Running Cosmological Constant and Running Newton Constant in Modified Gravity Theories

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Abstract. We discuss how to extract information about the cosmological constant from the Wheeler-DeWitt equation, considered as an eigenvalue of a Sturm-Liouville problem in a de Sitter and Anti-de Sitter background. The equation is approximated to one loop with the help of a variational approach with Gaussian trial wave functionals. A canonical decomposition of modes is used to separate transverse-traceless tensors (graviton) from ghosts and scalar. We show that no ghosts appear in the final evaluation of the cosmological constant. A zeta function regularization is used to handle with divergences. A renormalization procedure is introduced to remove the infinities together with a renormalization group equation. We apply this procedure on the induced cosmological constant Λ and, as an alternative, on the Newton constant G. A brief discussion on the extension to a $f(R)$ theory is considered.

INTRODUCTION

The Friedmann-Robertson-Walker model of the universe, based on the Einstein's field equations gives an explanation of why the Universe is in an acceleration phase. However such an expansion must be supported by almost 76% of what is known as *Dark Energy*[1]. The dark component problem results from an increasing number of independent cosmological observations, such as measurements to intermediate and high redshift supernova Ia (SNIa), measurements of the Cosmic Microwave Background (CMB) anisotropy, and the current observations of the Large-Scale Structure (LSS) in the universe. The simplest candidate to explain Dark Energy is based on the equation of state $P = \omega\rho$ (where P and ρ are the pressure of the fluid and the energy density, respectively). When $\omega < -1/3$, we are in the Dark energy regime, while we have a transition to *Phantom Energy* when $\omega < -1$. In particular, the case of $\omega = -1$ corresponding to a cosmological constant seems to be a good candidate for the Dark Energy problem. Globally, this is known as ΛCDM model. Nevertheless the ΛCDM model fails in explaining why the observed cosmological constant is so small. Indeed, there exist 120 order of difference between the estimated cosmological constant and observation. Basically, the theoretically prediction is based on the computation of Zero Point Energy (ZPE). One possibility of computing ZPE in the context of the cosmological constant is given by the Wheeler-DeWitt equation (WDW)[8], which is described by

$$\mathscr{H}\Psi = \left[(2\kappa)\, G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}\big(^3R - 2\Lambda\big)\right]\Psi = 0. \tag{1}$$

$\kappa = 8\pi G$, G_{ijkl} is the super-metric and ^3R is the scalar curvature in three dimensions. The main reason to work with a WDW equation becomes more transparent if we formally re-write the WDW equation as[12]

$$\frac{1}{V}\frac{\int \mathscr{D}\,[g_{ij}]\,\Psi^*\,[g_{ij}]\int_\Sigma d^3x\hat{\Lambda}_\Sigma\Psi\,[g_{ij}]}{\int \mathscr{D}\,[g_{ij}]\,\Psi^*\,[g_{ij}]\,\Psi\,[g_{ij}]} = \frac{1}{V}\frac{\left\langle \Psi \left| \int_\Sigma d^3x\hat{\Lambda}_\Sigma \right| \Psi \right\rangle}{\langle\Psi|\Psi\rangle} = -\frac{\Lambda}{\kappa}, \tag{2}$$

where

$$V = \int_\Sigma d^3x\sqrt{g} \tag{3}$$

is the volume of the hypersurface Σ and

$$\hat{\Lambda}_\Sigma = (2\kappa)\, G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{g}^3R/(2\kappa). \tag{4}$$

Eq.(2) represents the Sturm-Liouville problem associated with the cosmological constant. In this form the ratio Λ_c/κ represents the expectation value of $\hat{\Lambda}_\Sigma$ without matter fields. The related boundary conditions are dictated by the choice of the trial wave functionals which, in our case are of the Gaussian type. Different types of wave functionals correspond to different boundary conditions. The choice of a Gaussian wave functional is justified by the fact that we would like to explain the cosmological constant (Λ_c/κ) as a ZPE effect. To fix ideas, we will work with the following form of the de Sitter metric (dS)

$$ds^2 = -\left(1 - \frac{\Lambda_{dS}}{3}r^2\right)dt^2 + \frac{dr^2}{1 - \frac{\Lambda_{dS}}{3}r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{5}$$

and its counterpart the Anti-de Sitter metric (AdS)

$$ds^2 = -\left(1 + \frac{\Lambda_{AdS}}{3}r^2\right)dt^2 + \frac{dr^2}{1 + \frac{\Lambda_{AdS}}{3}r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{6}$$

which are different expressions of the FRW background. It is interesting to observe that Eq.(2) can be extended to the so-called modified gravity theories. Basically, one modifies the Einstein-Hilbert action with the following replacement[4]

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R + S^{matter} \quad \rightarrow \quad S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S^{matter}. \tag{7}$$

It is clear that other more complicated choices could be done in place of $f(R)$[6]. In particular, one could consider $f\left(R, R_{\mu\nu}R^{\mu\nu}, R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}, \dots\right)$ or $f(R, G)$ where G is the Gauss-Bonnet invariant or any combination of these quantities[1]. Even if one of the prerogatives of a $f(R)$ theory is the explanation of the cosmological constant, we are interested in a more general context, where a combination of Λ with a general $f(R)$ theory is considered. As a first step, we begin to decompose the gravitational perturbation in such a way to obtain the graviton contribution enclosed in Eq.(2).

