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On the geometric origin of the bi-Hamiltonian structure of the Calogero-Moser system

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Abstract

We show that the bi-Hamiltonian structure of the rational n-particle (attractive) Calogero-Moser system can be obtained by means of a double projection from a very simple Poisson pair on the cotangent bundle of $\mathfrak{gl}(n, \mathbb{R})$. The relation with the Lax formalism is also discussed.

1 Introduction

In 1971 Francesco Calogero [3] solved the quantum system consisting of n unit-mass particles moving on the line and pairwise interacting via a (repulsive) potential that is proportional to the inverse of the squared distance. (The case $n = 3$ was treated earlier [2] by Calogero himself). The integrability of the classical counterpart was conjectured in [3] and proved by Moser in [17]. Later, this system was showed to be superintegrable [22]. It is also worthwhile to mention that the classical 3-particle case appeared in the works of Jacobi [12]. More information on the (quantum and classical) Calogero-Moser system can be found in [4]. Recently, this system gained an important role in pure mathematics too. We just cite its relations with quiver varieties [9] and double affine Hecke algebras [6], referring to [5] for a more complete list.

Although a lot of papers were devoted to the many facets of the Calogero-Moser system, only a few results concerning its bi-Hamiltonian formulation were found. In [16] (see also [15]) a bi-Hamiltonian structure was constructed with the help of the Lax representation of the system. A $(2n-1)$-dimensional family of compatible Poisson tensors — apparently unrelated with the above mentioned Poisson pair — was found in [10], in the context of superintegrable systems.

In this paper we explain where the bi-Hamiltonian structure of [16] comes from. The spirit is very close to that of the fundamental paper [13], where the Calogero-Moser system is shown to be the Marsden-Weinstein reduction of a trivial system on the cotangent bundle of $\mathfrak{su}(n)$. In the same vein, we show that the bi-Hamiltonian structure can be obtained — by means of two projections — from a Poisson pair belonging to a wide class of bi-Hamiltonian structures on cotangent bundles. Such class is recalled in Section 2, while in Section 3 the particular example related to the Poisson brackets of the (attractive) Calogero-Moser system is considered. In Section 4 a first reduction is performed, corresponding to the action given by the simultaneous conjugation. A second projection, leading to the phase space of the Calogero-Moser system, is described in Section 5. Finally, Section 6 is devoted to the example of the 2-particle system (trivial from the physical point of view, but not from the mathematical one) and Section 7 to some final remarks.

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2 Bi-Hamiltonian structures on cotangent bundles

In this section we recall from [20] (see also [11]) that a torsionless (1,1) tensor field on a smooth manifold $\mathcal{Q}$ gives rise to a (second) Poisson structure on the cotangent space $T^*\mathcal{Q}$, compatible with the canonical one. More information on the geometry of bi-Hamiltonian manifolds can be found, e.g., in [15].

Let $L : T\mathcal{Q} \to T\mathcal{Q}$ be a type $(1,1)$ tensor field on $\mathcal{Q}$, whose Nijenhuis torsion vanishes. This means that

$$T(L)(X,Y) := [LX, LY] - L([LX,Y] + [X,LY] - L[X,Y]) = 0 \qquad (1)$$

for all pairs of vector fields X, Y on $\mathcal{Q}$. Let θ be the Liouville 1-form on $T^*\mathcal{Q}$ and $\omega = d\theta$ the standard symplectic 2-form on $T^*\mathcal{Q}$, whose associated Poisson tensor will be denoted with P_0. One can deform the Liouville 1-form to a 1-form θ_L:

$$\langle \theta_L, Z\rangle_\alpha = \langle \alpha, L(\pi_* Z)\rangle_{\pi(\alpha)} ,$$

for any vector field Z on $T^*\mathcal{Q}$ and for any 1-form α on $\mathcal{Q}$, where $\pi : T^*\mathcal{Q} \to \mathcal{Q}$ is the canonical projection. If we choose local coordinates $(x^1, \ldots, x^n)$ on $\mathcal{Q}$ and consider the corresponding symplectic coordinates $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ on $T^*\mathcal{Q}$, we get the local expression $\theta_L = L^i_j y_i dx^j$. Now, it is well-known that the canonical Poisson bracket is defined by

$$\{F, G\}_0 = \omega(X_F, X_G) \qquad\qquad F, G \in C^\infty(T^*\mathcal{Q}) ,$$

where X_F, X_G are the Hamiltonian vector fields associated to F, G with respect to the symplectic form ω. A second composition law on $C^\infty(T^*\mathcal{Q})$ is given by

$$\{F, G\}_1 = \omega_L(X_F, X_G) , \qquad (2)$$

where $\omega_L := d\theta_L$. It is easily seen that

$$\{x^i, x^j\}_1 = 0 , \quad \{y_i, x^j\}_1 = L^j_i , \quad \{y_i, y_j\}_1 = \left(\frac{\partial L^k_j}{\partial x^i} - \frac{\partial L^k_i}{\partial x^j}\right) y_k .$$

Moreover, the vanishing of the torsion of L entails that (2) is a Poisson bracket too, and that it is compatible with $\{\cdot, \cdot\}_0$. Thus we have a bi-Hamiltonian structure on $T^*\mathcal{Q}$.

Remark 1 Since P_0 is invertible, one can introduce the so-called recursion operator $N := P_1 P_0^{-1}$, whose Nijenhius torsion also vanishes. It turns out to be the *complete lift* of L (see, e.g., [23]), and it is uniquely determined by the condition

$$d\theta_L(X,Y) = \omega(NX,Y)$$

for all vector fields X, Y on $T^*\mathcal{Q}$. An easy computation shows that

$$\begin{aligned} N\left(\frac{\partial}{\partial x_k}\right) &= L^i_k \frac{\partial}{\partial x_i} - y_l \left(\frac{\partial L^l_i}{\partial x_k} - \frac{\partial L^l_k}{\partial x_i}\right)\frac{\partial}{\partial y_i} \\ N\left(\frac{\partial}{\partial y_k}\right) &= L^k_i \frac{\partial}{\partial y_i} \ . \end{aligned}$$

As pointed out in [11] (see also [1, 7] and the references cited therein), the geometry of such bi-Hamiltonian manifolds — often called ωN-manifolds — can be successfully exploited to characterize the Hamiltonian system that are separable in canonical coordinates in which N is diagonal.

