000$
25	4,800000	3,540541	4,690000

Tables 6.4-6.6 report the results of the bootstrapping procedure for three ages (x = 30, 40, 50) and for each insurance company. The first column in the table contains the maturity (in years) of each contract while the other columns contain for each insurance company the pure premiums (Q_i) and the values of the term structure of mortality rates obtained by the bootstrapping procedure $(\bar{\mu}_x(t, T_i))$. We can see from Table 2 that the term structures of mortality rates increase exponentially across time for each age and this result is consistent with the biological concept of organism senescence.

Beginning with the term structure of mortality rates bootstrapped as explained in Section 6.2, the Vasicek, Cox-Ingersoll-Ross, and jump-extended Vasicek models were calibrated. The mean-square error (MSE) and the euro calibration error $(ECE)^{14}$ for each model are reported in the Tables 6.7-6.9, along with optimal values for the parameters. The MSE and ECE are very low in all the models investigated, indicating a good fitting of the survival probability implied in the quotes.

bond price as

$$P(t,T_i) = \left(1 + \bar{r}(t,T_i)\right)^{-\tau(t,T_i)}$$

where $\bar{r}(t, T_i)$ is the deterministic and piecewise constant risk-free interest rate.

¹⁴The ECE is the euro difference between (1) the value of a contract with an insured amount of euro 1,000 and maturity of 20 years that is computed by applying the mortality rates derived by bootstrapping and (2) the value of a contract in which the mortality rates are derived by the model.

Table 6.3 :	Premiums	in eu	ros fo	an :	insured	amount	of	euro	$1,\!000.$	Age =	= 50
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	AXA MPS	CATTOLICA	GENERTEL LIFE
Year	$Q(t,T_j,x)$	$Q(t,T_j,x)$	$Q(t, T_j, x)$
5	4,382609	3,315315	4,280000
10	5,791304	4,414414	5,660000
15	7,452174	5,810811	7,280000
20	9,730435	7,459459	9,510000
25	12,573913	9,297297	12,290000

Table 6.4: Bootstrapping procedure results: interpolated premiums and term structure of mortality rates. Age = 30

	AXA	MPS	CATTO	DLICA	GENERTEL LIFE		
Year	$Q(t,T_i,x)$	$\bar{\mu}_x(t,T_i)$	$Q(t,T_i,x)$	$\bar{\mu}_x(t,T_i)$	$Q(t,T_i,x)$	$\bar{\mu}_x(t,T_i)$	
5	1,017391	0,001045	0,918919 0,000944		0,990000	0,001017	
6	5 1,038249 0,001071		0,913516	0,000940	1,009989	0,001042	
7	1,059165	0,001097	0,908099	0,000935	1,030033	0,001066	
8	1,080023	0,001123	0,902697	0,000930	1,050022	0,001092	
9	1,100881	0,001149	0,897294	0,000926	1,070011	0,001117	
10	1,121739	0,001176	0,891892	0,000920	1,090000	0,001143	
11	1 1,154837 0,001219		0,904525	0,000938	1,124056	0,001187	
12	1,187844	0,001263	0,917124	0,000956	1,158019	0,001232	
13	1,220851	0,001307	0,929723	0,000974	1,191981	0,001278	
14	1,253859	0,001353	0,942322	0,000992	1,225944	0,001324	
15	1,286957	0,001398	0,954955	0,001010	1,260000	0,001371	
16	1,342578	0,001479	0,992772	0,001066	1,313970	0,001450	
17	1,398200	0,001561	1,030589	0,001123	1,367941	0,001529	
18	1,453822	0,001644	1,068406	0,001180	1,421911	0,001609	
19	1,509596	0,001728	1,106327	0,001238	1,476030	0,001691	
20	1,565217	0,001813	1,144144	0,001296	1,530000	0,001773	
21	1,645174	0,001941	1,199969	0,001386	1,607957	0,001898	
22	1,725130	0,002071	1,255795	0,001476	$1,\!685915$	0,002024	
23	1,805305	0,002202	1,311773	1,311773 0,001568		0,002151	
24	1,885261	0,002334	1,367598 0,001660		1,842043	0,002280	
25	1,965217	0,002468	1,423423	0,001753	1,920000	0,002410	

6.4 Conclusions

In this chapter, we demonstrate how the use of term assurance contracts premiums can be utilized to derive the term structure of implied mortality rates and how to calibrate the parameters of affine stochastic mortality models. We provide a new procedure for estimating affine models based on insurance contract premiums.

In the approach we present, term assurance contracts are viewed as a swap in which a policyholder (or the investor) exchanges cash flows with an insurer (or a new counterparty) as in a generic interest rate swap or credit default swap. By employing the bootstrapping procedure that we propose, the term structure of mortality rates can be derived from insurance contract premiums. The technique, analogous to the bootstrapping procedure used to generate the term structure of default rates, is then used to calibrate the parameters of affine stochastic mor-

	AXA	MPS	CATTO	DLICA	GENERTEL LIFE		
Year	$Q(t,T_i,x)$	$\bar{\mu}_x(t,T_i)$	$Q(t,T_i,x)$	$\bar{\mu}_x(t,T_i)$	$Q(t,T_i,x)$	$\bar{\mu}_x(t,T_i)$	
5	1,704348	0,001751	1,126126 $0,001157$		1,670000	0,001716	
6	6 1,799948 0,001864		1,205362 0,001249		1,761950	0,001824	
7	1,895809	0,001977	1,284815	0,001342	1,854151	0,001933	
8	1,991409	0,002092	1,364051	0,001437	1,946101	0,002044	
9	2,087009	0,002210	1,443287	0,001534	2,038050	0,002157	
10	2,182609	0,002329	1,522523	0,001632	2,130000	0,002272	
11	2,306290	0,002488	1,618175	0,001754	2,252200	0,002429	
12	2,429633	0,002650	1,713566	0,001878	2,374067	0,002589	
13	2,552976	0,002816	1,808957	0,002005	2,495933	0,002753	
14	2,676319	0,002985	1,904348	0,002134	$2,\!617800$	0,002920	
15	2,800000	0,003157	2,000000	0,002267	2,740000	0,003090	
16	2,977294	0,003417	2,135061	0,002462	2,913905	0,003344	
17	3,154588	0,003681	2,270122	0,002661	3,087809	0,003604	
18	3,331882	0,003951	2,405183	0,002864	3,261714	0,003868	
19	3,509662	0,004226	2,540615	0,003070	$3,\!436095$	0,004137	
20	3,686957	0,004506	2,675676	0,003280	$3,\!610000$	0,004412	
21	3,909443	0,004873	2,848554	0,003560	3,825882	0,004767	
22	4,131930	0,005246	3,021432	0,003845	4,041763	0,005128	
23	4,355026	0,005628	3,194784	0,004135	4,258237	0,005498	
24	4,577513	0,006017	3,367662	0,004429	4,474118	0,005874	
25	4,800000	0,006414	3,540541	0,004729	4,690000	0,006257	

Table 6.5: Bootstrapping procedure results: interpolated premiums and term structure of mortality rates. Age = 40

Table 6.6: Bootstrapping procedure results: interpolated premiums and term structure of mortality rates. Age = 50

	AXA	MPS	CATTO	DLICA	GENERT	EL LIFE
Year	$Q(t,T_i,x)$	$\bar{\mu}_x(t,T_i)$	$Q(t,T_i,x)$	$\bar{\mu}_x(t,T_i)$	$Q(t,T_i,x)$	$\bar{\mu}_x(t,T_i)$
5	4,382609	0,004509	3,315315	0,003409	4,280000	0,004404
6	4,664194 0,004842		3,535015 0,003668		4,555849	0,004729
7	4,946550	0,005178	3,755316	0,003929	4,832453	0,005059
8	5,228135	0,005522	3,975016	0,004195	5,108302	0,005395
9	5,509720	0,005874	4,194715	0,004467	$5,\!384151$	0,005739
10	5,791304	0,006233	4,414414	0,004745	$5,\!660000$	0,006091
11	6,124024	0,006670	4,694152	0,005110	5,984532	0,006516
12	6,455834	0,007118	4,973126	0,005483	6,308177	0,006953
13	6,787644	0,007578	5,252099	0,005865	$6,\!631823$	0,007401
14	7,119455	0,008051	5,531073	0,006257	6,955468	0,007861
15	7,452174	0,008537	5,810811	0,006660	7,280000	0,008335
16	7,907577	0,009241	6,140360	0,007155	7,725756	0,009023
17	8,362979	0,009966	6,469909	0,007662	$8,\!171512$	0,009731
18	8,818382	0,010714	6,799458	0,008184	8,617267	0,010460
19	9,275032	0,011486	7,129910	0,008720	9,064244	0,011214
20	9,730435	0,012284	7,459459	0,009271	9,510000	0,011992
21	10,298819	0,013343	7,826826	0,009911	10,065696	0,013022
22	10,867203	0,014443	8,194192	0,010570	10,621391	0,014092
23	11,437145	0,015592	8,562565	0,011250	$11,\!178609$	0,015208
24	12,005529	0,016792	8,929931	0,011952	11,734304	0,016372
25	12,573913	0,018052	9,297297	0,012677	12,290000	0,017593

	AXA MPS			CATTOLICA			GENERTEL LIFE		
	VAS	CIR	JVAS	VAS	CIR	JVAS	VAS	CIR	JVAS
μ_0	0,0010	0,0010	0,0001	0,0008	0,0008	0,0000	0,0009	0,0009	0,0001
k	- 0,1660	-0,1509	- 0,1696	- 0,2183	- 0,1931	-0,1970	- 0,1660	-0,1352	-0,1705
θ	0,0008	0,0008	0,0008	0,0008	0,0008	0,0007	0,0008	0,0007	0,0007
σ	0,0002	0,0057	0,0002	0,0001	0,0025	0,0001	0,0002	0,0014	0,0002
λ_u	-	-	0,0001	-	-	0,0000	-	-	0,0001
η_u	-	-	0,0001	-	-	0,0002	-	-	0,0001
λ_d	-	-	0,0000	-	-	0,0001	-	-	0,0000
η_d	-	-	0,0001	-	-	0,0000	-	-	0,0001
MSE	0,0006	0,0007	0,0004	0,0150	0,0074	0,0070	0,0003	0,0009	0,0004
ECE	0,1796	0,0866	0,1388	0,1244	0,0587	0,0724	0,1448	0,0585	0,1535

Table 6.7: Calibrated parameters. Age = 30 (VAS = Vasicek model; CIR = Cox-Ingersoll-Ross model; JVAS = jump-extended Vasicek model).

Table 6.8: Calibrated parameters. Age = 40 (VAS = Vasicek model; CIR = Cox-Ingersoll-Ross model; JVAS = jump-extended Vasicek model).

	AXA MPS			0	CATTOLIC	A	GENERTEL LIFE		
	VAS	CIR	JVAS	VAS	CIR	JVAS	VAS	CIR	JVAS
μ_0	0,0014	0,0014	0,0009	0,0009	0,0009	0,0009	0,0014	0,0014	0,0007
k	- 0,1269	-0,1269	- 0,1300	- 0,1160	-0,1156	-0,1159	- 0,1299	- 0,1321	-0,1405
θ	0,0005	0,0005	0,0005	0,0000	0,0000	0,0000	0,0005	0,0005	0,0000
σ	0,0006	0,0130	0,0006	0,0005	0,0125	0,0005	0,0006	0,0141	0,0006
λ_u	-	-	0,0000	-	-	0,0000	-	-	0,0007
η_u	-	-	0,0001	-	-	0,0001	-	-	0,0929
λ_d	-	-	0,0000	-	-	0,0000	-	-	0,0000
η_d	-	-	0,0002	-	-	0,0001	-	-	0,0000
MSE	0,0006	0,0006	0,0006	0,0009	0,0010	0,0009	0,0005	0,0006	0,0005
ECE	0,2010	0,1883	0,2274	0,2098	0,2026	0,2043	0,1792	0,1776	0,2364

Table 6.9: Calibrated parameters. Age = 50 (VAS = Vasicek model; CIR = Cox-Ingersoll-Ross model; JVAS = jump-extended Vasicek model).

