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An inequality for local unitary Theta correspondence

by

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1 Introduction, notations

This section recalls the local theta correspondence as in [Kud2] and cites some of the results of [HKS].

We fix once and for all a non archimedean local field F of residual characteristic different from 2.

The application Δ will always be a diagonal embedding, usually from G to $G \times G$ except in one point where it will be precised.

1.1 Heisenberg group

Let W be a vector space with a symplectic form $\langle ., . \rangle$ on which the group $\mathrm{GL}(W)$ will act on the right – accordingly, if f and g are two endomorphisms of W, we will denote $f \circ g$ the endomorphism such that $(f \circ g)(w) = g(f(w))$. We will denote, as usual,

$$\mathrm{Sp}(W) = \{ g \in \mathrm{GL}(W) \mid \forall (x, y) \in W^2, \langle xg, yg \rangle = \langle x, y \rangle \}$$

its isometry group.

Definition 1.1 *The* Heisenberg group *of W if the group $H(W) = W \ltimes F$ with product*

$$(w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2} \langle w_1, w_2 \rangle).$$

The centre of $H(W)$ is $\{ (0, t) \mid t \in F \}$ and $\mathrm{Sp}(W)$ acts on $H(W)$ via its action on W:

$$(w, t)^g = (wg, t).$$

We recall

Theorem 1.2 (Stone–von Neumann theorem) *Let ψ be a non trivial unitary character of F. There exists, up to isomorphism, one smooth irreducible representation (ρ_ψ, S) of $H(W)$ such that*

$$\rho_\psi\big((0, t)\big) = \psi(t) \cdot \mathrm{id}_S.$$

If we fix such a representation (ρ_ψ, S), for any $g \in \mathrm{Sp}(g)$, the representation $h \longmapsto \rho_\psi^g(h) = \rho_\psi(h^g)$ is a representation of $H(W)$ with the same central character, which means that it must be isomorphic to ρ_ψ. Hence there is an isomorphism $A(g) \in \mathrm{GL}(S)$, unique up to a scalar, such that

$$\forall h \in H, \quad A(g)^{-1}\rho_\psi(h)A(g) = \rho_\psi^g(h). \tag{1}$$

The group

$$\mathrm{Mp}(W) = \big\{\big(g, A\left(g\right)\big) \mid \text{equation (1) holds}\big\}$$

is independent of the choice of ψ and is a central extension of $\mathrm{Sp}(W)$ by $\mathbf{C}^\times$:

$$0 \longrightarrow \mathbf{C}^\times \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1.$$

The group $\mathrm{Mp}(W)$ has a natural representation, called the Weil representation, ω_ψ on S given by

$$\begin{aligned}
\omega_\psi: \quad \mathrm{Mp}(W) &\longrightarrow \mathrm{End}(S) \\
\big(g, A\left(g\right)\big) &\longmapsto A(g)
\end{aligned}$$

1.2 The Schrödinger model of the Weil representation

The application $\big(g, A\left(g\right)\big) \mapsto A(g)$ defines a representation of $\mathrm{Mp}(W)$ of which there are several models. We are interested in the so-called Schrödinger model.

Let Y be a Lagrangian of W, i.e. a maximal isotropic subspace of W and $W = X \oplus Y$ a complete polarisation of W. We consider Y as a degenerate symplectic space and see $H(Y) = Y \ltimes F$ as a maximal abelian subgroup of $H(W)$. We consider the extension ψ_Y of the character ψ from F to $H(Y)$ defined by $\psi_Y(y,t) = \psi(t)$. Let

$$S_Y = \mathrm{Ind}_{H(Y)}^{H(W)} \psi_Y.$$

We recall that S_Y is the space of those $f : H(W) \longrightarrow \mathbf{C}$ such that

$$\forall h_1 \in H(Y), f(h_1 h) = \psi_Y(h_1)f(h)$$

and such that there exists a compact open subgroup L of W such that

$$\forall l \in L, f\big(h(l,0)\big) = f(h).$$

We fix an isomorphism of S_Y with the space $S(X)$ of Schwartz functions on X by

$$\begin{aligned}
S_Y &\longrightarrow S(X) \\
f &\longmapsto \varphi : X \to \mathbf{C} \\
&\qquad x \mapsto \varphi(x) = f(x, 0).
\end{aligned}$$

The group $H(W)$ acts on S_Y by right translation while it acts on $\varphi \in S(X)$ by

$$\big(\rho(x+y,t)\varphi\big)(x_0) = \psi\left(t + \langle x_0, y\rangle + \frac{1}{2}\langle x, y\rangle\right)\varphi(x_0 + x)$$

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where $x + y \in W$ is such that $x \in X$ and $y \in Y$. Then (see [MVW]) $(\rho, S(X))$ is a model for the Weil representation.

We specify the operator ω_ψ as follows. We identify an element $w \in W$ with the row vector $(x, y) \in X \oplus Y$. An element $g \in \mathrm{Sp}(W)$ will be of the form $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a \in \mathrm{End}(X)$, $b \in \mathrm{Hom}(X, Y)$, $c \in \mathrm{Hom}(Y, X)$ and $d \in \mathrm{End}(Y)$. Let $P_Y = \{g \in \mathrm{Sp}(W) \mid c = 0\}$ be the maximal parabolic subgroup of $\mathrm{Sp}(W)$ that stabilises Y and $N_Y = \{g \in P_Y \mid d = \mathrm{id}_Y\}$ its unipotent radical. We have a Levy subgroup $M_Y = \{g \in P_Y \mid b = 0\}$ of P_Y and $P_Y = M_Y N_Y$.

We define the following natural applications:

$$m : \mathrm{GL}(X) \longrightarrow M_Y \qquad\qquad n : \mathrm{Her}(X, Y) \longrightarrow N_Y$$
$$a \longmapsto m(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^\vee \end{smallmatrix}\right) \qquad\qquad b \longmapsto n(b) = \left(\begin{smallmatrix} \mathrm{id}_X & b \\ 0 & \mathrm{id}_Y \end{smallmatrix}\right)$$

where $a^\vee$ is the inverse of the dual of a and $\mathrm{Her}(X, Y)$ is the subset of those $b \in \mathrm{Hom}(X, Y)$ which are Hermitian (in both cases we identify the dual of $X \oplus Y$ with $Y \oplus X$ using $\langle ., . \rangle$).

Proposition 1.3 ([Kud2, Proposition 2.3, p8]) *Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(g)$. The operator $r(g)$ of $S(X)$ defined by*

$$r(g)(\varphi)(x) = \int_{\mathrm{Ker}\, c \backslash Y} \psi\left(\frac{1}{2}\langle xa, xb\rangle - \langle xb, yc\rangle + \frac{1}{2}\langle yc, yd\rangle\right) \varphi(xa + yc)\, \mathrm{d}\mu_g(y)$$

is proportional to $A(g)$ and moreover is unitary for a unique Haar measure $\mathrm{d}\mu_g(y)$ on $\mathrm{Ker}\, c \backslash Y$.

1.3 Dual reductive pairs

Definition 1.4 A dual reductive pair (G, G') in $\mathrm{Sp}(W)$ *is a pair of subgroups of $\mathrm{Sp}(W)$ such that both G and G' are reductive and*

$$\mathrm{Cent}_{\mathrm{Sp}(W)}(G) = G' \quad and \quad \mathrm{Cent}_{\mathrm{Sp}(W)}(G') = G.$$

If (G, G') is a dual reductive pair in $\mathrm{Sp}(W)$, we denote $\widetilde{G}$ and $\widetilde{G}'$ the pullbacks of the subgroups in $\mathrm{Mp}(W)$. As seen in [MVW], there exists a natural morphism

$$j : \widetilde{G} \times \widetilde{G}' \longrightarrow \mathrm{Mp}(W)$$

such that the restriction of j to $\mathbf{C}^\times \times \mathbf{C}^\times$ is the product.

We consider the pullback $(j^*(\omega_\psi), S)$ of ω_ψ to $\widetilde{G} \times \widetilde{G}'$. We note that the central character for both $\widetilde{G}$ and $\widetilde{G}'$ is the identity:

$$\omega_\psi(j(z_1, z_2)) = z_1 z_2 \cdot \mathrm{id}_S.$$

Let π be an irreducible admissible representation of $\widetilde{G}$ such that the central character of π is the identity. Then if

$$\mathcal{N}(\pi) = \bigcap_{\lambda \in \mathrm{Hom}_{\widetilde{G}}(S,\pi)} \mathrm{Ker}\,\lambda$$

$S(\pi) = S/\mathcal{N}(\pi)$ is the largest quotient of S on which $\widetilde{G}$ acts by π. The action of $\widetilde{G}'$ on S commutes with the action of $\widetilde{G}$ so that $\widetilde{G}'$ acts on $S(\pi)$ and thus $S(\pi)$ is a representation of $\widetilde{G} \times \widetilde{G}'$. There exists (see [MVW]) a smooth representation $\Theta_\psi(\pi)$ of G', unique up to isomorphism, such that

$$S(\pi) \simeq \pi \otimes \Theta_\psi(\pi).$$

The principal result is the following

Theorem 1.5 (Howe duality principle) *Let F be a non archimedean local field with residual characteristic different from 2 and let π be an irreducible admissible representation of $\widetilde{G}$. Then*

i) *If $\Theta_\psi(\pi) \neq 0$, then it is an admissible representation of $\widetilde{G}'$ of finite length.*

ii) *If $\Theta_\psi(\pi) \neq 0$, there exists a unique $\widetilde{G}'$-submodule $\Theta_\psi^0(\pi)$ such that the quotient*

$$\theta_\psi(\pi) = \Theta_\psi(\pi)/\Theta_\psi^0(\pi)$$

is irreducible. If $\Theta_\psi(\pi) = 0$, we let $\theta_\psi(\pi) = 0$.

iii) *If two irreducible admissible representations π_1 and π_2 of $\widetilde{G}$ are such that $\theta_\psi(\pi_1) \simeq \theta_\psi(\pi_2) \neq 0$ then $\pi_1 \simeq \pi_2$.*

1.4   The unitary case

Let E/F be a quadratic extension and $\epsilon_{E/F}$ the corresponding quadratic character of $F^\times$.

Let V be a quadratic space of dimension m with Hermitian form

$$(.|.) : V \times V \longrightarrow E$$

(linear in the second argument). We will denote

$$G(V) = \{g \in \mathrm{GL}(V) \mid \forall v, w \in V, (gv|gw) = (v|w)\}$$

the isometry group of V.

Let W be a quadratic space of dimension n with skew-Hermitian form

$$\langle .,. \rangle : W \times W \longrightarrow E$$

(linear in the second argument). We will denote $G(W)$ its isometry group.