EXTRACTING THE GRAVITON CONTRIBUTION

We can gain more information if we consider $g_{ij} = \bar{g}_{ij} + h_{ij}$, where $\bar{g}_{ij}$ is the background metric and h_{ij} is a quantum fluctuation around the background. Thus Eq.(2) can be expanded in terms of h_{ij}. Since the kinetic part of $\hat{\Lambda}_\Sigma$ is quadratic in the momenta, we only need to expand the three-scalar curvature $\int d^3x\sqrt{g^3}R$ up to the quadratic order. However, to proceed with the computation, we also need an orthogonal decomposition on the tangent space of 3-metric deformations[13, 14]:

$$h_{ij} = \frac{1}{3}\left(\sigma + 2\nabla\cdot\xi\right)g_{ij} + (L\xi)_{ij} + h_{ij}^\perp. \tag{8}$$

The operator L maps ξ_i into symmetric tracefree tensors

$$(L\xi)_{ij} = \nabla_i\xi_j + \nabla_j\xi_i - \frac{2}{3}g_{ij}(\nabla\cdot\xi), \tag{9}$$

$h_{ij}^\perp$ is the traceless-transverse component of the perturbation (TT), namely $g^{ij}h_{ij}^\perp = 0$, $\nabla^i h_{ij}^\perp = 0$ and h is the trace of h_{ij}. It is immediate to recognize that the trace element $\sigma = h - 2(\nabla\cdot\xi)$ is gauge invariant. If we perform the same decomposition also on the momentum π^{ij}, up to second order Eq.(2) becomes

$$\frac{1}{V}\frac{\left\langle\Psi\left|\int_\Sigma d^3x\left[\hat{\Lambda}_\Sigma^\perp + \hat{\Lambda}_\Sigma^\xi + \hat{\Lambda}_\Sigma^\sigma\right]^{(2)}\right|\Psi\right\rangle}{\langle\Psi|\Psi\rangle} = -\frac{\Lambda}{\kappa}\Psi[g_{ij}]. \tag{10}$$

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[1] For a recent riview on $f(R)$, see Refs.[5, 6, 7], while a recent review on the problem of $f(G)$ and $f(R, G)$ can be found in Ref.[9, 10].

Concerning the measure appearing in Eq.(2), we have to note that the decomposition (8) induces the following transformation on the functional measure $\mathscr{D}h_{ij} \rightarrow \mathscr{D}h_{ij}^\perp \mathscr{D}\xi_i \mathscr{D}\sigma J_1$, where the Jacobian related to the gauge vector variable ξ_i is

$$J_1 = \left[\det \left(\triangle g^{ij} + \frac{1}{3} \nabla^i \nabla^j - R^{ij} \right) \right]^{\frac{1}{2}}. \tag{11}$$

This is nothing but the famous Faddev-Popov determinant. It becomes more transparent if ξ_a is further decomposed into a transverse part ξ_a^T with $\nabla^a \xi_a^T = 0$ and a longitudinal part $\xi_a^\parallel$ with $\xi_a^\parallel = \nabla_a \psi$, then J_1 can be expressed by an upper triangular matrix for certain backgrounds (e.g. Schwarzschild in three dimensions). It is immediate to recognize that for an Einstein space in any dimension, cross terms vanish and J_1 can be expressed by a block diagonal matrix. Since $\det AB = \det A \det B$, the functional measure $\mathscr{D}h_{ij}$ factorizes into

$$\mathscr{D}h_{ij} = \left(\det \triangle_V^T \right)^{\frac{1}{2}} \left(\det \left[\frac{2}{3} \triangle^2 + \nabla_i R^{ij} \nabla_j \right] \right)^{\frac{1}{2}} \mathscr{D}h_{ij}^\perp \mathscr{D}\xi^T \mathscr{D}\psi \tag{12}$$

with $\left(\triangle_V^{ij} \right)^T = \triangle g^{ij} - R^{ij}$ acting on transverse vectors, which is the Faddeev-Popov determinant. In writing the functional measure $\mathscr{D}h_{ij}$, we have here ignored the appearance of a multiplicative anomaly[11]. Thus the inner product can be written as

$$\int \mathscr{D}h_{ij}^\perp \mathscr{D}\xi^T \mathscr{D}\sigma \Psi^* \left[h_{ij}^\perp \right] \Psi^* \left[\xi^T \right] \Psi^* \left[\sigma \right] \Psi \left[h_{ij}^\perp \right] \Psi \left[\xi^T \right] \Psi \left[\sigma \right] \left(\det \triangle_V^T \right)^{\frac{1}{2}} \left(\det \left[\frac{2}{3} \triangle^2 + \nabla_i R^{ij} \nabla_j \right] \right)^{\frac{1}{2}}. \tag{13}$$

