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HIGHER MOMENTS FOR THE COGARCH(1,1) MODEL

by

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Higher Moments for the Cogarch(1,1) model

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Abstract

The COGARCH (COntinuous Generalized Auto-Regressive Conditional Heteroschedastic) model can be considered as a continuous version of the well known GARCH discrete time model. They are driven by general Lévy processes and the resulting volatility process satisfies a stochastic differential equation. The main difference between COGARCH models and other stochastic volatility models is that there is only one source of randomness (the Lévy process) and all the stylized feature are captured by the dependance structure of the model as in the GARCH models. A general method to calculate the moment of higher order of the COGARCH(1,1) model is presented. A general formula to calculate all the joint and the conditional moments is also provided. The explicit form of the higher moment is useful to apply some prediction based estimation function (PBEF) methods to estimate the parameters of the COGARCH models and in particular to find an optimal PBEF.

Keywords: cogarch model, stochastic volatility models, prediction based estimating functions, parameter estimation

1 Introduction

The COGARCH model with order (1,1) was introduced as a continuous version of the GARCH(1,1) model in [Klüppelberg et al., 2004]. It is driven by a general Lévy process through the equation $dG_t = \sigma_{t-} dL_t$ and the resulting volatility process σ_t satisfies the stochastic differential equation $d\sigma_t^2 = (\beta - \eta\sigma_{t-}^2)dt + \phi\sigma_{t-}^2 d[L]_t^d$ where the parameters fulfill $\beta > 0$, $\eta \geq 0$ and $\phi \geq 0$ and $[L]_t^d$ is the discrete part of the quadratic variation of the Lévy process $L = (L_t)_{t\geq 0}$. Financial log-returns are modelled by the increments of the process $G_{t,h} = G_{t+h} - G_t$.

The main difference between COGARCH models and other stochastic volatility model is that the Lévy process is the sole source of randomness and when it jumps both the price and the volatility jump at the same time.

For a more thorough presentation of such model, for the relation between GARCH sequences and the COGARCH process, for a comparison with other continuos time models with the same aim and for how this model is able to capture the stylized facts about financial data we refer the reader to the following papers [Klüppelberg et al., 2004, Klüppelberg et al., 2011, Haug et al., 2007, Kallsen and Vesenmayer, 2009,Maller et al., 2008,Buchmann and Mueller, 2012]. In the last few years many generalizations of the COGARCH model have been proposed. Among them COGARCH processes of order (p,q) [Brockwell et al., 2006] and multivariate COGARCH(1,1) [Stelzer, 2010] seem to be the most interesting.

A few methods for the estimation of the model parameters from a sample of equally spaced returns $G_{ir,r} = G_{(i+1)r} - G_{ir}$ are currently available.

In [Haug et al., 2007] explicit estimators have been derived from a method of moments (MM). In [Maller et al., 2008] a pseudo maximum likelihood (PML) method has been proposed that allows also for non equally spaced observations, and in [Müller, 2010] an MCMC-based estimation method has been presented for the model driven by a compound Poisson process.

Our guess is that the method of Prediction Based Estimating Functions (PBEFs) introduced in [Sørensen, 2000] is applicable to the COGARCH(1,1) model and that its performances are better then the other available procedures. The general theory of PBEFs allows to find an optimal PBEF if the joint moments of the observation are explicitly known up to a certain order.

Motivated by the search for an optimal PBEF, the aim of the present paper is to present a recursive formula for the moments $\mathbb{E}(G_t^{2i}\sigma_t^{2(k-i)})$ and $\mathbf{E}_v[G_{s,h}^{2i}\sigma_{s+h}^{2(k-i)}]$ whenever they exist. Explicit expressions for any total order $2k$ and any integer $i \leq k$ and for any $t, h > 0$ and $s > v > 0$ are given. $\mathbf{E}_v$ denotes conditional expectation with respect to the natural filtration $\mathcal{F}_v$.

Explicit expression for the joint moments $\mathbb{E}(G_{t_h,r}^{2i_h} G_{t_{h-1},r}^{2i_{h-1}} \cdots G_{t_2,r}^{2i_2} G_{t_1,r}^{2i_1})$ are also provided for any integers $i_1 \cdots i_h$ and hence any total order $k = i_1 + \cdots + i_h$ and for any times $t_h \cdots t_1$ such that $t_i - t_{i-1} \geq r$.

Up to the order four ($k = 2$) our formulae coincide with those of [Haug et al., 2007], but explicit expressions for the higher orders are provided as a new result whose interest might go beyond the statistical methodology proposed.

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Up to the order four ($k = 2$) our formulae coincide with those of [Haug et al., 2007], but explicit expressions for the higher orders are provided as a new result whose interest might go beyond the statistical methodology proposed.

The paper is organized as follows. In Section 2 the definition and the properties of COGARCH(1,1) model are presented. In section 3 and in all its Subsection the higher moments are derived and explicit formulae are given. In Section 5 some further developments are discussed, in particular how can the moment of the process be used to find an optimal PBEF.