We conclude this section by recalling that the functions

$$H_k := \frac{1}{k}\mathrm{tr} L^k = \frac{1}{2k}\mathrm{tr} N^k \tag{3}$$

form a bi-Hamiltonian hierarchy on $T^*\mathcal{Q}$, that is, $P_1 dH_k = P_0 dH_{k+1}$ for all $k \geq 1$. This follows from $N^* dH_k = dH_{k+1}$, where N^* is the transpose of N, and is well-known to imply the involutivity (with respect to both Poisson brackets) of the H_k.

3 A Bi-Hamiltonian structure on $T^*\mathfrak{gl}(n)$

In this section we consider a particular case of the general construction described in the previous section. The manifold $\mathcal{Q}$ is the set $\mathfrak{gl}(n)$ of real $n \times n$ matrices, and the $(1,1)$ torsionless tensor field is defined as

$$L_A : V \mapsto AV \ , \tag{4}$$

where $A \in \mathfrak{gl}(n)$ and $V \in T_A\mathfrak{gl}(n) \simeq \mathfrak{gl}(n)$. It is known that the torsion of L vanishes (and one can easily check it by writing (1) for constant vector fields). The cotangent bundle $T^*\mathfrak{gl}(n) \simeq \mathfrak{gl}(n) \times \mathfrak{gl}(n)^*$ can be identified with $\mathfrak{gl}(n) \times \mathfrak{gl}(n)$ by means of the pairing given by the trace of the product.

Thus on $\mathfrak{gl}(n)\times\mathfrak{gl}(n)$ we have a bi-Hamiltonian structure, whose first Poisson bracket is associated with the canonical symplectic form $\omega_0 = \mathrm{tr}\,(dB\wedge dA)$, where $(A,B)\in\mathfrak{gl}(n)\times\mathfrak{gl}(n)$. In order to determine the second Poisson tensor, we have to consider the 1-form $\theta_L = \mathrm{tr}\,(BA\,dA)$ and to compute

$$\omega_L = d\theta_L = \mathrm{tr}\,(dB\wedge A\,dA + B\,dA\wedge dA)\ .$$

Let F_1, F_2 be two real functions on $\mathfrak{gl}(n)\times\mathfrak{gl}(n)$, and let (ξ_1,η_1) and (ξ_2,η_2) be their differentials. Since the canonical Poisson tensor P_0 acts on a covector (ξ,η) as

$$P_0 : \begin{pmatrix}\xi\\ \eta\end{pmatrix} \mapsto \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\eta\\ -\xi\end{pmatrix}\ , \tag{5}$$

the corresponding Hamiltonian vector fields (with respect to ω) are $X_{F_i} = (\eta_i, -\xi_i)$, for $i=1,2$. Therefore

$$\{F_1,F_2\}_1 = \omega_L(X_{F_1},X_{F_2}) = \mathrm{tr}\ (A(\eta_1\xi_2 - \eta_2\xi_1) + B[\eta_1,\eta_2])\ , \tag{6}$$

so that the second Poisson tensor is

$$P_1 : \begin{pmatrix}\xi\\ \eta\end{pmatrix} \mapsto \begin{pmatrix}0 & A\cdot\\ -\cdot A & [B,\cdot]\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}A\eta\\ -\xi A + [B,\eta]\end{pmatrix}\ , \tag{7}$$

and the recursion operator $N = P_1 {P_0}^{-1}$ and its transpose are given by

$$N = \begin{pmatrix}A\cdot & 0\\ [B,\cdot] & \cdot A\end{pmatrix}\ ,\qquad N^* = \begin{pmatrix}\cdot A & [\cdot,B]\\ 0 & A\cdot\end{pmatrix}\ .$$

From the expression of N^* it is evident that the functions $H_k = \frac{1}{k}\mathrm{tr}\,A^k$, for $k\geq 1$, form a bi-Hamiltonian hierarchy. It can be easily checked that this hierarchy coincide with the one mentioned at the end of Section 2, namely, that $\mathrm{tr}\,A^k = \mathrm{tr}\,L^k$. The corresponding vector fields $X_k := -P_0 dH_k$ are given by $(X_k)_{(A,B)} = (0, A^{k-1})$.

Remark 2 One can check that if $T^*GL(n,\mathbb{R})$ is seen as an open subset of $\mathfrak{gl}(n)\times\mathfrak{gl}(n)$ by means of left translations, then the canonical symplectic structure of $T^*GL(n,\mathbb{R})$ takes the form (7).

4 The first projection

The main aim of this paper is to show that the bi-Hamiltonian structure of the Calogero-Moser system is a reduction of the one presented in the previous section. The first step is to notice that the Poisson pair (P_0, P_1) is invariant with respect to the action of $G = GL(n, \mathbb{R})$ on $\mathfrak{gl}(n) \times \mathfrak{gl}(n)$ given by the simultaneous conjugation:

$$(g, (A, B)) \mapsto \left(gAg^{-1}, gBg^{-1}\right) \ .$$

This is obvious for the canonical Poisson tensor P_0, since this action is the lifting to the cotangent bundle of the action $(g, A) \mapsto gAg^{-1}$ on $\mathfrak{gl}(n)$. Since the latter leaves invariant the tensor field L, the Poisson tensor P_1 is invariant too. We would like to obtain a nice quotient, so we consider G acting on the open subset $\mathcal{M} \subset \mathfrak{gl}(n) \times \mathfrak{gl}(n)$ formed by the pairs (A, B) such that:

- A and B have real distinct eigenvalues;
- if $\{v_i\}_{i=1,\dots,n}$ is an eigenvector basis of B, then $Av_i \notin \langle v_1, \dots, \hat{v}_j, \dots, v_n \rangle$ for all $j \neq i$. As usual, $\langle \dots \rangle$ denotes the linear span and $\hat{v}_j$ means that v_j is not included in the list;
- the same condition as before with A and B exchanged.