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Let $\mathbb{W} = \mathrm{R}_{E/F}(V \otimes_E W)$ with symplectic form

$$
\begin{aligned}
\langle\!\langle .,. \rangle\!\rangle : \qquad \mathbb{W} \otimes \mathbb{W} &\longrightarrow F \\
(v_1 \otimes w_1, v_2 \otimes w_2) &\longmapsto \langle\!\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle\!\rangle = \frac{1}{2} \mathrm{Tr}_{E/F}\left((v_1, v_2)\langle w_1, w_2 \rangle \right).
\end{aligned}
$$

The pair $\big(G(V), G(W)\big)$ is a dual reductive pair in $\mathrm{Sp}(\mathbb{W})$. We have a natural inclusion

$$
\begin{aligned}
\iota : G(V) \times G(W) &\longrightarrow \mathrm{Sp}(\mathbb{W}) \\
(g, h) &\longmapsto \iota(g, h) = g \otimes h.
\end{aligned}
$$

For any pair of characters $\chi = (\chi_m, \chi_n)$ of $E^\times$ such that

$$
\chi_n \big|_{F^\times} = \epsilon_{E/F}^n, \quad \chi_m \big|_{F^\times} = \epsilon_{E/F}^m,
$$

one can define a homomorphism

$$
\tilde{\iota}_\chi : G(V) \times G(W) \longrightarrow \mathrm{Mp}(\mathbb{W})
$$

lifting ι (the homomorphism $\tilde{\iota}_\chi$ *does* depend on χ). Since the context will usually make clear which of χ_m and χ_n is considered, we will often use χ instead of χ_m or χ_n. Moreover we define $\iota_{V,\chi}$ (resp. $\iota_{W,\chi}$) the restriction of ι_χ to $G(V) \times 1$ (resp. $1 \times G(W)$).

We will denote ω_ψ the Weil representation of $\mathrm{Mp}(\mathbb{W})$ and ω_χ its pullback through $\tilde{\iota}_\chi$. As before, if π is an irreducible admissible representation of $G(V)$, we get a representation $\Theta_\chi(\pi, V)$ of $G(W)$ such that

$$
S(\pi) \simeq \pi \otimes \Theta_\chi(\pi, V)
$$

and if $\Theta_\chi(\pi, V) \neq 0$, we say that π appears in the local theta correspondence for the pair $\big(G(V), G(W)\big)$. This condition depends on χ_m but not on χ_n. As above we define $\theta_\pi(\pi, V)$ to be the unique irreducible quotient of $\Theta_\chi(\pi, V)$ (or 0 if $\Theta_\chi(\pi, V) = 0$).

Witt towers   For a fixed dimension m, there are two equivalence classes of Hermitian spaces of dimension m over E. These two classes are distinguished by their Hasse invariant

$$
\epsilon(V) = \epsilon_{E/F}\big((-1)^{\frac{m(m-1)}{2}} \det V\big).
$$

We thus get two families of spaces $V_m^\pm$ where the sign is the sign of the Hasse invariant. As Hermitian spaces we have $V_{m+2}^\pm \simeq V_m^\pm \oplus V_{1,1}$, where $V_{1,1}$ is an hyperbolic plane and the direct sum is orthogonal. We thus get four so-called Witt towers

$$
V_{2r}^+ = V_0^+ \oplus (V_{1,1})^r, \ V_{2r+2}^- = V_2^- \oplus (V_{1,1})^r, \ V_{2r+1}^+ = V_1^+ \oplus (V_{1,1})^r, \ V_{2r+1}^- = V_1^+ \oplus (V_{1,1})^r
$$

where V_0^+ is the null vector space, V_2^- is an anisotropic 2-dimensional Hermitian space and $V_1^\pm$ are one dimensional anisotropic Hermitian spaces. In each case the integer r is the Witt index of the corresponding Hermitian space[1].

We have

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[1]We recall that the Witt index of a quadratic space is the dimension of a maximal totally isotropic subspace

Proposition 1.6 ([HKS],[Kud2]) *Consider a Witt tower $\{V_m^\epsilon\}$ with $\epsilon = \pm$.*

(i) (Persistence) If $\theta_\chi(\pi, V_m^\epsilon) \neq 0$ then $\theta_\chi(\pi, V_{m+2}^\epsilon) \neq 0$.

(ii) (Stable range) We have $\theta_\chi(\pi, V_m^\epsilon) \neq 0$ if the Weil index r_0 of V_m is such that $r_0 \geqslant n$.

We fix $m_0 \in \{0, 1\}$ and a character χ of $E^\times$ such that $\chi_{|F^\times} = \epsilon_{E/F}^{m_0}$ and we consider the two towers $V_m^\pm$ with m of the parity of m_0 (if $m_0 = 0$ we disregard V_0^- which does not exist). Let $m_\chi^\pm(\pi)$ be the smallest m such that

$$\theta_\chi(\pi, V_m^\pm) \neq 0.$$

Based on several examples, we have

Conjecture 1.7 (Conservation relation, [HKS, Speculations 7.5 and 7.6], [KR, Conjecture 3.6])
$$m_\chi^+(\pi) + m_\chi^-(\pi) = 2n + 2.$$

1.5 Aim of this paper

We prove here one of the inequalities of Conjecture 1.7:

Theorem 1.8 *Let π be an irreducible admissible representation of $G(W)$, then*

$$m_\chi^+(\pi) + m_\chi^-(\pi) \geqslant 2n + 2.$$

1.6 Degenerate principal series

Let W_+ and W_- be two copies of W with respectively the same form as W and its opposite. We keep our pair of characters $\chi = (\chi_m, \chi_n)$. We fix for the space $W_+ \oplus W_-$ the complete polarisation $X \oplus Y$ where $X = \{(w, -w) \mid w \in W\}$ and $Y = \{(w, w) \mid w \in W\} = \Delta(W)$ where Δ is the diagonal embedding of W in $W_+ \oplus W_-$. We let then

$$\begin{aligned}
\mathbb{W}_+ &= \mathrm{R}_{E/F}(V \otimes_E W_+) & \mathbb{W}_- &= \mathrm{R}_{E/F}(V \otimes_E W_-) \\
\mathbb{X} &= \mathrm{R}_{E/F}(V \otimes_E X) & \mathbb{Y} &= \mathrm{R}_{E/F}(V \otimes_E Y).
\end{aligned}$$

and we consider the representation $\omega_{V, W_+ \oplus W_-, \chi}$ of $G(V) \times G(W_+ \oplus W_-)$ induced by the Weil representation of $\mathbb{W}_+ \oplus \mathbb{W}_-$ on $S = S(\mathbb{X}) \simeq S(V^n)$. Let $R_n(V, \chi)$ be the maximal quotient of S on which $G(V)$ acts by the character χ_m. The space $R_n(V, \chi)$ can be seen as a representation of $G(W) \times G(W)$ via the natural embedding

$$i : G(W) \times G(W) = G(W_+) \times G(W_-) \hookrightarrow G(W_+ \oplus W_-).$$

From now on, we will denote $G = G_n = G(W)$ and $\tilde{G} = \tilde{G}_n = G(W_+ \oplus W_-)$ so that $i : G \times G \hookrightarrow \tilde{G}$.

We then have

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Proposition 1.9 ([HKS, Proposition 3.1 and discussion before]) *Let π be an irreducible admissible representation of $G(W)$,*

$$\Theta_\chi(\pi, V) \neq 0 \iff \mathrm{Hom}_{G \times G}(R_n(V, \chi), \pi \otimes (\chi_m \cdot \pi^\vee)) \neq 0.$$

Let P_Y be the parabolic subgroup of $\tilde{G}$ stabilising Y. We will denote M_Y its maximal Levi subgroup and N_Y its unipotent radical. Recall that M_Y and N_Y are parametrised respectively by $\mathrm{GL}(X)$ and $\mathrm{Her}(X, Y)$.

For $s \in \mathbf{C}$ and χ a character of $E^\times$, let

$$I_n(s, \chi) = \mathrm{Ind}_{P_Y}^{\tilde{G}} \chi | \cdot |^s$$

be the degenerate principal series (the induction is unitary and the elements of $I_n(s, \chi)$ are locally constant functions $\Phi(g, s)$).

We can identify $R_n(V, \chi)$ as a subspace of some $I_n(s, \chi)$ by sending an element $\phi \in S$ to the function $g \longmapsto \omega_\chi(g)\phi(0)$ – here we denote $\omega_\chi = \omega_\psi \circ \tilde{\iota}_{V,\chi}$. The spaces $R_n(V_m^\pm, \chi)$ allows us to decompose $I_n(s, \chi)$ as explained by the following proposition.

Proposition 1.10 ([KS, Theorem 1.2, p257]) *Let $V_m^\pm$ be an m-dimensional unitary space of dimension m and Hasse invariant $\pm$. Let $s_0 = \frac{m-n}{2}$ and χ a character of $E^\times$ such that $\chi|_{F^\times} = \epsilon_{E/F}^m$.*

   i) If $m \leqslant n$, i.e. if $s_0 \leqslant 0$, then the modules $R_n(V_m^\pm, \chi)$ are irreducible and $R_n(V_m^+, \chi) \oplus R_n(V_m^-, \chi)$ is the maximal completely reducible submodule of $I_n(s_0, \chi)$.

   ii) If $m = n$, i.e. if $s_0 = 0$, then $I_n(0, \chi) = R_n(V_n^+, \chi) \oplus R_n(V_n^-, \chi)$.

   iii) If $n < m < 2n$, i.e. if $0 < s_0 < \frac{n}{2}$, then $I_n(s_0, \chi) = R_n(V_m^+, \chi) + R_n(V_m^-, \chi)$ and $R_n(V_m^+, \chi) \cap R_n(V_m^-, \chi)$ is the unique irreducible submodule of $I_n(s_0, \chi)$.

   iv) If $m = 2n$, i.e. if $s_0 = \frac{n}{2}$, then $I_n(s_0, \chi) = R_n(V_{2n}^+, \chi)$, $R_n(V_{2n}^-, \chi)$ is of codimension 1 and is the unique irreducible submodule of $I_n(s_0, \chi)$.

   v) If $m > 2n$, i.e. if $s_0 > \frac{n}{2}$, then $I_n(s_0, \chi) = R_n(V_m^\pm, \chi)$ is irreducible.

In all other cases $I_n(s, \chi)$ is irreducible.

To understand better the decompositions above we begin with the Bruhat decomposition of $\tilde{G}$:

$$\tilde{G} = \coprod_{j=0}^{n} P_Y \omega_j P_Y, \quad \text{with } \omega_j = \begin{pmatrix} I_{n-j} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_j \\ 0 & 0 & I_{n-j} & 0 \\ 0 & -I_j & 0 & 0 \end{pmatrix}$$

and let us introduce, as in [Kud2, p19] and [Rao] the application

$$\begin{aligned} x: \quad \tilde{G} &\longrightarrow E^\times / \mathrm{N}_{E/F} E^\times \\ p_1 \omega_j^{-1} p_2 &\longmapsto \det(p_1 p_2|_Y) \mod \mathrm{N}_{E/F} E^\times \end{aligned}$$

Whenever $\chi|_{F^\times} = \mathbf{1}$ we can introduce the character $\chi_{\tilde{G}}$ of $\tilde{G}$

$$\chi_{\tilde{G}}(g) = \chi(x(g)).$$

We extend the definition of R_n as follows:

$$R_n(V_0^+, \chi) = R_n(0, \chi) = \mathbf{C} \cdot \chi_{\tilde{G}}$$

and $R_n(V_0^+, \chi)$ is a submodule of dimension 1 o f $I_n(-\frac{n}{2}, \chi)$ (we are, at least formally, in the case $i)$ of Proposition 1.10). As a last step, we define the intertwining operators

$$M_n(s, \chi) : I_n(s, \chi) \longrightarrow I_n(-s, \chi)$$

by the integral

$$M_n(s, \chi)(\Phi) = \int_{N_Y} \Phi(w_n u g, s) \, \mathrm{d}u = \int_{\mathrm{Her}(X,Y)} \Phi(w_n n(b) g, s) \, \mathrm{d}b,$$

which is convergent for $\mathrm{Re}\, s > \frac{n}{2}$ and by meromorphic continuation for $s \in \mathbf{C}$. The Haar measure $\mathrm{d}b$ is chosen self-dual with respect to the Fourier transform

$$\hat{\phi}(y) = \int \phi(b) \psi(\mathrm{Tr}(by)) \, \mathrm{d}b.$$

We normalise $M_n(s, \chi)$ using

$$a(s, \chi) = \prod_{j=0}^{n-1} L_F\left(2s + j - (n-1), \chi \epsilon_{E/F}^j\right)$$

and then $M_n^*(s, \chi) = \frac{1}{a(s,\chi)} M_n(s, \chi)$ is holomorphic and non zero (see [KS, Proposition 3.2]).