Nevertheless, since there is no interaction between ghost fields and the other components of the perturbation at this level of approximation, the Jacobian appearing in the numerator and in the denominator simplify. The reason can be found in terms of connected and disconnected terms. The disconnected terms appear in the Faddeev-Popov determinant and these ones are not linked by the Gaussian integration. This means that disconnected terms in the numerator and the same ones appearing in the denominator cancel out. Therefore, Eq.(10) factorizes into three pieces. The piece containing $\hat{\Lambda}_\Sigma^\perp$ is the contribution of the transverse-traceless tensors (TT): essentially is the graviton contribution representing true physical degrees of freedom. Regarding the vector term $\hat{\Lambda}_\Sigma^T$, we observe that under the action of infinitesimal diffeomorphism generated by a vector field ε_i, the components of (8) transform as follows[13]

$$\xi_j \longrightarrow \xi_j + \varepsilon_j, \qquad h \longrightarrow h + 2\nabla \cdot \xi, \qquad h_{ij}^\perp \longrightarrow h_{ij}^\perp. \tag{14}$$

The Killing vectors satisfying the condition $\nabla_i \xi_j + \nabla_j \xi_i = 0$, do not change h_{ij}, and thus should be excluded from the gauge group. All other diffeomorphisms act on h_{ij} nontrivially. We need to fix the residual gauge freedom on the vector ξ_i. The simplest choice is $\xi_i = 0$. This new gauge fixing produces the same Faddeev-Popov determinant connected to the Jacobian J_1 and therefore will not contribute to the final value. We are left with

$$\frac{1}{V} \frac{\left\langle \Psi^\perp \left| \int_\Sigma d^3 x \left[\hat{\Lambda}_\Sigma^\perp \right]^{(2)} \right| \Psi^\perp \right\rangle}{\left\langle \Psi^\perp | \Psi^\perp \right\rangle} + \frac{1}{V} \frac{\left\langle \Psi^\sigma \left| \int_\Sigma d^3 x \left[\hat{\Lambda}_\Sigma^\sigma \right]^{(2)} \right| \Psi^\sigma \right\rangle}{\left\langle \Psi^\sigma | \Psi^\sigma \right\rangle} = -\frac{\Lambda}{\kappa} \Psi [g_{ij}]. \tag{15}$$

Note that in the expansion of $\int_\Sigma d^3 x \sqrt{g} R$ to second order, a coupling term between the TT component and scalar one remains. However, the Gaussian integration does not allow such a mixing which has to be introduced with an appropriate wave functional. Extracting the TT tensor contribution from Eq.(2) approximated to second order in perturbation of the spatial part of the metric into a background term $\bar{g}_{ij}$, and a perturbation h_{ij}, we get

$$\hat{\Lambda}_\Sigma^\perp = \frac{1}{4V} \int_\Sigma d^3 x \sqrt{\bar{g}} G^{ijkl} \left[(2\kappa) K^{-1\perp} (x,x)_{ijkl} + \frac{1}{(2\kappa)} \left(\tilde{\triangle}_L \right)_j^a K^\perp (x,x)_{iakl} \right], \tag{16}$$

where

$$\left(\tilde{\triangle}_L h^\perp \right)_{ij} = \left(\triangle_L h^\perp \right)_{ij} - 4R_i^k h_{kj}^\perp + {}^3R h_{ij}^\perp \tag{17}$$

is the modified Lichnerowicz operator and $\triangle_L$ is the Lichnerowicz operator defined by

$$(\triangle_L h)_{ij} = \triangle h_{ij} - 2R_{ikjl} h^{kl} + R_{ik} h_j^k + R_{jk} h_i^k \qquad \triangle = -\nabla^a \nabla_a. \tag{18}$$

G^{ijkl} represents the inverse DeWitt metric and all indices run from one to three. Note that the term $-4R_i^k h_{kj}^\perp + {}^3R h_{ij}^\perp$ disappears in four dimensions. The propagator $K^\perp(x,x)_{iakl}$ can be represented as

$$K^\perp\left(\overrightarrow{x},\overrightarrow{y}\right)_{iakl} = \sum_\tau \frac{h_{ia}^{(\tau)\perp}\left(\overrightarrow{x}\right) h_{kl}^{(\tau)\perp}\left(\overrightarrow{y}\right)}{2\lambda(\tau)}, \tag{19}$$

where $h_{ia}^{(\tau)\perp}\left(\overrightarrow{x}\right)$ are the eigenfunctions of $\tilde{\triangle}_L$. τ denotes a complete set of indices and $\lambda(\tau)$ are a set of variational parameters to be determined by the minimization of Eq.(16). The expectation value of $\hat{\Lambda}_\Sigma^\perp$ is easily obtained by inserting the form of the propagator into Eq.(16) and minimizing with respect to the variational function $\lambda(\tau)$. Thus the total one loop energy density for TT tensors becomes

$$\frac{\Lambda}{8\pi G} = -\frac{1}{2}\sum_\tau\left[\sqrt{\omega_1^2(\tau)} + \sqrt{\omega_2^2(\tau)}\right]. \tag{20}$$

