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A note on Keller-Osserman conditions on Carnot groups

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Abstract

This paper deals with the study of differential inequalities with gradient terms on Carnot groups. We are mainly focused on inequalities of the form $\Delta_\varphi u \geq f(u)l(|\nabla_0 u|)$, where f, l and φ are continuous functions satisfying suitable monotonicity assumptions and Δ_φ is the φ-Laplace operator, a natural generalization of the p-Laplace operator which has recently been studied in the context of Carnot groups. We extend to general Carnot groups the results proved in [9] for the Heisenberg group, showing the validity of Liouville-type theorems under a suitable Keller-Osserman condition. In doing so, we also prove a maximum principle for inequality $\Delta_\varphi u \geq f(u)l(|\nabla_0 u|)$. Finally, we show sharpness of our results for a general φ-Laplacian.

Keywords: Keller-Osserman; Carnot groups; differential inequalities; maximum principle.

Mathematics Subject Classification (2010): 35R03; 35B53.

1 Introduction

Let G be a homogeneous Carnot group on $\mathbb{R}^N$, that is a Lie group with underlying manifold $\mathbb{R}^N$, equipped with a family of automorphisms $\{\delta_\lambda\}_{\lambda>0}$, called dilations, of the form

$$\delta_\lambda\Big(x^{(1)}, x^{(2)}, \dots, x^{(r)}\Big) = \Big(\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)}\Big),$$

where $x^{(i)} \in \mathbb{R}^{m_i}$ and $m_1 + \ldots + m_r = N$, and such that the Lie algebra of G is generated by the m_1 left-invariant vector fields $X_1, \dots, X_{m_1}$ that agree with $\partial/\partial x_i^{(1)}$ at the origin.
The vector fields $\{X_1, \dots, X_{m_1}\}$ are homogeneous of degree 1 with respect to

the dilations δ_λ and their linear span is called the horizontal layer of the algebra of G.
The canonical sub-Laplacian on G is the differential operator

$$\Delta_G = \sum_{i=1}^{m_1} X_i^2,$$

which is hypoelliptic by Hörmander's theorem (see [7]). We refer the interested reader to [4] for a detailed introduction to Carnot groups and sub-Laplacians.
If Γ denotes the fundamental solution of the sub-Laplacian and Q is the homogeneous dimension of G, defined as $Q = m_1 + 2m_2 + \ldots + rm_r$, then the function

$$d = \Gamma^{\frac{1}{2-Q}}$$

is continuous and smooth out of the origin and it is a symmetric homogeneous norm, i.e. $d(\delta_\lambda(x)) = \lambda d(x)$, $d(x) > 0$ iff $x \neq 0$ and $d(x^{-1}) = d(x)$.
Setting $d(x,y) = d(y^{-1} \cdot x)$, one can verify that $d(x,y) = d(y,x)$, $d(x,y) = 0$ iff $x = y$ and that a pseudo-triangle inequality is satisfied:

$$\exists c > 0 : \quad d(x,y) \leq c[d(x,z) + d(z,y)] \quad \forall x,y,z \in G. \tag{1}$$

For $u \in C^1(G)$, we define the horizontal gradient $\nabla_0 u$ as the horizontal vector field

$$\nabla_0 u = \sum_{i=1}^{m_1} (X_i u) X_i.$$

For horizontal vector fields $Y = \sum y^i X_i$ and $W = \sum w^i X_i$, we can define

$$Y \cdot W = \sum_{i=1}^{m_1} y^i w^i,$$

so that by definition $|\nabla_0 u|^2 = \nabla_0 u \cdot \nabla_0 u$ and the Cauchy-Schwarz inequality holds. In particular, out of the origin we can consider the function $|\nabla_0 d|$, which is homogeneous of degree zero, and therefore bounded. Without loss of generality, up to rescaling the homogeneous norm by a constant, we can assume that $0 \leq |\nabla_0 d| \leq 1$. Finally, the horizontal divergence is defined, for horizontal vector fields as

$$\operatorname{div}_0 W = \sum_{i=1}^{m_1} X_i(w^i),$$

so that

$$\Delta_G u = \operatorname{div}_0 \nabla_0 u.$$

In recent years, the p-Laplace operator, a generalization of the sub-Laplacian defined, for $p \geq 2$, by

$$\Delta_p u = \mathrm{div}_0(|\nabla_0 u|^{p-2}\nabla_0 u),$$

has been studied by many authors in the setting of Carnot groups (see, for instance [6], [3], [2], [1]). In this paper, we consider a further generalization of the p-Laplacian called the φ-Laplace operator and defined as follows:

$$\Delta_\varphi u = \mathrm{div}_0\left(|\nabla_0 u|^{-1}\varphi(|\nabla_0 u|)\nabla_0 u\right),$$

where φ satisfies the structural conditions

$$\begin{cases} \varphi \in C^0(\mathbb{R}_0^+) \cap C^1(\mathbb{R}^+), \quad \varphi(0) = 0, \\ \quad \varphi' > 0 \qquad \text{on } \mathbb{R}^+. \end{cases} \tag{2}$$

This operator, which includes all the p-Laplacians, has been recently studied in the context of Riemannian geometry and Carnot groups (see [9], [11] and [5] and references therein).

In [9] the authors studied the existence of weak classical solutions of the differential inequality

$$\Delta_\varphi u \geq f(u)l(|\nabla_0 u|) \tag{3}$$

on the Heisenberg group and on $\mathbb{R}^n$, under suitable assumptions on f, l and φ. They introduced a generalized Keller-Osserman condition which ensures that (3) has no non-negative entire solutions. Moreover, they show that, in the special case of the p-Laplace operator, the Keller-Osserman condition is also necessary: when it is not satisfied, non-constant positive solutions of $\Delta_p u \geq f(u)l(|\nabla_0 u|)$ do, in fact, exist.

In this paper we extend the results introduced in [9] to every Carnot group under suitable assumptions on φ and, when the Keller-Osserman condition is not met, we also prove the existence of solutions for general φ-Laplacians.