Proposition 1.11 ([KS]) *Let $V_m^\pm$ be the m-dimensional unitary space of dimension m and Hasse invariant $\pm$. Let $s_0 = \frac{m-n}{2}$ and χ a character of $E^\times$ such that $\chi|_{F^\times} = \epsilon_{E/F}^m$.*

    i) If $m = 0$, i.e. if $s_0 = -\frac{n}{2}$, then $\mathrm{Ker}(M_n^*(-\frac{n}{2}, \chi)) = R_n(V_0^+, \chi)$ and $\mathrm{Im}(M_n^*(-\frac{n}{2}, \chi)) = R_n(V_{2n}^-, \chi)$.

    ii) If $1 \leqslant m < n$, i.e. if $-\frac{n}{2} < s_0 < 0$, then $\mathrm{Ker}(M_n^*(s_0, \chi)) = R_n(V_m^+, \chi) \oplus R_n(V_m^-, \chi)$ and $\mathrm{Im}(M_n^*(s_0, \chi)) = R_n(V_{2n-m}^+, \chi) \cap R_n(V_{2n-m}^-, \chi)$.

    iii) If $n \leqslant m < 2n$, i.e. if $0 \leqslant s_0 < \frac{n}{2}$, then $\mathrm{Ker}(M_n^*(s_0, \chi)) = R_n(V_m^+, \chi) \cap R_n(V_m^-, \chi)$, $M_n^*(s_0, \chi)(R_n(V_m^\pm, \chi)) = R_n(V_{2n-m}^\pm, \chi)$ thus we have $\mathrm{Im}(M_n^*(s_0, \chi)) = R_n(V_{2n-m}^+, \chi) \oplus R_n(V_{2n-m}^-, \chi)$.

    iv) If $m = 2n$, i.e. if $s_0 = \frac{n}{2}$, then $\mathrm{Ker}(M_n^*(\frac{n}{2}, \chi)) = R_n(V_{2n}^-, \chi)$ and $\mathrm{Im}(M_n^*(\frac{n}{2}, \chi)) = M_n^*(\frac{n}{2}, \chi)(R_n(V_{2n}^+), \chi) = R_n(V_0^+, \chi)$.

1.7 Local Zeta integral

The last element that we will use is the local Zeta integral of a representation. We fix π an irreducible admissible representation of $G(W)$.

Definition 1.12 *A* matrix coefficient *of π will be a linear combinations of functions of the form*

$$\phi(g) = \langle \pi(g)\xi, \xi^\vee \rangle$$

where ξ and $\xi^\vee$ are vectors of the space of respectively π and $\pi^\vee$.

Moreover if $\xi_\circ$ and $\xi_\circ^\vee$ are preassigned spherical vectors of π and $\pi^\vee$, we let

$$\phi^\circ(g) = \langle \pi(g)\xi_\circ, \xi_\circ^\vee \rangle.$$

We parametrise the space of matrix coefficients with the space of $\pi \otimes \pi^\vee$ through the obvious projection. If $s \in \mathbf{C}$ with $\operatorname{Re} s$ large enough, $\xi \in \pi$, $\xi^\vee \in \pi^\vee$, $\Phi \in I_n(s, \chi)$, let

$$Z(s, \chi, \pi, \xi \otimes \xi^\vee, \Phi) = \int_G \langle \pi(g)\xi, \xi^\vee \rangle \Phi(i(g, I_n), s) \, \mathrm{d}g$$

and extend it linearly to the space of matrix coefficients of π. We fix a maximal compact subgroup K of $\tilde{G}$ (for instance, one can fix a basis of $W_+ \oplus W_-$, see $\tilde{G}$ as a subgroup of $\mathrm{GL}(2n, E)$ and take $K = \tilde{G} \cap \mathrm{GL}(2n, \mathcal{O}_E)$).

Definition 1.13 *A* standard section *Φ is an application from $\mathbf{C}$ to the set of function from $\tilde{G}$ to $\mathbf{C}$ such that $\forall s \in \mathbf{C}$, $\Phi(g, s) = \Phi(s)(g) \in I_n(s, \chi)$ and, moreover, $\Phi(s)|_K$ is independent of s.*

It is rather obvious that any element $\Phi(g, s) \in I_n(s, \chi)$ can be inserted in a (unique) standard section. The Zeta integral above defines, for $\operatorname{Re} s$ sufficiently large, an intertwining operator

$$Z(s, \chi, \pi) \in \operatorname{Hom}_{G \times G}\big(I_n(s, \chi), \pi \otimes (\chi \cdot \pi^\vee)\big)$$

If Φ is a standard section, this operator can be meromorphically extended for all $s \in \mathbf{C}$ to an operator

$$Z^*(s, \chi, \pi) \in \operatorname{Hom}_{G \times G}\big(I_n(s, \chi), \pi \otimes (\chi \cdot \pi^\vee)\big).$$

2 Our results

2.1 Decomposition of the degenerate principal series

Let $\Omega(W_+ \oplus W_-)$ be the Grassmannian of the Lagrangians of $W_+ \oplus W_-$. We can identify

$$P_Y \backslash G(W_+ \oplus W_-) \simeq \Omega(W_+ \oplus W_-)$$

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using the map $P_Y \cdot g \longmapsto Yg$. There is a right action of $i(G(W) \times G(W))$ on $\Omega(W_+ \oplus W_-)$ which orbits are parametrised by the elements of the decomposition

$$G(W_+ \oplus W_-) = \coprod_{r=0}^{r_0} P_Y \delta_r i(G(W) \times G(W))$$

where r_0 is the Witt index of W. The aforementioned orbits are of the form

$$\Omega_r = P_Y \backslash P_Y \delta_r i(G(W) \times G(W)).$$

The orbit Ω_r is made of the Lagrangians Z such that $\dim Z \cap W_+ = \dim Z \cap W_- = r$. The only open orbit is that of Y, which is Ω_0, while the only closed one is that of Ω_{r_0} and the closure of the orbit Ω_r is
$$\overline{\Omega}_r = \coprod_{j \geqslant r} \Omega_j.$$

We consider the filtration

$$I_n(s, \chi) = I_n^{(r_0)}(s, \chi) \supset \cdots \supset I_n^{(1)}(s, \chi) \supset I_n^{(0)}(s, \chi),$$

where

$$I_n^{(r)}(s, \chi) = \{\Phi \in I_n(s, \chi) | \Phi|_{\overline{\Omega}_{r+1}} = 0\}.$$

Let

$$Q_n^{(r)}(s, \chi) = I_n^{(r)}(s, \chi)/I_n^{(r-1)}(s, \chi)$$

be the successive quotients of the filtration. All $I_n^{(r)}(s, \chi)$ and $Q_n^{(r)}(s, \chi)$ are $G \times G$-stable.

Let T_W be the Witt tower containing W. For any $W' \in T_W$ of dimension $n' = n - 2r \leqslant n$, let $G_{n'} = G(W')$. We identify W' with a subspace of W isomorphic to W'. There is a Witt decomposition
$$W = U' \oplus W' \oplus U$$

where U and U' are dual isotropic subspaces of dimension r. Let P_r be the parabolic subgroup of G stabilising U. The Levi subgroup of P_r is isomorphic to $\mathrm{GL}(U) \times G_{n'}$ so that, if we denote M_r its Levi component and N_r its unipotent radical, we have isomorphisms

$$
\begin{aligned}
M_r &\simeq \mathrm{GL}(U) \times G_{n'} & (2) \\
P_r &\simeq (\mathrm{GL}(U) \times G_{n'}) \ltimes N_r.
\end{aligned}
$$

Note in particular for $r = 0$ that $U = U' = \{0\}$, $W' = W$ and $P_0 = G_n = G$.

Let
$$\mathrm{St}_r = i^{-1}(\delta_r^{-1} P_Y \delta_r \cap i(G \times G))$$
be the stabiliser of $P_Y \delta_r$ in $i^{-1}(P_Y) \backslash G \times G$.

Lemma 2.1 *For a convenient choice of δ_r (specified in Equation (3) below), we have*

$$\mathrm{St}_r = (\mathrm{GL}(U) \times \mathrm{GL}(U) \times \Delta(G_{n'})) \ltimes (N_r \times N_r) \subset P_r \times P_r.$$

Moreover

$$Q_n^{(r)}(s, \chi) \simeq \mathrm{Ind}_{P_r \times P_r}^{G \times G} \left(\chi |.|^{s + \frac{r}{2}} \otimes \chi |.|^{s + \frac{r}{2}} \otimes \left(S(G_{n'}) \cdot (\mathbf{1} \otimes \chi) \right) \right)$$

where the action of $G_{n'} \times G_{n'}$ on the space $S(G_{n'}) \cdot (\mathbf{1} \otimes \chi)$ is given by $(g_1, g_2)\varphi(g) = \chi(\det g_2)\varphi(g_2^{-1} g g_1)$.

PROOF: We let $G' = G_{n'}$.

Recall the Witt decomposition

$$W = U' \oplus W' \oplus U$$

and consider the Lagrangian

$$Z = U \times \{0\} \oplus \Delta(W') \oplus \{0\} \times U$$

in $W_+ \oplus W_-$. Since the action of $\tilde{G}$ on $\Omega(W_+ \oplus W_-)$ is transitive, there exists $\delta_r \in \tilde{G}$ such that $Z = Y\delta_r$. Since any linear map from Y to Z can be extended to an element of $\tilde{G}$, we can furthermore require that

$$\begin{aligned} \forall v \in U', \delta_r|_{\Delta(U')}(v, v) &= (0, vd) \in \{0\} \times U \\ \delta_r|_{\Delta(W')} &= \mathrm{id}_{\Delta(W')} \\ \forall u \in U, \; \delta_r|_{\Delta(U)}(u, u) &= (u, 0) \in U \times \{0\} \end{aligned} \tag{3}$$

where $d : U' \longrightarrow U$ is an isomorphism. Note in particular that $\delta_0 = \mathrm{id}_G$. Following [Kud2, Proof of Proposition 2.1, p68], we find that there is a bijection between the orbit Ω_r of Z and the set

$$\{(Z_+, Z_-, \lambda)\}$$

where $Z_\pm$ is an isotropic subspace of $W_\pm$ of dimension r and

$$\lambda : Z_+^\perp / Z_+ \longrightarrow Z_-^\perp / Z_-$$

is an isometry[2]. The action of $(g_+, g_-) \in G \times G$ on this set is given by

$$(g_+, g_-)(Z_+, Z_-, \lambda) = (Z_+ g_+, Z_- g_-, g_+^{-1} \circ \lambda \circ g_-).$$

she stabiliser of (Z_+, Z_-, λ) is

$$\{(g_+, g_-) \in G \times G \mid g_\pm \text{ stabilises } Z_\pm \text{ and } g_+^{-1} \circ \lambda \circ g_- = \lambda\}.$$

---

[2]in [Kud2] it is an anti-isometry but, since W_- has the opposite of the form of W_+, here λ is an isometry.