		AXA MPS		CATTOLICA			GENERTEL LIFE		
	VAS	CIR	JVAS	VAS	CIR	JVAS	VAS	CIR	JVAS
μ_0	0,0035	0,0035	0,0033	0,0025	0,0025	0,0025	0,0034	0,0034	0,0014
k	- 0,1124	-0,1174	- 0,1117	- 0,1175	- 0,1183	-0,1176	- 0,1105	- 0,1104	- 0,1550
θ	0,0002	0,0005	0,0000	0,0000	0,0000	0,0000	0,0001	0,0001	0,0000
σ	0,0003	0,0091	0,0001	0,0013	0,0218	0,0013	0,0000	0,0001	0,0007
λ_u	-	-	0,0002	-	-	0,0000	-	-	0,0035
η_u	-	-	0,0021	-	-	0,0000	-	-	1,1122
λ_d	-	-	0,0000	-	-	0,0000	-	-	0,0000
η_d	-	-	0,0001	-	-	0,0001	-	-	0,0000
MSE	0,0009	0,0010	0,0009	0,0003	0,0002	0,0003	0,0009	0,0009	0,0003
ECE	0,7405	0,8538	0,7150	0,4217	0,3483	$0,\!4252$	0,6913	0,7018	0,6394

6.4. CONCLUSIONS

tality models by means of an optimization procedure.

Three affine models are investigated for fitting the term structure of mortality rates: the Vasicek, Cox-Ingersoll-Ross, and jump-extended Vasicek models. Because the survival probability implied in the affine models is expressed in closed form, by minimizing the differences between mortality rates implied in quotes and theoretical ones, we derive the value of the parameters of affine stochastic mortality models. Using term assurance premiums of three Italian insurance companies, we find support for fitting the term structure of mortality rates.

The calibrated affine stochastic mortality models can be used for the pricing of mortality/longevity-linked securities. In particular, it can be implemented in order to obtain a market-consistent assessment of the technical provisions of insurance contracts where the evaluation of embedded options is mandated under the IFRS market-consistent accounting framework and Solvency II requirements.

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Chapter 7

Pricing of extended coverage options embedded in life insurance policies

7.1 Introduction

The evaluation of the options embedded in life insurance policies is very present. In fact, under the new IAS/IFRS market-consistent accounting for insurance contracts (to be approval) and the risk-based Solvency II requirements for the European insurance market (enforcement to begin in 2013), insurance companies will have to identify all material contractual options embedded in life insurance policies.

An embedded option provides to the policyholder the right to modify the contract conditions during the contract term. Exercising it can affect the amount of the policy's cash flows and their payment time. Consequently, embedded options can represent a substantial value for the policyholder.¹

A particular type of embedded option is the *extended coverage option*. Such options are common in the European insurance market as embedded options in life insurance contracts. The extended coverage option gives to the policyholder the right to extend the policy's maturity at the expiry of the original contract maintaining the contractual conditions in force at the issue date without producing further evidence of health.²

¹See Gatzert (2009) for an overview on the options embedded in life insurance contracts. ²See European Commission (2010).

CHAPTER 7. PRICING OF EXTENDED COVERAGE OPTIONS EMBEDDED IN 120 LIFE INSURANCE POLICIES

In the recent actuarial literature, a growing attention has been devoted to the valuation of options embedded in life insurance contracts. Unfortunately, literature references related to the valuation of the extended coverage option are very poor. The valuation of such option involves considering many aspects in the pricing depending by the characteristics of the life insurance contract in which the option is embedded. For example, the valuation model has to take into account if the policyholder pays single or periodic premium. Furthermore, the value of the option depends not only by financial risk factors as in the classical financial options but it is affected also by actuarial risk factors such as lapse or mortality rates.

In this chapter, we develop a pricing model to evaluate the extended coverage option taking into account of interest rates and mortality rates as the main risk factors that impact the valuation. We provide a pricing model in closed-form where the well-known Black's model (1976) for option pricing is used. We assume that the premium (single or periodic) of the life insurance contract is a lognormal martingale under an appropriate probability's measure.

The chapter is organized as follows. In the next section, we describe the main characteristics of the extended coverage option. In Section 7.3 we present the pricing model and in Section 7.4 some numerical results are reported. Conclusive remarks are summarized in the last section.

7.2 Option design

The extended coverage option can be considered with respect to several types of life insurance contracts. However, we refer to an endowment life policy in which the extended coverage option is embedded.

An endowment is a life insurance contract which provides coverage for a limited period of time in exchange for premium payments. If the insured dies during the term, the death benefit will be paid to the beneficiary; if the insured does not dies during the term, the benefit is paid at the end of the period. If the extended coverage option is embedded in the contract, the policyholder has the right to extend the maturity of the contract for a certain period, postponing the payment of the final payout according to the deferment period. During the deferment period, the original contractual conditions are in force and the policyholder continues to benefit from the guarantees of the policy. The start date of the option coincides with the issue date of the policy while the exercise time coincides with its maturity date. Consequently, the maturity of the option coincides with the end date of the contract in which the option is embedded.

If the extended coverage option is exercised by the policyholder, an additional

premium has to be paid to the insurer in order to cover the additional guarantees in force during the deferment period. The additional premium will be single or periodic according to contractual conditions of the endowment contract.

Looking at the financial market, the extended coverage option can be considered as a European call option. In the case of the extended coverage option, the buyer is the policyholder, the seller is the insurer and the underlying is represented by the endowment life policy. The option gives its holder the right, but not the obligation, to enter in an endowment contract for a specified future time period and for a certain premium.

It is important to note that the extended coverage option is knocked out if the insured lapses the contract or deaths during the life of the option.

7.3 The proposed model

In order to evaluate the extended coverage option embedded in an endowment policy, we assume that the underlying of the option is represented by a *forward start endowment contract*. The start date of the forward contract coincides with the maturity of the endowment contract in which the option is embedded (the spot contract or the original contract) while the maturity is related to the deferment period. The forward contract has the same contractual conditions of the spot contract; the only difference is that the two contracts have different start and maturity dates.

In the case of endowment policy with single premium, the extended coverage option can be viewed as a bond option. Assuming that the single premium of the endowment is a lognormal martingale, we provide a closed-form pricing function using the well-known Black's model (1976) under an appropriate probability's measure. It is important to note that, unlike what occurs for the classical bond option where the underlying is represented by the forward price of the bond, in the case of the extended coverage option the underlying is represented by the forward single premium of the endowment policy.

In the case of endowment policy with periodic premium, we follow the approach suggested in Russo et al. (2011). We assume that a life insurance contract can be viewed as a swap, such as interest rate swap (IRS) or credit default swap (CDS). Consequently, an extended coverage option can be viewed as a European swap option, the so-called swaption, or a European credit default swap option. As in the previous case, we use the Black's model (1976) to quantify the value of the option assuming that the periodic premium is a lognormal martingale under an appropriate probability's measure. The approach we propose is similar to that of Jamshidian (1997) for the pricing of swaptions. We refer also to Hull and White (2003) and Schonbucher (2000) that propose a pricing model for the credit default swap option. However, unlike what occurs for the swaptions and credit

default swap options where the underlying is represented by a plain vanilla forward start IRS o CDS, in the case of the extended coverage option, the underlying is represented by the forward periodic premium of the endowment.

7.3.1 Assumptions

We refer to an endowment contract in which the extended coverage option is embedded. We consider both cases in which policyholder pays a single premium U or a constant periodic premium Q.³ We indicate by C the policy's benefit assuming that such benefit is constant. The quantity C represents the amount of money that the insurer pays to the policyholder in the case of the insured's death or at the maturity of the contract. Despite the benefit of an endowment policy is usually paid even if the policyholder decides to lapse the contract, we do not consider this case. Consequently, to achieve analytic tractability of our solution, we neglect the surrender option. Moreover, other characteristics of the endowment contracts as the profit sharing and the minimum guaranteed option are not considered.

The model we propose takes into account of interest rates and mortality rates as the main risk factors that affect the valuation of the extended coverage option. We assume that both interest rates and mortality rates are time-varying but deterministic. Furthermore, we use the convenient assumption that mortality rates are independent by interest rates.

7.3.2 Notation

In order to describe the model, we introduce and define the following quantities:

- x = reference age of the insured;
- t =value date;
- T_0 = issue date of the endowment contract, it coincides with the value date; it is also the issue date of the extended coverage option;
- T_n = maturity date of the spot endowment contract; it is also the maturity date of the extended coverage option; it coincides with the issue date of the forward endowment contract;
- T_m = maturity date of the forward endowment contract;
- $[T_n, T_m] = \text{deferment period};$

 $^{^3\}mathrm{Although}$ this form of insurance can have a fixed payment or one that changes over time, here we only consider the fixed payment case.

- $T_0, T_1, T_1, ..., T_i, ..., T_n$ = stream of payment dates of the spot endowment contract;
- $T_n, T_{n+1}, T_{n+2}, ..., T_i, ..., T_m$ = stream of payment dates of the forward endowment contract;
- $D_x(t, T_i)$ = death probability; it is the probability in t that an individual aged x dies within the period $[t, T_i]$; we have also that

$$D_x(T_{i-1}, T_i) = D_x(t, T_i) - D_x(t, T_{i-1});$$

• $S_x(t, T_i)$ = survival probability; it is the probability in t that an individual aged x dies after T_i and is such that

$$S_x(t,T_i) = 1 - D_x(t,T_i);$$

- $P(t, T_i)$ = value of a risk-free zero coupon bond with maturity in T_i ; the zero coupon bond pays one unit of cash at the maturity date;
- $\bar{P}(t, T_i)$ = value of a zero coupon longevity bond with maturity in T_i ; with reference to an individual aged x, the longevity zero coupon pays one unit of cash at the maturity date in case of life and zero in case of death; it is such that

$$\bar{P}(t,T_i) = P(t,T_i)S_x(t,T_i);$$

- $\tau(T_i, T_{i+1})$ = time measure as a fraction of the year between the dates T_i and T_{i+1} computed according to some convention;
- $A(t, T_n, x)$ = value of a temporary life annuity (at the beginning of the period) with respect to an insured age x and a fixed benefit equal to one unit of cash; the annuity starts at time t and matures at time T_n ; it is such that

$$A(t, T_n, x) = \tau(T_0, T_1) + \sum_{i=1}^{n-1} \tau(T_i, T_{i+1}) P(t, T_i) S_x(t, T_i)$$
$$= \tau(T_0, T_1) + \sum_{i=1}^{n-1} \tau(T_i, T_{i+1}) \bar{P}(t, T_i).$$

7.3.3 Endowment pricing

Spot endowment

We consider an endowment life policy related to an individual aged x. Following the approach suggested in Russo et al. (2011), the expected present value of

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the endowment at time t is the difference between the expected present value of the premium leg, denoted by $Leg_{pm}(t, T_n, x)$, and the expected present value of the protection leg, denoted by $Leg_{pr}(t, T_n, x)$. The expected present value of the endowment contract, viewed from the prospective of the insurance company and denoted by $E^{(c)}(t, T_n, x)$, is

$$E^{(c)}(t, T_n, x) = Leg_{pm}(t, T_n, x) - Leg_{pr}(t, T_n, x) = 0,$$

while from the prospective of the policyholder it is denoted by $E^{(p)}(t, T_n, x)$ and is such that

$$E^{(p)}(t, T_n, x) = Leg_{pr}(t, T_n, x) - Leg_{pm}(t, T_n, x) = 0.$$

The expected present value of the protection leg is

$$Leg_{pr}(t, T_n, x) = C \sum_{i=1}^{n} P(t, T_i) D_x(T_{i-1}, T_i) + P(t, T_n) S_x(t, T_n) C,$$

while the pricing function of the premium leg depends from the premium type. In the case of periodic premiums, indicating by $Q(t, T_n, x)$ the premium of the endowment, the expected present value of the premium leg is

$$Leg_{pm}(t,T_n,x) = Q(t,T_n,x)\tau(T_0,T_1) + Q(t,T_n,x)\sum_{i=1}^{n-1} P(t,T_i)\tau(T_i,T_{i+1})S_x(t,T_i).$$