We point out that, in dealing with this kind of problems in the more general setting of Carnot groups, while the framework of the proofs remains the same, some technical difficulties arise. As we shall see in the next section, most of these concern the radialization of the φ-Laplacian, whose expression is considerably more complicated than on the Heisenberg group. To overcome this difficulty we were forced to add one assumption on φ that unfortunately has the effect of making the φ-Laplacian close to a p-Laplacian. One of the main tools that we exploit in this paper is a maximum principle for (3), which, to the best of our knowledge, seems to be new and of independent interest and whose statement and proof had to be modified from their equivalents on the Heisenberg group,

as we shall see in Section 3.
Next, we introduce some notation and assumptions.
In this paper, we will consider weak classical solutions of (3), that is functions $u \in C^1(G)$ such that, for every $\zeta \in C_0^\infty(G)$, $\zeta \geq 0$,

$$-\int_{\mathbb{R}^N} |\nabla_0 u|^{-1}\varphi(|\nabla_0 u|)\nabla_0 u \cdot \nabla_0 \zeta \geq \int_{\mathbb{R}^N} f(u) l(|\nabla_0 u|)\zeta. \tag{4}$$

Our assumptions on f and l will be the following:

$$\begin{cases} f \in C^0(\mathbb{R}_0^+), \qquad f > 0 \text{ on } \mathbb{R}^+, \\ f \text{ is increasing on } \mathbb{R}_0^+; \end{cases} \tag{5}$$

$$\begin{cases} l \in C^0(\mathbb{R}_0^+), \quad l > 0 \text{ on } \mathbb{R}^+, \quad l(0) > 0, \\ l \text{ is } B\text{-monotone non-decreasing on } \mathbb{R}_0^+. \end{cases} \tag{6}$$

We recall that l is said to be *B-monotone non decreasing* on $\mathbb{R}_0^+$ if, for some $B \geq 1$,

$$\sup_{s\in[0,\,t]} l(s) \leq Bl(t), \qquad \forall\, t \in \mathbb{R}_0^+.$$

Clearly, if l is monotone non decreasing on $\mathbb{R}_0^+$, then it is 1-monotone non-decreasing on the same set; in fact the above condition allows a controlled oscillatory behavior of l on $\mathbb{R}_0^+$.
In order to be able to state the generalized Keller-Osserman condition, we also need to assume that

$$\frac{t\varphi'(t)}{l(t)} \notin L^1(+\infty). \tag{7}$$

We set

$$K(t) = \int_0^t \frac{s\varphi'(s)}{l(s)}\,\mathrm{d}s; \tag{8}$$

and observe that K is well defined since $l(0) > 0$. We also observe that $K : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a C^1-diffeomorphism with

$$K'(t) = \frac{t\varphi'(t)}{l(t)} > 0,$$

so that its inverse $K^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ exists and is also increasing. Finally we set

$$F(t) = \int_0^t f(s)\,\mathrm{d}s.$$

Definition 1.1. *The* **generalized Keller-Osserman condition** *for inequal-*

ity

$$\Delta_\varphi u \geq f(u)l(|\nabla_0 u|)$$

is the request:

$$\frac{1}{K^{-1}(F(t))} \in L^1(+\infty). \tag{KO}$$

This generalized Keller-Osserman condition was first introduced in [9] and, when $\varphi(t) = t$ and $l \equiv 1$, coincides with the classical Keller-Osserman condition as seen in [10] and [8].
In order to deal with the problems of radialization, we need to request the following conditions on φ and l:

$$\begin{array}{ll} (i) & s\varphi'(st) \leq Cs^\tau \varphi'(t) \\ (ii) & s^{\tau-1}l(t) \leq \Lambda l(st) \\ (iii) & t\varphi'(t) \leq C_1\varphi(t) \end{array} \tag{9}$$

for some constants $C, \Lambda, C_1 > 1$ and $\tau \geq 0$ and for every $s \in [0,1]$ and $t \in \mathbb{R}_0^+$.
We point out that condition (i) of (9) implies

$$\varphi(st) \leq Ds^\tau \varphi(t), \quad \forall t \in \mathbb{R}_0^+, \ s \in [0,1], \tag{10}$$

which will come in handy later on; we also remark that conditions (i) and (iii) of (9) imply that

$$at^p \leq \varphi(t) \leq bt^p$$

for some constants $a, b > 0$ and $p > 0$ and for every $t \in \mathbb{R}_0^+$.
We stress that (9) (ii) is a mild requirement: for example, it is satisfied by every $l(t)$ of the form

$$l(t) = \sum_{k=0}^{n} C_k t^{\nu_k}, \quad n \in \mathbb{N}, \quad C_k \geq 0, \quad \nu_k \in (-\infty, \tau - 1] \text{ for every } k.$$

The main results we are going to prove in this paper can be summerized in the following statement:

Theorem 5.2. *Assume the validity of* (2)*,* (5)*,* (6)*,* (9) *and of* (7)*. Then, the following are equivalent:*

(i) *there exists a non-negative, non-constant solution $u \in C^1(G)$ of inequality $\Delta_\varphi u \geq f(u)l(|\nabla_0 u|)$;*

(ii) $\dfrac{1}{K^{-1}(F(t))} \notin L^1(+\infty)$.

We observe that, as it will become apparent from the proof, several assumptions can be dropped if we only consider the implication $(ii) \Rightarrow (i)$. In this case we prove the existence of solutions in any Carnot group for any φ-Laplacian. Moreover, condition (iii) of (9) is unnecessary on every group of Heisenberg type and, more generally, in the class of polarizable group, which we shall discuss later on.

