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Noncommutative approach to the cosmological constant problem

Remo Garattini*

Facoltà di Ingegneria, Università degli Studi di Bergamo, Viale Marconi 5, 24044 Dalmine (Bergamo) Italy and INFN–Sezione di Milano, Milan, Italy

Piero Nicolini†

Frankfurt Institute for Advanced Studies (FIAS), Institut für Theoretische Physik, Johann Wolfgang Goethe-Universität, Ruth-Moufang-Strasse 1, 60438 Frankfurt am Main, Germany

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In this paper, we study the cosmological constant emerging from the Wheeler-DeWitt equation as an eigenvalue of the related Sturm-Liouville problem. We employ Gaussian trial functionals and we perform a mode decomposition to extract the transverse-traceless component, namely, the graviton contribution, at one loop. We implement a noncommutative-geometry–induced minimal length to calculate the number of graviton modes. As a result, we find regular graviton fluctuation energies for the Schwarzschild, de Sitter, and anti-de Sitter backgrounds. No renormalization scheme is necessary to remove infinities, in contrast to what happens in conventional approaches.

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I. INTRODUCTION

The emergence of a minimal length is widely accepted as a natural requirement when quantum features of spacetime are considered. Indeed, the spacetime structure at small distances is rather different from the conventional description in terms of a smooth differential manifold. When extreme energies probe spacetime, quantum gravitational fluctuations appear and prevent any measure of better accuracy than a natural length scale, e.g., the Planck length (see, for instance, [1]). Qualitatively, we can describe the spacetime in such an extreme regime as a *quantum foam*, namely, a complex turbulent storm-tossed sea which accounts for the seething fabric of the Universe [2]. The presence of a minimal length implies that singularities in general relativity and ultraviolet divergences in quantum field theory are nothing but spurious effects due to the inadequacy of the formalism at small scales/extreme energies, rather than actual physical phenomena. Along this line of reasoning, the renormalization procedure, too, even if very effective for its capacity of providing reliable and testable data, is nothing more than an artificial mechanism to get an *ad hoc* treatment for the bad short-distance behavior of quantum fields. As a further criticism to renormalization, there is also the well-known limitation of a systematic employment of regularization schemes when gravity is taken into account. A related problem is provided by the calculation of the cosmological constant: it is not yet clear what is the prescription which leads to a finite and reasonably small value, since trivial infinity subtractions are not viable in the presence of a gravitational coupling.

*Remo.Garattini@unibg.it
†nicolini@th.physik.uni-frankfurt.de

Given this background, great efforts have been devoted to implementing a minimal length in physical theories and curing the aforementioned pathologies or limitations of conventional approaches. For instance, we recall the route opened by the generalized uncertainty principle (GUP), according to which the Heisenberg commutation relation among coordinates and momenta would be deformed in order to include the effects of an ultraviolet (and/or an infrared) cutoff [3]. In the same spirit, several models of noncommutative geometry (NCG) have been extensively studied, i.e., geometries for which coordinate operators might fail to commute, giving rise to an effective graininess of the spacetime manifold (for general reviews on the topic, see [4]). Even if both the GUP and NCG are often regarded as mere effective tools or low-energy limits of more fundamental formulations [5], they turn out to be quite successful for their capacity of providing testable predictions and foreseeing new reliable scenarios [6]. Among the most relevant results, we recall that, with a minimal length induced by averaging noncommutative coordinate fluctuations [7–9], the curvature singularity of conventional black hole spacetimes has been tamed [10–12], and a new thermodynamically stable final stage of the Hawking evaporation has been determined [13,14] (for a review on these topics, see [15]).

In light of the above results, in this paper, we would like to do a step forward. In particular, we would like to apply some of the NCG properties to the computation of the cosmological constant. This procedure is based on the employment of the Wheeler-DeWitt (WDW) equation with the cosmological constant considered as an eigenvalue of a certain Sturm-Liouville problem. This approach has been initiated by one of us [16], with the purpose of computing the zero-point energy generated by the graviton fluctuations. In other words, zero-point energy is a

Casimir-like energy. We recall that, for calculating the Casimir energy, one generally invokes a subtraction procedure between zero-point energies having the same boundary condition. At the semiclassical level, one employs a zeta function regularization scheme to determine finite energy densities, when the graviton one-loop contribution to a classical energy is computed. As a goal of this paper, we want to implement in the WDW equation a NCG-induced minimal length and show how the resulting zero-point energies naturally arise as finite quantities without invoking any regularization scheme.