Consequently, the value of the periodic premium can be expressed as follows

$$Q(t,T_n,x) = \frac{C\sum_{i=1}^{n} P(t,T_i) D_x(T_{i-1},T_i) + P(t,T_n) S_x(t,T_n) C}{\tau(T_0,T_1) + \sum_{i=1}^{n-1} P(t,T_i) \tau(T_i,T_{i+1}) S_x(t,T_i)} = \frac{Leg_{pr}(t,T_n,x)}{A(t,T_n,x)}$$

Considering a contract with a single premium and denoting by $U(t, T_n, x)$ the premium of the endowment, it follows that

$$U(t, T_n, x) = Leg_{pr}(t, T_n, x).$$

Forward endowment

In order to evaluate the extended coverage option, we need to consider the value of a forward start endowment policy. The value in t of a forward start endowment with start date in T_n and maturity in T_m , viewed from the prospective of the policyholder and denoted by $E^p(t, T_n, T_m, x)$, is

$$E^{p}(t, T_{n}, T_{m}, x) = Leg_{pr}(t, T_{n}, T_{m}, x) - Leg_{pm}(t, T_{n}, T_{m}, x) = 0,$$

where the expected present value of the forward start protection leg is

$$Leg_{pr}(t, T_n, T_m, x) = C \sum_{i=n+1}^m P(t, T_i) D_x(T_{i-1}, T_i) + P(t, T_m) S_x(t, T_m) C.$$

As in the previous case, the pricing function of the forward start premium leg depends from the premium type. In the case of periodic premium, indicating by $Q(t, T_n, T_m, x)$ the *forward periodic premium* of the endowment, the expected present value of the forward start premium leg is

$$Leg_{pm}(t, T_n, T_m, x) = Q(t, T_n, T_m, x) \sum_{i=n}^{m-1} \tau(T_i, T_{i+1}) \bar{P}(t, T_i).$$

Consequently, it follows that

$$Q(t, T_n, T_m, x) = \frac{C \sum_{i=n+1}^{m} P(t, T_i) D_x(T_{i-1}, T_i) + P(t, T_m) S_x(t, T_m) C}{\sum_{i=n}^{m-1} \tau(T_i, T_{i+1}) \bar{P}(t, T_i)}$$

Considering a contract with a single premium and denoting by $U(t, T_n, T_m, x)$ the forward single premium of the endowment, it follows that

$$U(t, T_n, T_m, x) = \frac{Leg_{pr}(t, T_n, T_m, x)}{\bar{P}(t, T_n)}.$$

7.3.4 Option pricing in closed-form

Single premium contract

We assume that a single premium is paid by the policyholder at the inception date of the endowment contract. The extended coverage option is in-the-money for the policyholder when

$$\left[E^p(T_n, T_n, T_m, x)\right]^+ = \left[Leg_{pr}(T_n, T_n, T_m, x) - K\right]^+,$$

where K is the strike of the option and represents the premium leg of the forward contract. The strike is equal to the single premium paid by the policyholder and is fixed at the inception date of the contract. Since that

$$Leg_{pr}(T_n, T_n, T_m, x) = U(T_n, T_n, T_m, x),$$

it follows that

$$\left[E^{p}(T_{n}, T_{n}, T_{m}, x)\right]^{+} = \left[U(T_{n}, T_{n}, T_{m}, x) - K\right]^{+}.$$

In order to provide a pricing function for the extended coverage option, we assume the value of a zero coupon longevity bond as numeraire and define by $\mathcal{M}^{\bar{P}}$ the measure associated with this numeraire. Consequently, we compute the expectation of the above payoff under this measure

$$\mathbb{E}^{\bar{P}}\left[\left(U(T_n, T_n, T_n, x) - K\right)^+\right].$$

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We denote by $ECO_U(t, T_n, T_m, x, K)$ the value in t of the extended coverage option embedded in the endowment contract when a single premium is paid by the policyholder and the insured is aged x. We have that

$$ECO_U(t, T_n, T_m, x, K) = S_x(t, T_n)P(t, T_n)\mathbb{E}^{\bar{P}}\left[\left(U(T_n, T_n, T_m, x) - K\right)^+\right]$$
$$= \bar{P}(t, T_n)\mathbb{E}^{\bar{P}}\left[\left(U(T_n, T_n, T_m, x) - K\right)^+\right].$$

It is important to know that the option payoff is discounted using the price of a risk-free zero-coupon bond and multiplied by the survival probability. In fact, we consider the fact that the option is knocked out if the insured death during the life of the option.

We assume that the endowment single premium is lognormal. We assume also that, under the measure $\mathcal{M}^{\bar{P}}$, the premium is a martingale such that

$$dU(t, T_n, T_m, x) = U(t, T_n, T_m, x)\sigma_U dW^{\mathcal{P}},$$

where $W^{\bar{\mathcal{P}}}$ is a Brownian motion under the measure $\mathcal{M}^{\bar{\mathcal{P}}}$ and σ_U is the volatility of the single premium. Consequently, under the Black (1976) model, the value at time t of the extended coverage option is

$$ECO_U(t, T_n, T_m, x, K) = \bar{P}(t, T_n) \Big[U(t, T_n, T_m, x) N(d_1) - K N(d_2) \Big],$$

where

$$d_1 = \frac{\log\left(\frac{U(t,T_n,T_m,x)}{K}\right) + \frac{1}{2}\sigma_U^2(T_n - t)}{\sigma_U\sqrt{(T_n - t)}},$$

and

$$d_2 = d_1 - \sigma_U \sqrt{(T_n - t)}.$$

Periodic premium contract

Now, we consider the case of an endowment contract with periodic premiums. We assume that the extended coverage option is embedded in such contract. The pricing model is analogous to the case of single premium but, in this case, we have to consider a different numeraire. Since that the extended coverage option is in-the-money for the policyholder when

$$E^{p}(T_{n}, T_{n}, T_{m}, x) > 0, (7.-25)$$

the payoff of the extended coverage option can be formalized as follows

$$\left[E^{p}(T_{n}, T_{n}, T_{m}, x)\right]^{+} = \left[Leg_{pr}(T_{n}, T_{n}, T_{m}, x) - KA(T_{n}, T_{n}, T_{m}, x)\right]^{+},$$

where K is the strike of the option. The strike is equal to the periodic premium paid by the policyholder and is fixed at the inception date of the endowment contract. Multiplying and dividing, in the above formula, the protection leg by $A(T_n, T_n, T_m, x)$ we have that

$$\left[Q(T_n, T_n, T_m, x)A(T_n, T_n, T_m, x) - KA(T_n, T_n, T_m, x)\right]^+$$

= $\left[A(T_n, T_n, T_m, x)(Q(T_n, T_n, T_m, x) - K)\right]^+$
= $A(T_n, T_n, T_m, x)\left[Q(T_n, T_n, T_m, x) - K\right]^+.$

In order to provide a pricing function for the extended coverage option, we follow (1) the approach of Jamshidian (1997) for the pricing of swaptions and (2) the approaches of Hull and White (2003) and Schonbucher (2003) for the pricing of CDS options. However, unlike what occurs for the pricing of swaptions and credit default swap options, we assume the value of a temporary life annuity as numeraire and define by \mathcal{M}^A the measure associated with this numeraire. Consequently, we compute the expectation of the above payoff under this measure

$$\mathbb{E}^{A}\left[\left(Q(T_{n},T_{n},T_{n},x)-K\right)^{+}\right]$$

Denoting by $ECO_Q(t, T_n, T_m, x, K)$ the value in t of the extended coverage option embedded in the endowment contract when a periodic premium is paid, we have that

$$ECO_Q(t, T_n, T_m, x, K) = S_x(t, T_n)P(t, T_n)$$

$$\times A(T_n, T_n, T_m, x)\mathbb{E}^A \left[\left(Q(T_n, T_n, T_m, x) - K \right)^+ \right]$$

$$= \bar{P}(t, T_n)A(T_n, T_n, T_m, x)\mathbb{E}^A \left[\left(Q(T_n, T_n, T_m, x) - K \right)^+ \right].$$

Under the new measure \mathcal{M}^A , the endowment periodic premium is a martingale and under the assumption of log-normality it holds that

$$dQ(t, T_n, T_m, x) = Q(t, T_n, T_m, x)\sigma_Q dW^{\mathcal{A}},$$

where $W^{\mathcal{A}}$ is a Brownian motion under the measure $\mathcal{M}^{\mathcal{A}}$ and σ_Q is the volatility of the periodic premium. Applying the Black (1976) model, the value at time tof the extended coverage option is

$$ECO_Q(t, T_n, T_m, x, K) = A(t, T_n, T_m, x) \Big[Q(t, T_n, T_m, x) N(d_1) - KN(d_2) \Big],$$

where

$$d_1 = \frac{\log\left(\frac{Q(t,T_n,T_m,x)}{K}\right) + \frac{1}{2}\sigma_Q^2(T_n - t)}{\sigma_Q\sqrt{(T_n - t)}},$$

and

$$d_2 = d_1 - \sigma_Q \sqrt{(T_n - t)}.$$

7.4 Numerical results

In order to provide some numerical results, we consider endowment policies that mature, respectively, in 10 and 20 years. For each policy, we provide the results with respect to insured aged 30, 40 and 50 years. Both single and periodic premium (annual) are taken into account. We assume an insured amount equal to 1,000 EUR and, for the sake of simplicity, we did not allow for any costs. We assume that the endowment policy can be extended for 5 and 10 years after the maturity.

For our analysis, we use market data from October 31, 2011. The nominal term structure of interest rate is derived from traded instruments in the cash, futures and swap markets. We apply standard bootstrapping technique to derive the zero rates from the traded market instruments. As regards the mortality risk, we use the mortality rates bootstrapped using life insurance policies according to the method described in Russo et al. (2011). Mortality rates are derived by the term assurance pure premiums of Italian insurance companies in force during 2010.

We assume that the volatility of the endowment premium (single and periodic) is equal to 1%. This value was chosen on the base of an empirical analysis of the time series of the premiums related to Italian insurers.

Tables from 7.1 and 7.4 show the results. We have reported the value of the premium (single or periodic) of the endowment policy for both 10 and 20 years and with respect to the different ages. In the last column of the tables, the results on the pricing of the extended coverage option are reported for the two different level of maturity's extension. Moreover, we have reported strike level and forward premium's value. As in the case of the policy's premium, the option's value are expressed in Euro. It is worth to note that the option value is always a single value apart from the policy premium is single or periodic.

From the results arises that the value of this option is substantial. In particular, we can note that the option price is higher for older insured. We can note also that the option's value, mantaining constant the insured's age, increases with the maturity of the underlying contract.

Age	Endowment's	Maturity's	Strike	Forward	Option's
	premium	extension		premium	value
30	781,12	5	909,63	841,07	0,05
40	780,57	5	909,14	840,97	0,05
50	779,57	5	907,28	841,73	0,06
30	781,12	10	781,12	726,84	0,07
40	780,57	10	780,57	728,07	0,09
50	779,57	10	779.57	733,24	0.18

Table 7.1: Endowment that matured in 10 years with single premium.

Table 7.2: Endowment that matured in 20 years with single premium.

Age	Endowment's	Maturity's	Strike	Forward	Option's
	premium	extension		premium	value
30	570,20	5	909,63	883,97	3,46
40	572,97	5	909,14	883,96	3,35
50	583,71	5	907,28	884,49	$3,\!17$
30	570,20	10	781,12	795,67	12,32
40	572,97	10	780,57	805,62	$15,\!60$
50	583,71	10	779,57	834,35	24,87

Table 7.3: Endowment that matured in 10 years with periodic premium.