2 The COGARCH(1,1) model

Let us introduce on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the usual properties, a Lévy process $L = (L_t)_{t\geq 0}$ with Lévy triplet (γ, τ^2, ν) and Poisson random measure N (see [Applebaum, 2009, Kyprianou, 2006, Protter, 2005]). The COGARCH(1,1) model is defined as the solution $(G, \sigma^2) = \left(G_t, \sigma_t^2\right)_{t\geq 0}$ of the following system of stochastic differential equations (SDE) driven by the Lévy process L

$$\begin{cases} dG_t = \sigma_{t-} dL_t \\ d\sigma_t^2 = (\beta - \eta\sigma_{t-}^2)dt + \phi\sigma_{t-}^2 d[L]_t^d, \end{cases} \tag{1}$$

with initial value $G_0 = 0$ and σ_0 a random variable independent of the Lévy process $(L_t)_{t\geq 0}$. The parameter space $\Theta \subset \mathbb{R}^3$ is defined as $\theta = (\beta, \eta, \phi) \in \Theta$ if $\beta > 0$, $\eta \geq 0$ and $\phi \geq 0$. By $[L]_t^d$, for every $t \geq 0$ is the discrete part of the quadratic variation $[L]_t = \tau^2 t + [L]_t^d$ of the driving Lévy process L_t defined as

$$[L]_t^d = \int_{\mathbb{R}} x^2 N(t, dx) = \sum_{0<s\leq t} \Delta L_s^2$$

with $\Delta L_s = L_s - L_{s-}$.

The following is assumed throught all the paper.

Condition 2.1. $\mathbb{E}(L_1) = 0$ *and* $\mathbb{E}(L_1^2) = 1$.

If Conditions 2.1 holds, then L_t is a martingale and the volatility of the component G_t is given solely by σ_t.

Remark 2.1. *Under Condition 2.1, γ e τ^2 are not parameters of the model. Indeed since $\mathbb{E}(L_1) = 0$, $\gamma = \int_{|x|\geq 1} x^2 d\nu$. Moreover since, by the product formula, $[L]_t - L_t^2 = 2\int_0^t L_s dL_s$, we have*

$$\mathbb{E}[L]_1 = \tau^2 + \int_{\mathbb{R}} x^2 \nu(dx) = \mathbb{E}\left(L_1^2\right) = 1.$$

hence $\tau^2 = 1 - \int_{\mathbb{R}} x^2 \nu(dx)$. Let however remark that the Lévy measure ν may contain further parameters, that are supposed known.

For later convenience we list here without proof some properties of the COGARCH(1,1) that we will use later on.

An explicit solution

$$\sigma_t^2 = \beta \mathrm{e}^{-(X_t - X_u)} \int_u^t \mathrm{e}^{-(X_u - X_s)}\, ds + \mathrm{e}^{-(X_t - X_u)} \sigma_u^2. \tag{2}$$

of the second of equations (1) with initial condition σ_u^2 at time u is available in terms of the auxiliary process

$$X_t = \eta t + \sum_{0<s\leq t} \log(1 + \phi \Delta L_s)$$

whose Laplace transform can be written as

$$\mathbb{E}e^{-cX_t} = e^{t\Psi(c)}$$

for a function Ψ defined as

$$\Psi(c) = -\eta c + \int_R \left[(1+\phi\, x^2)^c - 1\right] \nu(dx) = \eta c + \sum_{i=1}^{c} \binom{c}{i} \phi^i \int_R x^{2i} \nu(dx). \quad (3)$$

The Laplace transform is finite at c if and only if L_1 has finite moments of order $2c$ and, together with $\Psi(c) < 0$, it is a sufficient condition for the process σ_t^2 to admit a stationary distribution (cf. [Klüppelberg et al., 2004]) with moments given by the following formula

$$\mathbb{E}\sigma_\infty^{2k} = k!\beta^k \prod_{l=1}^{k} \frac{-1}{\Psi(l)}. \quad (4)$$

In the COGARCH(1,1) model log-returns are represented as increments $G_{t,h} = G_{t+h} - G_t$ of the G process. The couple $\left(G_t, \sigma_t^2\right)_{t\geq 0}$ is a Markov process, but the single component $(G_t)_{t\geq 0}$ is not. It can be proved (see [Klüppelberg et al., 2004]) that if $\mathbb{E}(L_1^4) < \infty$ and if the parameters are such that $\Psi(2) < 0$, both the volatility process $(\sigma_t^2)_{t\geq 0}$ and the log-returns process $(G_{t,h})_{t\geq 0}$ are stationary (allows for a stationary density) and strongly mixing with an exponentially decreasing rate. We assume that σ_0^2 has the stationary distribution.

3 Higher moments

In this section we give conditions that assure the existence of simple and joint moments of the process $G_{t,r}$ up to any fixed order k, and we show how they can be calculated using an iterative procedure.

3.1 Notations

Whenever we refer to the quadratic variation of the driving Lévy process L_t we denote it simply $[L]_t$. We reserve the less compact standard notation $[M,N]_t$ for the quadratic covariation of two semimartingales M_t and N_t. Moreover, we often need to take quadratic covariations of quadratic variations and to this aim we introduce the following notation: quadratic variations of order $i+j$ are defined as $[L]_t^{(i+j)} = \left[[L]^{(i)}, [L]^{(j)}\right]_t$, with $[L]_t^{(1)} = L_t$ and hence $[L]_t^{(2)} = [L]_t$, $[L]_t^{(3)} = \left[L, [L]\right]_t = \left[[L], L\right]_t$ and so on so forth.