II. THE WHEELER-DEWITT EQUATION AND GRAVITON CONTRIBUTION

The WDW equation is a celebrated equation which formally extends to the quantum realm the Hamilton-Jacobi equation for general relativity, in the same fashion of what the Schrödinger equation does for quantum mechanics. It reads

$$\mathcal{H}\,\Psi = 0, \tag{1}$$

where Ψ is a functional of field configurations on all of spacetime, and the super-Hamiltonian $\mathcal{H}$ provides a Hamiltonian constraint, i.e., restricts Ψ to the physical configuration of the geometry and matter content of the Universe. The spacetime is supposed to be foliated into a family of spacelike hypersurfaces Σ. The Arnowitt-Deser-Misner variables offer a valid example of such a foliation. Explicitly, the metric background is written in the familiar form

$$ds^2 = -N^2 dt^2 + g_{ij}(N^i dt + dx^i)(N^j dt + dx^j). \tag{2}$$

N is called the *lapse function* N, and N_i is the *shift function*. The dynamical variables are, therefore, the three-dimensional metrics $g_{ij}(x^j, t)$, and their conjugate momenta π^{ij}, which are called supermomenta. The replacement of the dynamical variables with the corresponding quantum operators

$$\hat{g}_{\,ij}(t, x^k) \rightarrow g_{ij}(t, x^k), \tag{3}$$

$$\hat{\pi}^{\,ij}(t, x^k) \rightarrow -i\frac{\delta}{\delta g_{ij}(t, x^k)} \tag{4}$$

provides the quantization. In the following, for brevity, we shall skip the ˆ superscript for operator notation. In terms of dynamical variables, we can define the super-Hamiltonian, which reads

$$\mathcal{H} = (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}({}^3R - 2\Lambda), \tag{5}$$

where $\kappa = 8\pi G$, G_{ijkl} is the supermetric

$$G_{ijkl} = \frac{1}{2\sqrt{g}}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}),$$

and ${}^3R$ is the scalar curvature in three dimensions. The main reason to work with the WDW equation becomes more transparent if we formally rewrite it as

$$\frac{1}{V}\frac{\int \mathcal{D}[g_{ij}]\Psi^*[g_{ij}](\int_\Sigma d^3x\hat{\Lambda}_\Sigma)\Psi[g_{ij}]}{\int \mathcal{D}[g_{ij}]\Psi^*[g_{ij}]\Psi[g_{ij}]} = \frac{1}{V}\frac{\langle\Psi|\int_\Sigma d^3x\hat{\Lambda}_\Sigma|\Psi\rangle}{\langle\Psi|\Psi\rangle} = -\frac{\Lambda}{\kappa}, \tag{6}$$

where

$$V = \int_\Sigma d^3x\sqrt{g} \tag{7}$$

is the volume of the hypersurface Σ, and

$$\hat{\Lambda}_\Sigma = (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{g}\,{}^3R/(2\kappa). \tag{8}$$

Equation (6) represents the Sturm-Liouville problem associated with the cosmological constant. The related boundary conditions are dictated by the choice of the trial wave functionals which, in our case, are of Gaussian type. Different types of wave functionals correspond to different boundary conditions. We can gain more information if we consider

$$g_{ij} = \bar{g}_{ij} + h_{ij},$$

where $\bar{g}_{ij}$ is the background metric, and h_{ij} is a quantum fluctuation around the background. Thus, (6) can be expanded in terms of h_{ij}. Since the kinetic part of $\hat{\Lambda}_\Sigma$ is quadratic in the momenta, we only need to expand the three-scalar curvature $\int d^3x\sqrt{g}\,{}^3R$ up to the quadratic order. However, to extract the graviton contribution, we also need an orthogonal decomposition on the tangent space of three-metric deformations [17]

$$h_{ij} = \tfrac{1}{3}(\sigma + 2\nabla\cdot\xi)g_{ij} + (L\xi)_{ij} + h_{ij}^\perp. \tag{9}$$

The operator L maps the gauge vector ξ_i into symmetric trace-free tensors

$$(L\xi)_{ij} = \nabla_i\xi_j + \nabla_j\xi_i - \tfrac{2}{3}g_{ij}(\nabla\cdot\xi). \tag{10}$$

$h_{ij}^\perp$ is the traceless-transverse component of the perturbation (TT), namely,

$$g^{ij}h_{ij}^\perp = 0, \qquad \nabla^i h_{ij}^\perp = 0, \tag{11}$$

and h is the trace of h_{ij}. It is immediate to recognize that the trace element $\sigma = h - 2(\nabla\cdot\xi)$ is gauge-invariant. If we perform the same decomposition also on the momentum π^{ij}, up to second order, (6) becomes

$$\frac{1}{V}\frac{\langle\Psi|\int_\Sigma d^3x[\hat{\Lambda}_\Sigma^\perp + \hat{\Lambda}_\Sigma^\xi + \hat{\Lambda}_\Sigma^\sigma]^{(2)}|\Psi\rangle}{\langle\Psi|\Psi\rangle} = -\frac{\Lambda}{\kappa}. \tag{12}$$

Concerning the measure appearing in (6), we have to note that the decomposition (9) induces the following transformation on the functional measure $\mathcal{D}h_{ij} \to \mathcal{D}h_{ij}^{\perp}\mathcal{D}\xi_i\mathcal{D}\sigma J_1$, where the Jacobian related to the gauge-vector variable ξ_i is

$$J = [\det(\triangle g^{ij} + \tfrac{1}{3}\nabla^i\nabla^j - R^{ij})]^{1/2}. \tag{13}$$