Age	Endowment's	Maturity's	Strike	Forward	Option's
	premium	extension		premium	value
30	79,21	5	158,35	153,13	1,55
40	79,48	5	158,53	154,11	2,09
50	80,58	5	159,24	157,07	4,30
30	79,21	10	79,21	78,31	4,37
40	79,48	10	79,48	79,70	7,76
50	80,58	10	80,58	83,79	21,53

Table 7.4: Endowment that matured in 20 years with periodic premium.

Age	Endowment's	Maturity's	Strike	Forward	Option's
	premium	extension		premium	value
30	35,07	5	158,35	153,13	8,31
40	35,73	5	158,53	161,30	$12,\!65$
50	37,97	5	159,24	171,04	$27,\!60$
30	35,07	10	79,21	90,12	52,70
40	35,73	10	79,48	94,20	65,29
50	37,97	10	80,58	107,11	92,13

7.5 Conclusion

In this chapter, we have presented a pricing model for the extended coverage options embedded in endowment life insurance contracts.

The extended coverage option gives to the policyholder the right to extend the term of the policy after the original maturity maintaining the contractual conditions as valid. This type of option is common in the European insurance market but literature's references related to the evaluation of such type of option are very poor.

We have proposed a pricing model taking into account interest rates and mortality rates as the main risk factors of the option. We provide an evaluation method in closed-form in which the well-known Black (1976) option pricing formula is used with the assumption that the premiums (single or periodic) of life insurance contracts are lognormal martingales under an appropriate probability's measure. In the case of endowment with single premium, we assume the extended coverage option can be viewed as a bond option. In the case of endowment with periodic premium, we assume insurance contracts as a swap, as in the case of an interest rate swap (IRS) or credit default swap (CDS), following the approach of Russo et al. (2011). Consequently, we consider the analogy between the extended coverage option and the options diffused in the financial markets such as swaptions or credit default swap options. In this case we follow the approach of Jamshidian (1997), Hull and White (2003), and Schonbucher (2000). However, unlike what occurs for the classical financial options, where the underlying is represented by the forward price of the bond or the premium of IRS/CDS, in the case of the extended coverage option the underlying is represented by the forward premium (single or periodic) of the endowment life insurance policy.

With respect to the case of endowment contracts where the extended coverage option is embedded, some numerical results are presented. From the results arises that the value of this option is substantial. An important issue is that the value of the option increases with the age of the insured and the maturity of the underlying contract.

Under the new IAS/IFRS market-consistent accounting for insurance contracts (to be approval) and the risk-based Solvency II requirements for the European insurance market (enforcement to begin in 2013), insurance companies will have to identify all material contractual options embedded in life insurance policies. Consequently, the proposed model could be useful under the new accounting and solvency regimes in order to evaluate the extended coverage option.

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Chapter 8

Market-consistent approach for with-profit life insurance contracts and embedded options: a closed formula for the Italian policies

8.1 Introduction

Recently, market-consistent valuation of insurance liabilities are becoming relevant for accounting and solvency purposes. Being insurance liabilities not traded, insurance companies have to provide the market-consistent value of the policies by means of quantitative models applying the fair value principle.

Looking at the approaches proposed by the Solvency II requirements (enforcement to begin in 2013) and the new IAS/IFRS principles (to be approval), the value of the insurance contracts can be computed as expected present value of future cash flows (including the value of embedded options and guarantees), the so-called *best estimate of liabilities* (BEL), plus one or more additional margins. According to the Solvency II directive,¹ the economic value of the technical provisions have to be calculated as sum of the *best estimate* and the *risk margin*. The best estimate corresponds to the probability weighted average of future cash flows taking into account of the time value of money. The risk margin is defined as the cost of providing an amount of eligible own funds equal to the *Solvency*

¹See European Community (2009).

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Capital Requirement² (SCR) computed with respect to the non-hedgeable risks.³ Furthermore, significant improvements to the IAS/IFRS principles related to the insurance contracts are expected by the International Accounting Standards Board (IASB) with the so-called IFRS 4 (Phase 2) project. According to the IASB proposal,⁴ insurance companies should computing the balance-sheet value of the insurance liabilities quantifying (1) a current estimate of the future cash flows (taking into account a discount rate that adjusts those cash flows for the time value of money), (2) an explicit risk adjustment and (3) a residual margin.

Considering the life insurance business, very common life insurance policies are the so-called *with-profit* policies also known as *participating* or *profit-sharing* policies. In these type of policies, the benefits of the contract increase across time according to a return that is related to the performance of an asset's portfolio, the so-called *segregated fund*. An important issue is that with-profits policies provide guaranteed benefits which protect the policyholder against the volatility of the financial markets. Consequently, such contracts are characterized by a low risk for the policyholders and a competitive return with respect to other financial or insurance products. In this case, financial guarantees are embedded in the contract and such guarantees are in the form of financial options.

The market-consistent valuation of such policies involves considering many aspects in their pricing depending by the nature of these liabilities. In the recent actuarial literature, a growing attention has been devoted to the market-consistent valuation of with-profit life insurance contracts.⁵ With reference to the Italian policies, several models have been proposed by Bacinello (2001), Bacinello (2003a), Bacinello (2003b), Pacati (2003), Andreatta e Corradin (2003), Baione et al. (2006), Castellani et al. (2007), Floreani (2007).

In this chapter, we propose a closed formula to provide the market-consistent value for the Italian with-profit life insurance policies. We focus on the expected present value of the cash flows, the so-called best estimate of liabilities (BEL). Moreover, we provide the market-consistent value for the financial options embedded in these contracts. In particular, we are able to evaluate in closed-form the

²According to the Solvency II directive, insurers should hold an amount of capital, the socalled *Solvency Capital Requirement*, that enables them to absorb unexpected losses and meet their obligations to policyholders. The calculation of this requirement must be made on the basis of the value at risk (VaR) with a confidence level of 99.5% over a time horizon of one year.

³Insurance liabilities are considered as non-hedgeable if the future cash-flows associated with those obligations cannot be replicated using financial instruments.

⁴See International Accounting Standards Board (2010).

⁵ Several contributions addressing fair valuation of with-profit policies with guarantees have been suggested by Briys and De Varenne (1997), Grosen and Jorgenses (2000), Jensen et al. (2001), Hansen and Miltersen (2002), Miltersen and Persson (2003), Tanskanen and Lukkarinen (2003), Bernard et al. (2005), Ballotta (2005), Ballotta et al. (2006), Bauer et al. (2006).
minimum quaranteed option (MGO) embedded in the Italian with-profit policies. It is worth to compute separately the value of the minimum guaranteed option because insurance companies need to quantify the risk arising from the minimum guaranteed level defined contractually. Greater is the option's value, greater is the risk for the insurer to not match the obligations towards the policyholders. In addition, according to the new IAS/IFRS accounting principles and Solvency II requirements, insurance companies have to identify and quantify all material contractual options and financial guarantees embedded in their contracts. We are able to compute in closed-form also the expected present value of the so-called future discretionary benefits (FDB) as requested by the Solvency II requirements.⁶ According to the Solvency II requirements, the future discretionary benefits represent a specific component of the best estimate of liablities and it has to be computed separately. As requested by the Solvency II regime, in calculating the best estimate insurance companies should take into account future discretionary benefits which are expected to be made, whether or not those payments are contractually guaranteed.

The model we propose consists in a valuation method simpler than the commonly used benchmark or reference models. We propose a simplified model following the approaches of Bacinello (2001) and Castellani et al. (2007) where the classical Black and Scholes framework is applied for the evaluation of Italian participating policies. However, the drawback of this approach is that the classical Black and Scholes framework is appropriate when the asset's fund is composed by equities while in the case of Italian with-profit policies asset's funds are composed mainly by fixed income instruments. In order to take into account the main features of the segregated fund, we propose a stochastic dynamic where the effective asset allocation of the fund is taken into account (i.e. fixed income instruments for the main part and equities) according to the method proposed by Vellekoop et al. (2005). Then, we derive a new closed form approach under the Black and Scholes framework where the volatility to put in the formula is computed as a function of the effective asset allocation of the segregated fund.

In summary our model consists in: (1) defining the functional form of the typical payoff for the Italian policies; (2) assuming a specific stochastic dynamic for the segregated fund to which the policies are linked; (3) assuming a Black and Scholes pricing framework; (4) deriving the volatility to put in the pricing model as a function of the effective asset allocation of the segregated fund; (5) deriving the closed-form for best estimate of liabilities, minimum guaranteed option and future discretionary benefits.

Being the proposed model based on a simplified approach, it could be used in

⁶See European Community (2009) and European Commission (2010).

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order to quantify the best estimate and the related embedded options under Solvency II when simplifications are allowed according to the principle of proportionality. In fact, according to the Solvency II principles, in order to compute the best estimate undertakings should apply actuarial and statistical methodologies that are proportionate to the nature, scale and complexity of the underlying risks.⁷ In addition, it could be apply in order to evaluate the expected present value of the future cash flows when the new IAS/IFRS principles related to the insurance contracts will be in force.

The chapter is organized as follows. In the next section, we describe the characteristics of the Italian with-profit life insurance policies while in Section 8.3, we present the model. In Section 8.4, the calibration method for the model is discussed while in the Section 8.5 some numerical results are reported. Conclusive remarks are summarized in the last section.

8.2 Contract design

Looking at the Italian insurance market, with-profit policies represent a large component of the life insurance business.

In these type of policies, the premium paid by the policyholders is invested in an asset's portfolio, the segregated fund, for which life insurance company bears the investment's risk. The contractual benefits are paid to the policyholder at the maturity date of the contract or if the insured dies within the contract's term. The benefits can be paid also if the policyholder decides to lapse the contract when the lapse option is included in the contract.

The benefits of the contract are linked to the return of the segregated fund that is internally managed by the insurance company and not directly traded on the financial markets. The segregated fund is usually composed, for the main part, by fixed income instruments and it is totally managed by the insurer. Consequently, the performance of the asset's portfolio depends, unless market conditions, by the company's investment approach in terms of market views, asset allocation, etc...

The profits deriving from the asset's portfolio are shared between the insurer and the policyholder and periodically an interest rate is credited to the policy on the base of a specific distribution mechanism. This mechanism obviously plays a crucial role in the determination of the value of the contract. In the Italian policies, the return credited to the policyholders in a specified period is computed as a function of the net income and the average book value of the segregated fund in the period. Consequently, the benefits of the contracts are revaluated using a

⁷See European Community (2009) and European Commission (2010).

return rate that is not market-based.⁸

An important issue is that with-profits life insurance policies offer usually a guaranteed minimum interest rate. The policyholder receives the positive difference between the return of the segregated fund and the minimum guaranteed rate. This means that if the return of the segregated fund exceeds the minimum interest rate, the policyholder receives a percentage of that excess rate and the interest rate credited to the policyholder is ensured not to fall below some specified guaranteed level. In this case, a financial guarantee is embedded in the contract. Such guarantee is in the form of a financial option, the so-called *minimum guaranteed option (MGO)*.⁹ From a financial point of view, with-profit policies can be considered as a derivative contracts, where the underlying is the return of the segregated fund.

Looking at the Solvency II regime, it is worth to note that the present value of the excess returns which are expected to credit to the policyholders corresponds to the so-called *future discretionary benefits (FDB)*. Under Solvency II, when the best estimate for the with-profit contracts is computed, insurance companies have to estimate separately the value of the future discretionary benefits as a specific component of the entire stochastic reserve.¹⁰

Usually, two types of financial guarantees are embedded in the Italian Insurance life contracts: (1) the annual or multi-period guarantees and (2) the maturity guarantees.