It is clear that $\mathcal{M}$ is invariant under the action of G. Moreover, the description of the quotient $\mathcal{M}/G$ is made easy by the existence of a subset $\mathcal{P} \subset \mathcal{M}$ intersecting every orbit in one point. It is given by $\mathcal{P} = \bigcup_{\epsilon_i \in \{+,-\}} \mathcal{P}_{(\epsilon_1,\dots,\epsilon_{n-1})}$, where $\mathcal{P}_{(\epsilon_1,\dots,\epsilon_{n-1})}$ is the set of pairs $(A, B) \in \mathcal{M}$ such that B is diagonal with $B_{ii} < B_{jj}$ if $i < j$, and $A_{i+1,i} > 0$, $A_{i,i+1} = \epsilon_i A_{i+1,i}$ for all $i = 1, \dots, n-1$. For example, if $n = 2$ we have $\mathcal{P} = \mathcal{P}_+ \cup \mathcal{P}_-$, where the elements of $\mathcal{P}_+$ are those in $\mathcal{M}$ of the form

$$\left(\begin{pmatrix} A_{11} & A_{21} \\ A_{21} & A_{22} \end{pmatrix}, \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} \right) \ ,$$

with $A_{21} > 0$ and $B_{11} < B_{22}$, while the elements of $\mathcal{P}_-$ are those in $\mathcal{M}$ of the form

$$\left(\begin{pmatrix} A_{11} & -A_{21} \\ A_{21} & A_{22} \end{pmatrix}, \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} \right) \ ,$$

again with $A_{21} > 0$ and $B_{11} < B_{22}$.

Proposition 3 *Every orbit of G in $\mathcal{M}$ intersects $\mathcal{P}$ in just one point. Moreover, for all $(A,B) \in \mathcal{P}$ the tangent space $T_{(A,B)}\mathcal{M}$ is the direct sum of $T_{(A,B)}\mathcal{P}$ and the tangent space to the orbit.*

Proof. Given an orbit of G in $\mathcal{M}$, it is clearly possible to find a point (A,B) on such orbit with B diagonal and

$$B_{ii} < B_{jj} \qquad \text{for all } i < j. \tag{8}$$

Then, again by the definition of $\mathcal{M}$, one has that $A_{ij} \neq 0$ if $i \neq j$. Because of (8) we can still act on (A,B) only by an invertible diagonal matrix $g = \mathrm{diag}(d_1, \ldots, d_n)$. We have to show that one can choose the d_i in such a way that $(gAg^{-1}, gBg^{-1}) = (gAg^{-1}, B) \in \mathcal{P}_{(\epsilon_1,\ldots,\epsilon_{n-1})}$ for some $\epsilon_i = \pm 1$. Since $(gAg^{-1})_{ij} = d_i A_{ij} {d_j}^{-1}$, this means that

$$d_{i+1} A_{i+1,i} {d_i}^{-1} > 0 \ , \qquad d_i A_{i,i+1} {d_{i+1}}^{-1} = \epsilon_i d_{i+1} A_{i+1,i} {d_i}^{-1} \ .$$

Therefore ϵ_i is determined by the sign of $A_{i,i+1}/A_{i+1,i}$, and

$$\frac{d_i}{d_{i+1}} = \pm \sqrt{\epsilon_i \frac{A_{i+1,i}}{A_{i,i+1}}} \ ,$$

where the $\pm$ has to be chosen in such a way that $d_{i+1} A_{i+1,i} {d_i}^{-1} > 0$. In this way we have found the matrix g up to a multiple, and the first part of the claim is proved.

Let us fix now a point $(A,B) \in \mathcal{P}_{(\epsilon_1,\ldots,\epsilon_{n-1})}$ and a tangent vector $(V,W) \in T_{(A,B)}\mathcal{M}$. We have to show that (V,W) can be uniquely decomposed as

$$(V,W) = (\dot{A}, \dot{B}) + ([A,\xi],[B,\xi]) \ , \tag{9}$$

where $(\dot{A},\dot{B}) \in T_{(A,B)}\mathcal{P}_{(\epsilon_1,\ldots,\epsilon_{n-1})}$ and $\xi \in \mathfrak{gl}(n,\mathbb{R})$. Since $\dot{B}$ is diagonal, we immediately have that the off-diagonal entries of ξ are given by $\xi_{ij} = W_{ij}/(B_{ii} - B_{jj})$. Then we have to impose the conditions $\dot{A}_{i,i+1} = \epsilon_i \dot{A}_{i+1,i}$, getting the following equations,

$$\sum_{j=1}^{n} (\xi_{ij} A_{j,i+1} - A_{ij} \xi_{j,i+1}) = \epsilon_i \sum_{j=1}^{n} (\xi_{i+1,j} A_{ji} - A_{i+1,j} \xi_{ji}) - V_{i,i+1} + \epsilon_i V_{i+1,i} \ ,$$

for all $i = 1, \ldots, n-1$. Since $A_{i,i+1} = \epsilon_i A_{i+1,i}$, we obtain

$$2\epsilon_i A_{i+1,i}(\xi_{ii} - \xi_{i+1,i+1}) = - \sum_{j\neq i} (\xi_{ij} A_{j,i+1} + \epsilon_i \xi_{ji} A_{i+1,j}) + \sum_{j\neq i+1} (\xi_{j,i+1} A_{ij} + \epsilon_i \xi_{i+1,j} A_{ji}) - V_{i,i+1} + \epsilon_i V_{i+1,i} \ ,$$

for all $i = 1, \ldots, n-1$, namely, $(n-1)$ equations for the variables $\xi_{11}, \ldots, \xi_{nn}$. Thanks to the fact that $A_{i+1,i} > 0$, they can be solved, and the solution is unique up to a (common) additive constant. This shows the uniqueness of the vector $([A, \xi], [B, \xi])$, tangent to the orbit, and of the decomposition (9).

QED

Remark 4 It is clear from the previous proof that one can also identify $\mathcal{M}/G$ with the submanifold $\mathcal{P}' \subset \mathcal{M}$, whose definition is the same of $\mathcal{P}$, but with the matrices A and B exchanged.

Remark 5 The quotient space defined by simultaneous conjugation on k-tuples of matrices has been the subject of important investigations by Artin, Procesi, Razmyslov, and others (see, e.g., [14]). For our purposes, it is convenient to restrict to the open subset $\mathcal{M}$, and in this case an explicit description of the quotient is possible in terms of the transversal submanifold $\mathcal{P}$.