In our situation and with our choice of δ_r, we have $Z_+ = Z_- = U$, $Z_+^\perp/Z_+ = W'$ and $\lambda = \mathrm{id}_{W'}$. Hence, denoting $\mathrm{pr}_{W'}$ the projection on W' parallel to $U' \oplus U$,

$$\mathrm{St}_r = \left\{ (g_+, g_-) \in P_r \times P_r \Big| g_+|_{W'+U} \circ \mathrm{pr}_{W'} = g_-|_{W'+U} \circ \mathrm{pr}_{W'} \right\}$$
$$= (\mathrm{GL}(U) \times \mathrm{GL}(U) \times \Delta(G')) \ltimes (N_r \times N_r)$$

For further reference, an element of P_r has the form

$$\begin{pmatrix} a & b & c \\ 0 & e & b^* \\ 0 & 0 & a^\vee \end{pmatrix}$$

where b^* depends on b, a and e and where c satisfies an equation depending on a, b and e. We thus have

$$g_\pm = \begin{pmatrix} a_\pm & b_\pm & c_\pm \\ 0 & e_\pm & b_\pm^* \\ 0 & 0 & a_\pm^\vee \end{pmatrix} \tag{4}$$

and the condition $g_+|_{W'+U} \circ \mathrm{pr}_{W'} = g_-|_{W'+U} \circ \mathrm{pr}_{W'}$ is simply $e_+ = e_-$.

The description of the stabiliser allows us to describe the induced representations. If $\tilde{g} \in \mathrm{St}_r$, then $p(\tilde{g}) = \delta_r i(\tilde{g})\delta_r^{-1} = n \cdot m(a_r(\tilde{g})) \in P_Y$. Let $\xi_{s,r}$ be the character of St_r defined by $\xi_{s,r}(\tilde{g}) = \chi(a_r(\tilde{g}))|\det a_r(\tilde{g})|^{s+\frac{r}{2}}$. Consider the morphism of $G \times G$-modules

$$Q_n^{(r)}(s,\chi) \longrightarrow \mathrm{Ind}_{\mathrm{St}_r}^{G\times G}(\xi_{s,r})$$
$$\overline{f} \longmapsto \phi_{\overline{f}}(g_1, g_2) = \int_{N_r'} f(\delta_r n(u) i(g_1, g_2)) \, du$$

where $f \in I_n^{(r)}(s,\chi)$ is a representative of $\overline{f}$. This morphism is an isomorphism (see [HKS, Equation (4.9), p963]). Let $\tilde{g} = (g_+, g_-)$ be an element of St_r decomposed as in (4). Then $\det(a_r(\tilde{g})) = \det a_+ \det a_- \det e_+$ (where we recall that $e_+ = e_-$). Since $e_+ \in G'$, $|\det e_+| = 1$ hence

$$Q_n^{(r)}(s,\chi) \simeq \mathrm{Ind}_{\mathrm{St}_r}^{G\times G}(\chi|\cdot|^{s+\frac{r}{2}} \otimes \chi|\cdot|^{s+\frac{r}{2}} \otimes \chi)$$
$$\simeq \mathrm{Ind}_{P_r \times P_r}^{G\times G}\left(\mathrm{Ind}_{\mathrm{St}_r}^{P_r \times P_r}(\chi|\cdot|^{s+\frac{r}{2}} \otimes \chi|\cdot|^{s+\frac{r}{2}} \otimes \chi)\right)$$

The induction from St_r to $P_r \times P_r$ is an induction from $\Delta(G')$ to $G' \times G'$. Moreover, if $f \in \mathrm{Ind}_{\Delta(G')}^{G' \times G'} \chi$ then $f(h_1, h_2) = \chi(h_2)f(h_2^{-1}h_1, 1)$. Hence

$$\mathrm{Ind}_{\Delta(G')}^{G' \times G'} \chi \simeq S(G') \cdot (\mathbf{1} \otimes \chi)$$

where the action of $G' \times G'$ on $S(G') \cdot (\mathbf{1} \otimes \chi)$ is given by

$$\rho(g_1, g_2)\varphi(g) = \chi(\det g_2)\varphi(g_2^{-1}gg_1).$$

Hence

$$\mathrm{Ind}_{\mathrm{St}_r}^{P_r \times P_r}(\chi|\cdot|^{s+\frac{r}{2}} \otimes \chi|\cdot|^{s+\frac{r}{2}} \otimes \chi) \simeq \chi|\cdot|^{s+\frac{r}{2}} \otimes \chi|\cdot|^{s+\frac{r}{2}} \otimes \mathrm{Ind}_{\Delta(G')}^{G' \times G'} \chi$$
$$\simeq \chi|\cdot|^{s+\frac{r}{2}} \otimes \chi|\cdot|^{s+\frac{r}{2}} \otimes \left(S(G') \cdot (\mathbf{1} \otimes \chi)\right).$$

The result follows. $\qquad\square$

2.2 Simplicity of poles

We prove in our case the result of [KR, section 5]. We follow the same method. We denote χ_0 the trivial character of $F^\times$.

Proposition 2.2 *Let $\mathfrak{z}_s \in \mathcal{H}(G \,/\!/\, K) \otimes \mathbf{C}[q^s, q^{-s}]$ be the element defined by*

$$\mathfrak{z}_s = \prod_{i=1}^{r_0}(1 - q^{-s-\frac{1}{2}}t_i)(1 - q^{-s-\frac{1}{2}}t_i^{-1}).$$

For an unramified representation π of G, let $\pi(\mathfrak{z}_s)$ be the scalar by which $\mathfrak{z}_s$ acts on the unramified vector in π. Then for all matrix coefficients ϕ of π and all standard sections $\Phi(s) \in I_n(s)$, the function

$$\pi(\mathfrak{z}_s) \cdot Z(s, \chi_0, \pi, \phi, \Phi)$$

is an entire function of s.

PROOF: We divide the proof in several steps.

Step 1. By linearity of Z, we can limit ourselves to the case where ϕ is of the form

$$\phi(g) = \langle \pi(g)\pi(g_1)\xi_\circ, \pi^\vee(g_2)\xi_\circ^\vee \rangle$$

where $\xi_\circ$ and $\xi_\circ^\vee$ are spherical vectors in π and $\pi^\vee$ and g_1, $g_2 \in G$. We then have

$$Z(s, \chi_0, \pi, \phi, \Phi) = \int_G \langle \pi(g)\pi(g_1)\xi_\circ, \pi^\vee(g_2)\xi_\circ^\vee \rangle \Phi_s(i(g, I_n)) \, \mathrm{d}g \tag{5}$$

$$= \int_G \langle \pi(g)\xi_\circ, \xi_\circ^\vee \rangle \Phi_s(i(g_2 g g_1^{-1}, I_n)) \, \mathrm{d}g$$

$$= |\det g_2|^{s+r_0-\frac{1}{2}} \int_G \phi^\circ(g)\Phi_s(i(g, I_n)i(g_1^{-1}, g_2^{-1})) \, \mathrm{d}g$$

since $|\det g_2| = 1$ and ϕ° is bi-K invariant, for all k_1, $k_2 \in K$,

$$= \int_G \phi^\circ(g)\Phi_s(i(k_2^{-1}gk_1, I_n)i(g_1^{-1}, g_2^{-1})) \, \mathrm{d}g$$

$$= \int_G \phi^\circ(g)\Phi_s(i(g, I_n)i(k_1, k_2)i(g_1^{-1}, g_2^{-1})) \, \mathrm{d}g$$

and thus

$$= \int_G \phi^\circ(g)\Psi_s(i(g, I_n)) \, \mathrm{d}g$$

where, for any $h \in H = G_{2n}$,

$$\Psi_s(h) := \int_{K \times K} \Phi_s(hi(k_1, k_2)i(g_1^{-1}, g_2^{-1})) \, \mathrm{d}k_1 \mathrm{d}k_2. \tag{6}$$

Note that Ψ_s is $K \times K$-invariant section of $I_n(s)$ which is not necessarily standard.

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Step 2. We consider in the algebra

$$\mathcal{A} = \mathbf{C}[X, X^{-1}] \otimes \mathcal{H}(G /\!/ K) \simeq \mathbf{C}[X, X^{-1}] \otimes \mathbf{C}[t_1, \ldots, t_n]^{W_G},$$

where $\mathcal{H}(G /\!/ K)$ is the K-spherical Hecke algebra of G, the element

$$\mathfrak{z} = \prod_{i=1}^{r_0} (1 - Xq^{-\frac{1}{2}} t_i)(1 - Xq^{-\frac{1}{2}} t_i^{-1}).$$

We let $G \times G$ act on $I_n(s)$ through i, extend the action to $\mathcal{H}(G /\!/ K) \times \mathcal{H}(G /\!/ K)$ and let any $\phi \in \mathcal{H}(G /\!/ K)$ act as $(\phi, 1) \in \mathcal{H}(G /\!/ K) \times \mathcal{H}(G /\!/ K)$. We let $\mathcal{A}$ act on the space $I_n(s)^{K \times 1}$ of $K \times 1$-fixed vectors of $I_n(s)$ by the aforementioned action of $\mathcal{H}(G /\!/ K)$ and X acts by multiplication by q^{-s}. Note that action of $1 \times G$ commutes with the action of $\mathcal{A}$.

Proposition 2.3 *For any standard section Φ_s with associated section Ψ_s defined by (6), we have*

$$\Psi_s * \mathfrak{z} \in I_n^{(0)}(s)^{K \times K}.$$

PROOF: We want to show the the image of $\Psi_s * \mathfrak{z}$ in each $Q_n^{(r)}(s) = Q_n^{(r)}(s, \chi_0)$ is 0 for $0 < r \leqslant r_0$. We will, as an illustration, do the first step separately in the case of a split Hermitian space (in particular $n = 2r_0$). Consider the projection induced by restriction to the closed orbit:

$$\mathrm{pr}_{r_0} : I_n(s) = I_n^{(r_0)}(s) \longrightarrow Q_n^{(r_0)}(s) \simeq \mathrm{Ind}_{P_{r_0}}^G \left(|\,.\,|^{s + \frac{r_0}{2}} \right) \otimes \mathrm{Ind}_{P_{r_0}}^G \left(|\,.\,|^{s + \frac{r_0}{2}} \right)$$
$$\Phi_s \longmapsto \left((g_1, g_2) \mapsto \Phi_s(i(g_1, g_2)) \right).$$

We have

$$\mathrm{pr}_{r_0}(\Psi_s * \mathfrak{z}) = \mathrm{pr}_{r_0}(\Psi_s) * \mathfrak{z}$$

if we let $\mathfrak{z}$ act only on the first term of the tensor product on the right side. On the other hand, we have

$$\mathrm{Ind}_{P_{r_0}}^G \left(|\,.\,|^{s + \frac{r_0}{2}} \right) \subset \mathrm{Ind}_B^G(\lambda)$$

where B is the standard Borel subgroup of G and λ is the unramified principal series representation with Satake parameter[3]

$$(q^{s + r_0 - \frac{1}{2}}, q^{s + r_0 - \frac{3}{2}}, \ldots, q^{s + \frac{1}{2}}).$$

The element $\mathfrak{z}$ acts on the K-fixed vector of this representation by the scalar

$$\prod_{i=1}^{r_0} (1 - q^{-s - \frac{1}{2}} q^{s + r_0 + \frac{1}{2} - i})(1 - q^{-s - \frac{1}{2}} q^{-s - r_0 - \frac{1}{2} + i}) = 0.$$

This means that $\mathrm{pr}_{r_0}(\Psi_s * \mathfrak{z}) = 0$ i.e. that $\Psi_s * \mathfrak{z} \in I_n^{(r_0 - 1)}(s)$.