The above expression makes sense only for $\omega_i^2(\tau) > 0$, where ω_i are the eigenvalues of $\tilde{\triangle}_L$. Concerning the scalar contribution of Eq.(15), in Ref.[24] has been proved that the cosmological constant contribution is vanishing for a Schwarzschild background. If we follow the same procedure for dS and AdS metrics, we can show that the only consistent value is given by $\Lambda_{dS} = \Lambda_{dS} = 0$. In the next section, we will explicitly evaluate Eq.(20) for a specific background.

ONE LOOP ENERGY REGULARIZATION AND RENORMALIZATION FOR THE ORDINARY $f(R) = R$ THEORY

the dS and AdS metric can be cast into the following form

$$ds^2 = -N^2\left(r(x)\right)dt^2 + dx^2 + r^2(x)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{21}$$

where

$$dx = \pm\frac{dr}{\sqrt{1 - \frac{b(r)}{r}}} \tag{22}$$

with

$$b(r) = \frac{\Lambda_{dS}}{3}r^3; \qquad b(r) = -\frac{\Lambda_{AdS}}{3}r^3. \tag{23}$$

$\left(\tilde{\triangle}_L h^\perp\right)_{ij}$ can be reduced to

$$\left[-\frac{d^2}{dx^2} + \frac{l(l+1)}{r^2} + m_i^2(r)\right]f_i(x) = \omega_{i,l}^2 f_i(x) \quad i = 1,2 \quad, \tag{24}$$

with the help of Regge and Wheeler representation[15], where we have used reduced fields of the form $f_i(x) = F_i(x)/r$ and where we have defined two r-dependent effective masses $m_1^2(r)$ and $m_2^2(r)$

$$\begin{cases} m_1^2(r) = \frac{6}{r^2}\left(1 - \frac{b(r)}{r}\right) + \frac{3}{2r^2}b'(r) - \frac{3}{2r^3}b(r) \\ \\ m_2^2(r) = \frac{6}{r^2}\left(1 - \frac{b(r)}{r}\right) + \frac{1}{2r^2}b'(r) + \frac{3}{2r^3}b(r) \end{cases} \quad (r \equiv r(x)). \tag{25}$$

In order to use the WKB approximation, from Eq.(24) we can extract two r-dependent radial wave numbers

$$k_i^2\left(r,l,\omega_{i,nl}\right) = \omega_{i,nl}^2 - \frac{l(l+1)}{r^2} - m_i^2(r) \quad i = 1,2 \quad. \tag{26}$$

To further proceed we use the W.K.B. method used by 't Hooft in the brick wall problem[16] and we count the number of modes with frequency less than ω_i, $i = 1,2$. This is given approximately by

$$\tilde{g}(\omega_i) = \int_0^{l_{max}} \nu_i(l,\omega_i)(2l+1)dl, \tag{27}$$

where $v_i(l,\omega_i)$, $i = 1,2$ is the number of nodes in the mode with (l,ω_i), such that $(r \equiv r(x))$

$$v_i(l,\omega_i) = \frac{1}{\pi}\int_{-\infty}^{+\infty}dx\sqrt{k_i^2(r,l,\omega_i)}. \tag{28}$$

Here it is understood that the integration with respect to x and l_{max} is taken over those values which satisfy $k_i^2(r,l,\omega_i) \geq 0$, $i = 1,2$. With the help of Eqs.(27,28), Eq.(20) becomes

$$\frac{\Lambda}{8\pi G} = -\frac{1}{\pi}\sum_{i=1}^{2}\int_0^{+\infty}\omega_i\frac{d\tilde{g}(\omega_i)}{d\omega_i}d\omega_i. \tag{29}$$