In the annual or multi-period guarantees, the minimum rate of return is credited during every period, and not only at maturity. From a financial point of view, the option embedded in the contract is an option of *forward start cliquet (or ratchet)* type where the indexation rule is applied every year, consolidating the benefit level reached by the revaluation occurred in previous one. In this case, the benefit is gradually distributed at the end of a determined period of time; any excess return in previous periods can be used to build up a reserve for bad times. This type of guarantee is embedded in the so-called *cliquet* policies.

In the maturity guarantees, the minimum rate of return is credited only at maturity of the policy; any excess return realized in early period cannot be used in bad periods. This type of guarantee is embedded in the the so-called *best of* policies.

In the case of annual guarantees, denoted by $C(T_i)$ the policy's benefit accrued

 $^{^{8}}$ This is established in detail by the Italian insurance authority (ISVAP). See ISVAP (1986) n.71/1986.

⁹Usually, other types of contractual options are embedded in the Italian policies besides the minimum guaranteed option. In fact, in such policies can be included also the surrender option, the paid up option, the annuity conversion option, the extended coverage option.

¹⁰See European Community (2009) and European Commission (2010).

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at time T_i , for i = 1, 2, ..., n, the profit sharing rule is defined by the following recurrent equation

$$C(T_i) = C(T_{i-1}) [1 + W(T_{i-1}, T_i)],$$

where $W(T_{i-1}, T_i)$ is the so-called *revaluation rate* and $[1 + W(T_{i-1}, T_i)]$ is the so-called *revaluation factor*. Usually, the revaluation rate is defined as

$$W(T_{i-1}, T_i) = \frac{\max\left[\min\left[\beta Y(T_{i-1}, T_i); Y(T_{i-1}, T_i) - \alpha\right]; g\right] - h}{1+h},$$

where

- $Y(T_{i-1}, T_i)$ = it is the return of the segregated fund related to the period $[T_{i-1}, T_i]$,
- $\alpha \ge 0$ = it is the minimum return retained by the insurer,
- $\beta \in (0,1]$ = it is the participation coefficient,
- $g \ge 0$ = it is the guaranteed minimum interest rate for the policyholder,
- $h \ge 0$ = it is the technical interest rate.

Consequently, in case of annual guarantees,

• the benefit accrued to the policyholder at the maturity of the contract is

$$C(T_n) = C(T_0) \prod_{i=1}^n \left[1 + W(T_{i-1}, T_i) \right];$$

• indicating with T_d the death time¹¹ such that $T_0 < T_d < T_n$, the benefit accrued to the policyholder, if the insured dies before the maturity, is

$$C(T_d) = C(T_0) \prod_{i=1}^d \left[1 + W(T_{i-1}, T_i) \right];$$

• indicating with T_l the lapse time¹² such that $T_0 < T_l < T_n$, the benefit accrued to the policyholder, if the policy is lapsed, is

$$C(T_l) = C(T_0) \prod_{i=1}^{l} \left[1 + W(T_{i-1}, T_i) \right].$$

¹¹We assume that if the effective death time is comprised in the period $[T_{i-1}, T_i]$, the death benefit is accrued at time T_i . Consequently, $T_d = T_i$.

¹²We assume that if the effective lapse time is comprised in the period $[T_{i-1}, T_i]$, the lapse benefit is accrued at time T_i . Consequently, $T_l = T_i$.

In the case of maturity guarantees, the revaluation mechanism is a function of (1) the revaluation rate computed without considering the guaranteed minimum rate and (2) the revaluation rate computed taking into account the guaranteed minimum rate only. In the first case, the revaluation rate becomes

$$W_Y(T_{i-1}, T_i) = \frac{\min\left[\beta Y(T_{i-1}, T_i); Y(T_{i-1}, T_i) - \alpha\right] - h}{1+h},$$

while, in the second case, we have

$$W_g(T_{i-1}, T_i) = \frac{g-h}{1+h}.$$

Consequently, in case of maturity guarantees

• the benefit accrued to the policyholder at the maturity of the contract is

$$C(T_n) = C(T_0) \max\left\{\prod_{i=1}^n \left[1 + W_Y(T_{i-1}, T_i)\right]; \prod_{i=1}^n \left[1 + W_g(T_{i-1}, T_i)\right]\right\};$$

• indicating with T_d the death time such that $T_0 < T_d < T_n$, the benefit accrued to the policyholder, if the insured dies before the maturity, is

$$C(T_d) = C(T_0) \max\left\{\prod_{i=1}^d \left[1 + W_Y(T_{i-1}, T_i)\right]; \prod_{i=1}^d \left[1 + W_g(T_{i-1}, T_i)\right]\right\};$$

• indicating with T_l the lapse time such that $T_0 < T_l < T_n$, the benefit accrued to the policyholder, if the policy is lapsed, is

$$C(T_l) = C(T_0) \max \left\{ \prod_{i=1}^l \left[1 + W_Y(T_{i-1}, T_i) \right]; \prod_{i=1}^l \left[1 + W_g(T_{i-1}, T_i) \right] \right\}.$$

8.3 The proposed model

We propose a pricing model for the Italian with-profit life insurance policies where a simplified approach in closed-form is implemented. Despite the return credited to the policyholders is computed as a function of the net income and the average book value,¹³ we assume the performance of the fund to be market-based. We follow the approach proposed in Bacinello (2001) and Castellani (2007) where such simplification is adopted in the calculation of the segregated fund's return. In particular, Bacinello (2001) and Castellani et al. (2007) have proposed a

¹³Grosen and Jorgensen (2000), Jensen et al (2001), Taskanen and Lukkarinen (2003) and Bauer et al (2006) have proposed pricing models for participating policies where the interest rate credited to the policyholders depends on the book value of the reference portfolio. In particular, Floreani (2009) have suggested an approach based on the book value specifically for the Italian with-profit policies.

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market-based approach approximating the return of the Italian segregated fund as a percentage return of the segregated fund's market value. Assuming that such market value can be model by a standard geometric Brownian motion, they have proposed closed-form solution under the well-known Black and Scholes framework. However, the drawback of this approach is that the Black and Scholes pricing formula is appropriate when the underlying value considered as input of the pricing formula is an equity or a portfolio of equities. In the case of the Italian policies, instead, the classical Black and Scholes framework is not adequate because segregated funds are composed mainly by fixed income instruments and the equity component is marginal.

Following Vellekoop et al. (2005), we present a pricing model in which a particular stochastic dynamic is adopted in order to take into account the effective asset allocation of the Italian segregated fund. In particular, we develop a model in which a new functional form for the volatility is considered in the option pricing formula. The volatility we compute is able to take into account the features of the segregated fund in terms of the effective duration of the fixed income component and the weight of the equity component.

8.3.1 Model assumptions

In order to explain the features of the Italian with-profit policies, we consider a contract that starts at time $t = T_0$ and expires in $t = T_n$, with $T_n > T_0$. We consider the case in which the policyholder pays to the insurer a single premium U at time $t = T_0$. We indicate by $C(T_i)$ the policy's benefit accrued at time T_i , for $i = 1, 2, ..., n_i$. The benefit accrued at time T_i is equal to the capital accrued at time T_{i-1} revaluated applying an interest rate that is the greater value between the guaranteed minimum rate and the return of the segregated fund. The return of the reference fund is computed taking into account the profit sharing mechanism and the technical interest rate. The quantity $C(T_n)$ represents the amount of money that the insurer pays to the policyholder taking into account the revaluation rule at the maturity of the contract. Although in the Italian with-profit policies the contract's benefits are paid also in the case of the insured's death or in the case the contract is lapsed, we assume that the benefit is paid only at the maturity of the contract. Consequently, to achieve analytic tractability of our solution, we neglect the mortality risk and the surrender option. Moreover, also other types of option usually embedded in the Italian contracts, other than the minimum guaranteed option, are not considered.

We assume the performance of the fund to be market-based and computed as percentage return of the segregated fund's market value.¹⁴ Denoting by $V(T_i)$ the

¹⁴The percentage return represents an approximation with respect to the real case. In fact, the segregated fund's return is computed as a function of the net income and the book value of the asset's portfolio.

market value of the fund at time T_i , we assume that the return of the segregated fund over the time interval $[T_{i-1}, T_i]$ is computed as

$$Y(T_{i-1}, T_i) = \frac{V(T_i) - V(T_{i-1})}{V(T_{i-1})} = \frac{V(T_i)}{V(T_{i-1})} - 1.$$

8.3.2 Payoff functions

We consider four different cases for the payoff of the Italian with-profit policies:

- annual guarantees with partecipation coefficient,
- annual guarantees with minimum return reteined by the insurer,
- maturity guarantees with partecipation coefficient,
- maturity guarantees with minimum return reteined by the insurer.

For each payoff, we provide a closed-formula thanks to which it it possible to compute the market consistent value of the contract.

Annual guarantees with partecipation coefficient

In the case of with-profit policies with annual guarantees and participation coefficient, the revaluation factor is

$$1 + W_{\beta}(T_{i-1}, T_i) = 1 + \frac{\max\left[\beta Y(T_{i-1}, T_i); g\right] - h}{1 + h}.$$

For single premium policies, the insured amount $C(T_i)$ raises according to the revaluation rate on the base of the following recursive equation,

$$C(T_i) = C(T_{i-1}) [1 + W_{\beta}(T_{i-1}, T_i)].$$

Consequently, the insured amount $C(T_n)$ is

$$C(T_n) = C(T_0) \prod_{i=1}^n \left[1 + W_\beta(T_{i-1}, T_i) \right].$$

At time T_i , the revaluation factor can be computed as

$$1 + W_{\beta}(T_{i-1}, T_i) = 1 + \frac{\max\left[\beta Y(T_{i-1}, T_i); g\right] - h}{1 + h} = \frac{1 + \max\left[\beta Y(T_{i-1}, T_i); g\right]}{1 + h}$$
$$= \frac{1 + \max\left[\beta Y(T_{i-1}, T_i) - g; 0\right] + g}{1 + h} = \frac{1}{1 + h} \left[\beta \left(Y(T_{i-1}, T_i) - \frac{g}{\beta}\right)^+ + 1 + g\right]$$
$$= \frac{1}{1 + h} \left[\beta \left(\frac{V(T_i)}{V(T_{i-1})} - 1 - \frac{g}{\beta}\right)^+ + 1 + g\right].$$

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Setting,

$$K=1+\frac{g}{\beta},$$

it follows that,

$$C(T_n) = C(T_0) \left(\frac{1}{1+h}\right)^n \prod_{i=1}^n \left[\beta \left(\frac{V(T_i)}{V(T_{i-1})} - K\right)^+ + 1 + g\right].$$

Following Bacinello (2001), we assume the stochastic independence of $W_{\beta}(T_{i-1}, T_i)$. Consequently, in order to compute the value of the contract at time $t < T_1$, we consider the following payoff

$$\hat{C}(T_n) = C(T_0) \prod_{i=1}^n \mathbb{E}^{T_i} \left[1 + W_\beta(T_{i-1}, T_i) \right] = C(T_0) \left(\frac{1}{1+h} \right)^n \prod_{i=1}^n \left\{ \beta \mathbb{E}^{T_i} \left[\left(\frac{V(T_i)}{V(T_{i-1})} - K \right)^+ \middle| \mathscr{F}_t \right] + 1 + g \right\},\$$

where \mathbb{E}^{T_i} refers to the T_i -forward risk-adjusted measure while \mathscr{F}_t is the sigma-field generated up to time t.

