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#### Abstract

ANALYSIS AND OPTIMIZATION OF THE GENERALIZED SCHWARZ METHOD FOR ELLIPTIC PROBLEMS WITH APPLICATION TO FLUID-STRUCTURE INTERACTION


## by

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# Analysis and optimization of the generalized Schwarz method for elliptic problems with application to fluid-structure interaction 

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#### Abstract

We propose a unified convergence analysis of the generalized Schwarz method applied to a linear elliptic problem for a general interface (flat, cylindrical or spherical) in any dimension. In particular, we provide the exact convergence set of the interface symbols related to the operators involved in the transmission conditions. We also provide a general procedure to obtain estimates of the optimized interface symbols within the constants. Finally, we apply such general results to a fluid-structure interaction model problem, and we assess the effectiveness of the theoretical findings through three-dimensional numerical experiments in the haemodynamic context.


Keywords Optimized Schwarz Method • Robin interface conditions • fluid-structure interaction • haemodynamics

Mathematics Subject Classification (2000) 65N12, 42B37

## 1 Introduction

The Optimized Schwarz Method (OSM) is a domain decomposition method based on the splitting of the computational domain into subdomains, on the linear combination of the interface conditions between subdomains through

[^0]the introduction of interface operators, and on the search of optimized interface operators in a proper subset (e.g. the constants) which guarantee good convergence properties [19, 22].

This method has been considered so far for many problems in the case of flat interfaces, such as the advection-reaction-diffusion problem [14, 20], the Helmholtz equation [15, 23], the shallow-water equations [28], the Maxwell's equations [6], the fluid-structure interaction problem [16] and the scattering problem [29]. Recently, in [17] OSM has been considered and analyzed for the reaction-diffusion problem in the case of cylindrical interfaces.

In this work, we consider a general framework to analyze OSM for linear elliptic problems. This will allow to consider several situations, namely the case of flat, cylindrical and spherical interfaces, in any dimension. This is done by applying a general Fourier transform to the linear elliptic problem allowing to obtain a synthetic expression of this equation covering all the cited cases, and to write explicitly its solution.

Once we have derived a general expression for the solution of the elliptic problem, we provide the exact convergence set of the interface symbols for the generalized Schwarz method, that is the iterative algorithm obtained for general, non-optimized interface operators. Then, we propose a new optimization strategy, based on looking for optimized constant interface values along a selected curve in the space of the parameters, which is supposed to lead to good convergence properties. This allows to obtain an optimization problem with respect to one scalar parameter and to write explicitly a range of such a parameter which guarantees that the reduction factor is below a given value.

Finally, we apply the proposed analysis and optimization procedure to the fluid-structure interaction (FSI) problem, obtaining new estimates for the interface parameters. We present also some 3D numerical results both in a simplified and in a real geometry inspired by the haemodynamic applications.

The outline of this work is as follows. In Section 2 we present the general solution of the linear elliptic problem, whereas in Section 3 we provide the exact convergence set of the interface symbols. In Section 4 we present the new optimization procedure, and in Section 5.1 and 5.2 we show two applications of our strategies to problems introduced so far in the literature. In Section 5.3 we apply our results to the FSI problem, and finally in Section 6 we present the numerical results.

## 2 General solution of the elliptic problem

In this section we provide a general discussion about the solution of a linear elliptic problem. In particular, let $n$ be a positive integer, and $d$ an integer between 1 and $n$. Let $\Omega$ be a subset of $\mathbb{R}^{n}$ of the form

$$
\Omega:=\left\{(\boldsymbol{x}, \boldsymbol{y}): \boldsymbol{x} \in \mathbb{R}^{d}, a<\|\boldsymbol{x}\|<b, \boldsymbol{y} \in \mathbb{R}^{n-d}\right\}
$$

where $d \geq 2$ and $0 \leq a<b \leq+\infty$. In the case $d=1$, we set

$$
\Omega:=\left\{(x, \boldsymbol{y}): a<x<b, \boldsymbol{y} \in \mathbb{R}^{n-1}\right\},
$$



Fig. 1 Possible domain configurations. Left: $n=2, d=1$; Right: $n=3, d=2$.
so that $\boldsymbol{x}$ is a real number and we will assume $-\infty \leq a<b \leq+\infty$. For example, if $n=3, \Omega$ is a spherical shell when $d=3$, a cylindrical shell when $d=2$, and a "thick vertical wall" when $d=1$, whereas if $n=2, \Omega$ is a circular crown for $d=2$ and a vertical stripe for $d=1$, see Figure 1 .

Given a function $u$, we will use the following notation:

$$
\begin{aligned}
& \Delta_{\boldsymbol{x}} u(\boldsymbol{x}, \boldsymbol{y})=\sum_{j=1}^{d} \frac{\partial^{2} u}{\partial x_{j}^{2}}(\boldsymbol{x}, \boldsymbol{y}), \quad \Delta_{\boldsymbol{y}} u(\boldsymbol{x}, \boldsymbol{y})=\sum_{j=1}^{n-d} \frac{\partial^{2} u}{\partial y_{j}^{2}}(\boldsymbol{x}, \boldsymbol{y}), \\
& \nabla_{\boldsymbol{x}} u(\boldsymbol{x}, \boldsymbol{y})=\left(\frac{\partial u}{\partial x_{1}}(\boldsymbol{x}, \boldsymbol{y}), \ldots, \frac{\partial u}{\partial x_{d}}(\boldsymbol{x}, \boldsymbol{y})\right)^{T}, \\
& \nabla_{\boldsymbol{y}} u(\boldsymbol{x}, \boldsymbol{y})=\left(\frac{\partial u}{\partial y_{1}}(\boldsymbol{x}, \boldsymbol{y}), \ldots, \frac{\partial u}{\partial y_{n-d}}(\boldsymbol{x}, \boldsymbol{y})\right)^{T} .
\end{aligned}
$$