For $i > 2$ the quadratic variations $[L]_t^{(i)}$ do not have any continuous component

$$[L]_t^{(i)} = \int_{\mathbb{R}} x^i N(t,dx) \qquad \mathbb{E}[L]_t^{(i)} = t\int_{\mathbb{R}} x^i \nu(dx).$$

However, in some iterative formula below where an index i ranges between different values we will write $[L]_t^{d(i)}$ to keep track of the fact that when $i = 2$ the right object to be meant is the discrete part of the quadratic variation.

3.2 Higher moments of COGARCH(1,1)

Let us start with two Lemma that will be used repetitively in the next sections.

Lemma 3.1. *For every integer k, it holds that*

$$\sigma_t^{2k} = k \int_0^t \sigma_s^{2k-2}(\beta - \eta\sigma_s^2)\, ds + \sum_{i=1}^{k} \binom{k}{i} \phi^i \int_0^t \sigma_s^{2k} d[L]_s^{d(2i)} \tag{5}$$

and

$$G_t^{2k} = \sum_{i=1}^{2k} \binom{2k}{i} \int_0^t G_s^{2k-i} \sigma_s^i d[L]_s^{(i)} \tag{6}$$

Proof. We prove formula (5) by induction. For $k = 1$ it is true. Let us suppose that it is true for $k - 1$:

$$\sigma_t^{2k-2} = (k-1) \int_0^t \sigma_s^{2k-4}(\beta - \eta\sigma_s^2)\, ds + \sum_{i=1}^{k-1} \binom{k-1}{i} \phi^i \int_0^t \sigma_s^{2k-2} d[L]_s^{d(2i)}.$$

Then by Ito product formula (cfr. [Applebaum, 2009] Theorem 4.4.13), the (1) and equation (4.15) in [Applebaum, 2009] p. 257 or Theorem 29 in [Protter, 2005] we obtain

$$\begin{aligned}
\sigma_t^{2k} &= \int_0^t \sigma_s^{2k-2} d\sigma_s^2 + \int_0^t \sigma_s^2 d\sigma_s^{2k-2} + [\sigma^{2k-2}, \sigma^2]_t = \\
&= k \int_0^t \sigma_s^{2k-2}(\beta - \eta\sigma_s^2)\, ds + \phi \int_0^t \sigma_s^{2k} d[L]_s^d + \\
&\quad + \sum_{i=1}^{k-1} \binom{k-1}{i} \phi^i \int_0^t \sigma_s^{2k} d[L]_s^{d(2i)} + \sum_{i=1}^{k-1} \binom{k-1}{i} \phi^{i+1} \int_0^t \sigma_s^{2k} d[L]_s^{d(2i+2)} = \\
&= k \int_0^t \sigma_s^{2k-2}(\beta - \eta\sigma_s^2)\, ds + k\phi \int_0^t \sigma_s^{2k} d[L]_s^d \\
&\quad + \sum_{i=2}^{k-1} \left[\binom{k-1}{i} + \binom{k-1}{i-1} \right] \phi^i \int_0^t \sigma_s^{2k} d[L]_s^{d(2i)} + \phi^k \int_0^t \sigma_s^{2k} d[L]_s^{d(2k)}
\end{aligned}$$

by the well known Pascal's rule for the binomial coefficients we obtain (5).

The identity (6) was proven for $k = 1$ and $k = 2$ in [Haug et al., 2007]. For any $k > 2$ it follows by induction writing G_t^{2k} as $G_t^2 G_t^{2(k-1)}$ and applying Ito's product formula. Algebraic manipulations with repeated use of Pascal's rule are needed to simplify the coefficients. □

Lemma 3.2. *Let us assume Condition 2.1 and let for every integer $k \geq 2$, $\mathbb{E}(L_1^{2k}) < \infty$, $\Psi(k) < 0$, and for any integer $2 \leq i < k$, $\int_{\mathbb{R}} x^{2i-1} d\nu(x) = 0$. Then for every integer $1 \leq i \leq k-1$ we have*

$$\mathbb{E}(G_t^{2i} \sigma_t^{2(k-i)}) = e^{t\Psi(k-i)} \int_0^t C_{ki}(s) e^{-s\Psi(k-i)} ds \tag{7}$$

where

$$C_{ki}(t) = \beta(k-i)\mathbb{E}\left(G_t^{2i}\sigma_t^{2(k-i)-2}\right) + \sum_{j=1}^{i}\binom{2i}{2j}\mathbb{E}\left(G_t^{2i-2j}\sigma_t^{2(k-i)+2j}\right)\mathbb{E}\left([L]_1^{(2j)}\right)$$
$$+\sum_{j=1}^{k-i}\binom{k-i}{j}\sum_{h=1}^{i}\binom{2i}{2h}\phi^j\,\mathbb{E}\left([L]_1^{(2j+2h)}\right)\mathbb{E}\left(G_t^{2i-2h}\sigma_t^{2(k-i)+2h}\right).$$