This is the one loop graviton contribution to the induced cosmological constant. The explicit evaluation of Eq.(29) gives

$$\frac{\Lambda}{8\pi G} = \rho_1 + \rho_2 = -\frac{1}{4\pi^2}\sum_{i=1}^{2}\int_{\sqrt{m_i^2(r)}}^{+\infty}\omega_i^2\sqrt{\omega_i^2 - m_i^2(r)}d\omega_i, \tag{30}$$

where we have included an additional 4π coming from the angular integration. The use of the zeta function regularization method to compute the energy densities ρ_1 and ρ_2 leads to

$$\rho_i(\varepsilon) = \frac{m_i^4(r)}{64\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{4\mu^2}{m_i^2(r)\sqrt{e}}\right)\right] \quad i = 1,2 \quad, \tag{31}$$

where we have introduced the additional mass parameter μ in order to restore the correct dimension for the regularized quantities. Such an arbitrary mass scale emerges unavoidably in any regularization scheme. The renormalization is performed via the absorption of the divergent part into the re-definition of a bare classical quantity. Here we have two possible choices: the induced cosmological constant Λ or the gravitational Newton constant G. In addition, we restrict our investigation to the case where

$$m_1^2(r) = m_2^2(r) = m_0^2(r), \tag{32}$$

because the dS and AdS backgrounds fall in this case.

Running Cosmological Constant

If we adopt to absorb the divergence using the cosmological constant Λ, we can re-define $\Lambda \to \Lambda_0 + \Lambda^{div}$, where

$$\Lambda^{div} = \frac{m_0^4(r)}{\varepsilon 32\pi^2}. \tag{33}$$

The remaining finite value for the cosmological constant reads

$$\frac{\Lambda_0}{8\pi G} = (\rho_1(\mu) + \rho_2(\mu)) = \rho_{eff}^{TT}(\mu,r), \tag{34}$$

where $\rho_i(\mu)$ has the same form of $\rho_i(\varepsilon)$ but without the divergence. The quantity in Eq.(34) depends on the arbitrary mass scale μ. It is appropriate the use of the renormalization group equation to eliminate such a dependence. To this aim, we impose that[17]

$$\frac{1}{8\pi G}\mu\frac{\partial\Lambda_0(\mu)}{\partial\mu} = \mu\frac{d}{d\mu}\rho_{eff}^{TT}(\mu,r). \tag{35}$$

Solving it we find that the renormalized constant Λ_0 should be treated as a running one in the sense that it varies provided that the scale μ is changing

$$\frac{\Lambda_0(\mu,r)}{8\pi G} = \frac{\Lambda_0(\mu_0,r)}{8\pi G} + \frac{m_0^4(r)}{16\pi^2}\ln\frac{\mu}{\mu_0}. \tag{36}$$

Substituting Eq.(36) into Eq.(34) we find

$$\frac{\Lambda_0\left(\mu_0,r\right)}{8\pi G}=-\frac{1}{32\pi^2}\left\{m_0^4\left(r\right)\left[\ln\left(\frac{m_0^2\left(r\right)\sqrt{e}}{4\mu_0^2}\right)\right]\right\}.$$ (37)

If we go back and look at Eq.(2), we note that what we have actually computed is the opposite of an effective potential (better an effective energy). Therefore, we expect to find physically acceptable solutions in proximity of the extrema. We find that Eq.(37) has an extremum when

$$\frac{1}{e}=\frac{m_0^2\left(\bar{r}\right)}{4\mu_0^2}\qquad\Longrightarrow\qquad\frac{\bar{\Lambda}_0\left(\mu_0,\bar{r}\right)}{8\pi G}=\frac{m_0^4\left(\bar{r}\right)}{64\pi^2}=\frac{\mu_0^4}{4\pi^2 e^2}.$$ (38)

Actually $\bar{\Lambda}_0\left(\mu_0,\bar{r}\right)$ is a maximum, corresponding to a minimum of the effective energy. The effect of the gravitational fluctuations is to shift the minimum of the effective energy away from the flat solution leading to an induced cosmological constant. Plugging Eq.(38) into Eq.(36), we find

$$\frac{\Lambda_0\left(\mu,r\right)}{8\pi G}=\frac{\bar{\Lambda}_0\left(\mu_0,\bar{r}\right)}{8\pi G}+\frac{m_0^4\left(r\right)}{16\pi^2}\ln\frac{\mu}{\mu_0}=\frac{m_0^4\left(\bar{r}\right)}{64\pi^2}\left(1+4\frac{m_0^4\left(r\right)}{m_0^4\left(\bar{r}\right)}\ln\frac{\mu}{\mu_0}\right)$$ (39)

which can be set to zero when

$$\frac{\Lambda_0\left(\tilde{\mu},r\right)}{8\pi G}=0\qquad\text{when}\qquad\tilde{\mu}=\exp\left(-\frac{m_0^4\left(\bar{r}\right)}{4m_0^4\left(r\right)}\right)\mu_0.$$ (40)

It is clear that the case is strongly dependent on the background choice. In this work we fix our attention on dS and AdS metrics written in static way

dS and AdS background

In the case of dS and AdS spaces, the effective masses are

$$m_1^2\left(r\right)=m_2^2\left(r\right)=m_0^2\left(r\right)=\begin{cases}\frac{6}{r^2}-\Lambda_{dS}&r\in\left(0,\sqrt{\frac{3}{\Lambda}}\right]\qquad\text{dS Case}\\[2mm]\frac{6}{r^2}+\Lambda_{AdS}&r\in\left(0,+\infty\right)\qquad\text{AdS Case}\end{cases}.$$ (41)