2 Radial supersolutions

The core of the proofs of the non-existence theorem and of the maximum principle relies on being able to find suitable supersolutions of (3) with certain properties. This is achieved by considering the expression of the φ-Laplacian of functions which are radial with respect to the homogeneous norm d, i.e. functions $v(x) = \alpha(d(x))$.
Keeping in mind the definition of the φ-Laplacian and the properties of the horizontal divergence, such as the following

$$\operatorname{div}_0(fW) = f \operatorname{div}_0 W + \nabla_0 f \cdot W,$$

together with the fact that

$$\Delta_G d = |\nabla_0 d|^2 \frac{Q-1}{d}$$

(see e.g. [4]), some computation yields the following expression for the φ-Laplacian of a radial function v:

$$\begin{aligned} \Delta_\varphi v =& \varphi'(\alpha'|\nabla_0 d|)\alpha''|\nabla_0 d|^2 + \varphi(\alpha'|\nabla_0 d|)|\nabla_0 d|\frac{Q-1}{d} \\ &+ \left[\varphi'(\alpha'|\nabla_0 d|)\alpha' - \frac{\varphi(\alpha'|\nabla_0 d|)}{|\nabla_0 d|}\right]\nabla_0|\nabla_0 d| \cdot \frac{\nabla_0 d}{|\nabla_0 d|}, \end{aligned} \tag{11}$$

where, for ease of notation, we have assumed α increasing. As we shall see, this is not restrictive for our purposes.

Remark 2.1. *The last term in* (11) *does not appear on* $\mathbb{R}^n$ *and on the Heisenberg group, where the homogeneous norm satisfies*

$$\nabla_0|\nabla_0 d| \cdot \nabla_0 d = 0. \tag{12}$$

This is what makes treating radial functions more complicated on general Carnot groups; it is also the reason why we need to assume hypothesis (9) (iii)*.*
Carnot groups where (12) *holds are called polarizable groups and have been studied in [3], where the authors proved that every group of Heisenberg type is polarizable.*

Because of this, every theorem in this paper can be restated on polarizable groups without assuming condition (9) *(iii).*

Lemma 2.2. *Let $\sigma \in (0,1]$; then the generalized Keller-Osserman condition (KO) implies*

$$\frac{1}{K^{-1}(\sigma F(t))} \in L^1(+\infty). \tag{13}$$

The proof of this lemma is achieved through a change of variable. For the details, we refer the reader to [9].
We pass now to the construction of radial supersolutions of (3), the first of which will be used in the proof of the maximum principle (Theorem 3.2).

Proposition 2.3. *Assume the validity of* (2), (5), (6), (7) *and* (9) *and fix $q \in G$, $0 < t_0 < t_1$, $0 < h < k$. Then there exist $\sigma > 0$ and a radial function $v = \alpha \circ d$ satisfying*

$$\begin{cases} \Delta_\varphi v \leq f(v)l(|\nabla_0 v|) & \text{in } \overline{B_{t_1}(q)} \setminus B_{t_0}(q) \\ v \geq h & \text{on } \partial B_{t_0}(q), \\ v = k & \text{on } \partial B_{t_1}(q) \end{cases}$$

and such that α is strictly increasing and convex.

Proof. Consider $\sigma \in (0,1]$ to be determined later and set

$$\Phi(z) = \int_z^k \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))}.$$

Then $\Phi(k) = 0$ and $\sup_{[0,k]} \Phi = \Phi(0)$, where $\Phi(0)$ may possibly be $+\infty$. Therefore, for a fixed t, there exists a unique $z > 0$ such that the equality

$$t_1 - t = \Phi(z)$$

is satisfied if and only if $t \in (t_1 - \Phi(0), t_1]$. Observing that $\Phi(0) \to +\infty$ as $\sigma \to 0$, up to choosing σ sufficiently small we can assume that $t_1 - \Phi(0) < t_0$. Thus we can define the implicit function $\alpha(t)$ by requiring

$$t_1 - t = \int_{\alpha(t)}^k \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))} \qquad \text{on } \ [t_0, t_1].$$

We observe that, by construction, $\alpha(t_1) = k$. Moreover, since the value $\alpha(t_0)$ increases as $\sigma \to 0$, up to choosing σ small enough, we can assume that $\alpha(t_0) \geq h$. A first differentiation yields

$$\frac{\alpha'}{K^{-1}(\sigma F(\alpha))} = 1,$$

hence α is monotone increasing and $\sigma F(\alpha) = K(\alpha')$. Differentiating once more we deduce
$$\sigma f(\alpha)\alpha' = K'(\alpha')\alpha'' = \frac{\alpha'\varphi'(\alpha')}{l(\alpha')}\alpha''.$$
Cancelling α' throughout, we obtain
$$[\varphi(\alpha')]' = \varphi'(\alpha')\alpha'' = \sigma f(\alpha) l(\alpha');$$
thus, integrating on $[t_0, t]$,
$$\varphi(\alpha'(t)) = \varphi(\alpha'(t_0)) + \sigma \int_{t_0}^{t} f(\alpha(s)) l(\alpha'(s))\,\mathrm{d}s.$$
Now we set $v = \alpha \circ d$ and observe that v is a C^2 radial function whose φ-laplacian can be computed through (11). We also note that $\nabla_0 d$ is homogeneous of degree 0 and therefore $\nabla_0|\nabla_0 d|$ is homogeneous of degree -1. Moreover, since
$$|\nabla_0|\nabla_0 d||^2 = \sum_{j=1}^{m_1} \frac{\left(\sum_{i=1}^{m_1}(X_j X_i d)(X_i d)\right)^2}{|\nabla_0 d|^2}$$
and, for every j, we have
$$\sum_{i=1}^{m_1}(X_j X_i d)(X_i d) \le \sqrt{\sum_{i=1}^{m_1}(X_j X_i d)^2}\sqrt{\sum_{i=1}^{m_1}(X_i d)^2} = \sqrt{\sum_{i=1}^{m_1}(X_j X_i d)^2}\,|\nabla_0 d|,$$
then we can bound $|\nabla_0|\nabla_0 d||^2$ by
$$|\nabla_0|\nabla_0 d||^2 \le \sum_{j=1}^{m_1} \frac{\sum_{i=1}^{m_1}(X_j X_i d)^2 |\nabla_0 d|^2}{|\nabla_0 d|^2} = \sum_{i,j=1}^{m_1}(X_j X_i d)^2.$$
which is smooth out of the origin, hence bounded on compact sets which do not contain the origin. Therefore, $|\nabla_0|\nabla_0 d||$ is bounded by $C_2 d^{-1}$ for some positive constant C_2. Using this fact, along with condition (iii) of (9), we can estimate the last term on the RHS of (11):
$$\left[\varphi'(\alpha'|\nabla_0 d|)\alpha' - \frac{\varphi(\alpha'|\nabla_0 d|)}{|\nabla_0 d|}\right]\nabla_0|\nabla_0 d| \cdot \frac{\nabla_0 d}{|\nabla_0 d|} \le \left[C_1 \frac{\varphi(\alpha'|\nabla_0 d|)}{|\nabla_0 d|} + \frac{\varphi(\alpha'|\nabla_0 d|)}{|\nabla_0 d|}\right]\frac{C_2}{d}$$
Therefore, using this estimate in combination with assumption (i) of (9), we