Proof. Indeed, for the Ito product formula and for Lemma 3.1 we can write

$$G_t^{2i}\sigma_t^{2(k-i)} = \int_0^t G_s^{2i}d\sigma_s^{2(k-i)} + \int_0^t \sigma_s^{2(k-i)}dG_s^{2i} + [G^{2i},\sigma^{2(k-i)}]_t$$
$$= (k-i)\int_0^t G_s^{2i}\sigma_s^{2(k-i)-2}(\beta-\eta\sigma_s^2)\,ds + \sum_{j=1}^{k-i}\binom{k-i}{j}\phi^j\int_0^t G_s^{2i}\sigma_s^{2(k-i)}d[L]_s^{d(2j)}$$
$$+\sum_{j=1}^{2i}\binom{2i}{j}\int_0^t G_s^{2i-j}\sigma_s^{2(k-i)+j}d[L]_s^{(j)}+$$
$$+\left[\sum_{j=1}^{k-i}\binom{k-i}{j}\phi^j\int_0^t\sigma_s^{2(k-i)}d[L]_s^{d(2j)},\sum_{h=1}^{2i}\binom{2i}{h}\int_0^t G_s^{2i-h}\sigma_s^h d[L]_s^{(h)}\right]$$
$$= -(k-i)\eta\int_0^t G_s^{2i}\sigma_s^{2(k-i)}\,ds + \sum_{j=1}^{k-i}\binom{k-i}{j}\phi^j\int_0^t G_s^{2i}\sigma_s^{2(k-i)}d[L]_s^{d(2j)}$$
$$+\beta(k-i)\int_0^t G_s^{2i}\sigma_s^{2(k-i)-2}\,ds \quad +\sum_{j=1}^{2i}\binom{2i}{j}\int_0^t G_s^{2i-j}\sigma_s^{2(k-i)+j}d[L]_s^{(j)}+ \tag{8}$$
$$+\sum_{j=1}^{k-i}\binom{k-i}{j}\sum_{h=1}^{2i}\binom{2i}{h}\phi^j\int_0^t\sigma_s^{2(k-i)+h}G_s^{2i-h}d[L]_s^{(2j+h)}.$$

Now, taking the expectation, applying the compensation formula (see for example [Kyprianou, 2006, Theorem 4.4]) differentiating with respect to t, remembering (3) and the hypothesis on the odd moments of the measure ν, we obtain

$$\frac{d}{dt}\mathbb{E}\left(G_t^{2i}\sigma_t^{2(k-i)}\right) =$$
$$= \Psi(k-1)\mathbb{E}\left(G_t^{2i}\sigma_t^{2(k-i)}\right)+$$
$$+\beta(k-i)\mathbb{E}\left(G_t^{2i}\sigma_t^{2(k-i)-2}\right)+\sum_{j=1}^{i}\binom{2i}{2j}\mathbb{E}\left(G_t^{2i-2j}\sigma_t^{2(k-i)+2j}\right)\mathbb{E}\left([L]_1^{(2j)}\right)+$$
$$+\sum_{j=1}^{k-i}\binom{k-i}{j}\sum_{h=1}^{i}\binom{2i}{2h}\phi^j\,\mathbb{E}\left(G_t^{2i-2h}\sigma_t^{2(k-i)+2h}\right)\int_{\mathbb{R}}x^{(2j+2h)}d\nu(x).$$

Simplyfing we obtain

$$\frac{d}{dt}\mathbb{E}\left(G_t^{2i}\sigma_t^{2(k-i)}\right) = \Psi(k-i)\mathbb{E}\left(G_t^{2i}\sigma_t^{2(k-i)}\right) + C_{ki}(t),$$

and a stationary solution of this ode with initial condition $\mathbb{E}\left(G_0^{2i}\sigma_0^{2(k-i)}\right) = 0$ exists if $\Psi(k-i) < 0$ and is given by (7). This completes the proof. □

Theorem 3.3. *Let us assume Condition 2.1. Moreover, for every* $k \geq 1$ *let* $\Psi(k) < 0$ *and* $\mathbb{E}(L_1^{2k}) < \infty$. *Let moreover, for every* $c \leq k$, $\mathbb{E}([L]^{2c-1}) = \int_{\mathbb{R}} x^{(2c-1)} d\nu(x) = 0$. *Then*

$$\mathbb{E}G_t^{2k} = \sum_{i=1}^{k} \binom{2k}{2i} \mathbb{E}([L]_1^{(2i)}) \int_0^t \mathbb{E}\big(G_s^{2k-2i}\sigma_s^{2i}\big) ds$$

Proof. The result follows from (6) taking the expectation and applying the compensation formula [Kyprianou, 2006, Theorem 4.4]). □

Remark 3.4. *For* $k = 1$ *and* $k = 2$ *the moments where already calculated (see formulae (9) and (10) in [Haug et al., 2007]). We recovered equivalent expressions. Explicit calculation to obtain the moments for* $k = 3$ *and* $k = 4$ *have been derived. Since their expression are very long we include as supporting material a software implementation, that allows to calculate and manipulate the expression.*

3.3 Higher conditional moments

Conditional moments of the product are necessary not only to derive joint moments of higher order of the log returns, as we will do in the next section, but could be useful by itself and for this reason the result is presented in this section.