The effective masses have a spurious dependence on r, which can be fixed by the extremum condition described in Eq.(38). This is the analogue dependence of the energy momentum tensor on the scale factor a in the Friedmann-Robertson-Walker model. Note that from Eqs.(41), $m_0^2\left(r\right)$ can never be vanishing, except for the trivial case of $\Lambda_{dS}=\Lambda_{AdS}=0$. It is interesting to evaluate the result in proximity of the cosmological throat $r_C=\sqrt{3/\Lambda_{dS}}$ for the dS solution. In this case, Eq.(38) leads to

$$\bar{r}^2=\frac{6e}{4\mu_0^2+e\bar{\Lambda}_{dS}}\to\frac{4\mu_0^2}{e}=\bar{\Lambda}_{dS}\qquad\Longrightarrow\qquad\frac{\bar{\Lambda}_0\left(\mu_0,\bar{\Lambda}_{dS}\right)}{8\pi G}=\frac{\bar{\Lambda}_{dS}^2}{64\pi^2}=\frac{\mu_0^4}{4\pi^2 e^2}$$ (42)

and Eq.(40) becomes

$$\frac{\Lambda_0\left(\tilde{\mu}_{dS},\Lambda_{dS}\right)}{8\pi G}=0\qquad\text{when}\qquad\tilde{\mu}_{dS}=\exp\left(-\frac{\bar{\Lambda}_{dS}^2}{4\Lambda_{dS}^2}\right)\mu_0.$$ (43)

For the AdS background, we take the analogue limit of the cosmological throat, namely $r\to\infty$, then Eq.(38) leads to

$$\bar{r}^2=\frac{6e}{4\mu_0^2-e\bar{\Lambda}_{AdS}}\to\frac{4\mu_0^2}{e}=\bar{\Lambda}_{AdS}\qquad\Longrightarrow\qquad\frac{\bar{\Lambda}_0\left(\mu_0,\bar{\Lambda}_{AdS}\right)}{8\pi G}=\frac{\bar{\Lambda}_{AdS}^2}{64\pi^2}=\frac{\mu_0^4}{4\pi^2 e^2}$$ (44)

and Eq.(40) becomes

$$\frac{\Lambda_0\left(\tilde{\mu}_{AdS},\Lambda_{AdS}\right)}{8\pi G} = 0 \qquad \text{when} \qquad \tilde{\mu}_{AdS} = \exp\left(-\frac{\bar{\Lambda}_{AdS}^2}{4\Lambda_{AdS}^2}\right)\mu_0. \tag{45}$$

Note that at this level of approximation we are unable to distinguish contributions coming from a dS or AdS background. On the other hand, it is interesting to observe that if

$$\Lambda_{dS} \ll \bar{\Lambda}_{dS} \qquad \text{and} \qquad \Lambda_{AdS} \ll \bar{\Lambda}_{AdS} \tag{46}$$

then

$$\tilde{\mu}_{dS} \ll \mu_0 \qquad \text{and} \qquad \tilde{\mu}_{AdS} \ll \mu_0. \tag{47}$$

This means that if we start from μ_0 at the Planck scale, we can fine tune a vanishing cosmological constant for small energy scales. Nevertheless to obtain this behavior, we need to assume small initial background parameters in such a way that condition (46) is satisfied.

Running Newton constant

If we adopt to absorb the divergence using the Newton constant G, we have to consider the following substitution

$$\frac{1}{G} \to \frac{1}{G_0\left(\mu\right)} + \frac{m_0^4\left(r\right)}{\Lambda\varepsilon 4\pi}. \tag{48}$$

Nevertheless, we have to say that this procedure is not immediate for the Schwarzschild metric and related generalizations. Indeed in this case, Eq.(48) becomes

$$\frac{1}{G} \to \frac{1}{G_0\left(\mu\right)} + \left(\frac{3MG_0\left(\mu\right)}{r^3}\right)^2 \frac{1}{\Lambda\varepsilon 4\pi}, \tag{49}$$

which means that the divergence is not removed. Therefore, it appears that this procedure is well defined only for the dS and AdS cases. The remaining finite value for the cosmological constant reads now

$$\frac{\Lambda}{8\pi G_0\left(\mu\right)} = \left(\rho_1\left(\mu\right) + \rho_2\left(\mu\right)\right) = \rho_{eff}^{TT}\left(\mu,r\right), \tag{50}$$

where $\rho_i\left(\mu\right)$ has the same form of $\rho_i\left(\varepsilon\right)$ but without the divergence. We eliminate the dependence on the arbitrary mass scale μ, by imposing[17]

$$\frac{\Lambda}{8\pi}\mu\frac{\partial\left(G_0^{-1}\left(\mu\right)\right)}{\partial\mu} = \mu\frac{d}{d\mu}\rho_{eff}^{TT}\left(\mu,r\right). \tag{51}$$