With this notation, given $\mu>0, \xi \in \mathbb{R}$ and $\boldsymbol{\beta} \in \mathbb{R}^{n}$, we can introduce the operator $\mathcal{L}$ as follows

$$
\mathcal{L} u=-\mu \Delta u+\boldsymbol{\beta} \cdot \nabla u+\xi u=-\mu \Delta_{\boldsymbol{x}} u-\mu \Delta_{\boldsymbol{y}} u+\boldsymbol{\beta}_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{x}} u+\boldsymbol{\beta}_{\boldsymbol{y}} \cdot \nabla_{\boldsymbol{y}} u+\xi u
$$

where $\boldsymbol{\beta}=\left(\boldsymbol{\beta}_{\boldsymbol{x}}, \boldsymbol{\beta}_{\boldsymbol{y}}\right)$, with $\boldsymbol{\beta}_{\boldsymbol{x}}=\left(\beta_{1}, \ldots, \beta_{d}\right)$ and $\boldsymbol{\beta}_{\boldsymbol{y}}=\left(\beta_{d+1}, \ldots, \beta_{n}\right)$. Throughout the paper we will assume $\boldsymbol{\beta}_{\boldsymbol{x}}=\mathbf{0}$, so that the operator reduces to

$$
\mathcal{L} u=-\mu \Delta_{\boldsymbol{x}} u-\mu \Delta_{\boldsymbol{y}} u+\boldsymbol{\beta}_{\boldsymbol{y}} \cdot \nabla_{\boldsymbol{y}} u+\xi u .
$$

We want now to write an explicit expression of the solution of the equation $\mathcal{L} u=0$. Due to the particular shape of $\Omega$, it is natural to write the Laplacian $\Delta_{\boldsymbol{x}}$ in terms of the $d$-dimensional spherical coordinates. Thus, when $d \geq 2$, writing $\boldsymbol{x}=r \boldsymbol{x}^{\prime}$, where $r=\|\boldsymbol{x}\|$ and $\boldsymbol{x}^{\prime}=\boldsymbol{x} /\|\boldsymbol{x}\|$, we have (see, e.g., [10], Lemmas 2.62 and 2.63)

$$
\begin{equation*}
\Delta_{\boldsymbol{x}} u\left(r \boldsymbol{x}^{\prime}, \boldsymbol{y}\right)=\frac{\partial^{2} u}{\partial r^{2}}\left(r \boldsymbol{x}^{\prime}, \boldsymbol{y}\right)+\frac{d-1}{r} \frac{\partial u}{\partial r}\left(r \boldsymbol{x}^{\prime}, \boldsymbol{y}\right)+\frac{1}{r^{2}}\left(\Delta_{S^{d-1}} u(r \cdot, \boldsymbol{y})\right)\left(\boldsymbol{x}^{\prime}\right), \tag{1}
\end{equation*}
$$

where $\Delta_{S^{d-1}}$ is the Laplace-Beltrami operator on the $d$-1-dimensional sphere

$$
S^{d-1}=\left\{\boldsymbol{x} \in \mathbb{R}^{d}:\|\boldsymbol{x}\|=1\right\} .
$$

If instead $d=1$, we call $r$ the variable $\boldsymbol{x}$ and we simply have

$$
\Delta_{\boldsymbol{x}} u=\frac{\partial^{2} u}{\partial r^{2}}
$$

Now, let

$$
\left\{P_{m, l}\left(\boldsymbol{x}^{\prime}\right)\right\}_{m=0, l=1}^{+\infty, k_{m}}
$$

be an orthonormal basis of spherical harmonics of the sphere $S^{d-1}$, where $k_{m}$ is the dimension of the eigenspace associated with the eigenvalue $\lambda_{m}$, whose expression is given by $\lambda_{m}=m(m+d-2), m=0, \ldots,+\infty$, see, e.g., [10], Corollary 2.55 . We have $k_{m}=(2 m+d-2) \frac{(m+d-3)!}{m!(d-2)!}$, which in the case $d=2$ leads to $k_{m}=2$ for any $m$. Then, for any function $u(\boldsymbol{x}, \boldsymbol{y})$, let

$$
\begin{equation*}
\widehat{u}(r, m, l, \boldsymbol{k})=\int_{\mathbb{R}^{n-d}}\left(\int_{S^{d-1}} u\left(r \boldsymbol{x}^{\prime}, \boldsymbol{y}\right) \overline{P_{m, l}\left(\boldsymbol{x}^{\prime}\right)} d \sigma\left(\boldsymbol{x}^{\prime}\right)\right) e^{-i \boldsymbol{y} \cdot \boldsymbol{k}} d \boldsymbol{y} \tag{2}
\end{equation*}
$$

be its Fourier transform with respect to $\boldsymbol{x}^{\prime}$ and $\boldsymbol{y}$. Notice that the frequency variable $\boldsymbol{k}$ related to the spatial variable $\boldsymbol{y}$ is continuous, whereas the frequency variable $m$ related to the spatial variable $\boldsymbol{x}^{\prime}$ is discrete, since $S^{d-1}$ is a compact manifold.