This is nothing but the famous Faddeev-Popov determinant. It becomes more transparent if ξ_a is further decomposed into a transverse part ξ_a^T, with $\nabla^a\xi_a^T = 0$, and a longitudinal part $\xi_a^{\parallel}$, with $\xi_a^{\parallel} = \nabla_a\psi$. Then, J can be expressed by an upper triangular matrix for certain backgrounds (e.g., Schwarzschild in three dimensions). It is immediate to recognize that, for an Einstein space in any dimension, cross terms vanish, and J can be expressed by a block diagonal matrix. Since $\det AB = \det A \det B$, the functional measure $\mathcal{D}h_{ij}$ factorizes into

$$\mathcal{D}h_{ij} = (\det\triangle_V^T)^{1/2}(\det[\tfrac{2}{3}\triangle^2 + \nabla_i R^{ij}\nabla_j])^{1/2} \times \mathcal{D}h_{ij}^{\perp}\mathcal{D}\xi^T\mathcal{D}\psi, \tag{14}$$

leading to the Faddeev-Popov determinant with $(\triangle_V^{ij})^T = \triangle g^{ij} - R^{ij}$ acting on transverse vectors. In writing the functional measure $\mathcal{D}h_{ij}$, we have here ignored the appearance of a multiplicative anomaly [18]. Thus, the inner product can be written as

$$\int \mathcal{D}h_{ij}^{\perp}\mathcal{D}\xi^T\mathcal{D}\sigma\Psi^*[h_{ij}^{\perp}]\Psi^*[\xi^T]\Psi^*[\sigma]\Psi[h_{ij}^{\perp}]\Psi[\xi^T] \times \Psi[\sigma](\det\triangle_V^T)^{1/2}(\det[\tfrac{2}{3}\triangle^2 + \nabla_i R^{ij}\nabla_j])^{1/2}. \tag{15}$$

Nevertheless, since there is no interaction between ghost fields and the other components of the perturbation at this level of approximation, the Jacobian appearing in the numerator and in the denominator simplify. The reason can be found in terms of connected and disconnected terms. The disconnected terms appear in the Faddeev-Popov determinant, and the above ones are not linked by the Gaussian integration. This means that disconnected terms in the numerator and the same ones appearing in the denominator cancel out. Therefore, (12) factorizes into three pieces. The piece containing $E_\Sigma^\perp$, the contribution of the TT tensors , is essentially the graviton contribution representing true physical degrees of freedom. Regarding the vector operator $\hat{\Lambda}_\Sigma^T$, we observe that, under the action of infinitesimal diffeomorphism generated by a vector field ϵ_i, the components of (9) transform as follows [17]:

$$\xi_j \to \xi_j + \epsilon_j, \qquad h \to h + 2\nabla\cdot\xi, \qquad h_{ij}^{\perp} \to h_{ij}^{\perp}. \tag{16}$$

The Killing vectors satisfying the condition $\nabla_i\xi_j + \nabla_j\xi_i = 0$ do not change h_{ij} and thus should be excluded from the gauge group. All other diffeomorphisms act on h_{ij} nontrivially. We need to fix the residual gauge freedom on the vector ξ_i. The simplest choice is $\xi_i = 0$. This new gauge fixing produces the same Faddeev-Popov determinant connected to the Jacobian J and, therefore, will not contribute to the final value. We are left with

$$\frac{1}{V}\frac{\langle\Psi^\perp|\int_\Sigma d^3x[\hat{\Lambda}_\Sigma^\perp]^{(2)}|\Psi^\perp\rangle}{\langle\Psi^\perp|\Psi^\perp\rangle} + \frac{1}{V}\frac{\langle\Psi^\sigma|\int_\Sigma d^3x[\hat{\Lambda}_\Sigma^\sigma]^{(2)}|\Psi^\sigma\rangle}{\langle\Psi^\sigma|\Psi^\sigma\rangle} = -\frac{\Lambda}{\kappa}. \tag{17}$$

Note that, in the expansion of $\int_\Sigma d^3x\sqrt{g}R$ to second order, a coupling term between the TT component and the scalar one remains. However, the Gaussian integration does not allow such a mixing, which has to be introduced with an appropriate wave functional. By extracting the TT tensor contribution from (6) within second-order perturbation theory in h_{ij} onto the background $\bar{g}_{ij}$, we get

$$[\hat{\Lambda}_\Sigma^\perp]^{(2)} = \frac{1}{4V}\int_\Sigma d^3x\sqrt{\bar{g}}G^{ijkl}\Big[(2\kappa)K^{-1\perp}(x,x)_{ijkl} + \frac{1}{(2\kappa)}(\tilde{\triangle}_L)_j^a K^\perp(x,x)_{iakl}\Big], \tag{18}$$

where $\tilde{\triangle}$ is the modified Lichnerowicz operator

$$(\tilde{\triangle}_L h^\perp)_{ij} = (\triangle_L h^\perp)_{ij} - 4R_i^k h_{kj}^\perp + {}^3R h_{ij}^\perp, \tag{19}$$

defined in terms of the Lichnerowicz operator

$$(\triangle_L h)_{ij} = \triangle h_{ij} - 2R_{ikjl}h^{kl} + R_{ik}h_j^k + R_{jk}h_i^k, \quad \triangle = -\nabla^a\nabla_a. \tag{20}$$