Theorem 3.5. *For every* k *and for any* $0 \leq i \leq k$, *for* $h > 0$, $s > 0$ *and given* $0 < v < s$, *we have*

$$\mathbf{E}_v\left[G_{s,h}^{2i}\sigma_{s+h}^{2(k-i)}\right] = \sum_{r=0}^{k} J_{kir}(h, s-v)\,\sigma_v^{2r} \tag{9}$$

where $G_{s,h}^{2i} = (G_{s+h} - G_s)^{2i}$ *and the coefficients* $J_{kir}(h,d)$ *can be calculated recursively as follows.*

First

$$J_{k0k}(h,d) = e^{(h+d)\Psi(k)}, \tag{10}$$

then for every $1 \leq i \leq k$

$$\begin{aligned} J_{k0(k-i)}(h,d) =& \frac{k!}{(k-i)!}\beta^i \int_0^{h+d} ds_i \int_0^{s_i} ds_{i-1} \cdots \\ &\cdots \int_0^{s_2} e^{(h+d-s_i)\Psi(k)+(s_i-s_{i-1})\Psi(k-1)+\cdots+s_1\Psi(k-i)} ds_1. \end{aligned} \tag{11}$$

For any fixed k and $i<k$ the coefficients $J_{kir}(h,d)$ can be derived as follows

$$J_{kik}(h,d) = e^{h\Psi(k-i)} \int_0^h e^{-w\Psi(k-i)} \Bigg[\sum_{j=1}^{i} \mathbb{E}\left([L]_1^{(2j)}\right) \binom{2i}{2j} J_{k(i-j)k}(w,d) + \\ + \sum_{m=1}^{i} \binom{2i}{2m} J_{k(i-m)k}(w,d) \sum_{j=1}^{k-i} \binom{k-i}{j} \phi^j \mathbb{E}\left([L]_1^{(2j+2m)}\right) \Bigg] \, dw. \tag{12}$$

For any $r<k$

$$J_{kir}(h,d) = e^{h\Psi(k-i)} \int_0^h e^{-w\Psi(k-i)} \Bigg[(k-i)\beta J_{(k-1)ir}(w,d) + \\ + \sum_{j=1}^{i} \mathbb{E}\left([L]_1^{(2j)}\right) \binom{2i}{2j} J_{k(i-j)r}(w,d) + \\ + \sum_{m=1}^{i} \binom{2i}{2m} J_{k(i-m)r}(w,d) \sum_{j=1}^{k-i} \binom{k-i}{j} \phi^j \mathbb{E}\left([L]_1^{(2j+2m)}\right) \Bigg] \, dw. \tag{13}$$

Finally, for any $r \le k$ we have

$$J_{kkr}(h,d) = \sum_{j=0}^{k-1} \binom{2k}{2(k-j)} \mathbb{E}[L]_1^{(2(k-j))} \int_0^h J_{kjr}(u,d) du.$$

Proof. Fix k. Let us start to prove equation (9) for $i=0$. To calculate $\mathbf{E}_v\left(\sigma_t^{2k}\right)$ we apply formula (2) with initial condition at time v. For every $v \le s_1 \le \cdots \le s_k \le t$, we have

$$\begin{aligned}
\sigma_t^{2k} &= \left(\beta \int_v^t \mathrm{e}^{-(X_t - X_s)}\, ds + \mathrm{e}^{-(X_t-X_v)} \sigma_v^2 \right)^k \\
&= \mathrm{e}^{-k(X_t-X_v)} \sigma_v^{2k} \\
&+ \sum_{i=1}^{k} \binom{k}{i} \mathrm{e}^{-(k-i)(X_t-X_v)} \sigma_v^{2(k-i)} \beta^i \int_v^t \mathrm{e}^{-(X_t-X_{s_1})}\, ds_1 \cdots \int_v^t \mathrm{e}^{-(X_t - X_{s_i})}\, ds_i \\
&= \mathrm{e}^{-k(X_t-X_v)} \sigma_v^{2k} + \sum_{j=1}^{k} \binom{k}{j} \sigma_v^{2(k-j)} \beta^j \cdot j! \cdot \\
&\int_v^t ds_j \int_v^{s_j} ds_{j-1} \cdots \int_v^{s_2} \mathrm{e}^{-k(X_t - X_{s_j})} \mathrm{e}^{-(k-1)(X_{s_j} - X_{s_{j-1}})} \cdots \mathrm{e}^{-(k-j)(X_{s_1}-X_v)} ds_1.
\end{aligned}$$

The increments $X_t - X_v$ are independent of $\mathcal{F}_v$ and of σ_v^2 which is $\mathcal{F}_v$-measurable. Time homogeneity of X_t ensures that $X_t - X_v \overset{D}{=} X_{t-v}$. Then taking the conditional expectation with respect to $\mathcal{F}_v$, by equation (3), we get

$$\mathbf{E}_v\left(\sigma_t^{2k}\right) = \mathrm{e}^{(t-v)\Psi(k)}\sigma_v^{2k} + \sum_{j=1}^{k}\binom{k}{j}\sigma_v^{2(k-j)}\beta^j \cdot j! \cdot$$

$$\cdot \int_v^t ds_j \int_v^{s_j} ds_{j-1}\cdots\int_v^{s_2} \mathrm{e}^{(t-s_j)\Psi(k)}\mathrm{e}^{(s_j-s_{j-1})\Psi(k-1)}\cdots\mathrm{e}^{(s_1-v)\Psi(k-j)}ds_1.$$

that gives the thesis once observed that the coefficients in (9) with $i = 0$ are actually dependent only on the sum of their arguments.