Applying the transform (2) to (1), we obtain

$$
\begin{aligned}
& \widehat{\Delta_{x} u}(r, m, l, \boldsymbol{k})= \\
& \qquad \frac{\partial^{2} \widehat{u}}{\partial r^{2}}(r, m, l, \boldsymbol{k})+\frac{d-1}{r} \frac{\partial \widehat{u}}{\partial r}(r, m, l, \boldsymbol{k})-\frac{m(m+d-2)}{r^{2}} \widehat{u}(r, m, l, \boldsymbol{k}),
\end{aligned}
$$

and then

$$
\begin{aligned}
& \widehat{\mathcal{L u} u}(r, m, l, \boldsymbol{k}) \\
& \begin{aligned}
&=-\mu\left(\frac{\partial^{2} \widehat{u}}{\partial r^{2}}(r, m, l, \boldsymbol{k})+\frac{d-1}{r} \frac{\partial \widehat{u}}{\partial r}(r, m, l, \boldsymbol{k})-\frac{m(m+d-2)}{r^{2}} \widehat{u}(r, m, l, \boldsymbol{k})\right) \\
&+\|\boldsymbol{k}\|^{2} \mu \widehat{u}(r, m, l, \boldsymbol{k})-i \boldsymbol{\beta}_{\boldsymbol{y}} \cdot \boldsymbol{k} \widehat{u}(r, m, l, \boldsymbol{k})+\xi \widehat{u}(r, m, l, \boldsymbol{k}) .
\end{aligned}
\end{aligned}
$$

Then, the equation $\mathcal{L}=0$ becomes

$$
\begin{equation*}
\frac{\partial^{2} \widehat{u}}{\partial r^{2}}+\frac{d-1}{r} \frac{\partial \widehat{u}}{\partial r}-\left(\frac{m(m+d-2)}{r^{2}}+\alpha^{2}\right) \widehat{u}=0 \tag{3}
\end{equation*}
$$

where

$$
\alpha=\left(\|\boldsymbol{k}\|^{2}+\frac{\xi}{\mu}-i \frac{\boldsymbol{\beta}_{\boldsymbol{y}}}{\mu} \cdot \boldsymbol{k}\right)^{1 / 2}
$$

where $\gamma^{1 / 2}$ is the square root of $\gamma \in \mathbb{C}$ satisfying the condition $0 \leq \arg (\gamma)<\pi$. Of course, if $d=n$, the Fourier transform in the variable $\boldsymbol{y}$ disappears and we have to take $\boldsymbol{k}=\mathbf{0}$, so that $\alpha=\left(\frac{\xi}{\mu}\right)^{1 / 2}$ and the equation (3) becomes

$$
\frac{\partial^{2} \widehat{u}}{\partial r^{2}}+\frac{d-1}{r} \frac{\partial \widehat{u}}{\partial r}-\left(\frac{m(m+d-2)}{r^{2}}+\frac{\xi}{\mu}\right) \widehat{u}=0
$$

Notice that we considered homogeneous forcing term since in the convergence analyses reported in the next sections we will analyze without loss of generality the convergence to the zero solution. Observe also that the equation $\mathcal{L}=0$ needs to be equipped with suitable boundary conditions on $\partial \Omega$, see Section 3. Finally, we notice that in the case $d=2$ (cylindrical domain for $n=3$ ) equation (3) is exactly the one discussed in [17] whose solutions are the modified Bessel functions, see Section 2.2.

### 2.1 Solution of the case $d=1$.

When $d=1$, the Fourier transform in the variable $\boldsymbol{x}^{\prime}$ disappears and in (3) we have to take $m=0$, obtaining

$$
\frac{\partial^{2} \widehat{u}}{\partial r^{2}}-\alpha^{2} \widehat{u}=0
$$

If $\alpha \neq 0$, the solutions of this equation are simply

$$
\widehat{u}(r, \boldsymbol{k})=X_{1}(\boldsymbol{k}) e^{\alpha r}+X_{2}(\boldsymbol{k}) e^{-\alpha r}
$$

for suitable functions $X_{1}$ and $X_{2}$ determined by the boundary conditions. In particular, if $\boldsymbol{\beta}_{\boldsymbol{y}}=\mathbf{0}$, we have

$$
\widehat{u}(r, \boldsymbol{k})=X_{1}(\boldsymbol{k}) e^{r\left(\|\boldsymbol{k}\|^{2}+\frac{\xi}{\mu}\right)^{1 / 2}}+X_{2}(\boldsymbol{k}) e^{-r\left(\|\boldsymbol{k}\|^{2}+\frac{\xi}{\mu}\right)^{1 / 2}}
$$