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[3]A vérifier

More generally, if we restrict the orbit of a section to Ω_r, we obtain a map

$$\mathrm{pr}_r : I_n(s) \longrightarrow \mathrm{Ind}_{P_r \times P_r}^{G \times G}\left(|\,.\,|^{s+\frac{r}{2}} \otimes |\,.\,|^{s+\frac{r}{2}} \otimes C(G_{n-2r})\right) =: B_r(s)$$

where $C(G_{n-2r})$ is the space of smooth functions on G_{n-2r}. There is a non-degenerate pairing between $Q_n^{(r)}(s)$ and $B_r(-s-r)$ given by

$$\langle f_1, f_2 \rangle = \int_{P_r \times P_r \backslash G \times G} \langle f_1(g_1, g_2), f_2(g_1, g_2) \rangle_{G_{n-r}} \, \mathrm{d}\mu(g_1) \mathrm{d}\mu(g_2),$$

where the internal pairing is the integration over G_{n-r} and the external integral is the invariant functional for functions which transform on the left according to the square of the modulus character. A straightforward density argument shows that $\phi \in Q_n^{(r)}(s)$ is 0 if and only if it pairs to zero against all elements of the subspace $Q_n^{(r)}(-s-r) \subset B_r(-s-r)$. In addition if $\phi \in Q_n^{(r)}(s)^{K \times K}$ we can limit ourselves to elements of $Q_n^{(r)}(-s-r)^{K \times K}$. Let $f_s \in Q_n^{(r)}(-s-r)^{K \times K}$ and $\mathfrak{z}_s = \mathfrak{z}\big|_{X=q^{-s}}$. We have

$$\langle \mathrm{pr}_r(\Psi_s * \mathfrak{z}), f_2 \rangle = \langle \mathrm{pr}_r(\Psi_s) * \mathfrak{z}_s, f_s \rangle = \langle \mathrm{pr}_r(\Psi_s), f_s * \mathfrak{z}_s^{\vee} \rangle.$$

Lemma 2.4 *For any $f_s \in Q_n^{(r)}(-s-r)^{K \times K}$ we have*

$$f_s * \mathfrak{z}_s^{\vee} = 0.$$

PROOF: Since f_s is element of a parabolic induction and fixed by a maximal compact, it is determined by its value at the identity element I_n. It is not difficult to see that $f_s(I_n) \in S(G)^{K_{n-r} \times K_{n-r}}$ where $K_{n-r} = G_{n-r} \cap K$. Let τ be an irreducible admissible representation of G_{n-r}. The action of $S(G_{n-r})$ on τ determines a $G_{n-r} \times G_{n-r}$-equivariant map

$$\mu_\tau : S(G_{n-r}) \longrightarrow \mathrm{Hom}^{\mathrm{smooth}}(\tau, \tau) \simeq \tau^{\vee} \otimes \tau$$

where $\mathrm{Hom}^{\mathrm{smooth}}$ is the space of vector-space homomorphisms fixed by a compact open subgroup of $G_{n-r} \times G_{n-r}$. The two factors of $G_{n-r} \times G_{n-r}$ act respectively by pre- and post-multiplication on the elements of $\mathrm{Hom}^{\mathrm{smooth}}(\tau, \tau)$ so that each has finite dimensional image. A function $\phi \in S(G_{n-r})^{K_{n-r} \times K_{n-r}}$ is nonzero if and only if there exists an irreducible admissible representation τ such that $\tau(\phi) \neq 0$, i.e. such that $\mu_\tau(\phi) \neq 0$.

Consider $f_s * \mathfrak{z}_s^{\vee}$. Let τ be, as above, an irreducible admissible representation of G_{n-r}. The map μ_τ induces

$$\mathrm{Ind}(\mu_\tau) : \mathrm{Ind}_{P_r \times P_r}^{G \times G}\left(|\,.\,|^{-s-\frac{r}{2}} \otimes |\,.\,|^{-s-\frac{r}{2}} \otimes S(G_{n-r})\right) \longrightarrow \mathrm{Ind}_{P_r \times P_r}^{G \times G}\left(|\,.\,|^{-s-\frac{r}{2}} \otimes |\,.\,|^{-s-\frac{r}{2}} \otimes \tau^{\vee} \otimes \tau\right)$$

which verifies $\mathrm{Ind}(\mu_\tau)(f_s)(I_n) = \mu_\tau(f_s(I_n))$. The latter induced representation is isomorphic to

$$\mathrm{Ind}_{P_r}^{G}\left(|\,.\,|^{-s-\frac{r}{2}} \otimes \tau^{\vee}\right) \otimes \mathrm{Ind}_{P_r}^{G}\left(|\,.\,|^{-s-\frac{r}{2}} \otimes \tau\right)$$

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which can be embedded in

$$\mathrm{Ind}_B^G \lambda_1 \otimes \mathrm{Ind}_B^G \lambda_2$$

where the Satake parameters[4] are

$$\lambda_1 = (q^{-s-\frac{1}{2}}, q^{-s-\frac{3}{2}}, \ldots, q^{-s+\frac{1}{2}-r}, q^{-\nu_1}, \ldots, q^{-\nu_{n-r}})$$

$$\lambda_2 = (q^{-s-\frac{1}{2}}, q^{-s-\frac{3}{2}}, \ldots, q^{-s+\frac{1}{2}-r}, q^{\nu_1}, \ldots, q^{\nu_{n-r}})$$

(where $(q^{\nu_1}, \ldots, q^{\nu_{n-r}})$ is the Satake parameter of τ). The operator $\mathfrak{z}_s^\vee$ acts on the unique line of $K \times K$-invariant vectors of this representation by the scalar

$$\prod_{i=1}^r (1 - q^{-s} q^{-\frac{1}{2}} q^{s-\frac{1}{2}+i})(1 - q^{-s} q^{-\frac{1}{2}} q^{-s+\frac{1}{2}-i}) \cdot (\text{factor}) = 0.$$

But $\mathrm{Ind}(\mu_\tau)(f_s)$ is a $K \times K$-invariant vector in this representation so that $\mathrm{Ind}(\mu_\tau)(f_s) * \mathfrak{z}_s = 0$ and

$$\begin{aligned}
\mu_\tau(f_s * \mathfrak{z}_s^\vee(I_n)) &= \mathrm{Ind}(\mu_\tau)(f_s * \mathfrak{z}_s^\vee)(I_n) \\
&= (\mathrm{Ind}(\mu_\tau)(f_s * \mathfrak{z}_s^\vee))(I_n) \\
&= 0.
\end{aligned}$$

Since this is true for all τ, we have $f_s * \mathfrak{z}_s^\vee(I_n) = 0$ and thus $f_s * \mathfrak{z}_s^\vee = 0$. $\quad\square$ Lemma 2.4

We have $\mathrm{pr}_r(\Psi_s * \mathfrak{z}) = 0$ for all $r > 0$, which means that the support of $\Psi_s * \mathfrak{z}$ is included in Ω_0, which concludes the proof. $\quad\square$ Proposition 2.3

Step 3. Consider the isomorphism

$$\mathrm{pr}_0 : I_n(s) \longrightarrow Q_n^{(0)}(G) \simeq S(G).$$

Proposition 2.3 shows that, for a fixed s, we have $\mathrm{pr}_0(\Psi_s * \mathfrak{z}) \in S(G)^{K \times K}$. Its support could vary with s. The following proposition shows that the support of $\mathrm{pr}_0(\Psi_s * \mathfrak{z})$ is bounded uniformly in s.

Lemma 2.5

$$\mathrm{pr}_0(\Psi_s * \mathfrak{z}) \in \mathbf{C}[q^s, q^{-s}] \otimes S(G)^{K \times K} = \mathbf{C}[q^s, q^{-s}] \otimes \mathcal{H}(G /\!\!/ K).$$

PROOF: Using the Cartan decomposition, write

$$\mathrm{pr}_0(\Psi_s * \mathfrak{z}) = \sum_{\lambda \in \Lambda} c_\lambda(s) L_\lambda,$$

where L_λ is the characteristic function of the double coset $K g_\lambda K$ and Λ is the usual semigroup.

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[4]A vérifier

Lemma 2.6

$$c_\lambda(s) \in \mathbf{C}[q^s, q^{-s}]$$

and thus is an entire function of s.

PROOF: We have

$$c_\lambda(s) \cdot \|L_\lambda\|^2 = \int_G (\Psi_s * \mathfrak{z})(i(g, I_n)) \cdot L_\lambda(g)\, \mathrm{d}g. \tag{7}$$

The integral on the right is a (finite) linear combination, with coefficients in $\mathbf{C}[q^s, q^{-s}]$ of integrals of the form

$$\int_G \int_G (\Psi_s * \mathfrak{z})(i(g, I_n) i(g_0, I_n)) \cdot L_\mu(g_0)\, \mathrm{d}g_0 \cdot L_\lambda(g)\, \mathrm{d}g \tag{8}$$

$$= \int_G \int_G (\Psi_s * \mathfrak{z})(i(g_0, I_n)) \cdot L_\mu(g^{-1} g_0) \cdot L_\lambda(g)\, \mathrm{d}g_0 \mathrm{d}g$$

$$= \int_G \int_G (\Psi_s * \mathfrak{z})(i(g_0, I_n)) \cdot \varphi(g_0)\, \mathrm{d}g_0$$

where φ is a function depending on λ and μ. Since this function is a (finite) linear combination of characteristic functions of cosets gK, the integral is the last line of (8) is a (finite) linear combination with coefficients in $\mathbf{C}[q^s, q^{-s}]$ of integrals of the form

$$\int_K \int_{K \times K} \Phi_s\big(i(gk, I_n) i(k_1, k_2) i(g_1^{-1}, g_2^{-1})\big)\, \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k.$$

But Φ_s is standard, hence it is right-invariant under a fixed compact open subgroup H, uniformly in s. This means that the set of g necessary to obtain the full integral (7) is finite and fixed. The elements g_1 and g_2 are fixed by the matrix coefficient ϕ we are considering and thus the integral (7) is a (finite) linear combination of $q^{\ell s}$ with $\ell \in \mathbf{Z}$. $\qquad\square$