The metric G^{ijkl} represents the inverse DeWitt supermetric, and all indices run from one to three. Note that the term

$$-4R_i^k h_{kj}^\perp + {}^3R h_{ij}^\perp \tag{21}$$

disappears in four dimensions when we use a background which is a solution of the Einstein field equations without matter contribution. The propagator $K^\perp(x,x)_{iakl}$ can be represented as

$$K^\perp(\vec{x},\vec{y})_{iakl} = \sum_\tau \frac{h_{ia}^{(\tau)\perp}(\vec{x})h_{kl}^{(\tau)\perp}(\vec{y})}{2\lambda(\tau)}, \tag{22}$$

where $h_{ia}^{(\tau)\perp}(\vec{x})$ are the eigenfunctions of $\tilde{\triangle}_L$. The parameter τ denotes a complete set of indices, and $\lambda(\tau)$ are a set of variational parameters to be determined by the minimization of (18). The expectation value of $\hat{\Lambda}_\Sigma^\perp$ is easily obtained by inserting the form of the propagator into (18) and minimizing with respect to the variational function $\lambda(\tau)$. As a result, the expectation value of $\hat{\Lambda}_\Sigma^\perp$ can be written in terms of the eigenvalues $\omega_i^2(\tau)$ of $\tilde{\triangle}_L$. By means of (17), we obtain a cosmological term due to the TT tensor one-loop energy density

$$\frac{\Lambda}{8\pi G} = -\frac{1}{2V}\sum_{\tau}\left[\sqrt{\omega_1^2(\tau)} + \sqrt{\omega_2^2(\tau)}\right], \tag{23}$$

provided $\omega_i^2(\tau) > 0$. The above expression is interpreted as the expectation value of graviton fluctuations on a given background. In the above calculation, we did not consider the scalar contribution coming from Λ_Σ^σ, since, in the physically relevant cases, it is possible to show that it does not contribute. To complete the picture, we need to specify the form of the background $\bar{g}_{ij}$. In the next section, we will work within the spherically symmetric case.

III. THE SPHERICALLY SYMMETRIC BACKGROUND

The line element (2) can be recast in the following form:

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{24}$$

where $b(r)$ is termed the "*shape function*." With the help of the Regge and Wheeler representation, $(\tilde{\triangle}_L h^\perp)_{ij}$ reduces to

$$\left[-\frac{d^2}{dx^2} + \frac{l(l+1)}{r^2} + m_i^2(r)\right] f_i(x) = \omega_{i,l}^2 f_i(x), \quad i = 1,2 \qquad [r \equiv r(x)], \tag{25}$$

where we have used reduced fields of the form $f_i(x) = F_i(x)/r$ and where we have defined two r-dependent effective masses $m_1^2(r)$ and $m_2^2(r)$:

$$m_1^2(r) = \frac{6}{r^2}\left[1-\frac{b(r)}{r}\right] + \frac{3}{2r^2}b'(r) - \frac{3}{2r^3}b(r),$$
$$m_2^2(r) = \frac{6}{r^2}\left[1-\frac{b(r)}{r}\right] + \frac{1}{2r^2}b'(r) + \frac{3}{2r^3}b(r),$$

with x as the proper distance from the throat at $r_t = b(r_t)$, i.e.,

$$dx = \pm\frac{dr}{\sqrt{1-\frac{b(r)}{r}}}.$$

There are two interesting cases where a symmetry in the masses appears. The first case is the Schwarzschild metric with $r_t = b(r_t) = 2MG$. Thus, masses $m_1^2(r)$ and $m_2^2(r)$ read

$$\begin{aligned} m_1^2(r) &= \frac{6}{r^2}\left(1-\frac{2MG}{r}\right) - \frac{3MG}{r^3}, \\ m_2^2(r) &= \frac{6}{r^2}\left(1-\frac{2MG}{r}\right) + \frac{3MG}{r^3}. \end{aligned} \tag{26}$$

In the range where $r \in [2MG, 5MG]$, we have

$$m_1^2(r) = -m_2^2(r) = m_0^2(r). \tag{27}$$

The second case comes from the de Sitter (dS) [anti-de Sitter (AdS)] metric with $b(r) = \frac{\Lambda_{dS}}{3}r^3$ $(-\frac{\Lambda_{AdS}}{3}r^3)$. Thus, $m_1^2(r)$ and $m_2^2(r)$ become

$$m_1^2 = m_2^2 = m_{dS}^2 = \frac{6}{r^2}\left(1-\frac{\Lambda_{dS}}{3}r^2\right) + \Lambda_{dS},$$
$$m_1^2 = m_2^2 = m_{AdS}^2 = \frac{6}{r^2}\left(1+\frac{\Lambda_{AdS}}{3}r^2\right) - \Lambda_{AdS}.$$