Annual guarantees with minimum return reteined by the insurer

In the case of with-profit policies with annual guarantees and minimum return reteined by the insurer, the revaluation factor is

$$1 + W_{\alpha}(T_{i-1}, T_i) = 1 + \frac{\max\left[Y(T_{i-1}, T_i) - \alpha; g\right] - h}{1 + h},$$

Also in this case, for single premium policies, the insured amount $C(T_i)$ raises according to the revaluation rate on the base of the following recursive equation,

$$C(T_i) = C(T_{i-1}) [1 + W_{\alpha}(T_{i-1}, T_i)].$$

Consequently, the insured amount $C(T_n)$ is

$$C(T_n) = C(T_0) \prod_{i=1}^n \left[1 + W_\alpha(T_{i-1}, T_i) \right].$$

At time T_i , the revaluation factor can be computed as

$$1 + W_{\alpha}(T_{i-1}, T_i) = 1 + \frac{\max\left[Y(T_{i-1}, T_i) - \alpha; g\right] - h}{1 + h} = \frac{1 + \max\left[Y(T_{i-1}, T_i) - \alpha; g\right]}{1 + h}$$
$$= \frac{1 + \max\left[Y(T_{i-1}, T_i) - \alpha - g; 0\right] + g}{1 + h} = \frac{1}{1 + h} \left[\left(Y(T_{i-1}, T_i) - \alpha - g\right)^+ + 1 + g\right]$$
$$= \frac{1}{1 + h} \left[\left(\frac{V(T_i)}{V(T_{i-1})} - 1 - \alpha - g\right)^+ + 1 + g\right].$$

Setting,

$$K = 1 + \alpha + g,$$

it follows that,

$$C(T_n) = C(T_0) \left(\frac{1}{1+h}\right)^n \prod_{i=1}^n \left[\left(\frac{V(T_i)}{V(T_{i-1})} - K\right)^+ + 1 + g \right].$$

In order to compute the value of the contract at time $t < T_1$, we have to consider the following payoff,

$$\hat{C}(T_n) = C(T_0) \prod_{i=1}^n \mathbb{E}^{T_i} \left[1 + W_\alpha(T_{i-1}, T_i) \right] = C(T_0) \left(\frac{1}{1+h} \right)^n \prod_{i=1}^n \left\{ \mathbb{E}^{T_i} \left[\left(\frac{V(T_i)}{V(T_{i-1})} - K \right)^+ \middle| \mathscr{F}_t \right] + 1 + g \right\}.$$

Maturity guarantees with partecipation coefficient

The case of maturity guarantees is quite typical in the Italian bancassurance companies.

For single premium policies, the insured amount $C(T_n)$ is such that,

$$C(T_n) = C(T_0) \max\left[\prod_{i=1}^n \left[1 + W_{Y,\beta}(T_{i-1}, T_i)\right]; \prod_{i=1}^n \left[1 + W_g(T_{i-1}, T_i)\right]\right],$$

where,

$$\prod_{i=1}^{n} \left[1 + W_g(T_{i-1}, T_i) \right] = \prod_{i=1}^{n} \left(1 + \frac{g-h}{1+h} \right) = \left(1 + \frac{g-h}{1+h} \right)^n = \left(\frac{1+g}{1+h} \right)^n,$$

and

$$\begin{split} &\prod_{i=1}^{n} \left[1 + W_{Y,\beta}(T_{i-1}, T_i) \right] = \prod_{i=1}^{n} \left(1 + \frac{\beta Y(T_{i-1}, T_i) - h}{1 + h} \right) = \prod_{i=1}^{n} \left(\frac{1 + \beta Y(T_{i-1}, T_i)}{1 + h} \right) \\ &= \left(\frac{1}{1 + h} \right)^n \prod_{i=1}^{n} \left(1 + \beta Y(T_{i-1}, T_i) \right) = \left(\frac{\beta}{1 + h} \right)^n \prod_{i=1}^{n} \left(\frac{1}{\beta} + Y(T_{i-1}, T_i) \right) \\ &= \left(\frac{\beta}{1 + h} \right)^n \prod_{i=1}^{n} \left(\frac{1}{\beta} + \frac{V(T_i)}{V(T_{i-1})} - 1 \right) \\ &= \left(\frac{\beta}{1 + h} \right)^n \prod_{i=1}^{n} \left(\frac{V(T_i)}{V(T_{i-1})} + \frac{1 - \beta}{\beta} \right). \end{split}$$

In order to simplify the payoff function, we adopt the following approximation¹⁵

$$\prod_{i=1}^{n} \left(\frac{V(T_i)}{V(T_{i-1})} + \frac{1-\beta}{\beta} \right) \cong \frac{V(T_n)}{V(T_0)} \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1-\beta}{\beta} \right)^j.$$

 15 We use the binomial theorem in order to approximate the payoff's function.

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Consequently,

$$C(T_{n}) = C(T_{0}) \max\left[\prod_{i=1}^{n} W_{Y,\beta}(T_{i-1}, T_{i}); \prod_{i=1}^{n} W_{g}(T_{i-1}, T_{i})\right]$$

= $C(T_{0}) \left\{ \max\left[\left(\frac{\beta}{1+h}\right)^{n} \frac{V(T_{n})}{V(T_{0})} \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1-\beta}{\beta}\right)^{j} - \left(\frac{1+g}{1+h}\right)^{n}; 0\right] + \left(\frac{1+g}{1+h}\right)^{n} \right\}$
= $C(T_{0}) \left\{ \left(\frac{\beta}{1+h}\right)^{n} \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1-\beta}{\beta}\right)^{j}$
 $\times \max\left[\frac{V(T_{n})}{V(T_{0})} - \frac{(1+g)^{n}}{\beta^{n} \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1-\beta}{\beta}\right)^{j}}; 0\right] + \left(\frac{1+g}{1+h}\right)^{n} \right\}.$

Setting,

$$K = \frac{(1+g)^n}{\beta^n \sum_{j=0}^n \binom{n}{j} \left(\frac{1-\beta}{\beta}\right)^j},\tag{8.-41}$$

it follows that,

$$C(T_n) = C(T_0) \left[\left(\frac{\beta}{1+h}\right)^n \sum_{j=0}^n \binom{n}{j} \left(\frac{1-\beta}{\beta}\right)^j \left(\frac{V(T_n)}{V(T_0)} - K\right)^+ + \left(\frac{1+g}{1+h}\right)^n \right].$$

In order to compute the value of the contract at time $t < T_1$, we have to consider the following payoff,

$$\hat{C}(T_n) = C(T_0) \left\{ \left(\frac{\beta}{1+h}\right)^n \sum_{j=0}^n \binom{n}{j} \left(\frac{1-\beta}{\beta}\right)^j \times \mathbb{E}^{T_i} \left[\left(\frac{V(T_n)}{V(T_0)} - K\right)^+ \middle| \mathscr{F}_t \right] + \left(\frac{1+g}{1+h}\right)^n \right\}.$$

Maturity guarantees with minimum reteined by the insurer

For single premium policies, the insured amount $C(T_n)$ is such that

$$C(T_n) = C(T_0) \max\left[\prod_{i=1}^n \left[1 + W_{Y,\alpha}(T_{i-1}, T_i)\right]; \prod_{i=1}^n \left[1 + W_g(T_{i-1}, T_i)\right]\right],$$

where

$$\begin{split} &\prod_{i=1}^{n} \left[1 + W_{Y,\alpha}(T_{i-1}, T_i) \right] = \prod_{i=1}^{n} \left(1 + \frac{Y(T_{i-1}, T_i) - \alpha - h}{1 + h} \right) = \prod_{i=1}^{n} \left(\frac{1 + Y(T_{i-1}, T_i) - \alpha}{1 + h} \right) \\ &= \left(\frac{1}{1 + h} \right)^n \prod_{i=1}^{n} \left(1 + Y(T_{i-1}, T_i) - \alpha \right) = \left(\frac{1}{1 + h} \right)^n \prod_{i=1}^{n} \left(1 + \frac{V(T_i)}{V(T_{i-1})} - 1 - \alpha \right) \\ &= \left(\frac{1}{1 + h} \right)^n \prod_{i=1}^{n} \left(\frac{V(T_i)}{V(T_{i-1})} - \alpha \right). \end{split}$$

In order to simplify the payoff function, we adopt the following approximation,¹⁶

$$\prod_{i=1}^{n} \left(\frac{V(T_i)}{V(T_{i-1})} - \alpha \right) \cong \frac{V(T_n)}{V(T_0)} \sum_{j=0}^{n} \binom{n}{j} (-\alpha)^j.$$

Consequently,

$$\begin{split} C(T_n) &= C(T_0) \max\left[\prod_{i=1}^n W_{Y,\alpha}(T_{i-1},T_i);\prod_{i=1}^n W_g(T_{i-1},T_i)\right] \\ &= C(T_0) \left\{ \max\left[\left(\frac{1}{1+h}\right)^n \frac{V(T_n)}{V(T_0)} \sum_{j=0}^n \binom{n}{j} (-\alpha)^j - \left(\frac{1+g}{1+h}\right)^n; 0\right] + \left(\frac{1+g}{1+h}\right)^n \right\} \\ &= C(T_0) \left\{ \left(\frac{1}{1+h}\right)^n \sum_{j=0}^n \binom{n}{j} (-\alpha)^j \\ &\times \max\left[\frac{V(T_n)}{V(T_0)} - \frac{(1+g)^n}{\sum_{j=0}^n \binom{n}{j} (-\alpha)^j}; 0\right] + \left(\frac{1+g}{1+h}\right)^n \right\}. \end{split}$$

Setting,

$$K = \frac{(1+g)^n}{\sum_{j=0}^n {n \choose j} (-\alpha)^j},$$

it follows that,

$$C(T_n) = C(T_0) \left[\left(\frac{1}{1+h} \right)^n \sum_{j=0}^n \binom{n}{j} (-\alpha)^j \left(\frac{V(T_n)}{V(T_0)} - K \right)^+ + \left(\frac{1+g}{1+h} \right)^n \right].$$

In order to compute the value of the contract at time $t < T_1$, we have to consider the following payoff,

$$\hat{C}(T_n) = C(T_0) \left\{ \left(\frac{1}{1+h} \right)^n \sum_{j=0}^n \binom{n}{j} (-\alpha)^j \\ \times \mathbb{E}^{T_i} \left[\left(\frac{V(T_n)}{V(T_0)} - K \right)^+ \middle| \mathscr{F}_t \right] + \left(\frac{1+g}{1+h} \right)^n \right\}.$$

8.3.3 Asset allocation and stochastic dynamic for the segregated fund

We consider that case in which the segregated fund is composed by bonds and equities.¹⁷

Denoting the bond's portfolio by $B_p(t)$, we assume that the fund is managed in such a way that the sensitivity with respect to the interest rates is deterministic

 $^{^{16}\}mathrm{We}$ use the binomial theorem in order to approximate the payoff's function.

 $^{^{17}{\}rm This}$ is a realistic assumption. In fact, the Italian segregated fund are usually composed by bonds for the 80/90% and equities for the remain part.

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but time-varying. We use the *effective duration* as sensitivity measure as defined in Fabozzi (1996). We denote the effective duration by δ . In order to indicate the value of the effective duration across the time we denote it by $\delta(t)$.

We indicate by $E_p(t)$ the market value of the equity's portfolio and denote by ω the equity's percentage in the segregated fund. We assume that ω is deterministic but time-varying indicating by $\omega(t)$ the equity's weight at time t.

Following Vellekoop et al. (2005), we assume that, under the *T*-forward riskadjusted measure denoted by \mathcal{M}^T , the value of the segregated fund evolves according to the following stochastic dynamic

$$\frac{dV(t)}{V(t)} = \left[1 - \omega(t)\right] \frac{dB(t)}{B(t)} + \omega(t) \frac{dE(t)}{E(t)}$$
$$= r(t)dt - \left[1 - \omega(t)\right] \delta(t)\sigma dW^{T}(t) + \omega(t)\nu dZ^{T}(t)$$

where

- r(t) =it is instantaneous short rate,
- σ = it is the volatility of the short rate,
- $\delta(t) =$ it is the effective duration at time t,
- ν = it is the volatility of the equities,
- $\omega(t) = \text{it is the equity's weight at time } t$,

and where W^T and Z^T are correlated Brownian motions. Considering independent Brownian motions, we have

$$dW^{T}(t) = d\tilde{W}^{T}(t),$$

$$dZ^{T}(t) = \rho d\tilde{W}^{T}(t) + \sqrt{1 - \rho^{2}} d\tilde{Z}^{T}(t),$$

where \tilde{W}^T and \tilde{Z}^T are independent and ρ is the correlation coefficient between interest rates and equities.