Now let us prove equation (9) for $i = k = 1$. By the Ito product formula

$$G_{s,h}^2 = (G_{s+h}-G_s)^2 = \left(\int_s^{s+h}\sigma_u dL_u\right)^2 = 2\int_0^h (G_{s+u}-G_s)\sigma_{s+u}dL_{s+u} + \int_0^h \sigma_{s+u}^2 d[L]_{s+u}$$

and by the compensation formula for the conditional expectation [Kyprianou, 2006, Corollary 4.5]

$$\mathbf{E}_r G_{s,h}^2 = \mathbb{E}([L]_1)\int_0^h \mathbf{E}_r(\sigma_{s+u}^2)du.$$

Now let us assume as inductive hypothesis that (9) holds for any given integer value $k \le a-1$ and for all $i \le k$. We have to show that it holds also for $k = a$ and all $i \le k$. Let us start to notice that for $k = a$ and $i = 0$ this has already been proved. So it is enough to prove that equation (9) for $k = a$ and all $i \le b-1 < a$ implies equation (9) with $k = a$ and $i = b$. For every k, by writing $G_{s,h}^{2k} = G_{s,h}^{2(k-1)}G_{s,h}^2$ and applying the Ito product formula, we have in analogy with (6)

$$G_{s,h}^{2k} = \sum_{i=1}^{2k}\binom{2k}{i}\int_0^h (G_{s+u}-G_s)^{2k-i}\sigma_{s+u}^i d[L]_{s+u}^{(i)}. \tag{14}$$

With the analogous calculations that lead to formula (8), Ito product formula guarantees that

$$\begin{aligned} G_{s,h}^{2b}\sigma_{s+h}^{2(a-b)} &= -(a-b)\eta\int_0^h G_{s,u}^{2b}\sigma_{s+u}^{2(a-b)}\,du + \sum_{j=1}^{a-b}\binom{a-b}{j}\phi^j\int_0^h G_{s,u}^{2b}\sigma_{s+u}^{2(a-b)}d[L]_{s+u}^{d(2j)} \\ &+ \beta(a-b)\int_0^h G_{s,u}^{2b}\sigma_{s+u}^{2(a-b-1)}\,du \quad + \sum_{j=1}^{2b}\binom{2b}{j}\int_0^h G_{s,u}^{2b-j}\sigma_{s+u}^{2(a-b)+j}d[L]_{s+u}^{(j)} + \\ &+ \sum_{j=1}^{(a-b)}\binom{a-b}{j}\sum_{h=1}^{2b}\binom{2b}{h}\phi^j\int_0^h G_{s,u}^{2b-h}\sigma_{s+u}^{2(a-b)+h}d[L]_{s+u}^{(2j+h)}. \end{aligned}$$

again analogously to the proof of Lemma 3.2 we can show that $\mathbf{E}_v\left(G_{s,h}^{2b}\sigma_{s+h}^{2(a-b)}\right)$ solves the following ode

$$\frac{d}{dh}\mathbf{E}_v\left(G_{s,h}^{2b}\sigma_{s+h}^{2(a-b)}\right) = \Psi(a-1)\mathbf{E}_v\left(G_{s,h}^{2b}\sigma_{s+h}^{2(a-b)}\right) + C_{ab}(h,s,v)$$

with

$$C_{ab}(h,s,v) = (a-b)\beta\mathbf{E}_v\left(G_{s,h}^{2b}\sigma_{s+h}^{2(a-b-1)}\right) + \sum_{j=1}^{b}\binom{2b}{2j}\mathbf{E}_v\left(G_{s,h}^{2(b-j)}\sigma_{s+h}^{2(a-b+j)}\right)\mathbb{E}\left([L]_1^{(2j)}\right) +$$
$$+ \sum_{m=1}^{b}\binom{2b}{2m}\mathbf{E}_v\left(G_{s,h}^{2(b-m)}\sigma_{s+h}^{2(a-b+m)}\right)\sum_{j=1}^{a-b}\binom{a-b}{j}\phi^j\mathbb{E}\left([L]_1^{(2j+2m)}\right)$$

with initial condition $\mathbf{E}_v\left(G_{s,0}^{2b}\sigma_s^{2(a-b)}\right) = 0$. Solving the ode we get

$$\mathbf{E}_v(G_{s,h}^{2b}\sigma_{s+h}^{2(a-b)}) = e^{h\Psi(a-b)}\int_0^h C_{ab}(u,s,v)e^{-u\Psi(a-b)}du \tag{15}$$