find that

$$\begin{aligned}\Delta_\varphi v &\le C|\nabla_0 d|^{\tau+1}\left[[\varphi(\alpha')]' + \frac{Q-1}{d}\varphi(\alpha')\right] + \left[C_1\frac{\varphi(\alpha'|\nabla_0 d|)}{|\nabla_0 d|} + \frac{\varphi(\alpha'|\nabla_0 d|)}{|\nabla_0 d|}\right]\frac{C_2}{d} \\ &\le C|\nabla_0 d|^{\tau-1}\left[[\varphi(\alpha')]' + \frac{Q-1}{d}\varphi(\alpha')\right] + |\nabla_0 d|^{\tau-1}\varphi(\alpha')\frac{C_3}{d} \\ &\le C|\nabla_0 d|^{\tau-1}\left[[\varphi(\alpha')]' + \frac{\widetilde{C}}{d}\varphi(\alpha')\right]\end{aligned}$$

for some constant $\widetilde{C}$. Now we use the properties of α that we discussed above and perform some further manipulation:

$$\begin{aligned}\Delta_\varphi v &\le C|\nabla_0 d|^{\tau-1}\left[\sigma f(\alpha)l(\alpha') + \frac{\widetilde{C}}{d}\varphi(\alpha'(t_0)) + \frac{\widetilde{C}}{d}\sigma\int_{t_0}^{d} f(\alpha(s))l(\alpha'(s))\,\mathrm{d}s\right] \\ &= C|\nabla_0 d|^{\tau-1}\left[\sigma + \frac{\widetilde{C}}{d}\frac{\varphi(\alpha'(t_0))}{f(\alpha(d))l(\alpha'(d))} + \frac{\widetilde{C}}{d}\frac{\sigma\int_{t_0}^{d} f(\alpha(s))l(\alpha'(s))\,\mathrm{d}s}{f(\alpha(d))l(\alpha'(d))}\right]f(\alpha)l(\alpha') \\ &\le C|\nabla_0 d|^{\tau-1}\left[\sigma + \frac{\widetilde{C}}{d}\frac{B\varphi(\alpha'(t_0))}{f(\alpha(t_0))l(\alpha'(t_0))} + \frac{\widetilde{C}}{d}\frac{\sigma B f(\alpha(d))l(\alpha'(d))(d-t_0)}{f(\alpha(d))l(\alpha'(d))}\right]f(\alpha)l(\alpha'),\end{aligned}$$

that is,

$$\Delta_\varphi v \le \widetilde{C}|\nabla_0 d|^{\tau-1}\left[\frac{\varphi(\alpha'(t_0))}{t_0 f(\alpha(t_0))l(\alpha'(t_0))} + \sigma\right]f(\alpha)l(\alpha'), \tag{14}$$

for some constant $\widetilde{C}$. Since $K(0) = 0$ and $\alpha(t_0) \le k$, we deduce that $\alpha'(t_0) = K^{-1}(\sigma F(\alpha(t_0))) \le K^{-1}(\sigma F(k)) \to 0$ as $\sigma \to 0$; moreover, since $\alpha(t_0) \ge k - \delta > 0$, so that $f(\alpha(t_0))$ is bounded away from zero. Therefore, choosing σ small enough, we can estimate the whole square bracket with $\frac{1}{\widetilde{C}\Lambda}$, so that

$$\Delta_\varphi v \le \widetilde{C}|\nabla_0 d|^{\tau-1}\frac{1}{\widetilde{C}\Lambda}f(\alpha)l(\alpha') \le f(v)l(|\nabla_0 v|). \tag{15}$$

□

In the next proposition we construct a supersolution which will be needed in the proof of Theorem 4.1, the non-existence result.

Proposition 2.4. *Assume the validity of* (2), (5), (6), (7) *and* (9) *and fix* $q \in G$, $0 < t_0 < t_1$, $0 < \varepsilon < \eta < A$, *where* A *may possibly be equal to* $+\infty$ *if* (KO) *holds. Then there exist* $\sigma > 0$, $T > r_1$ *and a radial function* $v = \alpha \circ d$ *satisfying*

$$\begin{cases}\Delta_\varphi v \le f(v)l(|\nabla_0 v|) \ \text{ on } B_T(q)\setminus B_{t_0}(q), \\ v \equiv \varepsilon \ \text{ on } \partial B_{t_0}(q); \quad v = A \ \text{ on } \partial B_T(q), \\ \varepsilon \le v \le \eta \ \text{ on } B_{t_1}(q)\setminus B_{t_0}(q).\end{cases}$$

and such that α is strictly increasing and convex.

Proof. Consider $\sigma \in (0,1]$ to be determined later and choose $T_\sigma > t_0$ such that

$$T_\sigma - t_0 = \int_\varepsilon^A \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))}.$$

Note that, when $A = +\infty$ and (KO) holds, the RHS is well defined by Lemma 2.2. Moreover, since the RHS diverges as $\sigma \to 0^+$, up to choosing σ sufficiently small we can shift T_σ in such a way that $T_\sigma > t_1$. We implicitly define the C^2-function $\alpha(t)$ by requiring

$$T_\sigma - t = \int_{\alpha(t)}^A \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))} \qquad \text{on} \ \ [t_0, T_\sigma).$$