Let then Λ_1 be the set of $\lambda \in \Lambda$ such that $c_\lambda \neq 0$ and for $\lambda \in \Lambda$ let

$$D_\lambda = \{s \in \mathbf{C} : c_\lambda(s) = 0\}.$$

If $\lambda \in \Lambda_1$ then D_λ is a numerable subset of $\mathbf{C}$. Hence $\bigcup_{\lambda \in \Lambda_1} D_\lambda$ is numerable and thus different from $\mathbf{C}$. Let $s_0 \in \mathbf{C}$ be such that $\forall \lambda \in \Lambda_1,\ c_\lambda(s_0) \neq 0$. Since

$$\mathrm{pr}_0(\Psi_{s_0} * \mathfrak{z}) = \sum_{\lambda \in \Lambda_1} c_\lambda(s_0) \cdot L_\lambda$$

has compact support, Λ_1 is finite and thus for all $s \in \mathbf{C}$, $\mathrm{pr}_0(\Psi_s * \mathfrak{z})$ has support in $\cup_{\lambda \in \Lambda_1} L_\lambda$. $\qquad\square$ Lemma 2.5

Step 4. Returning to the Zeta integral in (5), we define

$$Z^*(s, \chi_0, \pi, \phi, \Phi) = \int_G \phi^\circ(g)(\Psi_s * \mathfrak{z})(i(g, I_n)) \, dg.$$

This integral is equal to the scalar by which $\mathrm{pr}_0(\Psi_s * \mathfrak{z})$ acts on $\xi_\circ$ and is thus an entire function of s because it is an element of $\mathbf{C}[q^s, q^{-s}]$. On the other hand, if $\mathrm{Re}(s)$ is large enough we can unfold

$$Z^*(s, \chi_0, \pi, \phi, \Phi) = \pi(\mathfrak{z}_s) \int_G \phi^\circ(g) \Psi_s(i(g, I_n)) \, dg$$

$$= \pi(\mathfrak{z}_s) Z(s, \chi_0, \pi, \phi, \Phi).$$

where $\pi(\mathfrak{z}_s)$ is the scalar by which $\mathfrak{z}_s = \mathfrak{z}\big|_{X=q^{-s}}$ acts on the spherical vector of π. Since $Z^*(s, \chi_0, \pi, \phi, \Phi)$ is an entire function of s, this completes the proof. $\qquad \square$ Proposition 2.2

2.3 The conjecture holds for the trivial representation in the even dimensional tower

Definition 2.7 ([HKS, Definition 4.6, p963]) *For $s_0 \in \mathbf{C}$, χ a character and π and irreducible admissible representation of G, we say that π occurs in the boundary at the point $s = s_0$ if*

$$\mathrm{Hom}_{G \times G}(Q_n^{(r)}(s_0, \chi), \pi \otimes (\chi \cdot \pi^\vee)) \neq 0$$

for some $r > 0$.

Proposition 2.8 *Let $\pi = \mathbf{1}$ the trivial representation of G, ϖ_E an uniformiser of E and $q_E = |\varpi_E|$. We will denote $X^u(E^\times)$ the set of unramified characters of $E^\times$. Let*

$$X(\mathbf{1}) = \left\{ (s, \chi) \in \mathbf{C} \times X^u(E^\times) \,\middle|\, \chi(\varpi_E) = (-1)^k, s = \frac{n}{2} - r - \frac{ki\pi}{\log q_E}, 1 \leqslant r \leqslant r_0 \right\}$$

with $1 \leqslant r \leqslant r_0$ and $k \in \mathbf{Z}$.

Then $\mathbf{1}$ appears in the boundary at s if and only if $(s, \chi) \in X(\mathbf{1})$. Moreover if $(s_0, \chi) \notin X(\mathbf{1})$, for any standard section Φ the operator $Z(s, \chi, \mathbf{1})$ is holomorphic at $s = s_0$ and

$$\mathrm{Hom}_{G \times G}(I_n(s_0, \chi), \mathbf{1} \otimes \chi) = \mathbf{C} \cdot Z(s, \chi, \mathbf{1}).$$

PROOF: We know from Lemma 2.1 that

$$\mathrm{Hom}_{G \times G}(Q_n^{(r)}(s, \chi), \mathbf{1} \otimes \chi) = \mathrm{Hom}_{G \times G} \Big(\mathrm{Ind}_{P_r \times P_r}^{G \times G} \Big(\chi |.|^{s+\frac{r}{2}} \otimes \chi |.|^{s+\frac{r}{2}} \otimes \big(S(G') \cdot (\mathbf{1} \otimes \chi) \big) \Big),$$

$$\mathbf{1} \otimes \chi \Big)$$

$$\simeq \mathrm{Hom}_{G \times G} \Big(\mathbf{1} \otimes \chi^{-1},$$

$$\mathrm{Ind}_{P_r \times P_r}^{G \times G} \Big(\chi^{-1} |.|^{-s-\frac{r}{2}} \otimes \chi^{-1} |.|^{-s-\frac{r}{2}} \otimes \big(C^\infty(G') \cdot (\mathbf{1} \otimes \chi^{-1}) \big) \Big) \Big)$$

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$$\simeq \mathrm{Hom}_{M_r \times M_r}\big(\mathbf{1} \otimes \chi^{-1},$$
$$\chi^{-1}|.|^{-s-\frac{r}{2}+\frac{n-r}{2}} \otimes \chi^{-1}|.|^{-s-\frac{r}{2}+\frac{n-r}{2}} \otimes \big(\mathrm{C}^\infty(G') \cdot (\mathbf{1} \otimes \chi^{-1})\big)\big)$$

because the Jacquet module for $\mathbf{1} \otimes \chi^{-1}$ is $\mathbf{1} \otimes \chi^{-1}$ (as a representation of M_r)

$$\simeq \mathrm{Hom}_{\mathrm{GL}(U) \times \mathrm{GL}(U)}\big(\mathbf{1} \otimes \chi^{-2}, \chi^{-1}|.|^{-s+\frac{n}{2}-r} \otimes \chi^{-1}|.|^{-s+\frac{n}{2}-r}\big)$$

because if g corresponds to (a, g') in Equation (2) then $\det g = \det a \overline{\det a^{-1}} \det g'$ so that $\chi(\det g) = \chi(\det a)^2 \chi(\det g')$ and because $\dim \mathrm{Hom}_{G' \times G'}(\mathbf{1} \otimes \chi^{-1}, \mathrm{C}^\infty(G') \cdot (\mathbf{1} \otimes \chi^{-1})) = 1$ (see [HKS, end of section 4, p964] for general π).

It follows that π occurs in the boundary at s if and only if χ is unramified, $\chi(\varpi_E) = (-1)^k$ and $(s - \frac{n}{2} + r) \log q_E + ki\pi = 0$, as required.

Suppose $(s_0, \chi) \notin X(\mathbf{1})$, i.e. that $\mathbf{1}$ does not appear in the boundary. Let k be the maximum order of the pole of the Z integral in $s = s_0$ (as Φ varies). Thus

$$Z(s, \chi, \mathbf{1}, \Phi) = \frac{\tau_{-k}(s, \chi, \mathbf{1}, \Phi)}{(s - s_0)^k} + \cdots + \tau_0(s, \chi, \mathbf{1}, \Phi) + \cdots$$

where the τ_i are holomorphic functions of s in a neighbourhood of s_0 and τ_{-k} is non-zero. The leading term τ_{-k} is itself an intertwining operator. If we had $k > 0$, that is, if the Z integral had a pole in $s = s_0$, the restriction of τ_{-k} to $I_n^{(0)}(s_0, \chi)$ would be zero because the Z integral is convergent on

$$I_n^{(0)}(s_0, \chi) = Q_n^{(0)}(s, \chi) =\simeq S(G) \cdot (\mathbf{1} \otimes \chi)$$

thus convergent for every standard section $\Phi(s)$ such that $\Phi \in I_n^{(0)}(s, \chi)$. This means that we would have a non-zero intertwining operator in $\mathrm{Hom}_{G \times G}(Q_n^{(r)}(s, \chi), \mathbf{1} \otimes \chi)$ for some $r > 0$, which is impossible by hypothesis. Thus $k \geqslant 0$, i.e. the integral is entire for any $\Phi \in I_n(s_0, \chi)$. Moreover, $Z(s_0, \chi, \mathbf{1})$ is a non-zero intertwining operator between $I_n^{(0)}(s_0, \chi)$ and $\mathbf{1} \otimes \chi$, which means that $\mathrm{Hom}_{G \times G}(I_n^{(0)}(s_0, \chi), \mathbf{1} \otimes \chi)$ is non zero and thus has dimension 1 and that $Z(s_0, \chi, \mathbf{1})$ is its basis.

Let $\lambda \in \mathrm{Hom}_{G \times G}(I_n(s_0, \chi), \mathbf{1} \otimes \chi)$. Its restriction $\bar{\lambda}$ to $I_n^{(0)}(s_0, \chi)$ is a multiple of $Z(s_0, \chi, \mathbf{1})$. Since $\mathbf{1}$ is supposed not to appear in the boundary, if $\lambda \neq 0$, then $\bar{\lambda} \neq 0$, i.e. $\bar{\lambda} = cZ(s_0, \chi, \mathbf{1})$ for some $c \neq 0$. Since $\lambda - cZ(s_0, \chi, \mathbf{1})$ is zero on $I_n^{(0)}(s_0, \chi)$, it must be zero everywhere, i.e. $\lambda = cZ(s_0, \chi, \mathbf{1})$. $\qquad\square$

Theorem 2.9 *Let m be an even integer and χ_0 the trivial character of $E^\times$, then*

$$\forall m \leqslant 2n, \quad \mathrm{Hom}_{G \times G}(R_n(V_m^-, \chi_0), \mathbf{1}) = 0,$$

so that by (ii) of Proposition 1.6

$$\mathrm{Hom}_{G \times G}(R_n(V_{2n+2}^-, \chi_0), \mathbf{1}) \neq 0$$

and thus $m_{\chi_0}^-(\mathbf{1}) = 2n + 2$. Since $m_{\chi_0}^+(\mathbf{1}) = 0$, we have

$$m_{\chi_0}^+(\mathbf{1}) + m_{\chi_0}^-(\mathbf{1}) = 2n + 2.$$

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PROOF: By (i) of Proposition 1.6, it suffices to prove that

$$\text{Hom}_{G \times G}(R_n(V_{2n}^-, \chi_0), \mathbf{1}) = 0.$$

From Proposition 2.8 we know that

$$\text{Hom}_{G \times G}\left(I_n\left(-\frac{n}{2}, \chi_0\right), \mathbf{1}\right)$$

is non zero and generated by

$$Z\left(-\frac{n}{2}, \chi_0, \mathbf{1}\right)$$

which is holomorphic at $-\frac{n}{2}$. The element of $I_n(-\frac{n}{2}, \chi_0)$ equal to 1 on K is $\chi_{0,\tilde{G}}$. As seen in [Li, Theorem 3.1, p186] and [LR, Proposition 3, p333] we have

$$Z\left(-\frac{n}{2}, \chi_0, \mathbf{1}, \phi^\circ, \chi_{0,\tilde{G}}\right) \neq 0$$

and thus $Z(-\frac{n}{2}, \chi_0, \mathbf{1})(\chi_{0,\tilde{G}}) \neq 0$. Let

$$\phi \in \text{Hom}_{G \times G}(R_n(V_{2n}^-, \chi_0), \mathbf{1})$$

and

$$\tilde{\phi} = \phi \circ M_n^*\left(-\frac{n}{2}, \chi_0\right) \in \text{Hom}_{G \times G}\left(I_n\left(-\frac{n}{2}, \chi_0\right), \mathbf{1}\right).$$