Next we consider the vector fields X_k of the bi-Hamiltonian hierarchy on $\mathcal{M}$, that is, $(X_k)_{(A,B)} = \left(0, A^{k-1}\right)$. Since their Hamiltonians $H_k = \frac{1}{k}\mathrm{tr}\, A^k$ and the bi-Hamiltonian structure are invariant with respect to the action of G, the X_k can be projected on $\mathcal{M}/G$. Their projections are the vector fields associated with the Hamiltonians H_k (seen as functions on the quotient) and the reduced bi-Hamiltonian structure. We can exploit the identification between $\mathcal{M}/G$ and the submanifold $\mathcal{P} \subset \mathcal{M}$ in order to explicitly write the projected vector fields. Indeed, we have just seen that we can uniquely find $(\partial_k A, \partial_k B) \in T_{(A,B)}\mathcal{P}$ and $([A, \xi_k], [B, \xi_k])$, tangent to the orbit passing through $(A, B) \in \mathcal{P}$, such that

$$(0, A^{k-1}) = (\partial_k A, \partial_k B) + ([A, \xi_k], [B, \xi_k]) \ .$$

This shows that

$$\partial_k A = [\xi_k, A] \ , \qquad \partial_k B = [\xi_k, B] + A^{k-1} \ , \tag{10}$$

i.e., the projected flows possess a Lax representation. In the next section we will perform a second reduction and we will show that the flows (10) give rise to the (attractive) Calogero-Moser flows.

Remark 6 The deduction of the Lax equations (10) is well-known (see, e.g., [18]). Notice however that such equations describe flows on the (n^2+1)-dimensional manifold $\mathcal{P} \simeq \mathcal{M}/G$ and so they are an extension of the usual Lax representation of the Calogero-Moser system.

We close this section with an interesting description of the bi-Hamiltonian structure on the quotient $\mathcal{M}/G$ (see [5], where only the first Poisson structure is considered). Let $F_1 = \operatorname{tr}(a_1 \cdots a_r)$ and $F_2 = \operatorname{tr}(b_1 \cdots b_s)$, where a_i and b_j are either A or B. Then

$$dF_1 = \left(\sum_{i:a_i=A} a_{i+1} \cdots a_r a_1 \cdots a_{i-1}, \sum_{j:a_j=B} a_{j+1} \cdots a_r a_1 \cdots a_{j-1} \right)$$

and therefore we have the so-called necklace bracket formula

$$\begin{aligned} \{F_1, F_2\}_0 = & \sum_{(i,j):a_i=B,b_j=A} \operatorname{tr}\left(a_{i+1} \cdots a_r a_1 \cdots a_{i-1} b_{j+1} \cdots b_s b_1 \cdots b_{j-1}\right) \\ & - \sum_{(i,j):a_i=A,b_j=B} \operatorname{tr}\left(b_{j+1} \cdots b_s b_1 \cdots b_{j-1} a_{i+1} \cdots a_r a_1 \cdots a_{i-1}\right) . \end{aligned} \tag{11}$$

As far as the second Poisson bracket is concerned, we have from (6) that

$$
\begin{aligned}
\{F_1, F_2\}_1 = & \sum_{(i,j):a_i=B,b_j=A} \mathrm{tr}\ (A a_{i+1}\cdots a_r a_1 \cdots a_{i-1} b_{j+1}\cdots b_s b_1 \cdots b_{j-1}) \\
& - \sum_{(i,j):a_i=A,b_j=B} \mathrm{tr}\ (A b_{j+1}\cdots b_s b_1 \cdots b_{j-1} a_{i+1}\cdots a_r a_1 \cdots a_{i-1}) \\
& + \sum_{(i,j):a_i=B,b_j=B} \mathrm{tr}\ (B[a_{i+1}\cdots a_r a_1 \cdots a_{i-1}, b_{j+1}\cdots b_s b_1 \cdots b_{j-1}]) \\
& = \sum_{(i,j):a_i=B,b_j=A} \mathrm{tr}\ (a_i a_{i+1}\cdots a_r a_1 \cdots a_{i-1} b_{j+1}\cdots b_s b_1 \cdots b_{j-1}) \\
& - \sum_{(i,j):a_i=A,b_j=B} \mathrm{tr}\ (b_{j+1}\cdots b_s b_1 \cdots b_{j-1} a_{i+1}\cdots a_r a_1 \cdots a_{i-1} a_i) \\
& + \sum_{(i,j):a_i=B,b_j=B} \mathrm{tr}\ (a_i a_{i+1}\cdots a_r a_1 \cdots a_{i-1} b_{j+1}\cdots b_s b_1 \cdots b_{j-1}) \\
& - \sum_{(i,j):a_i=B,b_j=B} \mathrm{tr}\ (b_j b_{j+1}\cdots b_s b_1 \cdots b_{j-1} a_{i+1}\cdots a_r a_1 \cdots a_{i-1})\ .
\end{aligned}
\tag{12}
$$

Now let us pass from the (n^2+1)-dimensional quotient $\mathcal{M}/G \simeq \mathcal{P}$ to the phase space of the Calogero-Moser system.

5 The second projection

In this section we will perform a second reduction of the bi-Hamiltonian structure on $\mathcal{M}/G \simeq \mathcal{P}$. The starting point is the observation that the invariant functions

$$
I_k(A,B) = \frac{1}{k}\mathrm{tr}\, A^k = H_k(A,B)\ , \quad J_k(A,B) = \mathrm{tr}\,(A^{k-1}B)\ , \quad \text{for } k=1,\ldots,n,
$$

form a Poisson subalgebra with respect to both Poisson brackets. Indeed, by direct computation or using the necklace bracket formulas (11-12), one finds that

$$
\begin{aligned}
&\{I_k, I_l\}_0 = 0\ , \quad \{J_l, I_k\}_0 = (k+l-2) I_{k+l-2}\ , \quad \{J_k, J_l\}_0 = (l-k) J_{k+l-2}\ , \\
&\{I_k, I_l\}_1 = 0\ , \quad \{J_l, I_k\}_1 = (k+l-1) I_{k+l-1}\ , \quad \{J_k, J_l\}_1 = (l-k) J_{k+l-1}\ ,
\end{aligned}
\tag{13}
$$

with the exception that $\{J_1, I_1\}_0 = n$. In any case, the Cayley-Hamilton theorem implies that all the right-hand sides of (13) can be written in terms of $I_1, \ldots, I_n, J_1, \ldots, J_n$. Thus both Poisson brackets can be further projected on the quotient space defined by the map $\pi : \mathcal{M}/G \simeq \mathcal{P} \to \mathbb{R}^{2n}$ whose components are the functions $I_1, \ldots, I_n, J_1, \ldots, J_n$.