We observe that, by construction, $\alpha(t_0) = \varepsilon$ and, since $K^{-1} > 0$, $\alpha(t) \uparrow A$ as $t \to T_\sigma$. As in the previous lemma, a first differentiation yields

$$\frac{\alpha'}{K^{-1}(\sigma F(\alpha))} = 1,$$

hence α is monotone increasing and $\sigma F(\alpha) = K(\alpha')$. With the same computation as in the proof of Proposition 2.3 we arrive to

$$\Delta_\varphi v \leq \widetilde{C} |\nabla_0 d|^{\tau - 1} \left[\frac{\varphi(\alpha'(t_0))}{t_0 f(\alpha(t_0)) l(\alpha'(t_0))} + \sigma \right] f(\alpha) l(\alpha'), \tag{16}$$

for some uniform constant $\widetilde{C}$. Since $K(0) = 0$, $\alpha(t_0) = \varepsilon$ and $\alpha'(t_0) = K^{-1}(\sigma F(\varepsilon)) \to 0$ as $\sigma \to 0$, choosing σ small enough, we can again estimate the whole square bracket with $\frac{1}{\widetilde{C}\Lambda}$, so that

$$\Delta_\varphi v \leq \widetilde{C} |\nabla_0 d|^{\tau-1} \frac{1}{\widetilde{C}\Lambda} f(\alpha) l(\alpha') \leq f(v) l(|\nabla_0 v|). \tag{17}$$

The only thing left to prove is that, possibly with a further reduction of σ, $\alpha(t_1) \leq \eta$. From the trivial identity

$$\int_{\alpha(t_1)}^A \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))} = T_\sigma - t_1 = (T_\sigma - t_0) + (t_0 - t_1) = \int_\varepsilon^A \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))} + (t_0 - t_1)$$

we deduce

$$\int_\varepsilon^{\alpha(r_1)} \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))} = t_1 - t_0.$$

It suffices to choose σ such that $\int_\varepsilon^\eta \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))} > t_1 - t_0$; then obviously $\alpha(t_1) \leq \eta$.

Therefore, fixing a value for σ which satisfies all the above requirements and renaming T the corresponding T_σ, we have proved the claim. $\square$

3 The maximum principle

We begin this section by stating a comparison principle for the φ-laplacian on Carnot groups.

Proposition 3.1 (Comparison principle)**.** *Let $\Omega \subset\subset G$ be a relatively compact domain with C^1 boundary and assume the validity of* (2)*. Let $u, v \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ satisfy*

$$\begin{cases} \Delta_\varphi u \geq \Delta_\varphi v & \text{on } \Omega \\ u \leq v & \text{on } \partial\Omega. \end{cases} \tag{18}$$

Then $u \leq v$ on Ω.

The proof of this comparison principle is achieved in the same way as for the Heisenberg group (see [9]), so we omit it.
Next, we prove a maximum principle for inequality (3).

Theorem 3.2 (Maximum principle)**.** *Let $\Omega \subset G$ be a domain. Assume the validity of* (2)*,* (5)*,* (6)*,* (7) *and* (9)*. Let $u \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ satisfy*

$$\Delta_\varphi u \geq f(u) l(|\nabla_0 u|) \quad \text{in } \Omega \tag{19}$$

and let $u^* = \sup_\Omega u$. If $u(q_M) = u^*$ for some $q_M \in \Omega$, then $u \equiv u^*$.

Proof. By contradiction, assume there exist a solution u of (19) and $q_M \in \Omega$ such that $u(q_M) = u^*$, but $u \not\equiv u^*$. Set $\Gamma = \{q \in \Omega : u(q) = u^*\}$. Let $\delta > 0$ and define

$$\Omega^+ = \{q \in \Omega : u^* - \delta < u(q) < u^*\}; \qquad \Gamma_\delta = \{q \in \Omega : u(q) = u^* - \delta\}; \tag{20}$$

note that $\partial\Omega^+ \cap \Omega = \Gamma \cup \Gamma_\delta$. Let $q' \in \Omega^+$ be such that

$$d(q', \Gamma) < d(q', \Gamma_\delta), \qquad d(q', \Gamma) < d(q', \partial\Omega) \tag{21}$$

This is possible provided q' is sufficiently close to q_M. Indeed, if q' is chosen so that

$$d(q', q_M) < \frac{1}{2c} \min\{d(q_M, \Gamma_\delta), d(q_M, \partial\Omega)\},$$

where c denotes the constant appearing in the pseudo-triangle inequality (1), then we have

$$d(q_M, \Gamma_\delta) \leq c[d(q_M, q') + d(q', \Gamma_\delta)],$$

which implies

$$d(q',\Gamma_\delta) \geq \frac{d(q_M,\Gamma_\delta)}{c} - d(q_M,q') > d(q_M,q') \geq d(q',\Gamma)$$

and likewise for $d(q',\partial\Omega)$.

Let now $B_R(q')$ be the largest ball centered at q' and contained in Ω^+. Then, by construction $u < u^*$ in $B_R(q')$ while $u(q_0) = u^*$ for some $q_0 \in \partial B_R(q')$. Since q_0 is an absolute maximum for u in Ω, we have $\nabla u(q_0) = 0$. Now we construct an auxiliary function by means of Proposition 2.3. Towards this aim, we consider the annular region

$$E_R(q') = \overline{B_R(q')} \setminus B_{R/2}(q') \subset \overline{\Omega^+}; \tag{22}$$

and define a radial function $v = \alpha \circ d$ such that

$$\begin{cases} \Delta_\varphi v \leq f(v)l(|\nabla_0 v|) \quad \text{in } E_R(q') \\ v \geq \max_{\partial B_{R/2}(q')} u \quad \text{on } \partial B_{R/2}(q'), \\ v = u^* \quad \text{on } \partial B_R(q'). \end{cases} \tag{23}$$