We have $\chi_{0,\tilde{G}} \in R_n(V_0^+, \chi_{0,\tilde{G}}) = \ker M_n^*(-\frac{n}{2}, \chi_0)$ so that $\tilde{\phi}(\chi_{0,\tilde{G}}) = 0$. This means that $\tilde{\phi} = 0$ because it is a multiple of $Z\left(-\frac{n}{2}, \chi_0, \mathbf{1}\right)$. We know from Proposition 1.11 that the application

$$M_n^*\left(-\frac{n}{2}, \chi_0\right) : I_n\left(-\frac{n}{2}, \chi_0\right) \longrightarrow R_n(V_{2n}^-, \chi_0)$$

is surjective so that $\phi = 0$. $\qquad \square$

2.4   Half of the conjecture

Theorem 2.10 *Let π be an irreducible admissible representation of $G(W)$, then*

$$m_\chi^+(\pi) + m_\chi^-(\pi) \geqslant 2n + 2.$$

PROOF: Fix $m_0 \in \{0, 1\}$, a character χ of $E^\times$ such that $\chi|_{F^\times} = \epsilon_{E/F}^{m_0}$ and suppose we have two Hermitian spaces V_a^+ and V_b^- such that

$$\theta_\chi(\pi, V_a^+) \neq 0 \quad \text{and} \quad \theta_\chi(\pi, V_b^-) \neq 0,$$

with $\dim V_a^+ = a$, $\dim V_b^- = b$, a and b of the parity of m_0, $\epsilon(V_a^+) = 1$ and $\epsilon(V_b^-) = -1$. Let $V_{b,-}^-$ be the same space as V_b^- with opposite form and

$$\mathbb{W}_a = V_a^+ \otimes W, \quad \mathbb{W}_b = V_b^- \otimes W, \quad \mathbb{W}_{b,-} = V_{b,-}^- \otimes W.$$

We denote $\omega_{a,\chi}$ (resp. $\omega_{b,\chi}$, $\omega_{b,-,\chi}$) the representations of G induced by the representations $\omega_{a,\psi}$ (resp. $\omega_{b,\psi}$, $\omega_{b,-,\psi}$) of $\mathrm{Mp}(\mathbb{W}_a)$ (resp. $\mathrm{Mp}(\mathbb{W}_b)$, $\mathrm{Mp}(\mathbb{W}_{b,-})$). By hypothesis on V_a^+ and V_b^- we have two non-zero (and thus surjective) elements

$$\lambda \in \mathrm{Hom}_G(\omega_{a,\chi}, \pi), \quad \mu \in \mathrm{Hom}_G(\omega_{b,\chi}, \pi).$$

Let $g_0 \in \mathrm{GL}_F(W)$ be an F-automorphism of W which is conjugate-linear as an E-morphism. Then $\mathrm{Ad}(g_0)$ is a MVW involution on G. Conjugating μ and π by $\mathrm{Ad}(g_0)$ we get a non-zero morphism

$$\mu^\vee \in \mathrm{Hom}_G(\omega_{b,\chi}^\vee, \pi^\vee)$$

and thus a surjective

$$\nu_0 = \lambda \otimes \mu^\vee \in \mathrm{Hom}_{G \times G}(\omega_{a,\chi} \otimes \omega_{b,\chi}^\vee, \pi \otimes \pi^\vee).$$

Composing ν_0 with Δ and projecting on the trivial subquotient produces a non-zero element

$$\nu \in \mathrm{Hom}_G(\omega_{a,\chi} \otimes \omega_{b,\chi}^\vee, \mathbf{1}).$$

We have

$$\omega_{b,\psi}^\vee \simeq \omega_{b,\overline{\psi}} \simeq \omega_{b,-,\psi}.^{[5]}$$

On the other hand we can identify $\mathrm{Mp}(\mathbb{W}_b)$ and $\mathrm{Mp}(\mathbb{W}_{b,-})$ in which case we get

Lemma 2.11

$$\tilde{\iota}_{b,\chi} \simeq \tilde{\iota}_{b,-,\chi^{-1}}.$$

Where we added a subscript to $\tilde{\iota}$ to remember which Hermitian space is involved.

PROOF: The space V_b^- can be decomposed as an orthogonal direct sum of a split space and zero, one or two anisotropic lines. Since the splitting $\tilde{\iota}$ is additive, we consider separately the split and the anisotropic case.

We first consider the case in which V_b^- is split. We will need some additional notations (see [HKS, n.10, p950]). For any additive character η of F and $a \in F$ we will let η_a be the character such that $\eta_a(x) = \eta(ax)$, $\gamma_F(\eta) \in \mu_8$ the Weil index of the quadratic character $x \longmapsto \eta(x^2)$ and $\gamma_F(a, \eta) = \frac{\gamma_F(\eta_a)}{\gamma_F(\eta)}$. Recall that (see [HKS, n.11, p950])

$$\gamma_F(ab, \eta) = (a, b)_F \gamma_F(a, \eta) \gamma_F(b, \eta).$$

Let η be the character such that $\eta(x) = \psi(\frac{1}{2}x)$ (i.e. $\eta = \psi_{\frac{1}{2}}$). For $g \in G$, we denote $j(g)$ the integer such that $i(g, I_n) \in P_Y \delta_{j(g)} i(G \times G)$. Since V_b^- is split we have (see [HKS, 1.15, p953]),

$$\tilde{\iota}_{b,\chi}(g) = (\iota_b(g), \beta_{V_b^-,\chi}(g))$$

with

$$\beta_{V_b^-,\chi}(g) = \chi(x(g)) \gamma_F(\eta \circ RV)^{-j(g)}$$

---

[5]The first isomorphism because $\omega_{b,\psi}$ is unitary, the second because of the definition of $r(g)$ in 1.3

where and
$$\gamma_F(\eta \circ RV) = (\Delta, \det V_b^-)_F \gamma_F(-\Delta, \eta)^b \gamma_F(-1, \eta)^{-b}.^{[6]}$$

Let
$$\varphi : \mathrm{Sp}(\mathbb{W}_b) \times \mathbf{C}^1 \simeq \mathrm{Mp}(\mathbb{W}_b) \longrightarrow \mathrm{Sp}(\mathbb{W}_{b,-}) \times \mathbf{C}^1 \simeq \mathrm{Mp}(\mathbb{W}_{b,-})$$
$$(g \ , \ z) \qquad\qquad \longmapsto \qquad\qquad (g \ , \ \overline{z})$$

be the identification. Then $\overline{\chi(x(g))} = \chi^{-1}(x(g))$ and

$$\begin{aligned}
\overline{\gamma_F(-\Delta, \eta)\gamma_F(-1, \eta)^{-1}} = \overline{\left(\frac{\gamma_F(\eta_{-\Delta})}{\gamma_F(\eta_{-1})}\right)} &= \frac{\gamma_F(\eta_\Delta)}{\gamma_F(\eta_1)} = \gamma_F(\Delta, \eta)\gamma_F(1, \eta)^{-1} \\
&= (\Delta, -1)_F \gamma_F(-\Delta, \eta)(-1, -1)_F \gamma_F(-1, \eta)^{-1} \\
&= (\Delta, -1)_F \gamma_F(-\Delta, \eta)\gamma_F(-1, \eta)^{-1}
\end{aligned}$$

thus, since $\det V_{b,-}^- = (-1)^b \det V_b^-$, we have $\overline{\beta_{V_b^-, \chi}(g)} = \beta_{V_{b,-}^-, \chi^{-1}}(g)$ and

$$\varphi \circ \tilde{\imath}_{b,\chi} = \tilde{\imath}_{b,-,\chi^{-1}}$$

as claimed.

We now consider the case in which V_b^- is an anisotropic line. We identify V_b^- with E and if $(x, y) \in E^2$, we have $\langle x, y \rangle = \mathbf{a}\overline{x}y$ for some $\mathbf{a} \in F$. If $g \in G(V_b^-) = E^1$, we decompose $g = x + \delta y$ (with $x, y \in F$) and we have (see [Kud1, Proposition 4.8, p396])

$$\begin{aligned}
\beta_{V_b^-, \chi}(g) &= \chi(\delta(g-1))\gamma_F(2\mathbf{a}y(x-1), \eta)\gamma_F(\eta)(\Delta, -2y(1-x))_F \\
&= \chi(\delta(g-1))\gamma_F(\eta_{2\mathbf{a}y(x-1)})(\Delta, -2y(1-x))_F
\end{aligned}$$

and
$$\beta_{V_{b,-}^-, \chi}(g) = \chi(\delta(g-1))\gamma_F(\eta_{-2\mathbf{a}y(x-1)})(\Delta, -2y(1-x))_F.$$

It is immediate that $\overline{\beta_{V_{b,-}^-, \chi^{-1}}(g)} = \beta_{V_b^-, \chi}(g)$ and

$$\varphi \circ \tilde{\imath}_{b,\chi} = \tilde{\imath}_{b,-,\chi^{-1}}$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let
$$V_{a,b,-} = V_a^+ \oplus V_{b,-}^-, \quad \mathbb{W}_{a,b,-} = \mathbb{W}_a \oplus \mathbb{W}_{b,-}$$

and, as before χ_0 the trivial character of $E^\times$. We denote, as above, $\omega_{a,b,-,\chi_0}$ the representation of G induced by the Weil representation $\omega_{a,b,-,\psi}$. Let

$$\tilde{i} : \mathrm{Mp}(\mathbb{W}_a) \times \mathrm{Mp}(\mathbb{W}_{b,-}) \longrightarrow \mathrm{Mp}(\mathbb{W}_{a,b,-})$$

---

[6]for this single proof, $\Delta \in F^\times$ is the square of an element $\delta \in E^\times - F^\times$ which is used to identify the Hermitian and skew-Hermitian spaces

be the natural map whose restriction to $\mathbf{C}^1$ is the product. Then[7]

$$\tilde{i}^* \omega_{a,b,-,\psi} = \omega_{a,\psi} \otimes \omega_{b,-,\psi}.$$

According to [HKS, Lemma 5.2, p964],

$$\tilde{\iota}_{a,b,-,\chi_0} = \tilde{i} \circ (\tilde{\iota}_{a,\chi} \times \tilde{\iota}_{b,-,\chi^{-1}}) \circ \Delta : G \longrightarrow \mathrm{Mp}(\mathbb{W}_{a,b,-}).$$

Thus as a representation of G we have

$$\omega_{a,\chi} \otimes \omega_{b,-,\chi^{-1}} \simeq \omega_{a,b,-,\chi_0}.$$

We thus have a non-zero element

$$\nu \in \mathrm{Hom}_G(\omega_{a,\chi} \otimes \omega_{b,\chi}^\vee, \mathbf{1}) \simeq \mathrm{Hom}_G(\omega_{a,b,-,\chi_0}, \mathbf{1}).$$

We have $\dim V_{a,b,-} = a + b$ even. Let us compute $\epsilon(V_{a,b,-})$:

$$
\begin{aligned}
\epsilon(V_{a,b,-}) &= (-1)^{\frac{(a+b)(a+b-1)}{2}} \det V_{a,b,-} \\
&= (-1)^{\frac{a(a-1)+ab+ba+b(b-1)}{2}} \det V_a^+ \det V_{b,-}^- \\
&= (-1)^{\frac{a(a-1)+b(b-1)}{2}+ab} \det V_a^+ (-1)^b \det V_b^- \\
&= (-1)^{ab+b}(-1)^{\frac{a(a-1)}{2}} \det V_a^+ (-1)^{\frac{b(b-1)}{2}} \det V_b^- \\
&= (-1)^{ab+b} \epsilon(V_a^+) \epsilon(V_b^-).
\end{aligned}
$$