Remark 7 The Poisson brackets (13) appeared in [16] (see also [15]), in the context of the repulsive Calogero-Moser system. In that paper the construction of the bi-Hamiltonian structure heavily relies on the Lax representation of the system. On the contrary, here we will show that equations (10) imply the existence of such a representation.

Proposition 8 *The map π is a submersion, i.e., its differential is surjective at every point of $\mathcal{M}/G$.*

Proof. It is convenient to identify $\mathcal{M}/G$ with $\mathcal{P}'$ and to consider, among the coordinates on $\mathcal{P}'$, the diagonal entries $(\lambda_1, \ldots, \lambda_n)$ of the (diagonal) matrix A and the diagonal entries $(\mu_1, \ldots, \mu_n)$ of B. Then

$$I_k = \frac{1}{k} \sum_{l=1}^{n} {\lambda_l}^k , \qquad J_k = \sum_{l=1}^{n} \mu_l {\lambda_l}^{k-1} ,$$

which implies that

$$\det \begin{pmatrix} \frac{\partial I}{\partial \lambda} & \frac{\partial J}{\partial \lambda} \\ \frac{\partial I}{\partial \mu} & \frac{\partial J}{\partial \mu} \end{pmatrix} \neq 0$$

since the λ_i are distinct. This shows that the differential of π is surjective.

QED

The previous proposition entails that the image of π is an open subset $U \subset \mathbb{R}^{2n}$, which is in 1-1 correspondence with the second quotient space (by its very definition). Our final step is to prove that the projection on U gives rise to the phase space of the attractive Calogero-Moser system, with its bi-Hamiltonian flows. To do this, we recall once more that the (first) quotient $\mathcal{M}/G$ can be identified with the submanifold $\mathcal{P} \subset \mathcal{M}$ and we restrict to its connected component $\mathcal{P}_{(-,\ldots,-)}$, that is, the set of pairs $(A, B) \in \mathcal{M}$ such that B is diagonal with $B_{ii} < B_{jj}$ if $i < j$, and $A_{i+1,i} > 0$, $A_{i,i+1} = -A_{i+1,i}$ for all $i = 1, \ldots, n-1$. Then we introduce a submanifold $\mathcal{Q} \subset \mathcal{P}_{(-,\ldots,-)}$ which will be shown to be in 1-1 correspondence with an open subset of the second quotient space. The elements of $\mathcal{Q}$ are the pairs $(L, \mathrm{diag}(x_1, \dots, x_n)) \in \mathcal{P}$ such that $x_i < x_j$ if $i < j$, and $L_{ij} = \frac{1}{x_i - x_j}$ if $i \neq j$. If we put $L_{ii} = y_i$, we obtain the Lax matrix of the attractive Calogero-Moser system:

$$L = \begin{pmatrix} y_1 & \frac{1}{x_1-x_2} & \cdots & \frac{1}{x_1-x_n} \\ \frac{1}{x_2-x_1} & y_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{x_n-x_1} & \frac{1}{x_n-x_2} & \cdots & y_n \end{pmatrix} .$$

Let us also introduce the submanifold $\mathcal{Q}' \subset \mathcal{P}'$ whose elements are the pairs $(\mathrm{diag}(\lambda_1, \dots, \lambda_n), L') \in \mathcal{P}'$ such that $\lambda_i < \lambda_j$ if $i < j$, and $L'_{ij} = \frac{1}{\lambda_j - \lambda_i}$ if $i \neq j$. In order to identify $\mathcal{Q}$ with a subset of the second quotient space, we need the following result. It is a restatement of Proposition 2.6 in [5], but we give its proof for the reader's sake.

Proposition 9 *If $\rho : \mathcal{M} \to \mathcal{M}/G$ is the canonical projection, then $\rho(\mathcal{Q})$ coincides with $\rho(\mathcal{Q}')$, and is formed by the orbits of the pairs (A, B) such that the rank of $[B, A] + I$ is 1.*

Proof. We notice that

$$\mathcal{Q} = \{(A, B) \in \mathcal{P} \mid [B, A] = \mu\} \text{ and } \mathcal{Q}' = \{(A, B) \in \mathcal{P}' \mid [B, A] = \mu\} ,$$

where $\mu_{ij} = 1 - \delta_{ij}$. Thus, the elements (A, B) in the orbits passing through $\mathcal{Q}$ and $\mathcal{Q}'$ satisfy the condition $\mathrm{rank}([B, A] + I) = 1$. Conversely, let us suppose that $(A, B) \in \mathcal{M}$ and the above condition holds. We can also suppose that B has already been diagonalized. Since the rank of $K := [B, A] + I$ is 1, there exist $a_i, b_i \in \mathbb{R}$, $i = 1, \dots, n$, such that $K_{ij} = a_i b_j$. From $[B, A]_{ij} = (B_{ii} - B_{jj}) A_{ij}$ we have that $K_{ii} = 1$ and therefore $b_i = {a_i}^{-1}$. By acting with $\mathrm{diag}(a_1, \dots, a_n)$, the entries of the matrix K all become 1 and so (A, B) is mapped into $\mathcal{Q}$. This shows that the orbits in $\rho(\mathcal{Q})$ are precisely those of the pairs (A, B) such that $\mathrm{rank}([B, A] + I) = 1$. Diagonalizing A instead of B, one proves that the same is true for $\rho(\mathcal{Q}')$.