Consequently, we have that

$$\frac{dV(t)}{V(t)} = [1 - \omega(t)]\frac{dB(t)}{B(t)} + \omega(t)\frac{dE(t)}{E(t)}$$
$$= r(t)dt - [1 - \omega(t)]\delta(t)\sigma dZ_1(t) + \omega(t)\nu\rho d\tilde{W}^T(t) + \omega(t)\nu\sqrt{1 - \rho^2}d\tilde{Z}^T(t).$$

We assume also that interest rates are stochastic. In particular, we assume that under the *T*-forward risk-adjusted measure the dynamic of the instantaneous short rate is given by the Hull and White model¹⁸. Consequently,

$$\begin{aligned} r(t) &= \varphi(t) + x(t), \\ dx(t) &= -ax(t)dt + \sigma dW^T(t), \qquad x(0) = 0, \end{aligned}$$

 $^{^{18}}$ See Hull and White (1990).

where k and σ are positive constants while the deterministic function $\varphi(t)$ is such that

$$\varphi(t) = f^M(0,t) + \frac{\sigma^2}{2a^2} (1 - e^{-at})^2.$$

The quantity $f^{M}(0,t)$ is the market instantaneous forward rate at time 0 for the maturity t defined as

$$f(0,t) = -\frac{\delta \log P(0,t)}{\delta t},$$

where $P^{M}(0,t)$ is the market value at time 0 for a zero-coupon bond that expires in t. Under the Hull-White model, the dynamic of the zero coupon bond price is

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt - H(t,T)\sigma dW^{T}(t),$$

where

$$H(t,T) = \frac{1}{k} \left[1 - e^{-a(T-t)} \right].$$

In conclusion, the model we use to define the stochastic dynamic for the segregated fund can be summarized as

$$\frac{dV(t)}{V(t)} = r(t)dt - [1 - \omega(t)]\delta(t)\sigma dZ_1(t) + \omega(t)\nu\rho d\tilde{W}^T(t) + \omega(t)\nu\sqrt{1 - \rho^2}d\tilde{Z}^T(t)$$

$$r(t) = \varphi(t) + x(t)$$

$$dx(t) = -ax(t)dt + \sigma dW^T(t)$$

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt - H(t,T)\sigma dW^T(t).$$

8.3.4 Closed-form solution

In order to provide the market-consistent value for with profit policies, we have to compute the following expectation

$$\mathbb{E}^{T_i}\left[\left(\frac{V(T_i)}{V(T_{i-1})} - K\right)^+ \middle| \mathscr{F}_t\right].$$

Since that the ratio $V(T_i)/V(T_{i-1})$ conditional on \mathscr{F}_t is lognormally distributed under the T_i -forward risk-adjusted measure denoted by \mathcal{M}^{T_i} , the expected value can be derived from the properties of the lognormal distribution. In fact, if $\log(X)$ is normally distributed with $\mathbb{E}[X] = m$ and $\mathbb{V}[ln(X)] = s^2$, adopting the wellknown Black-Scholes standard approach used for pricing of financial options¹⁹, we have

$$\frac{\mathbb{E}[X-K]^{+} = m\Phi\left[\frac{\log\frac{m}{K} + \frac{1}{2}s^{2}}{s}\right] - K\Phi\left[\frac{\log\frac{m}{K} - \frac{1}{2}s^{2}}{s}\right]}{s}$$

 $^{^{19}\}mathrm{See}$ Black and Scholes (1973).

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In order to use the previous formula for the pricing of with-profit policies, we have to compute the expected value and the variance of the ratio $V(T_i)/V(T_{i-1})$ under the T_i -forward risk-adjusted measure.

The expected value can be immediately obtained as

$$m = \mathbb{E}^{T_i} \left[\frac{V(T_i)}{V(T_{i-1})} \middle| \mathscr{F}_t \right] = \frac{P(t, T_{i-1})}{P(t, T_i)},$$

while the variance is

$$s^{2} = \mathbb{V}^{T_{i}}\left[\log\frac{V(T_{i})}{V(T_{i-1})}\middle|\mathscr{F}_{t}\right] = \Sigma_{V}^{2}(t, T_{i-1}, T_{i}).$$

Consequently, we solve the expected value as

$$\mathbb{E}^{T_i}\left[\left(\frac{V(T_i)}{V(T_{i-1})} - K\right)^+ \middle| \mathscr{F}_t\right] = \left[\frac{P(t, T_{i-1})}{P(t, T_i)}\Phi(d_1) - K\Phi(d_2)\right],$$

where

$$d_1 = \frac{\log\left[\frac{P(t,T_{i-1})}{P(t,T_i)}\frac{1}{K}\right] + \frac{1}{2}\Sigma_V^2(t,T_{i-1},T_i)}{\Sigma_V(t,T_{i-1}-T_i)}$$

and

$$d_2 = \frac{\log\left[\frac{P(t,T_{i-1})}{P(t,T_i)}\frac{1}{K}\right] - \frac{1}{2}\Sigma_V^2(t,T_{i-1},T_i)}{\Sigma_V(t,T_{i-1},T_i)}$$

For what concerns the variance of the logarithm of the ratio $V(T_i)/V(T_{i-1})$, we need to compute such variance under the *T*-forward risk-adjusted measure. However, it can be computed under the risk-neutral measure since that the change of measure produces only a deterministic additive term which has no impact in the variance calculation.²⁰ We find that

$$\Sigma_V^2(t, T_{i-1}, T_i) = \int_{T_{i-1}}^{T_i} \sigma_{V/P}^2(u, T) du + \int_t^{T_{i-1}} \left[\sigma_P(u, T_i) - \sigma_P(u, T_{i-1}) \right]^2 du,$$

where the first integral is the variance under the risk-neutral measure and the second one represents the deterministic additive term.

In order to solve the integrals, it results that

$$\sigma_{V|P}^2(t,T) = \sigma_V^2 + \sigma_P^2(t,T) - 2\sigma_V \sigma_P(t,T),$$

 $^{^{20}{\}rm We}$ use the approach used in Brigo and Mercurio (2006) for the pricing of the Inflation-Indexed Caplets/Floorlets using the Jarrow Yildirim model.

where the variance of the zero-coupon bond is

$$\sigma_P^2(t,T) = \frac{\sigma^2}{a^2} \left[1 - \exp\left[-a(T-t) \right] \right]^2,$$

while the variance of the segregated fund consists in

$$\sigma_V^2 = \left[\rho\omega(T_{i-1})\nu - [1 - \omega(T_{i-1})]\delta(T_{i-1})\sigma\right]^2 + (1 - \rho^2)\omega(T_{i-1})^2\nu^2$$

= $[1 - \omega(T_{i-1})]^2\delta(T_{i-1})^2\sigma^2 - 2\rho\omega(T_{i-1})[1 - \omega(T_{i-1})]\nu\delta(T_{i-1})\sigma + \omega(T_{i-1})^2\nu^2.$

Consequently, we have that

$$\begin{split} & \Sigma_V^2(t, T_{i-1}, T_i) \\ &= \int_{T_{i-1}}^{T_i} \sigma_V^2 du + \int_{T_{i-1}}^{T_i} \sigma_P^2(t, T) du - 2 \int_{T_{i-1}}^{T_i} \sigma_V \sigma_P(t, T) du + \int_t^{T_{i-1}} \sigma_P^2(u, T_i) du \\ &+ \int_t^{T_{i-1}} \sigma_P^2(u, T_{i-1}) du - 2 \int_t^{T_{i-1}} \sigma_P(u, T_i) \sigma_P(u, T_{i-1}) du. \end{split}$$

Substituting and integrating, it follows that,

$$\begin{split} & \Sigma_V^2(t, T_{i-1}, T_i) \\ &= \omega(T_{i-1})^2 \nu^2(T_i - T_{i-1}) + 2\rho \omega(T_{i-1}) \nu \frac{\sigma}{a} \bigg[T_i - T_{i-1} - \frac{1 - \exp[-a(T_i - T_{i-1})]}{a} \bigg] \\ &- 2\rho \omega(T_{i-1}) [1 - \omega(T_{i-1})] \nu \delta(T_{i-1}) \sigma(T_i - T_{i-1}) \\ &+ \frac{\sigma^2}{a^2} \bigg[T_i - T_{i-1} + \frac{2}{a} \exp[-a(T_i - T_{i-1})] - \frac{1}{2a} \exp[-2a(T_i - T_{i-1})] - \frac{3}{2a} \bigg] \\ &+ [1 - \omega(T_{i-1})]^2 \delta(T_{i-1})^2 \sigma^2(T_i - T_{i-1}) \\ &- 2[1 - \omega(T_{i-1})] \delta(T_{i-1}) \sigma \frac{\sigma}{a} \bigg[T_i - T_{i-1} - \frac{1 - \exp[-a(T_i - T_{i-1})]}{a} \bigg] \\ &+ \frac{\sigma^2}{2a^3} \bigg[1 - \exp[-a(T_i - T_{i-1})] \bigg]^2 \bigg[1 - \exp[-2a(T_{i-1} - t)] \bigg]. \end{split}$$

8.3.5 Best estimate of liabilities (BEL) for Italian with-profit policies

Using the proposed model, we provide a closed-formula for the best estimate of liabilities in the case of Italian with-profit policies. In this section, we consider the case of annual guarantees and partecipation coefficient. Analogous results can be found for the other types of payoff we have presented.

Standard no-arbitrage pricing theory implies that the best estimate of liabilities computed at time $t < T_{i-1}$ for the maturity T_n , denoted by $BEL(t, T_n)$, is

$$\begin{split} BEL(t,T_n) &= P(t,T_n)\hat{C}(T_n) = P(t,T_n)C(T_0)\prod_{i=1}^n \left\{ \mathbb{E}^{T_i} \bigg[1 + W_\beta(T_{i-1},T_i) \bigg] \right\} \\ &= P(t,T_n)C(T_0) \bigg(\frac{1}{1+h} \bigg)^n \prod_{i=1}^n \left\{ \beta \mathbb{E}^{T_i} \bigg[\bigg(\frac{V(T_i)}{V(T_{i-1})} - K \bigg)^+ \bigg| \mathscr{F}_t \bigg] + 1 + g \right\}. \end{split}$$

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Applying the proposed model, the BEL of the contract can be computed as,

$$BEL(t,T_n) = P(t,T_n)\hat{C}(T_n) = P(t,T_n)C(T_0)\prod_{i=1}^n \left\{ \mathbb{E}^{T_i} \left[1 + W(T_{i-1},T_i) \right] \right\}$$
$$= P(t,T_n)C(T_0) \left(\frac{1}{1+h}\right)^n \prod_{i=1}^n \left\{ \beta \left[\frac{P(t,T_{i-1})}{P(t,T_i)} \Phi(d_1) - K\Phi(d_2) \right] + 1 + g \right\}.$$

8.3.6 Embedded options

We use the proposed model to compute the market-consistent value of the financial options embedded in the Italian with-profit life insurance contracts.

In order to compute the value of the minimum guaranteed option, we adopt the so-called *put option approach* according to which it is possible to decompose the best estimate of the contract into two part: (1) the contract's value where the minimum guaranteed option is not considered (2) an additional value reflecting the fact that the the benefit for the policyholder is subject to a certain minimum value. The optional component (put) is the minimum guaranteed option. This approach is discussed in a paper of the Financial Service Authority²¹ (FSA) and its application for the Italian policies is due to De Felice and Moriconi (2002).

An alternative approach, known as *call option approach*, consists in considering the entire contract's value as the sum of a guaranteed component and an optional component that represents the cost of a call option. The optional component (call) represents the excess of assets return over contractual guarantees and corresponds to the future discretionary benefits. This alternative approach is discussed in Hare et al. (2003) and Dullaway and Needleman (2003) while the application to the Italian policies is due to De Felice and Moriconi (2002). We adopt this approach to quantify the expected present value of the future discretionary benefits, namely the fair value of the call option embedded in the Italian contracts.