Let us now observe that by the inductive hypothesis formula (9) is true for all the conditional expectations appearing in $C_{ab}(u,s,v)$, thus

$$\mathbf{E}_v\left(G_{s,h}^{2b}\sigma_{s+h}^{2(a-b-1)}\right) = \sum_{r=0}^{a-1}J_{(a-1)br}(h,s-v)\sigma_v^{2r}$$
$$\mathbf{E}_v\left(G_{s,h}^{2(b-j)}\sigma_{s+h}^{2(a-b+j)}\right) = \sum_{r=0}^{a}J_{a(b-j)r}(h,s-v)\sigma_v^{2r}$$
$$\mathbf{E}_v\left(G_{s,h}^{2(b-m)}\sigma_{s+h}^{2(a-b+m)}\right) = \sum_{r=0}^{a}J_{a(b-m)r}(h,s-v)\sigma_v^{2r}.$$

Substituting in (15) we get that $\mathbf{E}_v(G_{s,h}^{2b}\sigma_{s+h}^{2(a-b)})$ is itself a polynomial in σ_v^2 of highest order a with coefficients given by formula (13) if $r \neq k$ of according to formula (12) if $r = k$.

To conclude the proof we need to show that if (9) is true for $k = a$ and $i \leq a-1$ then it is also true for $k = i = a$. To this aim we rewrite (14), with $k = a$

$$G_{s,h}^{2a} = \sum_{i=1}^{2a}\binom{2a}{i}\int_s^{s+h}(G_u - G_s)^{2a-i}\sigma_u^i d[L]_u^{(i)}.$$

Redefining the index of the sum as $j = a - i$ we have for all $v < s$ and $h > 0$

$$\begin{aligned}\mathbf{E}_v\left(G_{s,h}^{2a}\right) &= \sum_{j=0}^{a-1}\binom{2a}{2(a-j)}\mathbb{E}[L]_1^{(2(a-j))}\int_s^{s+h}\mathbf{E}_v\left[G_{s,h}^{2j}\sigma_u^{2(a-j)}\right]du \\ &= \sum_{r=1}^{a}\sigma_v^{2r}\left(\sum_{j=0}^{a-1}\binom{2a}{2(a-j)}\mathbb{E}[L]_1^{(2(a-j))}\int_s^{s+h}J_{ajr}(h,s-v)du\right)\end{aligned}$$

hence

$$J_{aaj}(h, s-v) = \sum_{j=0}^{a-1} \binom{2a}{2(a-j)} \mathbb{E}[L]_1^{(2(a-j))} \int_s^{s+h} J_{ajr}(h, s-v) du$$

□

Remark 3.6. *In the coefficients given by* (10) *and* (11) *the dependence from the time lags* h *and* d *came just throw the total time lag* $h + d$.

3.4 Joint Moments

We are now ready to state the main result that motivated us throw the paper

Theorem 3.7. *Fix any integer* $k \geq 1$. *Let* $\Psi(k) < 0$, $\mathbb{E}(L_1^{2k}) < \infty$ *and for every* $c \leq k$ *let* $\mathbb{E}([L]^{2c-1}) = \int_{\mathbb{R}} x^{(2c-1)} d\nu(x) = 0$. *For any integer* $h \geq 2$ *and any set of integers* $i_j \geq 0$, $j = 1, \ldots, h$ *such that* $i_1 + i_2 + \ldots + i_h = k$ *we have for every* $0 \leq t_1 < t_2 < \ldots < t_h < T$, *and* $t_j > t_{j-1} + r$ *for any* j,

$$\mathbb{E}\left(G_{t_h,r}^{2i_h} G_{t_{h-1},r}^{2i_{h-1}} \cdots G_{t_2,r}^{2i_2} G_{t_1,r}^{2i_1}\right) = \sum_{r_1=0}^{i_h} \Bigg(J_{i_h i_h r_1}(r, s_{h-1} - s_{h-2} - r)$$
$$\cdot \sum_{r_2=0}^{r_1+i_{h-1}} \Bigg\{ J_{(r_1+i_{h-1}) i_{h-1} r_2}(r, s_{h-2} - s_{h-3} - r)$$
$$\cdot \sum_{r_3=0}^{r_2+i_{h-2}} \Bigg[J_{(r_2+i_{h-2}) i_{h-2} r_3}(r, s_{h-3} - s_{h-4} - r) \cdots$$
$$\cdots \sum_{r_{h-1}=0}^{r_{h-2}+i_2} \left(J_{(r_{h-2}+i_2) i_2 r_{h-1}}(r, s_1 - r) \mathbb{E}\left(\sigma_r^{2r_{h-1}} G_r^{2i_1}\right) \right) \Bigg] \Bigg\} \Bigg),$$

where $s_{j-1} = t_j - t_1$, $j = h, \ldots, 2$.