We point out that the function α is strictly increasing on the interval $[R/2, R]$. Let us now assume that the maximum of $u - v$ on E_R be positive. Then it has to be internal, and therefore there must exist p_0 in the interior of E_R such that $u(p_0) > v(p_0)$ and $\nabla_0 u(p_0) = \nabla_0 v(p_0)$, which, since $l(0) > 0$ and f is strictly increasing, implies that

$$f(u(p_0))l(|\nabla_0 u(p_0)|) > f(v(p_0))l(|\nabla_0 v(p_0)|). \tag{24}$$

Now set $\mu = \max_{E_R}(u - v)$ and let Λ_μ be the connected component of

$$\{q \in E_R : u(q) - v(q) = \mu\}$$

containing p_0. Observe that, by continuity, (24), which holds at every point of Λ_μ, implies that

$$\Delta_\varphi u \geq \Delta_\varphi v$$

on a neighborhood U of Λ_μ. Fix $0 < \rho < \mu$ and let Ω_ρ be the connected component containing p_0 of

$$\big\{q \in E_R^0 : u(q) > v(q) + \rho\big\}.$$

We observe that $p_0 \in \Omega_\rho$ for every ρ and that Ω_ρ is a nested sequence as ρ tends to μ. We claim that if ρ is close to μ, then $\overline{\Omega}_\rho \subset U$. This can be shown by a compactness argument such as the following: since Λ_μ is closed and bounded, there exists $\varepsilon > 0$ such that $d(U^c, \Lambda_\mu) \geq \varepsilon$. Suppose, by contradiction, that there exist sequences $\rho_n \uparrow \mu$ and $\{q_n\}$ such that $q_n \in \Omega_{\rho_n}$ and $q_n \notin U$, therefore $d(q_n, \Lambda_\mu) > \varepsilon$. Then, we can assume that the sequence is contained in Ω_{ρ_0} which, by construction, has compact closure; passing to a subsequence converging to some $\overline{q}$, we have by continuity

$$d(\overline{q}, \Lambda_\mu) \geq \varepsilon, \tag{25}$$

but, on the other hand, $(u-v)(\overline{q}) = \lim_n (u-v)(q_n) \geq \lim_n \rho_n = \mu$, hence $\overline{q} \in \Lambda_\mu$ and this contradicts (25). Therefore, $d(\partial\Omega_\rho, \Lambda_\mu) \to 0$ as $\rho \to \mu$, and the claim is proved.
Therefore, on Ω_ρ we have

$$\Delta_\varphi u \geq \Delta_\varphi v = \Delta_\varphi(v+\rho)$$

and $u = v+\rho$ on $\partial\Omega_\rho$ which, by the comparison principle, implies that $u \leq v+\rho$ on Ω_ρ, a contradiction since $u(p_0) = v(p_0)+\mu$. This shows that the maximum of $u-v$ on E_R has to be nonpositive, that is, $u - v \leq 0$ on E_R.
We point out that, while the horizontal gradient of the homogeneous norm may vanish out of the origin (and in fact it does in every nontrivial Carnot group), its Euclidean gradient does not. Postponing for a while the proof of this simple fact, we conclude the proof of the maximum principle. In the light of this, there exists a positive constant $\lambda > 0$ such that

$$\langle \nabla v, \nabla d\rangle = \alpha'(d)|\nabla d|^2 \geq \lambda > 0 \quad \text{on} \quad \partial E_R(q'). \tag{26}$$

Going back to the function $v-u$, we found that it satisfies $v-u \geq 0$ on $E_R(q')$ and $v(q_0) - u(q_0) = u^* - u^* = 0$, so that $\langle \nabla(v-u), \nabla d\rangle(q_0) \leq 0$. Therefore

$$0 = \langle \nabla u, \nabla d\rangle(q_0) \geq \langle \nabla v, \nabla d\rangle(q_0) > 0, \tag{27}$$

a contradiction.
Finally, to prove that the Euclidean gradient of the homogeneous norm does not vanish out of the origin, fix $x_0 \in G$ and consider the composition $g(t) = d(\delta_t x_0) = td(x_0)$. By elementary calculus, renaming for convenience of notation $\delta_t x_0 = \gamma(t)$ we get

$$d(x_0) = g'(1) = \nabla d(x_0) \cdot \dot{\gamma}(1),$$

which cannot vanish out of the origin. $\square$

Remark 3.3. *If G is a group of Heisenberg type or, more generally, a polarizable* *group, then a strong maximum principle for inequality*

$$\Delta_\varphi u \geq 0$$

can be stated and the proof can be adapted from the one in [9] with no effort.

4 Non existence results

This section is devoted to proving some Liouville-type results for inequality (3): as the next theorem states, this inequality has no nontrivial entire non-negative solution if the Keller-Osserman condition is satisfied.

Theorem 4.1. *Let φ, f, l satisfy* (2)*,* (5)*,* (6) *and* (7)*. Assume also the validity of* (9)*. If the generalized Keller-Osserman condition* (KO) *holds, then every solution* $0 \leq u \in C^1(G)$ *of*

$$\Delta_\varphi u \geq f(u)l(|\nabla_0 u|) \qquad \text{on } G \tag{28}$$

is identically zero.

Actually, we can prove that inequality (3) does not possess any non-negative entire bounded solutions regardless of whether the Keller-Osserman condition be satisfied or not. This is stated in the next

Theorem 4.2. *Let φ, f, l satisfy* (2)*,* (5)*,* (6)*,* (7) *and* (9)*. Then every non-negative bounded C^1-solution u of* (28) *vanishes identically.*

Proof of Theorems 4.1 and 4.2. Having proved Proposition 2.4 for a general Carnot group G, the proof of the non-existence theorems follows the same outline as those for the Heisenberg group presented in [9]. However, we reproduce the steps here for the sake of completeness. We first prove Theorem 4.2 under the assumptions (2), (5), (6), (7) and (9). Later on, under the additional hypothesis (KO), we will also prove the constancy of possibly unbounded solutions u of (28).

Therefore, we denote by $u^* = \sup u$ and we first assume that $u^* < +\infty$. We reason by contradiction and assume $u \not\equiv u^*$; by Proposition 3.2 $u < u^*$ on G. Choose $r_0 > 0$ and define

$$u_0^* = \sup_{\overline{B}_{r_0}} u < u^*.$$

Fix $\eta > 0$ sufficiently small such that $u^* - u_0^* > 2\eta$ and choose $\widetilde{q} \in G \setminus \overline{B}_{r_0}$ such that $u(\widetilde{q}) > u^* - \eta$. Choose also $0 < \varepsilon < \eta$ and A in such a way that $A > 2\eta + \varepsilon$. We then set $r_1 = d(\widetilde{q})$ and, for our choice of $r_0, r_1, A, \varepsilon, \eta$ we construct a radial function $v(q) = \alpha(d(q))$ on $B_T \setminus B_{r_0}$ as in Proposition 2.4, so that

$$\begin{cases} \Delta_\varphi v \le f(v) l(|\nabla_0 v|) \text{ on } B_T \setminus B_{r_0}, \\ v \equiv \varepsilon \text{ on } \partial B_{r_0}; \quad v = A \text{ on } \partial B_T, \\ \varepsilon \le v \le \eta \text{ on } B_{r_1} \setminus B_{r_0}. \end{cases}$$