QED

Corollary 10 *The restriction to $\mathcal{Q}$ of the map $\pi = (I_1, \dots, I_n, J_1, \dots, J_n)$ is injective.*

Proof. Since π is an invariant map, we can exploit the identification between $\mathcal{Q}$ and $\mathcal{Q}'$ given by the previous proposition and show that π is injective on $\mathcal{Q}'$. If $(D', L') \in \mathcal{Q}'$, with

$$D' = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \ , \qquad L' = \begin{pmatrix} \mu_1 & \frac{1}{\lambda_2-\lambda_1} & \cdots & \frac{1}{\lambda_n-\lambda_1} \\ \frac{1}{\lambda_1-\lambda_2} & \mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{\lambda_1-\lambda_n} & \frac{1}{\lambda_2-\lambda_n} & \cdots & \mu_n \end{pmatrix} ,$$

then

$$I_k(D', L') = \frac{1}{k} \sum_{l=1}^{n} {\lambda_l}^k \ , \qquad J_k(D', L') = \sum_{l=1}^{n} \mu_l {\lambda_l}^{k-1} \ .$$

To conclude that π is injective, we simply have to recall that $\lambda_i < \lambda_j$ if $i < j$.

QED

Remark 11 From the proof of Proposition 9 it is also clear that

$$\rho(\mathcal{Q}) = \rho(\mathcal{Q}') = \{\text{orbits of the pairs } (A, B) \text{ such that } [B, A] = \mu\}.$$

We have thus shown that an open subset of the quotient defined by the map π can be identified with the submanifold $\mathcal{Q}$, that is, the phase space of the Calogero-Moser system.

Now we consider the (projected) bi-Hamiltonian hierarchy on $\mathcal{P}_{(-,\ldots,-)} \subset \mathcal{M}/G$. Since the Hamiltonians are just the functions $H_k = I_k$, this hierarchy further projects on the second quotient space. In particular, it gives rise to bi-Hamiltonian vector fields on $\mathcal{Q}$, which we will soon see to be those of the attractive Calogero-Moser system. In principle, to write these vector fields we should project the flows (10), as we did after Proposition 3. But they are already tangent to $\mathcal{Q}$, as shown in

Proposition 12 *Let $(A, B) \in \mathcal{Q}$ and let $(\partial_k A, \partial_k B) \in T_{(A,B)}\mathcal{P}$ be given by (10), that is,*

$$\partial_k A = [\xi_k, A] \ , \qquad \partial_k B = [\xi_k, B] + A^{k-1} \ .$$

Then $(\partial_k A, \partial_k B) \in T_{(A,B)}\mathcal{Q}$ for all $k \geq 1$.

Proof. We know that

$$\mathcal{Q} = \{(A, B) \in \mathcal{P} \mid [B, A] = \mu\} \ ,$$

where $\mu_{ij} = 1 - \delta_{ij}$. This entails that $(\partial_k A, \partial_k B) \in T_{(A,B)}\mathcal{Q}$ if and only if $[\partial_k A, B] + [A, \partial_k B] = 0$ at the points of $\mathcal{Q}$. But this is equivalent to the assertion that $[\xi_k, \mu] = 0$. Let us prove this fact, introducing a matrix ξ such that $\xi_{ij} = (\xi_k)_{ij} = \left(A^{k-1}\right)_{ij}/(x_i - x_j)$ for $i \neq j$, and

$$\xi_{ii} = -\frac{1}{2}\sum_{l \neq i}(\xi_{il} + \xi_{li}) \ . \tag{14}$$

Since B is diagonal, we have that

$$\partial_k B = [\xi_k, B] + A^{k-1} = [\xi, B] + A^{k-1} \ . \tag{15}$$

At the end, it will turn out that ξ is a possible choice for ξ_k (recall from the proof of Proposition 3 that ξ_k is determined up to a multiple of the identity matrix). Now let us show that $[\xi, \mu] = 0$. Indeed,

$$[\xi, \mu] = [\xi, [B, A]] = [A, [B, \xi]] + [B, [\xi, A]] = -[A, \partial_k B] + [B, [\xi, A]] \ ,$$

so that, putting $\partial_k B = \mathrm{diag}(\dot{x}_1, \ldots, \dot{x}_n)$, we have

$$[\xi, \mu]_{ij} = (\dot{x}_i - \dot{x}_j)A_{ij} + (x_i - x_j)[\xi, A]_{ij} \ . \tag{16}$$

Thus $[\xi, \mu]_{ii} = 0$ for all $i = 1, \ldots, n$. Moreover, (14) implies that, for $i \neq j$,

$$[\xi, \mu]_{ij} = \frac{1}{2}\sum_{l}(\xi_{il} - \xi_{li}) + \frac{1}{2}\sum_{m}(\xi_{jm} - \xi_{mj}) = [\xi, \mu]_{ii} + [\xi, \mu]_{jj} \ .$$

Therefore we have that $[\xi, \mu] = 0$. In order to finish the proof, we have to show that ξ is a possible choice for ξ_k, i.e., that $[\xi, A]_{i,i+1} = -[\xi, A]_{i+1,i}$ for all $i = 1, \ldots, n-1$. But from (16), the vanishing of $[\xi, \mu]$, and the fact that $A_{ij} = -A_{ji}$ for $i \neq j$, we have that

$$0 = (x_i - x_j)\left([\xi, A]_{ij} + [\xi, A]_{ji}\right)$$

and therefore $[\xi, A]_{ij} = -[\xi, A]_{ji}$.

QED

As it is well-known, the Calogero-Moser system is given by the second flow $\partial_2 A = [\xi_2, A]$, where $(\xi_2)_{ij} = 1/(x_i - x_j)^2$ for $i \neq j$ and $(\xi_2)_{ii} = -\sum_{j\neq i}(\xi_2)_{ij}$. In general ξ_k is not symmetric for $k > 2$.

Remark 13 In the repulsive case, corresponding to the connected component $\mathcal{P}_{(+,\ldots,+)} \subset \mathcal{M}/G$, one can still introduce the submanifold

$$\mathcal{Q}_+ = \left\{ (L, \mathrm{diag}(x_1, \ldots, x_n)) \mid x_i < x_j \text{ and } L_{ij} = L_{ji} = \frac{1}{x_i - x_j} \text{ if } i < j \right\}$$

and show that it can be identified with (an open subset of) the second quotient space. However, if $n > 2$ the flows (10) are not tangent to $\mathcal{Q}_+$ and therefore they need to be projected on $\mathcal{Q}_+$, where they assume a more complicated form.