If $\xi>0$, then $\|\boldsymbol{k}\|^{2}+\frac{\xi}{\mu}>0$ and the solutions are

$$
\widehat{u}(r, \boldsymbol{k})=X_{1}(\boldsymbol{k}) e^{r \sqrt{\|\boldsymbol{k}\|^{2}+\frac{\xi}{\mu}}}+X_{2}(\boldsymbol{k}) e^{-r \sqrt{\|\boldsymbol{k}\|^{2}+\frac{\xi}{\mu}}}
$$

where we have used the symbol $\sqrt{\gamma}$ to indicate the square root of a real nonnegative number $\gamma$. If, on the contrary, $\xi \leq 0$, then we have three possibilities, according to $\|\boldsymbol{k}\|$ :

$$
\widehat{u}(r, \boldsymbol{k})= \begin{cases}X_{1}(\boldsymbol{k}) e^{r \sqrt{\|\boldsymbol{k}\|^{2}+\frac{\xi}{\mu}}}+X_{2}(\boldsymbol{k}) e^{-r \sqrt{\|\boldsymbol{k}\|^{2}+\frac{\xi}{\mu}}} & \text { if }\|\boldsymbol{k}\|^{2}>-\frac{\xi}{\mu} \\ X_{1}(\boldsymbol{k})+X_{2}(\boldsymbol{k}) r & \text { if }\|\boldsymbol{k}\|^{2}=-\frac{\xi}{\mu} \\ X_{1}(\boldsymbol{k}) \cos \left(r \sqrt{-\|\boldsymbol{k}\|^{2}-\frac{\xi}{\mu}}\right) & \\ \quad+X_{2}(\boldsymbol{k}) \sin \left(r \sqrt{-\|\boldsymbol{k}\|^{2}-\frac{\xi}{\mu}}\right) & \text { if }\|\boldsymbol{k}\|^{2}<-\frac{\xi}{\mu}\end{cases}
$$

2.2 Solution of the case $d>1$.

The equation in this case is (3). If $\alpha \neq 0$, with the change of variables

$$
\widehat{u}(r)=v(\alpha r) r^{-\frac{d-2}{2}}
$$

this equation becomes

$$
v^{\prime \prime}(t)+\frac{1}{t} v^{\prime}(t)-\left(1+\frac{\left(m+\frac{d-2}{2}\right)^{2}}{t^{2}}\right) v(t)=0
$$

This is the modified Bessel equation, and the solutions are

$$
v(t)=X_{1} I_{m+\frac{d-2}{2}}(t)+X_{2} K_{m+\frac{d-2}{2}}(t),
$$

where $I_{\nu}$ and $K_{\nu}$ are the modified Bessel functions, see [21]. Thus, we have

$$
\begin{equation*}
\widehat{u}(r, m, l, \boldsymbol{k})=X_{1}(m, l, \boldsymbol{k}) \frac{I_{m+\frac{d-2}{2}}(\alpha r)}{r^{\frac{d-2}{2}}}+X_{2}(m, l, \boldsymbol{k}) \frac{K_{m+\frac{d-2}{2}}(\alpha r)}{r^{\frac{d-2}{2}}} \tag{4}
\end{equation*}
$$

Once again, let us look closely to the case $\boldsymbol{\beta}_{\boldsymbol{y}}=\mathbf{0}$, so that $\alpha=\left(\|\boldsymbol{k}\|^{2}+\frac{\xi}{\mu}\right)^{1 / 2}$. If $\xi>0$, then $\alpha=\sqrt{\|\boldsymbol{k}\|^{2}+\frac{\xi}{\mu}}$ and the solutions are

$$
\begin{aligned}
& \widehat{u}(r, m, l, \boldsymbol{k}) \\
= & X_{1}(m, l, \boldsymbol{k}) \frac{I_{m+\frac{d-2}{2}}\left(r \sqrt{\|\boldsymbol{k}\|^{2}+\frac{\xi}{\mu}}\right)}{r^{\frac{d-2}{2}}}+X_{2}(m, l, \boldsymbol{k}) \frac{K_{m+\frac{d-2}{2}}\left(r \sqrt{\|\boldsymbol{k}\|^{2}+\frac{\xi}{\mu}}\right)}{r^{\frac{d-2}{2}}} .
\end{aligned}
$$

If, on the contrary, $\xi \leq 0$, then we have three possibilities, according to $\|\boldsymbol{k}\|$ : if $\|\boldsymbol{k}\|^{2}>-\frac{\xi}{\mu}$, then

$$
\begin{aligned}
& \widehat{u}(r, m, l, \boldsymbol{k}) \\
= & X_{1}(m, l, \boldsymbol{k}) \frac{I_{m+\frac{d-2}{2}}\left(r \sqrt{\|\boldsymbol{k}\|^{2}+\frac{\xi}{\mu}}\right)}{r^{\frac{d-2}{2}}}+X_{2}(m, l, \boldsymbol{k}) \frac{K_{m+\frac{d-2}{2}}\left(r \sqrt{\|\boldsymbol{k}\|^{2}+\frac{\xi}{\mu}}\right)}{r^{\frac{d-2}{2}}} ;
\end{aligned}
$$

if $\|\boldsymbol{k}\|^{2}=-\frac{\xi}{\mu}$, then

$$
\begin{aligned}
\widehat{u}(r, m, l, \boldsymbol{k})=X_{1}(m, l, \boldsymbol{k}) \frac{r^{m+\frac{d-2}{2}}}{r^{\frac{d-2}{2}}}+X_{2}(m, l, \boldsymbol{k}) \frac{r^{-\left(m+\frac{d-2}{2}\right)}}{r^{\frac{d-2}{2}}} \\
=X_{1}(m, l, \boldsymbol{k}) r^{m}+X_{2}(m, l, \boldsymbol{k}) r^{-d-m+2}
\end{aligned}
$$

if $\|\boldsymbol{k}\|^{2}<-\frac{\xi}{\mu}$, then

$$
\begin{aligned}
& \widehat{u}(r, m, l, \boldsymbol{k})=X_{1}(m, l, \boldsymbol{k}) \frac{J_{m+\frac{d-2}{2}}\left(r \sqrt{-\|\boldsymbol{k}\|^{2}-\frac{\xi}{\mu}}\right)}{r^{\frac{d-2}{2}}} \\
&+X_{2}(m, l, \boldsymbol{k}) \frac{Y_{m+\frac{d-2}{2}}\left(r \sqrt{-\|\boldsymbol{k}\|^{2}-\frac{\xi}{\mu}}\right)}{r^{\frac{d-2}{2}}} .
\end{aligned}
$$