Therefore

$$u(\widetilde{q}) - v(\widetilde{q}) > u^* - \eta - \eta = u^* - 2\eta,$$

and, on ∂B_{r_0},

$$u(q) - v(q) \le u_0^* - \varepsilon < u^* - 2\eta - \varepsilon.$$

Since also

$$u(q) - v(q) \le u^* - A < u^* - 2\eta - \varepsilon \qquad \text{for } \ q \in \partial B_T,$$

the difference $u - v$ attains a positive maximum μ in $B_T \setminus \overline{B}_{r_0}$. Now the proof proceeds exactly as for the maximum principle: we consider a parameter $0 < \rho < \mu$ and, applying Proposition 3.1 (the comparison principle) to the functions u and $v + \rho$ on a suitable neighborhood of a point of maximum, we get the desired contradiction. This shows that $u \equiv c$, where c is a non-negative constant; since $l(0) > 0$ we have $0 = \Delta_\varphi c \ge f(c) l(0)$. This implies $f(c) = 0$, hence $c = 0$.
Assume now the validity of the Keller-Osserman condition (KO), and suppose that u is a solution of (28). By the previous arguments, if u is not constant then necessarily $u^* = +\infty$. Again, fix $r_0 > 0$ such that $u \not\equiv 0$ on B_{r_0}, and define $u_0^* = \sup_{\overline{B}_{r_0}} u$. Choose $\widetilde{q}, \eta, \varepsilon$ in such a way that $u(\widetilde{q}) > 2u_0^*$, $0 < \varepsilon < \eta < u_0^*$, and consider the function α defined as before with $A = +\infty$. Then, $v(q) = \alpha(d(q))$ is a supersolution of (28) and

$$\begin{aligned} &u(q) - v(q) \le u_0^* - \varepsilon && \text{on } \ \partial B_{r_0}, \\ &u(\widetilde{q}) - v(\widetilde{q}) > 2u_0^* - \eta > u_0^* && \\ &u(q) - v(q) \to -\infty && \text{as } \ r(q) \to T^-. \end{aligned}$$

Hence, $u - v$ attains a positive maximum in $B_T \setminus \overline{B}_{r_0}$. The proof now proceeds in the same way as in the previous case. □

Remark 4.3. *Theorem 4.1 can be restated for polarizable groups getting rid of condition* (iii) *of* (9).

5 Existence

When the Keller-Osserman condition is not satisfied, then inequality (3) admits entire, unbounded solutions. This result can be stated as follows.

Theorem 5.1. *Assume the validity of* (2), (5), (6) *and* (7). *Then, if the generalized Keller-Osserman condition* (KO) *is not satisfied, there exists a non-negative, non-constant solution* $u \in C^1(G)$ *of inequality* $\Delta_\varphi u \geq f(u)l(|\nabla_0 u|)$.

In light of this, Theorem 4.1 and Theorem 5.1 can be combined in the following statement.

Theorem 5.2. *Assume the validity of* (2),(5), (6), (9) *and of* (7). *Then, the following are equivalent:*

(i) *there exists a non-negative, non-constant solution* $u \in C^1(G)$ *of inequality* $\Delta_\varphi u \geq f(u)l(|\nabla_0 u|)$;

(ii) $\dfrac{1}{K^{-1}(F(t))} \notin L^1(+\infty)$.

Proof. First of all we observe that the sufficiency of the Keller-Osserman condition, i.e. implication $(i) \Rightarrow (ii)$, follows from Theorem 4.1. Our aim is therefore to provide existence of unbounded C^1-solutions of inequality (3) under the assumption that (KO) is not satisfied; this will be achieved by pasting together two subsolutions defined on complementary sets. Such solutions will be "radial' in the variables of the first layer, that is, functions of the form $v(x) = w\big(\big|x^{(1)}\big|\big)$, where

$$x = \left(x^{(1)}, \dots, x^{(r)}\right) \text{ and } \left|x^{(1)}\right| = \left(\sum_{i=1}^{m_1} \left(x_i^{(1)}\right)^2\right)^{\frac{1}{2}}.$$

For notational convenience, we set $z = x^{(1)}$. Straightforward computation shows that

$$|\nabla_0 |z|| \equiv 1, \qquad \Delta |z| = \frac{m_1 - 1}{|z|}, \tag{29}$$

and thus the expression of the φ-Laplacian for such functions is

$$\Delta_\varphi v = \varphi'(|w'(|z|)|)w''(|z|) + \frac{m_1 - 1}{|z|} \operatorname{sgn}(w'(|z|))\varphi(|w'(|z|)|). \tag{30}$$

Define implicitly the C^2-function w on $\mathbb{R}_0^+$ by setting

$$t = \int_1^{w(t)} \frac{\mathrm{d}s}{K^{-1}(F(s))}. \tag{31}$$

Note that w is well defined, $w(0) = 1$ and, by Lemma 2.2 and since the Keller-Osserman condition does not hold, $w(t) \to +\infty$ as $t \to +\infty$. Differentiating (31) yields

$$w' = K^{-1}(F(w(t))) > 0, \tag{32}$$

and a further differentiation gives

$$\varphi'(w')w'' = f(w)l(w'). \tag{33}$$

We fix $\bar{z} > 0$ to be specified later and set

$$A_{\bar{z}} = \left\{ \left(x^{(1)}, \dots, x^{(r)} \right) \in G : \left| x^{(1)} \right| < \bar{z} \right\},$$

and let $u_1(x)$ be the function defined on $G \setminus A_{\bar{z}}$ by the formula $u_1(x) = w(|z|)$. Since, by (29), $|\nabla_0 u_1| = w'$, using (30) and (33) we conclude that u_1 satisfies

$$\Delta_\varphi u_1 = \varphi'(w'(|x|))w''(|x|) + \frac{m-1}{|x|}\varphi(w'(|x|)) \geq f(u_1)l(|\nabla_0 u_1|) \tag{34}$$

on $G \setminus A_{\bar{z}}$.