In summary, the payoff representing the benefits for the policyholders can be expressed as (1) a non guaranteed benefit plus a put option representing the protection, or (2) a minimum guaranteed benefit plus a call option representing the extra benefits.

Minimum guaranteed option

We consider a with-profit policy with annual guarantee and partecipation coefficient. According to the put option approach, the value of the option is the difference between the value of the contract where the benefits are revaluated according to the contractual conditions and the value in which the revaluation factor does not take into account the guarantee.

²¹See Financial Service Authority (2003).

Consequently, we have to consider the quantity

$$BEL_Y(t,T_n) = P(t,T_n)\hat{C}(T_n) = P(t,T_n)C(T_0)\prod_{i=1}^n \left\{ \mathbb{E}^{T_i} \left[1 + W_{Y,\beta}(T_{i-1},T_i) \right] \right\},\$$

where

$$\left[1 + W_{Y,\beta}(T_{i-1}, T_i) \right] = \left(1 + \frac{\beta Y(T_{i-1}, T_i) - h}{1 + h} \right) = \left(\frac{1 + \beta Y(T_{i-1}, T_i)}{1 + h} \right)$$

= $\frac{\beta}{1 + h} \left(\frac{V(T_i)}{V(T_{i-1})} + \frac{1 - \beta}{\beta} \right).$

Denoting by $MGO(t, T_n)$ the value of the minimum guaranteed option computed at time $t < T_{i-1}$ for the maturity T_n , we have

$$MGO(t, T_n) = BEL(t, T_n) - BEL_Y(t, T_n).$$

Future discretionary benefits

As in the previous case, we assume annual guarantee and partecipation coefficient. According to the call option approach, we compute the value of the future discretionary benefits as difference between two quantities. In particular, the value of FDB is obtained as difference between the value of the contract where the benefits are revaluated according to the contractual conditions and the value of the contract revaluated taking into account the minimum garanteed only. In formula, it holds that

$$FDB(t, T_n) = BEL(t, T_n) - BEL_q(t, T_n),$$

where

$$BEL_g(t, T_n) = \prod_{i=1}^n \left[1 + W_g(T_{i-1}, T_i) \right] = \prod_{i=1}^n \left(1 + \frac{g-h}{1+h} \right) = \left(1 + \frac{g-h}{1+h} \right)^n = \left(\frac{1+g}{1+h} \right)^n$$

8.4 Calibration

In this section, we show how our model can be calibrated to market data. The objective of calibration is to choose the model parameters in such a way that the model prices are consistent with the market prices of simple instruments. The calibration process is then a matter of choosing particular values for the parameters and fitting them so as to match the prices of selected market instruments. The first stage in the calibration process is to derive the initial term structures of interest rates. The term structure is derived from traded instruments quoted in

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the cash, futures and swap markets. We apply standard bootstrapping technique to derive the zero rates from the traded market instruments. In order to derive the interest rates volatility parameters, we calibrate our model using caps quoted in the market. The calibration to caps is done by choosing the values of a and σ so as to minimize the sum of the square difference between market and model cap prices using the goodness-of-fit measure

$$\arg\min_{a,\sigma}\sum_{i=1}^{n} \left(Cap_{i}^{M}-Cap_{i}\right)^{2},$$

where Cap_i^M is the value of the caps quoted by the market while Cap_i represents the cap formula implied by Hull-White model. The number of calibrated instruments is n.

For the equity's component, we use the ATM volatility implied in the options quoted on the FTSE Mib index. We use the longest maturity available in the market. To quantify the effect deriving from the correlation between stock and interest rates we derive the correlation coefficient by historical estimation.

In order to provide numerical results, we have calibrated our model on the market data as at October 31, 2011. The results are the following

- a = 0,4487,- $\sigma = 0,0224,$ - $\nu = 0,4,$ - $\rho = 0,1.$

8.5 Numerical results

We consider with-profit policies with participation coefficient where both annual and maturity guarantees are taken into account. A single premium and a maturity of 10 years are assumed. We consider also different levels of partecipating coefficient and minimum guaranteed rate for each contract with and an initial insured amount of Euro 100.

With reference to the asset allocation of the segregated fund, we provide different values for effective duration and equity's weight.

For each case, we provide the expected present value for the entire contract (BEL) and separated values for the minimum guaranteed option (MGO) and future discretionary benefits (FDB). The results are reported in the Tables from 8.1 to 8.6.

We can appreciate that BEL, MGO and FDB are affected by the contractual parameters such as minimum guaranteed rate and participation coefficient. It

8.6. CONCLUSION

Table 8.1: Numerical results for a with-profit policy with annual guarantee and partecipation coefficient - technical rate of 0% - maturity of 10 years.

β	g	$\delta(t)$	$\omega(t)$	$BEL(t,T_n)$	$BEL_g(t, T_n)$	$FDB(t,T_n)$	$BEL_Y(t,T_n)$	$MGO(t, T_n)$
1	0	3	0	$135,\!45$	78,10	57,35	100	35,45
0,9	0	3	0	$128,\!37$	78,10	50,27	$97,\!59$	30,78
0,9	0,01	3	0	133,97	86,28	$47,\!68$	$97,\!59$	36,38
0,9	0,01	5	0	148,26	86,28	61,98	$97,\!59$	$50,\!67$
0,8	0,01	5	0,2	$140,\!65$	86,28	54,36	95,23	45,42
0,8	0,02	6	0,3	159,82	95,23	64,59	$95,\!23$	64,59

Table 8.2: Numerical results for a with-profit policy with annual guarantee and partecipation coefficient - technical rate of 4% - maturity of 10 years.

β	g	$\delta(t)$	$\omega(t)$	$BEL(t,T_n)$	$BEL_g(t, T_n)$	$FDB(t,T_n)$	$BEL_Y(t,T_n)$	$MGO(t,T_n)$
1	0	3	0	91,51	52,74	38,77	$67,\!56$	23,95
0,9	0	3	0	86,72	52,74	33,98	$65,\!93$	20,79
0,9	0,01	3	0	90,50	58,26	32,24	$65,\!93$	24,58
0,9	0,01	5	0	100, 16	58,26	41,90	$65,\!93$	34,23
0,8	0,01	5	0,2	95,02	58,26	36,75	$64,\!33$	$30,\!68$
0,8	0,02	6	0,3	$107,\!97$	64,30	43,66	64,33	43,63

Table 8.3: Numerical results for a with-profit policy with annual guarantee and minimum reteined - technical rate of 0% - maturity of 10 years.

α	<i>g</i>	$\delta(t)$	$\omega(t)$	$BEL(t,T_n)$	$BEL_g(t, T_n)$	$FDB(t,T_n)$	$BEL_Y(t,T_n)$	$MGO(t,T_n)$
0	0	3	0	135,45	78,10	$57,\!35$	100	35,45
0,01	0	3	0	128,53	78,10	$50,\!43$	$90,\!66$	37,87
0,01	0,01	3	0	134,58	86,28	48,30	$90,\!66$	43,92
0,01	0,01	5	0	150,78	86,28	64,50	$90,\!66$	60,12
0,015	0,01	5	$_{0,2}$	149,54	86,28	63,26	86,29	63,25
0,015	0,02	6	0,3	173,73	95,23	78,50	86,29	87,44

is important to note that the expected present value of the contract and option values are significantly influenced also by the asset allocation parameters such as effective duration and equity's weight.

8.6 Conclusion

We have developed a closed formula to provide the market-consistent value for the Italian with-profit life insurance policies. We focus on the expected present value of the cash flows, the so-called best estimate of liabilities (BEL). Furthermore, we have provided closed formula for financial options embedded in these contracts. In particular, we are able to evaluate separately the minimum guaranteed option (MGO) and the future discretionary benefits (FDB) embedded in the Italian with-profit life policies.

Our approach could be used in order to quantify the technical provisions for Solvency II (enforcement to begin in 2013) purposes. In addition, it could be useful

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Table 8.4: Numerical results for a with-profit policy with annual guarantee and minimum reteined - technical rate of 4% - maturity of 10 years.

α	g	$\delta(t)$	$\omega(t)$	$BEL(t,T_n)$	$BEL_g(t, T_n)$	$FDB(t,T_n)$	$BEL_Y(t,T_n)$	$MGO(t,T_n)$
0	0	3	0	91,51	52,74	38,77	$67,\!56$	23,95
0,01	0	3	0	86,83	52,74	34,09	61,25	$25,\!58$
0,01	0,01	3	0	90,92	58,26	32,65	61,25	$29,\!67$
0,01	0,01	5	0	101,86	58,26	43,60	61,25	$40,\!61$
0,015	0,01	5	0,2	101,03	58,26	42,76	58,30	42,73
0,015	0,02	6	0,3	117,36	$64,\!30$	53,06	58,30	59,07

Table 8.5: Numerical results for a with-profit policy with maturity guarantee and partecipation coefficient - technical rate of 0% - maturity of 10 years.

β	g	$\delta(t)$	$\omega(t)$	$BEL(t,T_n)$	$BEL_g(t,T_n)$	$FDB(t,T_n)$	$BEL_Y(t,T_n)$	$MGO(t,T_n)$
1	0	3	0	100,11	78,10	22,01	100	0,11
0,9	0	3	0	100,11	78,10	22,01	100	0,11
0,9	0,01	3	0	100,72	86,28	14,44	100	0,72
0,9	0,01	5	0	103,92	86,28	$17,\!64$	100	3,92
0,8	0,01	5	0,2	105,55	86,28	19,26	100	5,55
0,8	0,02	6	0,3	113,64	95,23	18,41	100	13,64

Table 8.6: Numerical results for a with-profit policy with maturity guarantee and minimum reteined - technical rate of 0% - maturity of 10 years.

α	g	$\delta(t)$	$\omega(t)$	$BEL(t,T_n)$	$BEL_g(t,T_n)$	$FDB(t,T_n)$	$BEL_Y(t,T_n)$	$MGO(t,T_n)$
0	0	3	0	100,11	78,10	22,01	100	0,11
0,01	0	3	0	91,10	78,10	13,00	90,44	$0,\!67$
0,01	0,01	3	0	93,15	86,28	6,87	90,44	2,71
0,01	0,01	5	0	97,26	86,28	10,97	90,44	6,82
0,015	0,01	5	0,2	96,39	86,28	10,11	85,97	10,42
0,015	0,02	6	0,3	105,82	95,23	10,59	85,97	19,84

under the new IAS/IFRS principles for insurance contracts (to be approval).

We assume a specific stochastic dynamic for the segregated fund such that the effective asset allocation of the fund can be taken into account. Then, we derive a closed-form approach under the well-known Black and Scholes framework where the volatility is computed as a function of the effective asset allocation of the segregated fund. In summary our model consists in: (1) defining the functional form of the typical payoff for the Italian policies; (2) assuming a specific stochastic dynamic for the segregated fund to which the policies are linked; (3) assuming a Black and Scholes pricing framework; (4) deriving the volatility to put in the pricing model as a function of the effective asset allocation of the segregated fund; (5) deriving the closed-form for best estimate of liabilities, minimum guaranteed option and future discretionary benefits.

To achieve analytic tractability in our model, we have made certain simplifying assumptions with respect to the interest rate crediting scheme. Moreover, we

8.6. CONCLUSION

have neglected of mortality risk and surrender option.

It is important to note that thank to our pricing model in closed-form it is possible to quantify the BEL as a function of the contractual parameters and asset allocation parameters of the segregated fund.

We have describe the model calibration procedure and provide some numerical results for the value of Italian with-profit life contracts and related embedded options. The pricing behaviors of Italian participating policies have been explored examining the impact of various parameters. We have found that the value of the contract is affected by the guaranteed minimum interest rate and the crediting scheme and that it is significantly influenced by the effective asset allocation of the segregated fund.

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