Observe that in the case $\|\boldsymbol{k}\|^{2}<-\frac{\xi}{\mu}$, in order to avoid complications due to the presence of complex valued functions, we have taken a different expression than the one coming from (4), based on the introduction of the Bessel functions $J_{\nu}$ and $Y_{\nu}$, see [21].

In all the cases considered in the previous two subsections, the solution of (3) has the general form

$$
\begin{equation*}
\widehat{u}(r, m, l, \boldsymbol{k})=X_{1}(m, l, \boldsymbol{k}) g_{1}(r, m, \boldsymbol{k})+X_{2}(m, l, \boldsymbol{k}) g_{2}(r, m, \boldsymbol{k}) . \tag{5}
\end{equation*}
$$

for suitable functions $X_{1}$ and $X_{2}$ determined by the boundary conditions.

## 3 Convergence analysis of the generalized Schwarz method

For any real number $L, a \leq L \leq b$, let $\Sigma_{L}$ be defined by

$$
\Sigma_{L}:=\left\{(\boldsymbol{x}, \boldsymbol{y}): \boldsymbol{x} \in \mathbb{R}^{d},\|\boldsymbol{x}\|=L, \boldsymbol{y} \in \mathbb{R}^{n-d}\right\}
$$

This is a surface for $n=3$ and a curve for $n=2$. We fix now a real number $R, a<R<b$. Then, $\Sigma_{R}$ divides $\Omega$ into two non-overlapping subdomains, namely

$$
\begin{aligned}
\Omega_{1} & :=\left\{(\boldsymbol{x}, \boldsymbol{y}): \boldsymbol{x} \in \mathbb{R}^{d}, a<\|\boldsymbol{x}\|<R, \boldsymbol{y} \in \mathbb{R}^{n-d}\right\}, \\
\Omega_{2} & :=\left\{(\boldsymbol{x}, \boldsymbol{y}): \boldsymbol{x} \in \mathbb{R}^{d}, R<\|\boldsymbol{x}\|<b, \boldsymbol{y} \in \mathbb{R}^{n-d}\right\},
\end{aligned}
$$

see Figure 2. In particular, we have the following cases: $n=2, d=1$, straight line interface; $n=3, d=1$, plane interface; $n=2, d=2$, circular interface; $n=3, d=2$, cylindrical interface; $n=3, d=3$, spherical interface.

We are here interested in the solution of the following problem

$$
\begin{cases}\mathcal{L} u=0 & (\boldsymbol{x}, \boldsymbol{y}) \in \Omega \\ \gamma u+\mu \frac{\partial u}{\partial r}=0 & (\boldsymbol{x}, \boldsymbol{y}) \in \Sigma_{a} \\ \gamma u+\mu \frac{\partial u}{\partial r}= & 0(\boldsymbol{x}, \boldsymbol{y}) \in \Sigma_{b}\end{cases}
$$




Fig. 2 Possible splitting of the domain. Left: $n=2, d=1$; Right: $n=3, d=2$.
where as usual $\mathcal{L}=-\mu \Delta+\boldsymbol{\beta} \cdot \nabla+\xi$ and with $\gamma \in \mathbb{R}$. We suppose that the domain $\Omega$ is subdivided into the two non-overlapping subdomains $\Omega_{1}$ and $\Omega_{2}$. Then, the previous problem is equivalent to the following coupled problem:

$$
\begin{cases}\mathcal{L}_{1} u_{1}=0 & (\boldsymbol{x}, \boldsymbol{y}) \in \Omega_{1}  \tag{6}\\ \gamma_{1} u_{1}+\mu_{1} \frac{\partial u_{1}}{\partial r}=0 & (\boldsymbol{x}, \boldsymbol{y}) \in \Sigma_{a} \\ u_{1}=\delta u_{2}+(1-\delta) \kappa_{D} \mu_{2} \frac{\partial u_{2}}{\partial r} & (\boldsymbol{x}, \boldsymbol{y}) \in \Sigma_{R} \\ \mu_{1} \frac{\partial u_{1}}{\partial r}=\delta \mu_{2} \frac{\partial u_{2}}{\partial r}+(1-\delta) \kappa_{N} u_{2} & (\boldsymbol{x}, \boldsymbol{y}) \in \Sigma_{R} \\ \gamma_{2} u_{2}+\mu_{2} \frac{\partial u_{2}}{\partial r}=0 & (\boldsymbol{x}, \boldsymbol{y}) \in \Sigma_{b} \\ \mathcal{L}_{2} u_{2}=0 & (\boldsymbol{x}, \boldsymbol{y}) \in \Omega_{2}\end{cases}
$$