Solving it we find that the renormalized constant G_0 should be treated as a running one in the sense that it varies provided that the scale μ is changing

$$G_0\left(\mu\right) = \frac{G_0\left(\mu_0\right)}{1 + \frac{m_0^4(r)}{32\pi^2}G_0\left(\mu_0\right)\ln\frac{\mu}{\mu_0}}. \tag{52}$$

Even in this case, it is interesting to consider the asymptotic part of $m_0^2\left(r\right)$ for both dS and AdS metrics. It appears from Eq.(52) that there is a Landau pole at the scale

$$\begin{cases} \mu_0\exp\left(-\frac{32\pi^2}{\Lambda_{dS}^2 G_0(\mu_0)}\right) = \mu & \text{dS Case} \\[4mm] \mu_0\exp\left(-\frac{32\pi^2}{\Lambda_{AdS}^2 G_0(\mu_0)}\right) = \mu & \text{AdS Case} \end{cases}, \tag{53}$$

invalidating the perturbative calculation[2, 3]. Substituting Eq.(52) into Eq.(50) we find that the expression of the induced cosmological constant is the same as the one in Eq.(37) with the replacement

$$\frac{\Lambda_0(\mu_0, r)}{8\pi G} \rightarrow \frac{\Lambda(r)}{8\pi G_0(\mu_0)}, \tag{54}$$

showing that in this case is the Newton constant that is running. Nevertheless, a fundamental difference in rinormalizing the Newton constant comes from the fact that we cannot find an appropriate scale where the cosmological constant can be very small or eventually zero. To this purpose, we try to generalize this approach including a generic $f(R)$ theory.

EXTENSION TO A GENERIC $f(R)$ THEORY

It is interesting to note that Eq.(2) can be generalized by replacing the scalar curvature R with a generic function of R. Although a $f(R)$ theory does not need a cosmological constant, rather it should explain it, we shall consider the following Lagrangian density describing a generic $f(R)$ theory of gravity

$$\mathscr{L} = \sqrt{-g}\left(f(R) - 2\Lambda\right), \qquad \text{with } f'' \neq 0, \tag{55}$$

where $f(R)$ is an arbitrary smooth function of the scalar curvature and primes denote differentiation with respect to the scalar curvature. A cosmological term is added also in this case for the sake of generality, because in any case, Eq.(55) represents the most general Lagrangian to examine. Obviously $f'' = 0$ corresponds to GR.[20]. The semi-classical procedure followed in this work relies heavily on the formalism outlined in Refs.[24, 19]. The main effect of this replacement is that at the scale μ_0, we have a shift of the old induced cosmological constant into

$$\frac{\Lambda_0'(\mu_0, r)}{8\pi G} = \frac{1}{\sqrt{h(R)}}\left[\frac{\Lambda_0(\mu_0, r)}{8\pi G} + \frac{1}{16\pi GV}\int_\Sigma d^3x\sqrt{g}\frac{Rf'(R) - f(R)}{f'(R)}\right], \tag{56}$$

where V is the volume of the system. Note that when $f(R) = R$, consistently it is $h(R) = 1$ with

$$h(R) = \frac{3f'(R) - 2}{f'(R)} \tag{57}$$

We can always choose the form of $f(R)$ in such a way $\Lambda_0(\mu_0, r) = 0$. This implies

$$\frac{\Lambda_0'(\mu_0, r)}{8\pi G} = \frac{1}{\sqrt{h(R)}}\frac{1}{16\pi GV}\int_\Sigma d^3x\sqrt{g}\frac{Rf'(R) - f(R)}{f'(R)}. \tag{58}$$

As an example we can examine the following model[2]

$$f(R) = AR^p\exp\left(-\alpha R\right). \tag{59}$$

With this choice, the integrated extra-potential becomes

$$\frac{1}{V}\int_\Sigma d^3x\sqrt{g}\frac{Rf'(R) - f(R)}{f'(R)} = \frac{1}{V}\int_\Sigma d^3x\sqrt{g}\frac{R(p - \alpha R - 1)}{p - \alpha R} \tag{60}$$

and the function $h(R)$ assumes the form

$$h(R) = \frac{3A\exp\left(-\alpha R\right)R^{p-1}(p - \alpha R) - 2}{A\exp\left(-\alpha R\right)R^{p-1}(p - \alpha R)}. \tag{61}$$

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[2] Several models of $f(R)$ theories are examined in Ref.[21]