Note that, in the case of the dS background, $r \in [0, \sqrt{3/\Lambda_{dS}}]$, while, for the AdS background, one works in the range $r \in [0,\infty)$. In order to use the WKB approximation along the lines of the 't Hooft brick wall problem [19], we can extract two r-dependent radial wave numbers from (25):

$$k_i^2(r,l,\omega_{i,nl}) = \omega_{i,nl}^2 - \frac{l(l+1)}{r^2} - m_i^2(r), \qquad i = 1,2. \tag{28}$$

It is now possible to explicitly evaluate (23) in terms of the effective masses. To further proceed, we have to count the number of modes with frequency less than ω_i, $i = 1,2$. This is given approximately by

$$\tilde{g}(\omega_i) = \int_0^{l_{max}} \nu_i(l,\omega_i)(2l+1)dl, \tag{29}$$

where $\nu_i(l,\omega_i)$, $i = 1,2$ is the number of nodes in the mode with (l,ω_i), such that

$$\nu_i(l,\omega_i) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dx\sqrt{k_i^2(r,l,\omega_i)}. \tag{30}$$

Here, it is understood that the integration with respect to x and l is taken over those values which satisfy $k_i^2(r,l,\omega_i) \geq 0$, $i = 1,2$. However, (29) is based on the classical Liouville counting number of nodes

$$dn = \frac{d^3\vec{x}d^3\vec{k}}{(2\pi)^3}. \tag{31}$$

The procedure leads to divergent results. Conventionally, one performs a renormalization absorbing the divergent parts into the redefinition of bare classical quantities. In the spirit of any efficient quantum gravity approach, such a procedure must be reviewed. Indeed, both GUP and NCG formulations predict a deformation of the integration measure in momentum space,

$$1 = \int \frac{d^n k}{[1+\mathcal{F}(\vec{k}^2)]}|k\rangle\langle k|. \tag{32}$$

The function $\mathcal{F}(\vec{k}^2)$ depends on positive powers of the argument. As a result, $\mathcal{F}(\vec{k}^2)$ accounts for the suppression in the UV region, when an effective minimal length models the quantum gravity uncertainty. As shown in [13,14], NCG in coherent-state formalism provides a specific form for the function $\mathcal{F}(\vec{k}^2)$. Thus, the number of states reads

$$dn = \frac{d^3\vec{x}d^3\vec{k}}{(2\pi)^3} \Rightarrow dn_i = \frac{d^3\vec{x}d^3\vec{k}}{(2\pi)^3} \exp\left(-\frac{\theta}{4}k_i^2\right), \tag{33}$$

with

$$k_i^2 = \omega_{i,nl}^2 - m_i^2(r), \qquad i = 1, 2. \tag{34}$$

This deformation corresponds to an effective cutoff on the background geometry (24). The UV cutoff is triggered only by higher-momenta modes $\gtrsim 1/\sqrt{\theta}$ which propagate over the background geometry. The virtue of this kind of deformation lies in the fact that the exponential damping not only fulfils the general requirement of UV completeness for fields $f_i(x)$, but also provides the strongest possible suppression of higher momenta. Even if we are dealing with an effective approach that, strictly speaking, can reliably work only until scales $\sim\sqrt{\theta}$, this exponential profile lets us have at least a glimpse at smaller scales. To this purpose, we recall that this kind of deformation of the integration measure has been already successfully employed in taming the nonperturbative behavior of the gravitational field: curvature singularities in general relativity have been cured, giving rise to new quantum corrected regular geometries also at black hole centers without any breakdown at small scales [10]. Plugging (30) into (29) and taking account of (33), the number of modes with frequency less than ω_i, $i = 1, 2$ is given by

$$\tilde{g}(\omega_i) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dx \int_0^{l_{\max}} \sqrt{\omega_{i,nl}^2 - \frac{l(l+1)}{r^2} - m_i^2(r)}(2l+1) \times \exp\left(-\frac{\theta}{4}k_i^2\right)dl. \tag{35}$$

After integration over modes, one gets

$$\tilde{g}(\omega_i) = \frac{2}{3\pi}\int_{-\infty}^{+\infty} dxr^2\left(\frac{3}{2}\sqrt{[\omega_{i,nl}^2 - m_i^2(r)]^3} \times \exp\left\{-\frac{\theta}{4}[\omega_{i,nl}^2 - m_i^2(r)]\right\}\right). \tag{36}$$

This form of $\tilde{g}(\omega_i)$ allows an integration by parts in (23), leading to

$$\frac{\Lambda}{8\pi G} = -\frac{1}{4\pi^2 V}\sum_{i=1}^{2}\int_0^{+\infty}\omega_i\frac{d\tilde{g}(\omega_i)}{d\omega_i}d\omega_i = \frac{1}{4\pi^2 V}\sum_{i=1}^{2}\int_0^{+\infty}\tilde{g}(\omega_i)d\omega_i. \tag{37}$$

This is the graviton contribution to the induced cosmological constant at one loop. To get this result, we have used (27) and we have included an additional 4π coming from the angular integration. As a result for the Schwarzschild case, we find for the energy

$$4\pi\int_{-\infty}^{+\infty} dxr^2\left[\frac{\Lambda}{8\pi G} - \frac{1}{4\pi^2}\sum_{i=1}^{2}\int_0^{+\infty}\tilde{g}(\omega_i)d\omega_i\right] = 0. \tag{38}$$