where $\mathcal{L}_{i}:=-\mu_{i} \Delta+\boldsymbol{\beta}_{i} \cdot \nabla+\xi_{i}, i=1,2, \mu_{i}, \boldsymbol{\beta}_{i}, \xi_{i}, \gamma_{i}$ are constant within each subdomain, but they could in principle assume different values in the two subdomains, $\kappa_{D}, \kappa_{N} \in \mathbb{R}$ account for possible heterogeneous couplings, $\delta=0,1$, and $\partial / \partial r$ is the directional derivative with respect to the outward unit normal to $\Sigma_{a}, \Sigma_{R}$ or $\Sigma_{b}$. The interface conditions $(6)_{3-4}$ state the continuity of $u=\left(u_{1}, u_{2}\right)$ and of the tractions. Observe that the case $\delta=0$ arises for example when coupling the Darcy problem with the wave equation, see Section 5.3. We observe also that we prescribed Robin conditions on the physical boundaries $\Sigma_{a}$ and $\Sigma_{b}$, to make the discussion as general as possible. If $d<n, \Omega_{1}$ and $\Omega_{2}$ are unbounded in the $\boldsymbol{y}$ directions, so that we require that the corresponding solution decays for $\|\boldsymbol{y}\| \rightarrow+\infty$. Analogously, if $\Omega_{1}$ and/or $\Omega_{2}$ are unbounded in the $\boldsymbol{x}$ directions, that is if $d=1$ and $a=-\infty$, or $b=+\infty$, we again require that the corresponding solution decays at infinity $\left(\gamma_{i}=+\infty\right)$. When $d \geq 2$ and $a=0$, condition $(6)_{2}$ on $\Sigma_{a}$ should be replaced with

$$
\begin{equation*}
\int_{\mathbb{R}^{n-d}} \int_{S^{d-1}}\left|u_{1}\left(r \boldsymbol{x}^{\prime}, \boldsymbol{y}\right)\right| d \sigma\left(\boldsymbol{x}^{\prime}\right) d \boldsymbol{y} \quad \text { bounded as } r \rightarrow 0^{+} . \tag{7}
\end{equation*}
$$

By linearly combining the interface conditions $(6)_{3-4}$, through the linear operators $\mathcal{S}_{i}, i=1,2$, acting in the tangential direction to $\Sigma_{R}$, we obtain the following equivalent coupled problem [8,22]:

$$
\begin{cases}\mathcal{L}_{1} u_{1}=0 & (\boldsymbol{x}, \boldsymbol{y}) \in \Omega_{1}  \tag{8}\\ \gamma_{1} u_{1}+\mu_{1} \frac{\partial u_{1}}{\partial r}=0 & (\boldsymbol{x}, \boldsymbol{y}) \in \Sigma_{a} \\ \mathcal{S}_{1} u_{1}+\mu_{1} \frac{\partial u_{1}}{\partial r} & \\ \quad=\delta \mathcal{S}_{1} u_{2}+(1-\delta) \mathcal{S}_{1} \kappa_{D} \mu_{2} \frac{\partial u_{2}}{\partial r}+\delta \mu_{2} \frac{\partial u_{2}}{\partial r}+(1-\delta) \kappa_{N} u_{2}(\boldsymbol{x}, \boldsymbol{y}) \in \Sigma_{R} \\ \delta \mathcal{S}_{2} u_{2}+(1-\delta) \mathcal{S}_{2} \kappa_{D} \mu_{2} \frac{\partial u_{2}}{\partial r}+\delta \mu_{2} \frac{\partial u_{2}}{\partial r}+(1-\delta) \kappa_{N} u_{2} & \\ \quad=\mathcal{S}_{2} u_{1}+\mu_{1} \frac{\partial u_{1}}{\partial r} & (\boldsymbol{x}, \boldsymbol{y}) \in \Sigma_{R} \\ \gamma_{2} u_{2}+\mu_{2} \frac{\partial u_{2}}{\partial r}=0 & (\boldsymbol{x}, \boldsymbol{y}) \in \Sigma_{b} \\ \mathcal{L}_{2} u_{2}=0 & (\boldsymbol{x}, \boldsymbol{y}) \in \Omega_{2}\end{cases}
$$

To solve problem (8) we consider the following generalized Schwarz method at iteration $j$ :

$$
\begin{align*}
& \begin{cases}\mathcal{L}_{1} u_{1}^{j}=0 & (\boldsymbol{x}, \boldsymbol{y}) \in \Omega_{1}, \\
\gamma_{1} u_{1}^{j}+\frac{\partial u_{1}^{j}}{\partial r}=0 & (\boldsymbol{x}, \boldsymbol{y}) \in \Sigma_{a}, \\
\mathcal{S}_{1} u_{1}^{j}+\mu_{1} \frac{\partial u_{1}^{j}}{\partial r} & \\
=\delta \mathcal{S}_{1} u_{2}^{j-1}+(1-\delta) \mathcal{S}_{1} \kappa_{D} \mu_{2} \frac{\partial u_{2}^{j-1}}{\partial r}+\delta \mu_{2} \frac{\partial u_{2}^{j-1}}{\partial r}+(1-\delta) \kappa_{N} u_{2}^{j-1}(\boldsymbol{x}, \boldsymbol{y}) \in \Sigma_{R},\end{cases} \\
& \begin{cases}\mathcal{L}_{2} u_{2}^{j}=0 & (\boldsymbol{x}, \boldsymbol{y}) \in \Omega_{2}, \\
\gamma_{2} u_{2}^{j}+\mu_{2} \frac{\partial u_{2}^{j}}{\partial r}=0 & (\boldsymbol{x}, \boldsymbol{y}) \in \Sigma_{b} \\
\delta \mathcal{S}_{2} u_{2}^{j}+(1-\delta) \mathcal{S}_{2} \kappa_{D} \mu_{2} \frac{\partial u_{2}^{j}}{\partial r}+\delta \mu_{2} \frac{\partial u_{2}^{j}}{\partial r}+(1-\delta) \kappa_{N} u_{2}^{j} & (\boldsymbol{x}, \boldsymbol{y}) \in \Sigma_{R} .\end{cases} \tag{9}
\end{align*}
$$