Remark 14 It is well-known [13, 5, 21] that the first Poisson structure P_0 can be reduced on the phase space of the Calogero-Moser system by means of the Marsden-Weinstein reduction and that it gives rise to the canonical structure in the coordinates (x_i, y_j). Our reduction consists in a double projection, and has the same effect on P_0. However, we can reduce also the second Poisson structure P_1, on which the Marsden-Weinstein reduction cannot be performed, since it employs the moment map $(A, B) \to [A, B]$ of P_0. Notice that this moment map appears in the proof of Proposition 9.

Remark 15 We have used two projections to obtain the bi-Hamiltonian structure of the Calogero-Moser system from the one on $T^*\mathfrak{gl}(n)$. Of course, such projections can be composed and we could have found directly the Poisson pair on $\mathcal{Q}$. We decided to study also the first quotient $\mathcal{M}/G$ because it is more natural and we think that it might be of interest on its own.

6 Example: $n = 2$

In this section we consider the 2-particle Calogero-Moser system in order to exemplify our construction. The starting point is the set $\mathcal{M}$ whose elements are pairs (A, B) of matrices in $\mathfrak{gl}(2, \mathbb{R})$ such that A and B have real distinct eigenvalues and no common eigenvector. The first quotient space $\mathcal{M}/G$ can be identified with the 5-dimensional manifold $\mathcal{P} = \mathcal{P}_+ \cup \mathcal{P}_-$ (see Section 4) or with $\mathcal{P}' = \mathcal{P}'_+ \cup \mathcal{P}'_-$, where the elements of $\mathcal{P}'_+$ are those in $\mathcal{M}$ of the form

$$\left(\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix} \right) ,$$

with $A_{11} < A_{22}$ and $B_{12} > 0$, while the elements of $\mathcal{P}'_-$ are those in $\mathcal{M}$ of the form

$$\left(\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \begin{pmatrix} B_{11} & B_{12} \\ -B_{12} & B_{22} \end{pmatrix} \right) ,$$

again with $A_{11} < A_{22}$ and $B_{12} > 0$. There are only two independent vector fields X_1 and X_2 in the bi-Hamiltonian hierarchy on $\mathcal{M}$, corresponding to the invariant functions $H_1 = \operatorname{tr} A$ and $H_2 = \frac{1}{2}\operatorname{tr} A^2$. The equations on $\mathcal{M}/G \simeq \mathcal{P}$ are given by (10), with $\xi_1 = I$ and

$$\xi_2 = \begin{pmatrix} 0 & \frac{A_{21}}{B_{22}-B_{11}} \\ \frac{A_{21}}{B_{22}-B_{11}} & 0 \end{pmatrix} \text{ on } \mathcal{P}_-, \quad \xi_2 = \begin{pmatrix} 0 & -\frac{A_{21}}{B_{22}-B_{11}} \\ \frac{A_{21}}{B_{22}-B_{11}} & 0 \end{pmatrix} \text{ on } \mathcal{P}_+.$$

The second projection $\pi : \mathcal{P} \to \mathbb{R}^4$ is given by $\pi = (I_1, I_2, J_1, J_2)$, where

$$I_1 = H_1 = \operatorname{tr} A, \quad I_2 = H_2 = \frac{1}{2}\operatorname{tr} A^2, \quad J_1 = \operatorname{tr} B, \quad J_2 = \operatorname{tr}(AB) .$$

One can check that the image of π is the set

$$U = \left\{ (I_1, I_2, J_1, J_2) \in \mathbb{R}^4 \text{ s.t. } 4I_2 - {I_1}^2 > 0, J_2 - \frac{1}{2} I_1 J_1 \neq 0 \right\} .$$

The restriction of π to

$$\mathcal{Q} = \left\{ \left(\begin{pmatrix} y_1 & \frac{1}{x_1-x_2} \\ \frac{1}{x_2-x_1} & y_2 \end{pmatrix}, \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \right) \text{ s.t. } x_1 < x_2, |y_1 - y_2|(x_2 - x_1) > 2 \right\}$$

is a bijection onto

$$V = \left\{ (I_1, I_2, J_1, J_2) \in \mathbb{R}^4 \text{ s.t. } 4I_2 - {I_1}^2 > 0, |J_2 - \frac{1}{2} I_1 J_1| > 1 \right\} .$$

The Poisson brackets on $\mathcal{Q}$ are given by

$$
\begin{aligned}
&\{I_1, I_2\}_0 = 0, \quad \{J_1, I_1\}_0 = 2, \quad \{J_1, I_2\}_0 = \{J_2, I_1\}_0 = I_1,\\
&\quad \{J_2, I_2\}_0 = 2I_2, \quad \{J_1, J_2\}_0 = J_1,\\
&\{I_1, I_2\}_1 = 0, \quad \{J_1, I_1\}_1 = I_1, \quad \{J_1, I_2\}_1 = \{J_2, I_1\}_1 = 2I_2,\\
&\quad \{J_2, I_2\}_1 = 3I_3 = 3I_1 I_2 - \tfrac{1}{2}{I_1}^3, \quad \{J_1, J_2\}_1 = J_2.
\end{aligned}
\tag{17}
$$

In terms of the physical coordinates (x_1, x_2, y_1, y_2), the first bracket is the canonical one, while

$$
\begin{aligned}
&\{x_1, x_2\}_1 = \tfrac{2x_{12}}{\Delta}, \quad \{x_1, y_1\}_1 = y_1 + \tfrac{(y_1-y_2){x_{12}}^2}{\Delta}, \qquad \{x_2, y_2\}_1 = y_2 - \tfrac{(y_1-y_2){x_{12}}^2}{\Delta},\\
&\{y_2, x_1\}_1 = \{x_2, y_1\}_1 = \tfrac{(y_1-y_2){x_{12}}^2}{\Delta}, \quad \{y_1, y_2\}_1 = -{x_{12}}^3.
\end{aligned}
\tag{18}
$$

where $x_{12} = 1/(x_1 - x_2)$ and $\Delta = 4{x_{12}}^2 - (y_1 - y_2)^2$.