Extracting the energy density, we find

$$\frac{\Lambda}{8\pi G} = \frac{1}{6\pi^2}\left\{\int_{\sqrt{m_0^2(r)}}^{+\infty}\sqrt{[\omega^2 - m_0^2(r)]^3}e^{-(\theta/4)[\omega^2 - m_0^2(r)]} + \int_0^{+\infty}\sqrt{[\omega^2 + m_0^2(r)]^3}e^{-(\theta/4)[\omega^2 + m_0^2(r)]}\right\}. \tag{39}$$

In the Appendix, we explicitly evaluate the previous integrals. Plugging the result of (A11) into (39), we get

$$\frac{\Lambda}{8\pi G} = \frac{1}{12\pi^2}\left(\frac{4}{\theta}\right)^2\left[y\cosh\left(\frac{y}{2}\right) - y^2\sinh\left(\frac{y}{2}\right)\right]K_1\left(\frac{y}{2}\right) + y^2\cosh\left(\frac{y}{2}\right)K_0\left(\frac{y}{2}\right), \tag{40}$$

where

$$y = \frac{m_0^2(r)\theta}{4} = \frac{3MG\theta}{4r^3}. \tag{41}$$

The asymptotic properties of (40) show that the one-loop contribution is regular everywhere. Indeed, when we rescale the radial coordinate to the wormhole throat

$$\rho \equiv \frac{r}{2MG},$$

with $\rho \in [1, 5/2]$, we have

$$y = \frac{1}{8\rho^3}\frac{\theta}{(MG)^2}. \tag{42}$$

This means that, when $MG \ll \theta$, we have $y \to \infty$. From the expression (A12), we find that, when $y \to +\infty$,

$$\frac{\Lambda}{8\pi G} \simeq \frac{1}{12\pi^2}\left(\frac{4}{\theta}\right)^2\left\{\frac{1}{8}\sqrt{\frac{\pi}{y}}[3 + (8y^2 + 6y + 3)\exp(-y)]\right\} \to 0; \tag{43}$$

namely, we recover the correct behavior, according to which, for a vanishing background gravity, i.e., $M = 0$, the one-loop energy must go to zero. Conversely, when $MG \gg \theta$, we have $y \to 0$ and, from expression (A13), we obtain

$$\frac{\Lambda}{8\pi G} \simeq \frac{1}{12\pi^2}\left(\frac{4}{\theta}\right)^2\left[2 - \left(\frac{7}{8} + \frac{3}{4}\ln\left(\frac{y}{4}\right) + \frac{3}{4}\gamma\right)y^2\right] \to \frac{8}{3\pi^2\theta^2}, \tag{44}$$

a finite value for Λ. This shows the effect of the NCG cutoff $\sqrt{\theta}$ at work.

For the dS and AdS cases, we find that the effective masses contribute in the same way at one loop. Thus, (37) becomes

$$\frac{\Lambda}{8\pi G} = 2 \times \frac{1}{6\pi^2} \int_{\sqrt{m_0^2(r)}}^{+\infty} \sqrt{[\omega^2 - m_0^2(r)]^3} e^{-(\theta/4)[\omega^2 - m_0^2(r)]}. \tag{45}$$

Plugging the result of (A3) into (37), we get

$$\frac{\Lambda}{8\pi G} = \frac{1}{6\pi^2}\left(\frac{4}{\theta}\right)^2 \left[\frac{1}{2} z(1-z) K_1\left(\frac{z}{2}\right) + \frac{1}{2} z^2 K_0\left(\frac{z}{2}\right)\right] \exp\left(\frac{z}{2}\right), \tag{46}$$

where

$$z = m_{\mathrm{dS}}^2(r)\theta/4 \quad \text{or} \quad z = m_{\mathrm{AdS}}^2(r)\theta/4. \tag{47}$$

To analyze these results, we recall that, in the de Sitter case, the radial coordinates $r \in [0, \sqrt{3/\Lambda_{\mathrm{dS}}}]$. Therefore, at short distances $r \ll \sqrt{\theta}$, we have

$$z = \frac{3}{2}\frac{\theta}{r^2} - \frac{\Lambda_{\mathrm{dS}}\theta}{4} \longrightarrow \infty.$$

From expansions (A7) and (A8), we find

$$\frac{\Lambda}{8\pi G} \simeq \frac{1}{6\pi^2}\left(\frac{4}{\theta}\right)^2 \frac{3}{8}\sqrt{\frac{\pi}{z}} \longrightarrow 0, \tag{48}$$

when $z \longrightarrow \infty$. This corresponds to the correct behavior in a spacetime region where the curvature vanishes. On the other hand, for $r \approx \sqrt{3/\Lambda_{\mathrm{dS}}} \gg \sqrt{\theta}$, we have

$$z \approx \frac{\Lambda_{\mathrm{dS}}\theta}{4} \longrightarrow 0,$$

which implies

$$\frac{\Lambda}{8\pi G} \simeq \frac{1}{6\pi^2}\left(\frac{4}{\theta}\right)^2 \left\{1 - \frac{z}{2} + \left[-\frac{7}{16} - \frac{3}{8}\ln\left(\frac{z}{4}\right) - \frac{3}{8}\gamma\right] z^2\right\} \longrightarrow \frac{8}{3\pi^2\theta^2}, \tag{49}$$

i.e., a finite value of the cosmological term. The same conclusion holds for the anti-de Sitter case.