By applying the transform (2) to the previous iterations and thanks to the boundary conditions, the solution of each of the two equations in (9) has the general form of (5), that is

$$
\begin{equation*}
{\widehat{u_{i}}}^{j}(r, m, l, \boldsymbol{k})=X_{i}^{j}(m, l, \boldsymbol{k}) g_{i}(r, m, \boldsymbol{k}), \quad i=1,2 \tag{10}
\end{equation*}
$$

for suitable functions $X_{i}^{j}$ and $g_{i}$. Then, we have the following results.
Proposition 1. Let

$$
\begin{align*}
A(m, \boldsymbol{k}) & :=\frac{-\delta \mu_{2} \frac{\partial g_{2}}{\partial r}(R, m, \boldsymbol{k})-(1-\delta) \kappa_{N} g_{2}(R, m, \boldsymbol{k})}{\delta g_{2}(R, m, \boldsymbol{k})+(1-\delta) \kappa_{D} \mu_{2} \frac{\partial g_{2}}{\partial r}(R, m, \boldsymbol{k})}  \tag{11}\\
B(m, \boldsymbol{k}) & :=-\frac{\mu_{1}}{g_{1}(R, m, \boldsymbol{k})} \frac{\partial g_{1}}{\partial r}(R, m, \boldsymbol{k})
\end{align*}
$$

Then, the reduction factor related to iterations (9) is given by

$$
\begin{equation*}
\rho(m, l, \boldsymbol{k})=\left|\frac{\sigma_{1}(m, l, \boldsymbol{k})-A(m, \boldsymbol{k})}{\sigma_{2}(m, l, \boldsymbol{k})-A(m, \boldsymbol{k})} \cdot \frac{\sigma_{2}(m, l, \boldsymbol{k})-B(m, \boldsymbol{k})}{\sigma_{1}(m, l, \boldsymbol{k})-B(m, \boldsymbol{k})}\right|, \tag{12}
\end{equation*}
$$

where $\sigma_{i}$ are the symbols related to the operators $\mathcal{S}_{i}, i=1,2$.
Proof We start by noticing that the reduction factor could be defined as $\rho=$ $\left|\frac{X_{2}^{j}}{X_{2}^{j-1}}\right|$, see, e.g., [14]. After the application of the Fourier transform to problem (9), we have that the interface conditions $(9)_{3,6} \mathrm{read}$ as follows:

$$
\begin{aligned}
& \sigma_{1}{\widehat{u_{1}}}^{j}(R)+\mu_{1} \frac{\partial{\widehat{u_{1}}}^{j}}{\partial r}(R) \\
& =\left(\delta \sigma_{1}+(1-\delta) \kappa_{N}\right){\widehat{u_{2}}}^{j-1}(R)+\left((1-\delta) \sigma_{1} \kappa_{D}+\delta\right) \mu_{2} \frac{\partial{\widehat{u_{2}}}^{j-1}}{\partial r}(R), \\
& \left(\delta \sigma_{2}+(1-\delta) \kappa_{N}\right){\widehat{u_{2}}}^{j}(R)+\left((1-\delta) \sigma_{2} \kappa_{D}+\delta\right) \mu_{2} \frac{\partial{\widehat{u_{2}}}^{j}}{\partial r}(R) \\
& =\sigma_{2}{\widehat{u_{1}}}^{j}(R)+\mu_{1} \frac{\partial{\widehat{u_{1}}}^{j}}{\partial r}(R) \text {. }
\end{aligned}
$$

Then, substituting the solutions (10) into the above interface conditions and eliminating $X_{1}^{j}$, the thesis easily follows.

In the following result, we provide the exact convergence sets for the symbols $\sigma_{1}$ and $\sigma_{2}$.

Theorem 1. Fix $m$ and $\boldsymbol{k}$. Then, under the assumption $A>B$, with $A$ and $B$ given by (11), the inequality

$$
\begin{equation*}
|\rho(m, l, \boldsymbol{k})|=\left|\frac{\sigma_{1}(m, l, \boldsymbol{k})-A(m, \boldsymbol{k})}{\sigma_{2}(m, l, \boldsymbol{k})-A(m, \boldsymbol{k})} \cdot \frac{\sigma_{2}(m, l, \boldsymbol{k})-B(m, \boldsymbol{k})}{\sigma_{1}(m, l, \boldsymbol{k})-B(m, \boldsymbol{k})}\right|<1 \tag{13}
\end{equation*}
$$

holds if and only if $\left(\sigma_{1}, \sigma_{2}\right) \in \Theta(A, B):=\Theta_{1}(A, B) \cup \Theta_{2}(A, B)$, where

$$
\begin{align*}
& \Theta_{1}(A, B)=\left\{\left(\sigma_{1}, \sigma_{2}\right): \sigma_{2}<\sigma_{1} \text { and }\left(\sigma_{1}-\frac{A+B}{2}\right)\left(\sigma_{2}-\frac{A+B}{2}\right)<\left(\frac{B-A}{2}\right)^{2}\right\} \\
& \Theta_{2}(A, B)=\left\{\left(\sigma_{1}, \sigma_{2}\right): \sigma_{2}>\sigma_{1} \text { and }\left(\sigma_{1}-\frac{A+B}{2}\right)\left(\sigma_{2}-\frac{A+B}{2}\right)>\left(\frac{B-A}{2}\right)^{2}\right\} \tag{14}
\end{align*}
$$