Proof. By stationarity of $G_{t,r}$ we can write

$$\mathbb{E}\left(G_{t_h,r}^{2i_h} G_{t_{h-1},r}^{2i_{h-1}} \cdots G_{t_2,r}^{2i_2} G_{t_1,r}^{2i_1}\right) = \mathbb{E}\left(G_{s_{h-1},r}^{2i_h} G_{s_{h-2},r}^{2i_{h-1}} \cdots G_{s_1,r}^{2i_2} G_r^{2i_1}\right)$$

Taking the conditional expectation repeatedly in the right hand side

$$\mathbb{E}\left[\mathbf{E}_r \left\{\cdots \mathbf{E}_{s_{h-3}+r} \left[\mathbf{E}_{s_{h-2}+r}\left(G_{s_{h-1},r}^{2i_h}\right) G_{s_{h-2},r}^{2i_{h-1}}\right] G_{s_{h-3},r}^{2i_{h-2}} \cdots G_{s_1,r}^{2i_2}\right\} G_r^{2i_1}\right]$$

applying Theorem 15 we can reduce the argument of the expectation (unconditional) to be a part deterministic and a part measurable with respect to $\mathcal{F}_r$ and we get the thesis. □

4 Examples

The class of driving Lévy processes that can be considered in the COGARCH(1,1) model is very general. Compound Poisson, Normal Inverse Gaussian, Variance Gamma and Meixner processes are families of Lévy processes that for some value of their parameters are such that the moments of the COGARCH(1,1) process associate exist. Details are presented only for the Variance Gamma family.

4.1 Variance Gamma

The Variance Gamma process V_t is an infinity activity pure jump Lévy process that has been used itself to model log returns [Madan and Seneta, 1990]. The characteristic function is given by

$$\mathbb{E}\left(e^{iuV_t}\right) = \left(1 + \frac{A^2u^2}{2C}\right)^{-tC},$$

where A and C are positive parameters. The Lévy measure has density

$$\nu_L(dx) = \frac{C}{|x|}\exp\left(-\frac{|x|}{A}\sqrt{2C}\right)dx \quad x \neq 0.$$

The Variance Gamma process has finite moments of any order and a symmetric density which cannot be expressed in a closed form. Its variance is given by A^2t. If we assume that it drives (without a Brownian component) a COGARCH(1,1) model, the first of Conditions 2.1 imposes $A = 1$, while the parameter C remains free.

5 Further developments

A few methods for the estimation of the model parameters from a sample of equally spaced returns $G_{ir,r} = G_{(i+1)r} - G_{ir}$ are currently available.

In [Haug et al., 2007] explicit estimators have been derived from a method of moments (MM). In [Maller et al., 2008] a pseudo maximum likelihood (PML) method has been proposed that allows also for non equally spaced observations, and in [Müller, 2010] an MCMC-based estimation method has been presented for the model driven by a compound Poisson process.

In a forthcoming paper the method of Prediction Based Estimating Functions (PBEFs) introduced in [Sørensen, 2000] is applyed to the COGARCH(1,1) model and its performances are compared with some of the other available procedures. The general theory of PBEFs allows to find an optimal PBEF if the joint moments of the observation are explicitly known up to a certain order.

This paper was motivated by the search for an optimal PBEF, where a recursive formula for the moments $\mathbb{E}(G_t^{2i}\sigma_t^{2(k-i)})$ and $\mathbf{E}_v[G_{s,h}^{2i}\sigma_{s+h}^{2(k-i)}]$ wow necessary. Explicit expressions for any total order $2k$ and any integer $i \leq k$ and for any $t, h > 0$ and $s > v > 0$ are given in this paper.

Explicit expression for the joint moments $\mathbb{E}(G^{2i_h}_{t_h,r}\, G^{2i_{h-1}}_{t_{h-1},r} \cdots G^{2i_2}_{t_2,r}\, G^{2i_1}_{t_1,r})$ are also provided for any integers $i_1 \cdots i_h$ and hence any total order $k = i_1 + \cdots + i_h$ and for any times $t_h \cdots t_1$ such that $t_i - t_{i-1} \geq r$.

Up to the order four ($k = 2$) our formulae coincide with those of [Haug et al., 2007], but explicit expressions for the higher orders are provided as a new result whose interest might go beyond the statistical methodology proposed.

Knowing all simple and joint moments up to the order *four*, such as, for example, $\mathbb{E}\big(G^2_{jr,r}G^2_{ir,r}\big)$, $\mathbb{E}\big(G^4_{jr,r}\big)$, for any integer i, j is essential to calculate the predictors and hence to calculate any estimating function. Such explicit expressions for the COGARCH(1,1) model are given in [Haug et al., 2007]. However the asymptotic variance of the estimates involves a matrix which depends on all the simple and joint moments up to the order *eight*, e.g. $\mathbb{E}\big(G^8_{jr,r}\big)$, $\mathbb{E}\big(G^6_{jr,r}\big)$, $\mathbb{E}\big(G^2_{jr,r}G^6_{ir,r}\big)$, $\mathbb{E}\big(G^2_{ir,r}G^2_{jr,r}G^2_{kr,r}G^2_{hr,r}\big)$ and similar. Explicit expressions for such moments are currently not available and finding such expressions is the goal of Section 3. The optimal weight matrix W^* depends on all the simple and joint moments up to the order *eight*. Explicit expressions for such moments are currently not available and finding such expressions is the goal of Section 3. In term of existence of higher moments, the condition requested for the optimality for the estimators obtained via the PBEF and via the MM is the same ($\mathbb{E}_\theta(G_1^{8+\delta}) < \infty$ for some $\delta > 0$), see [Haug et al., 2007]. What is needed to calculate explicitly the estimators are the moments till the order four for the MM estimator, the moments till the order eight for the PBEF.

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