Since both ab and b have the parity of m_0 we have $\epsilon(V_{a,b,-}) = \epsilon(V_a^+)\epsilon(V_b^-) = -1$. Thus, according to Theorem 2.9

$$a + b \geqslant 2n + 2$$

as needed. $\qquad\square$

2.5 Criterion

Definition 2.12 *For a given $m \in \{0, ..., 2n\}$, let $m' = 2n - m$. The space $V_{m'}^\pm$ is said to* be complementary *to $V_m^\pm$ (the space V_{2n}^- has no complementary).*

Remark 2.13 *If $V_m^\pm$ and $V_{m'}^\pm$ are complementary, then $s_0' = \frac{m'-n}{2} = \frac{2n-m-n}{2} = \frac{n-m}{2} = -s_0$.*

We give the composition series for $I_n(s_0, \chi)$ in each case where it is reducible, with indication of the action of the operators $M^*(s_0, \chi)$. Implicitely we have $m' = 2n - m$. All these

---

[7]Is it relevant ?

results are taken from [KS].

$$
\begin{array}{ccccc}
0 & \subset & R_n(V_0^+, \chi) & \subset & I(-\tfrac{n}{2}, \chi) \\
\| & & \| & & \\
R_n(V_0^-, \chi) & & M^*(\tfrac{n}{2}, \chi)(R_n(V_{2n}^+, \chi)) = M^*(\tfrac{n}{2}, \chi)(I(\tfrac{n}{2}, \chi)) \\
\| & & \| & & \\
M^*(\tfrac{n}{2}, \chi)(R_n(V_{2n}^+, \chi)) & & \mathrm{Ker}\, M^*(-\tfrac{n}{2}, \chi) & & m = 0,\; s_0 = -\tfrac{n}{2}
\end{array}
$$

$$
\begin{array}{c}
M^*(-s_0, \chi)(R_n(V_{m'}^+, \chi)) \\
\| \\
R_n(V_m^+, \chi) \\
\subset \qquad\qquad \subset \\
0 \qquad\qquad\qquad\qquad\qquad R_n(V_m^+, \chi) \oplus R_n(V_m^-, \chi) \;\subset\; I_n(s_0, \chi) \\
\subset \qquad\qquad \subset \qquad\qquad \| \\
R_n(V_m^-, \chi) \qquad\qquad \mathrm{Ker}\, M^*(s_0, \chi) \\
\| \\
M^*(-s_0, \chi)(R_n(V_{m'}^-, \chi)) \qquad\qquad 1 \leqslant m < n,\; -\tfrac{n}{2} < s_0 < 0
\end{array}
$$

$$
\begin{array}{c}
M^*(0, \chi)(R_n(V_n^+, \chi)) \\
\| \\
R_n(V_n^+, \chi) \\
\subset \qquad\qquad \subset \\
0 \qquad\qquad\qquad\qquad\qquad R_n(V_n^+, \chi) \oplus R_n(V_n^-, \chi) \;=\; I(0, \chi) \\
\| \qquad \subset \qquad\qquad \subset \\
\mathrm{Ker}\, M^*(0, \chi) \quad R_n(V_n^-, \chi) \\
\| \\
M^*(0, \chi)(R_n(V_n^-, \chi)) \qquad\qquad m = n,\; s_0 = 0
\end{array}
$$

$$
\begin{array}{c}
R_n(V_m^+, \chi) \\
\subset \qquad\qquad \subset \\
0 \subset R_n(V_m^+, \chi) \cap R_n(V_m^-, \chi) \qquad\qquad R_n(V_m^+, \chi) + R_n(V_m^-, \chi) \;=\; I_n(s_0, \chi) \\
\| \qquad \subset \qquad\qquad\qquad \subset \\
\mathrm{Im}\, M^*(-s_0, \chi) \qquad R_n(V_m^-, \chi) \\
\| \\
\mathrm{Ker}\, M^*(s_0, \chi) \qquad\qquad\qquad n < m < 2n,\; 0 < s_0 < \tfrac{n}{2}
\end{array}
$$

$$
\begin{array}{c}
0 \quad \subset R_n(V_{2n}^-, \chi) \quad \subset R_n(V_{2n}^+, \chi) = I_n\left(\tfrac{n}{2}, \chi\right) \\
\| \\
\mathrm{Im}\, M^*(-\tfrac{n}{2}, \chi) \\
\| \\
\mathrm{Ker}\, M^*(\tfrac{n}{2}, \chi) \qquad\qquad m = 2n,\; s_0 = \tfrac{n}{2}
\end{array}
$$

In each case an inclusion sign means that the quotient is non-zero and irreducible. Note that V_0^- does not exist, but we define the space $R_n(V_0^-, \chi)$ as the null space in $R_n(V_0^+, \chi)$.

Theorem 2.14 *Fix $m_0 \in \{0, 1\}$ and a character χ of $E^\times$ such that $\chi|_{F^\times} = \epsilon_{E/F}^{m_0}$. Suppose that*

$$\dim \operatorname{Hom}_{G \times G}(I_n(s_0, \chi), \pi \otimes (\chi \cdot \pi^\vee)) = 1$$

for all s_0 in

$$\begin{cases} \{-\frac{n}{2}, 1 - \frac{n}{2}, ..., \frac{n}{2} - 1, \frac{n}{2}\} & \text{if } m_0 = 0 \\ \{\frac{1-n}{2}, \frac{3-n}{2}, ..., \frac{n-3}{2}, \frac{n-1}{2}\} & \text{if } m_0 = 1, \end{cases}$$

i.e. for all $s_0 \in \frac{m_0}{2} + \mathbf{Z}$ such that $|s_0| \leqslant \frac{n}{2}$. Then

$$m_\chi^+(\pi) + m_\chi^-(\pi) = 2n + 2.$$

PROOF: Fix $m_0 \in \{0, 1\}$ and a character χ of $E^\times$ such that $\chi|_{F^\times} = \epsilon_{E/F}^{m_0}$. For $0 \leqslant m' \leqslant 2n$, we put $m = 2n - m'$ and recall that $s_0 = \frac{m-n}{2}$.

The case $m_\chi^+(\pi) = 0$ is immediate because it implies $\pi = \mathbf{1}$ and Theorem 2.9 says that $m_\chi^-(\pi) = 2n + 2$.

If $s_0 \geqslant 0$ we have $I_n(s_0, \chi) = R_n(V_m^+, \chi) + R_n(V_m^-, \chi)$ and thus, thanks to the hypothesis of the theorem, at least one of

$$\operatorname{Hom}_{G \times G}(R_n(V_m^\pm, \chi), \pi \otimes (\chi \cdot \pi^\vee))$$

is non zero. This in turn means, thanks to Proposition 1.9, that

$$\min(m_\chi^+(\pi), m_\chi^-(\pi)) \leqslant n + 1$$

(the bound is $n + 1$ and not n in case m and n have opposite parity). If $s_0 > \frac{n}{2}$ then $I_n(s_0, \chi)$ is irreducible and thus

$$R_n(V_m^\pm, \chi) = I_n(s_0, \chi).$$

By the persistence principle (see Proposition 1.6, point i) since we have $m > 2n > \min(m_\chi^+(\pi), m_\chi^-(\pi))$, one and thus both

$$\operatorname{Hom}_{G \times G}(R_n(V_m^\pm, \chi), \pi \otimes (\chi \cdot \pi^\vee)) \neq 0.$$

This means $\max(m_\chi^+(\pi), m_\chi^-(\pi)) \leqslant 2n + 2 - m_0$.

Let $\epsilon = \pm$ be such that $m_\chi^\epsilon(\pi) = \min(m_\chi^+(\pi), m_\chi^-(\pi))$. We let m' be $m_\chi^\epsilon(\pi)$ (and choose m and s_0 accordingly). As observed above, the case $m' = 0$ has already been proven. If $m' = 1$, then from Theorem 2.10 we have $m_\chi^{-\epsilon}(\pi) \geqslant 2n + 1$ and thus, thanks to the preceeding bound, $m_\chi^{-\epsilon}(\pi) = 2n + 1$ (observe that if $m' = 1$ then $m_0 = 1$).

We now suppose $2 \leqslant m' \leqslant n$, i.e. $0 \leqslant s_0 \leqslant \frac{n}{2} - 1$. By Theorem 2.10 we thus have $m_\chi^{-\epsilon}(\pi) \geqslant 2n + 2 - m' \geqslant n + 2$. Since m' is the minimum of $m_\chi^\pm(\pi)$, we have

$$\operatorname{Hom}_{G \times G}(R_n(V_{m'-2}^+, \chi) \oplus R_n(V_{m'-2}^-, \chi), \pi \otimes (\chi \cdot \pi^\vee)) = 0 \tag{9}$$

(here $R_n(V_0^-, \chi) = 0$ as defined above). This means that any element of $\mathrm{Hom}_{G \times G}(I_n(-s_0 - 1, \chi), \pi \otimes (\chi \cdot \pi^\vee))$ factors through

$$I_n(-s_0 - 1, \chi)/R_n(V_m^+, \chi) \oplus R_n(V_m^-, \chi) \simeq \mathrm{Im}\, M^*(-s_0 - 1, \chi)$$

and thus

$$\dim \mathrm{Hom}_{G \times G}(\mathrm{Im}\, M^*(-s_0 - 1, \chi), \pi \otimes (\chi \cdot \pi^\vee)) = 1.$$

On the other hand, let

$$\mu \in \mathrm{Hom}_{G \times G}(I_n(s_0 + 1, \chi), \pi \otimes (\chi \cdot \pi^\vee))$$

with $\mu \neq 0$. We know from (9) that

$$\mu \circ M^*(s_0 + 1, \chi) = 0$$

hence μ must be non-zero on $\mathrm{Ker}\, M^*(s_0 + 1, \chi) = \mathrm{Im}\, M^*(-s_0 - 1, \chi)$. Since $s_0 + 1 > 0$, the space $\mathrm{Im}\, M^*(-s_0 - 1, \chi)$ is a non-zero submodule of $R_n(V_{m+2}^{-\epsilon})$ and thus

$$\mu\big|_{R_n(V_{m+2}^{-\epsilon})} \neq 0,$$

hence

$$m_\chi^{-\epsilon}(\pi) \leqslant m + 2 = 2n + 2 - m'.$$

We thus have $m_\chi^+(\pi) + m_\chi^-(\pi) = 2n + 2$ as claimed.

The only remaining case is $m' = n + 1$. We thus have $m = n - 1$ and $s_0 = -\frac{1}{2}$. The proof is similar to the preceeding one. If

$$\mu \in \mathrm{Hom}_{G \times G}\left(I_n\left(\frac{1}{2}, \chi\right), \pi \otimes (\chi \cdot \pi^\vee)\right)$$

is non-zero, then its composition with $M^*(\frac{1}{2}, \chi)$ is zero, this means that the restriction of $\mathrm{Ker}\, M^*(\frac{1}{2}, \chi)$ must be non-zero. Hence, for the same reason as above, $m_\chi^+(\pi) = m_\chi^-(\pi) = n + 1$. $\qquad\square$

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