Note that the scalar curvature is four-dimensional like the argument in $f(R)$. One can choose, for example the Schwarzschild background to obtain

$$\frac{\Lambda_0'(\mu_0,r)}{8\pi G} = \nearrow \!\!\!\!\!\! \searrow \left\{ \begin{array}{ll} 0 & p \neq 0 \\ \sqrt{\frac{A\alpha}{3A\alpha+2}}\frac{1}{\alpha 16\pi G} & p = 0 \end{array} \right. . \tag{62}$$

while for the dS (AdS) case $R = \pm 4\Lambda$ and for the extra-potential one gets

$$\frac{1}{V}\int_\Sigma d^3x\sqrt{g}\frac{R(p-\alpha R-1)}{p-\alpha R} = \nearrow \!\!\!\!\!\! \searrow \left\{ \begin{array}{ll} \frac{4\Lambda(p-\alpha 4\Lambda-1)}{p-\alpha 4\Lambda} & \text{dS} \\ \frac{-4\Lambda(p+4\alpha\Lambda-1)}{p+4\alpha\Lambda} & \text{AdS} \end{array} \right. . \tag{63}$$

There exists a singularity when $p = \pm\alpha 4\Lambda$ for the dS (AdS) case, while $h(R)$ becomes

$$h(R) = \nearrow \!\!\!\!\!\! \searrow \left\{ \begin{array}{ll} \frac{3A\exp(-\alpha 4\Lambda)(4\Lambda)^{p-1}(p-\alpha 4\Lambda)-2}{A\exp(-\alpha 4\Lambda)(4\Lambda)^{p-1}(p-\alpha 4\Lambda)} & \text{dS} \\ \frac{3A\exp(\alpha 4\Lambda)(-4\Lambda)^{p-1}(p+\alpha 4\Lambda)-2}{A\exp(\alpha 4\Lambda)(-4\Lambda)^{p-1}(p+\alpha 4\Lambda)} & \text{AdS} \end{array} \right. . \tag{64}$$

From Eq.(63) we can see that

$$\frac{\Lambda_0'(\mu_0,r)}{8\pi G} = 0, \tag{65}$$

when

$$\left\{ \begin{array}{l} \Lambda_{dS} = \frac{p-1}{4\alpha} \\ \Lambda_{AdS} = \frac{1-p}{4\alpha} \end{array} \right. . \tag{66}$$

Since the modified cosmological constant Λ_0' follows an evolution equation of the form (36), it appears that

$$\frac{\Lambda_0'(\mu,r)}{8\pi G} = \nearrow \!\!\!\!\!\! \searrow \left\{ \begin{array}{l} \frac{\Lambda_{dS}^2}{16\pi^2}\ln\frac{\mu}{\mu_0} \\ \frac{\Lambda_{AdS}^2}{16\pi^2}\ln\frac{\mu}{\mu_0} \end{array} \right. \tag{67}$$

and for the value of $p = 1$ for both the dS and AdS background, we get a vanishing induced cosmological constant also at the scale μ. However, this result is valid when we assume that the main contribution of the effective masses is concentrated in proximity of Λ_{dS} or Λ_{AdS}. On the other hand, when we adopt to renormalize the Newton constant, because of Eq.(58) and Eqs.(66), the modified cosmological constant cannot be set to zero at any scale, because there is no a corresponding evolution equation similar to Eq.(67).

CONCLUSIONS

In this contribution, the effect of a ZPE on the cosmological constant has been investigated using two specific geometries such as dS and AdS metrics. The computation has been done by means of a variational procedure with a Gaussian Wave Functional which should be a good candidate for a ZPE calculation. We have found that only the graviton is relevant[18]. Actually, the appearance of a ghost contribution is connected with perturbations of the shift vectors[13]. In this work we have excluded such perturbations. As usual, in ZPE calculation we meet the problem of divergences which are regularized with zeta function techniques. After regularization , we have adopted to remove divergences by absorbing them into classical quantities: in particular the Newton constant G and the induced cosmological constant Λ. This procedure makes these constants running with the change of the scale μ appearing in the regularization scheme. There are two possibilities:

a) Λ is running. Then we find that it is possible to find some critical values of the renormalization scale μ where Λ can be set to zero. However, these points are strongly dependent on the background choice. The situation changes a little when we replace R with $f(R)$ even if the final result depends on a case to case. In the case under examination of the model (59) for the value of $p = 1$, we find a vanishing cosmological constant at any scale.

b) G is running. For this case, the induced cosmological constant of Eq.(50) cannot be set to zero at any scale. Actually, the ratio $\Lambda/(8\pi G_0(\mu))$ can be vanished. Nevertheless, the point where this happens is the Landau point, that it means that the procedure fails for that value. For a $f(R)$ theory, the problem has not yet examined.

A comment concerning our one loop computation is in order. This approach is deeply different from the one loop computation of Refs.[22, 23], where the analysis has been done expanding directly $f(R)$. In our case, the expansion involves only the three dimensional scalar curvature. Note that with the metric (21) and the effective masses (25), in principle, we can examine every spherically symmetric metric. Note also the absence of boundary terms in the evaluation of the induced cosmological constant.

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