7 Final remarks

In this paper we have shown that the Poisson pair of the (rational, attractive) Calogero-Moser system is a reduction of a very natural bi-Hamiltonian structure on $T^*\mathfrak{gl}(n, \mathbb{R})$. A first possible development of this result is to extend this construction to other Calogero-Moser systems, such as the trigonometric one (see also Remark 2) and those associated to (root systems of) simple Lie algebras [18]. Secondly, it would be interesting to investigate, from the point of view of bi-Hamiltonian geometry, the problem of duality between Calogero-Moser systems [19, 8] and, more generally, between integrable systems. In particular, on $\mathcal{M}$ there is another bi-Hamiltonian structure, obtained by exchanging A with B. (In other words, one can look at B as the "point" in $\mathfrak{gl}(n, \mathbb{R})$ and A as the "covector" in $T^*_B\mathfrak{gl}(n, \mathbb{R}) \simeq \mathfrak{gl}(n, \mathbb{R})$, and consider the (1,1) tensor field $V \mapsto BV$ on $\mathfrak{gl}(n, \mathbb{R})$.) The functions $H'_k = \frac{1}{k}\operatorname{tr} B^k$, for $k \geq 1$, form a bi-Hamiltonian hierarchy with respect to the new Poisson pair. Being invariant with respect to the action of G, such pair can be projected on the (first) quotient space $\mathcal{M}/G$, along with its hierarchy. However, they cannot be projected on the second quotient space, but one has to introduce the map $\pi' : \mathcal{M}/G \to \mathbb{R}^{2n}$ whose components are the functions

$$
I'_k(A, B) = \frac{1}{k}\operatorname{tr} B^k = H'_k(A, B)\,, \quad J'_k(A, B) = \operatorname{tr}(AB^{k-1})\,, \quad \text{for } k = 1, \ldots, n.
$$

The projection along π' gives rise again to the rational Calogero-Moser system, which is indeed well-known to be dual to itself. From the bi-Hamiltonian viewpoint, it is important to observe that the map $(A, B) \mapsto (-B, A)$ sends the Poisson pair (P_0, P_1) into the new Poisson pair.

References

[1] C. Bartocci, G. Falqui, M. Pedroni, *A geometric approach to the separability of the Neumann-Rosochatius system*, Diff. Geom. Appl. **21** (2004), 349–360, nlin.SI/0307021.

[2] F. Calogero, *Solution of a three-body problem in one dimension*, J. Math. Phys. **10** (1969), 2191–2196.

[3] F. Calogero, *Solution of the one-dimensional N-body problem with quadratic and/or inversely quadratic pair potentials*, J. Math. Phys. **12** (1971), 419–436 ("Erratum", ibidem **37** (1996), 3646).

[4] F. Calogero, *Calogero-Moser system*, Scholarpedia, 3(8):7216 (2008). Available at http://www.scholarpedia.org/article/Calogero-Moser_system

[5] P. Etingof, *Lectures on Calogero-Moser systems*, math.QA/0606233.

[6] P. Etingof, V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), 243–348.

[7] G. Falqui, M. Pedroni, *Separation of variables for bi-Hamiltonian systems*, Math. Phys. Anal. Geom. **6** (2003), 139–179.

[8] V. Fock, A. Gorsky, N. Nekrasov, V. Rubtsov *Duality in integrable systems and gauge theories*, J. High Energy Phys. **2000**, no. 7, Paper 28, 40 pp.

[9] V. Ginzburg, *Non-commutative symplectic geometry, quiver varieties, and operads*, Math. Res. Lett. **8** (2001), 377–400.

[10] C. Gonera, Y. Nutku, *Super-integrable Calogero-type systems admit maximal number of Poisson structures*, Phys. Lett. A **285** (2001), 301–306.

[11] A. Ibort, F. Magri, G. Marmo, *Bihamiltonian structures and Stäckel separability*, J. Geom. Phys. **33** (2000), 210–228.

[12] C. Jacobi, *Problema trium corporum mutuis attractionibus cubis distantiarium inverse proportionalibus recta linea se moventium*, Gesammelte Werke, Berlin, 4 (1866), 533–539.

[13] D. Kazhdan, B. Kostant, S. Sternberg, *Hamiltonian group actions and dynamical systems of Calogero type*, Comm. Pure Appl. Math. **31** (1978), 481–507.

[14] L. Le Bruyn, *Noncommutative geometry and Cayley-smooth orders*, Chapman & Hall/CRC, Boca Raton, 2008.

[15] F. Magri, P. Casati, G. Falqui, M. Pedroni, *Eight lectures on Integrable Systems*, In: Integrability of Nonlinear Systems (Y. Kosmann-Schwarzbach et al. eds.), Lecture Notes in Physics **495** (2nd edition), 2004, pp. 209-250.

[16] F. Magri, T. Marsico, *Some developments of the concept of Poisson manifold in the sense of A. Lichnerowicz*, in: Gravitation, Electromagnetism, and Geometric Structures (G. Ferrarese, ed.), Pitagora editrice, Bologna, 1996, pp. 207–222.

[17] J. Moser, *Three integrable Hamiltonian systems connected with isospectral deformations*, Adv. Math. **16** (1975), 197–220.

[18] A. M. Perelomov, *Integrable systems of classical mechanics and Lie algebras. Vol. I*, Birkhäuser, Basel, 1990.

[19] S. Ruijsenaars, *Action-angle maps and scattering theory for some finite-dimensional integrable systems. III. Sutherland type systems and their duals*, Publ. Res. Inst. Math. Sci. **31** (1995), 247–353.

[20] F. Turiel, *Structures bihamiltoniennes sur le fibré cotangent*, C. R. Acad. Sci. Paris SérI Math. **315** (1992), 1085–1088.

[21] G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian*, Invent. Math. **133** (1998), 1–41.

[22] S. Wojciechowski, *Superintegrability of the Calogero-Moser systems*, Phys. Lett. A **95** (1983), 279–281.

[23] K. Yano, S. Ishihara, *Tangent and cotangent bundles: differential geometry*, Marcel Dekker, New York, 1973.