IV. SUMMARY AND CONCLUSIONS

In this paper, we calculated the cosmological constant as an eigenvalue of the Sturm-Liouville problem related to the Wheeler-DeWitt equation. With the help of Gaussian trial functionals, we extracted the one-loop contribution of the transverse-traceless component, namely, the graviton. Instead of embarking in conventional regularization schemes, we implemented a natural UV cutoff in the background geometry, invoking a NCG-induced minimal length. As a result, we get a modified counting of graviton modes. This lets us obtain regular values everywhere for the cosmological constant, independently of the chosen background, which, nevertheless, is of a spherically symmetric type. We show this for the Schwarzschild, de Sitter, and anti-de Sitter backgrounds. The strength of our approach lies in the specific kind of integration measure deformation in momentum space we derived from NCG. This lets us overcome previous attempts which only led to mild effects and just a reduction of the degree of divergence [20,21]. Although the result seems to be promising, we have to note that the evaluation is at the Planck scale, and, even if Fig. 1 shows a vanishing behavior, one has to bear in mind that this behavior corresponds to the switching off of the Schwarzschild background. The paper is subjected to future developments. First, we restricted the attention only on spherically symmetric backgrounds like Schwarzschild or de Sitter/anti-de Sitter backgrounds. A further extension should be the inclusion of rotations, which considerably increase the technical difficulty level. Moreover, regarding the Schwarzschild background, we worked with the "*classical Schwarzschild*" and not with the smeared solution predicted by the noncommutative theory developed in configuration space, having a shape function $b(r)$ of the form

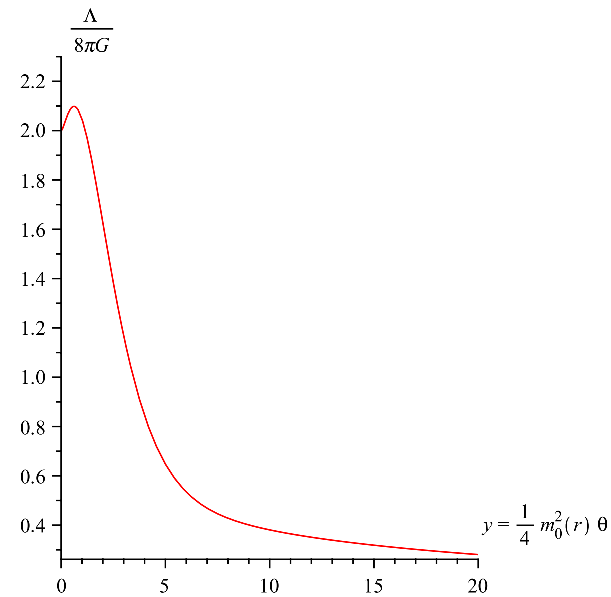

FIG. 1 (color online). Plot of $\Lambda/8\pi G$ in Planck units as a function of the scale-invariant y, which depends on the background choice. For dS and AdS backgrounds, the variable y is replaced by z.

$$b_{\mathrm{NC}}(r) = \frac{4MG}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right). \tag{50}$$

The use of $b_{\mathrm{NC}}(r)$ instead of $b(r)$ could introduce new features of the full noncommutative theory, allowing a better exploration of the wormhole throat. As a further point, we have to observe that, even if we have a finite value for the cosmological constant, it will still come too large with respect to its observed value. This seems to be a general fact, as far as one employs a UV natural cutoff [22]. A possible solution to this problem could be found in the fact that the cosmological constant might arise from fluctuations of vacuum energy [23], rather than from the vacuum energy itself. Therefore, we believe that the paper is opening a new route to further investigations.

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APPENDIX: INTEGRALS

In this Appendix, we explicitly compute the integrals coming from (37). We begin with

$$\begin{aligned}\int_{\sqrt{m_0^2(r)}}^{+\infty}&\sqrt{[\omega^2-m_0^2(r)]^3}e^{-(\theta/4)[\omega^2-m_0^2(r)]}d\omega\\ &\underset{\omega^2=x}{=}\frac{1}{2}\int_{\sqrt{m_0^2(r)}}^{+\infty}\sqrt{[x-m_0^2(r)]^3}e^{-(\theta/4)[x-m_0^2(r)]}\frac{dx}{\sqrt{x}}\\ &=\exp\left[\frac{m_0^2(r)\theta}{4}\right]\frac{1}{2}\left(\frac{\theta}{4}\right)^{-(3/2)}\sqrt{m_0^2(r)}\Gamma\left(\frac{5}{2}\right)\\ &\quad\times\exp\left[-\frac{m_0^2(r)\theta}{8}\right]W_{-1,-1}\left[\frac{m_0^2(r)\theta}{4}\right],\end{aligned}\tag{A1}$$