To produce a subsolution u_2 on $A_{\bar{z}}$, we fix constants β_0, $\Theta > 0$ to be determined later and define a function Ω (depending on Θ) through

$$\int_0^{\Omega(s)} \frac{dt}{l(\varphi^{-1}(t))} = \Theta s. \tag{35}$$

Note that Ω is well defined since $l(0) > 0$ and solves the differential equation

$$\Omega'(s) = \Theta\, l(\varphi^{-1}(\Omega(s))).$$

We set

$$\beta(t) = \int_0^t \varphi^{-1}(\Omega(s))ds + \beta_0$$

for $t \in [0, \bar{z}]$ and, observing that $\beta'(0) = 0$, we deduce that the function $u_2(x) = \beta(|x|)$ is C^1 on $\mathbb{R}^m$. Straightforward computation then shows that

$$\Delta_\varphi u_2(x) \geq \varphi'(\beta'(|x|))\beta''(|x|) = \Omega'(|x|) = \Theta\, l(\varphi^{-1}(\Omega(|x|))) = \Theta\, l(\beta'(|x|)). \tag{36}$$

So, by the monotonicity of f, it follows that, if

$$\Theta \geq f(\beta(\bar{z})), \tag{37}$$

then

$$\Delta_\varphi u_2 \geq f(u_2)l(|\nabla_0 u_2|) \quad \text{on } A_{\bar{z}}. \tag{38}$$

To join u_1 and u_2 so that the resulting function u is C^1, we shall choose the parameters $\bar{z}$, Θ, β_0 in such a way that (37) and

$$\begin{cases} \beta(\bar{z}) = w(\bar{z}) \\ \beta'(\bar{z}) = w'(\bar{z}) \end{cases} \tag{39}$$

are satisfied. Setting $w(\bar{z}) = \mu$, where $1 < \mu \leq 2$, this translates into

$$\begin{cases} \int_0^{\bar{z}} \varphi^{-1}(\Omega(s))ds + \beta_0 = \mu \\ \varphi^{-1}(\Omega(\bar{z})) = K^{-1}(F(\mu)) \\ \Theta \geq f(\beta(\bar{z})) \end{cases} \tag{40}$$

We observe that

$$\bar{z} = \int_1^{\mu} \frac{\mathrm{d}s}{K^{-1}(F(s))} \tag{41}$$

and $\bar{z} \to 0$ as $\mu \to 1^+$. Moreover, by the monotonicity of K^{-1} and F,

$$\frac{\mu - 1}{K^{-1}(F(2))} \leq \bar{z} \leq \frac{\mu - 1}{K^{-1}(F(1))}, \tag{42}$$

From the second equation of (40) we deduce that

$$\Omega(\bar{z}) = \varphi(K^{-1}(F(\mu))),$$

which in turn, by (35) yields

$$\Theta\bar{z} = \int_0^{\varphi(K^{-1}(F(\mu)))} \frac{dt}{l(\varphi^{-1}(t))}.$$

We use this last equation to define Θ and observe that

$$\begin{aligned} \Theta =& \frac{1}{\bar{z}} \int_0^{\varphi(K^{-1}(F(\mu)))} \frac{dt}{l(\varphi^{-1}(t))} \\ \geq& \frac{K^{-1}(F(1))}{\mu - 1} \int_0^{\varphi(K^{-1}(F(1)))} \frac{dt}{l(\varphi^{-1}(t))} \to +\infty \text{ as } \mu \to 1^+. \end{aligned}$$

Therefore the third of (40) is satisfied if μ is close enough to 1, since in this case we certainly have

$$\Theta \geq f(2) \geq f(\mu).$$

Finally, the first of (40) becomes

$$\beta_0 = \mu - \int_0^{\bar{z}} \varphi^{-1}(\Omega(s))ds,$$

which is well defined if

$$\int_0^{\bar{z}} \varphi^{-1}(\Omega(s))ds < 1.$$

But this is trivially true for μ sufficiently close to 1 since

$$\int_0^{\bar{z}} \varphi^{-1}(\Omega(s))ds \le \bar{z}\varphi^{-1}(\Omega(\bar{z})) = \bar{z}K^{-1}(F(\mu)) \le \frac{K^{-1}(F(2))}{K^{-1}(F(1))}(\mu - 1) \to 0 \text{ as } \mu \to 1^+.$$

Summing up, if μ is sufficiently close to 1, the function

$$u(x) = \begin{cases} u_1(x) & \text{on } G \setminus A_{\bar{z}} \\ u_2(x) & \text{on } A_{\bar{z}} \end{cases} \tag{43}$$

is a weak classical solution of $\Delta_\varphi u \ge f(u)l(|\nabla u|)$. Indeed, the weak inequality follows easily from the C^1-regularity of u on $\partial A_{\bar{z}}$. This concludes the proof. $\square$

Remark 5.3. *The previous result is all the more valid on* $\mathbb{R}^m$*, where the hypothesis* $l(0) > 0$ *and conditions* (9) *are unnecessary and can be replaced by the assumption:*

$$\frac{1}{l(\varphi^{-1}(t))} \in L^1(0^+), \tag{44}$$

which is needed in order to be sure that the function Ω *introduced in* (35) *is well-defined.*
We observe that if $l(t) = t^\theta$ *and* $\varphi(t) = t^{p-1}$*, the integrability condition* (44) *translates into the request* $\theta < p - 1$*. Lastly, we remark that on polarizable groups, and in particular on the Heisenberg group, condition* (9) *is, as usual, unnecessary. Therefore Theorem 5.1 and Theorem 5.2 are improvements of Theorem 1.3 of [9]*

Remark 5.4. *With the same assumptions, Theorem 5.5 and Theorem 5.7 of [9], which are concerned with the existence and non-existence of non-negative, non-constant solutions of*

$$\Delta_\varphi u \ge f(u) - h(u)g(|\nabla_0 u|),$$

can be also restated and proved with no effort in the context of Carnot groups.

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