where we have used the following relationship:

$$\begin{aligned}\int_u^{+\infty}&x^{\nu-1}(x-u)^{\mu-1}e^{-\beta x}dx\\ &=\beta^{-[(\nu+\mu)/2]}u^{(\nu+\mu-2)/2}\Gamma(\mu)\\ &\quad\times\exp\left(-\frac{\beta u}{2}\right)W_{(\nu-\mu)/2,(1-\nu-\mu)/2}(\beta u),\end{aligned}\tag{A2}$$

$$\mathrm{Re}\mu>0,\qquad \mathrm{Re}\beta u>0,$$

where $W_{\mu,\nu}(x)$ is the Whittaker function, and $\Gamma(\nu)$ is the gamma function. Further manipulation on (A1) leads to

$$\frac{1}{2}\left(\frac{\theta}{4}\right)^{-2}\left[\frac{1}{2}x(1-x)K_1\left(\frac{x}{2}\right)+\frac{1}{2}x^2K_0\left(\frac{x}{2}\right)\right]\exp\left(\frac{x}{2}\right),\tag{A3}$$

where

$$x=\frac{m_0^2(r)\theta}{4}.\tag{A4}$$

It is useful to write an asymptotic expansion for $K_0(\frac{x}{2})$ and $K_1(\frac{x}{2})$. We get

$$\begin{aligned}K_0(x/2)&\simeq\sqrt{\pi}e^{-x/2}x^{-(1/2)}\left(1-\frac{1}{4x}\right)+O[x^{-(5/2)}],\\ K_1(x/2)&\simeq\sqrt{\pi}e^{-x/2}x^{-(1/2)}\left(1+\frac{3}{4x}\right)+O[x^{-(5/2)}].\end{aligned}\tag{A5}$$

Plugging expansion (A5) into expression (A3), one obtains that the asymptotic behavior is given by

$$\begin{aligned}\frac{1}{2}\left(\frac{\theta}{4}\right)^{-2}&\sqrt{\pi}\left[\frac{1}{2}\sqrt{x}(1-x)\left(1+\frac{3}{4x}\right)+\frac{1}{2}\sqrt{x^3}\left(1-\frac{1}{4x}\right)\right]\\ &+O[x^{-(5/2)}]\end{aligned}\tag{A6}$$

and, after a further simplification, one gets

$$\frac{1}{2}\left(\frac{\theta}{4}\right)^{-2}\frac{3}{8}\sqrt{\frac{\pi}{x}},\tag{A7}$$

while, when $x\to0$, one gets

$$\frac{1}{2}\left(\frac{\theta}{4}\right)^{-2}\left\{1-\frac{x}{2}+\left[-\frac{7}{16}-\frac{3}{8}\ln\left(\frac{x}{4}\right)-\frac{3}{8}\gamma\right]x^2\right\}.\tag{A8}$$

For the other integral, we proceed in the same way and we get

$$\begin{aligned}\int_0^{+\infty}&\sqrt{[\omega^2+m_0^2(r)]^3}e^{-(\theta/4)[\omega^2+m_0^2(r)]}d\omega\\ &=\exp\left[-\frac{m_0^2(r)\theta}{8}\right]\frac{1}{2}\left(\frac{\theta}{4}\right)^{-(3/2)}\\ &\quad\times\sqrt{m_0^2(r)}\Gamma\left(\frac{1}{2}\right)W_{1,-1}\left[\frac{m_0^2(r)\theta}{4}\right].\end{aligned}\tag{A9}$$

Converting to Bessel functions, (A9) yields

$$\frac{1}{2}\left(\frac{\theta}{4}\right)^{-2}\left[\frac{x}{2}(1+x)K_1\left(\frac{x}{2}\right)+\frac{x^2}{2}K_0\left(\frac{x}{2}\right)\right]\exp\left(-\frac{x}{2}\right),\tag{A10}$$

whose sum with Eq. (A3) gives

$$\begin{aligned}\frac{1}{2}\left(\frac{\theta}{4}\right)^{-2}&\left[x\cosh\left(\frac{x}{2}\right)-x^2\sinh\left(\frac{x}{2}\right)\right]K_1\left(\frac{x}{2}\right)\\ &+x^2\cosh\left(\frac{x}{2}\right)K_0\left(\frac{x}{2}\right).\end{aligned}\tag{A11}$$

Thus, the asymptotic expansion for (A11) yields

$$\frac{1}{2}\left(\frac{\theta}{4}\right)^{-2}\left\{\frac{1}{8}\sqrt{\frac{\pi}{x}}[3+(8x^2+6x+3)\exp(-x)]\right\}.\tag{A12}$$

On the other hand, when $x\to0$, one gets

$$\frac{1}{2}\left(\frac{\theta}{4}\right)^{-2}\left\{2-\left[\frac{7}{8}+\frac{3}{4}\ln\left(\frac{x}{4}\right)+\frac{3}{4}\gamma\right]x^2\right\}.\tag{A13}$$

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