Furthermore, $|\rho|=0$ if and only if $\sigma_{1}(m, l, \boldsymbol{k})=\sigma_{1}^{o p t}(m, \boldsymbol{k}):=A(m, \boldsymbol{k})$ and $\sigma_{2}(m, l, \boldsymbol{k}) \neq A(m, \boldsymbol{k})$, or $\sigma_{2}(m, l, \boldsymbol{k})=\sigma_{2}^{o p t}(m, \boldsymbol{k}):=B(m, \boldsymbol{k})$ and $\sigma_{1}(m, l, \boldsymbol{k}) \neq$ $B(m, \boldsymbol{k})$, whereas $|\rho|=1$ if and only if $\left(\sigma_{1}, \sigma_{2}\right) \in \partial \Theta(A, B) \backslash\{(A, A),(B, B)\}$.

Proof Inequality (13) can be rewritten as

$$
\begin{aligned}
\left|\left(\sigma_{1}-A\right)\left(\sigma_{2}-B\right)\right| & <\left|\left(\sigma_{1}-B\right)\left(\sigma_{2}-A\right)\right|, \\
\left|\sigma_{1} \sigma_{2}-A \sigma_{2}-B \sigma_{1}+A B\right| & <\left|\sigma_{1} \sigma_{2}-A \sigma_{1}-B \sigma_{2}+A B\right| .
\end{aligned}
$$

Now, by analyzing the sign of the two terms, we have that the above inequality becomes

$$
\left\{\begin{array}{r}
(A-B)\left(\sigma_{1}-\sigma_{2}\right)<0 \\
\text { if }\left(\sigma_{1}-A\right)\left(\sigma_{2}-B\right) \geq 0 \text { and }\left(\sigma_{1}-B\right)\left(\sigma_{2}-A\right)>0 \\
2 \sigma_{1} \sigma_{2}-(A+B) \sigma_{1}-(A+B) \sigma_{2}+2 A B<0 \\
\text { if }\left(\sigma_{1}-A\right)\left(\sigma_{2}-B\right) \geq 0 \text { and }\left(\sigma_{1}-B\right)\left(\sigma_{2}-A\right)<0 \\
2 \sigma_{1} \sigma_{2}-(A+B) \sigma_{1}-(A+B) \sigma_{2}+2 A B>0 \\
\text { if }\left(\sigma_{1}-A\right)\left(\sigma_{2}-B\right)<0 \text { and }\left(\sigma_{1}-B\right)\left(\sigma_{2}-A\right)>0 \\
(A-B)\left(\sigma_{1}-\sigma_{2}\right)>0 \\
\text { if }\left(\sigma_{1}-A\right)\left(\sigma_{2}-B\right)<0 \text { and }\left(\sigma_{1}-B\right)\left(\sigma_{2}-A\right)<0
\end{array}\right.
$$

By exploiting the assumption $A>B$, we obtain

$$
\begin{cases}\sigma_{1}-\sigma_{2}<0 & \text { if } \sigma_{1} \geq A, \sigma_{2}>A \text { or } \sigma_{1}<B, \sigma_{2} \leq B  \tag{15}\\ 2 \sigma_{1} \sigma_{2}-(A+B) & \sigma_{1}-(A+B) \sigma_{2}+2 A B<0 \\ & \text { if } B \leq \sigma_{2}<A, \sigma_{1} \geq A \text { or } B<\sigma_{1} \leq A, \sigma_{2} \leq B \\ 2 \sigma_{1} \sigma_{2}-(A+B) & \sigma_{1}-(A+B) \sigma_{2}+2 A B>0 \\ & \text { if } B<\sigma_{2}<A, \sigma_{1}<B \text { or } B<\sigma_{1}<A, \sigma_{2}>A \\ \sigma_{1}-\sigma_{2}>0 & \text { if } \sigma_{1}>A, \sigma_{2}<B \text { or } \sigma_{1}<B, \sigma_{2}>A, \text { or } B<\sigma_{1}, \sigma_{2}<A .\end{cases}
$$

Now, if $\sigma_{2}<\sigma_{1}$ the above reduces to

$$
\begin{cases}\sigma_{1}-\sigma_{2}<0 & \text { if } \sigma_{1}>\sigma_{2}>A, \text { or } \sigma_{2}<\sigma_{1}<B \\ 2 \sigma_{1} \sigma_{2}-(A+B) & \sigma_{1}-(A+B) \sigma_{2}+2 A B<0 \\ & \text { if } B \leq \sigma_{2}<A, \sigma_{1} \geq A \text { or } B<\sigma_{1} \leq A, \sigma_{2} \leq B \\ \sigma_{1}-\sigma_{2}>0 & \text { if } \sigma_{1}>A, \sigma_{2}<B, \text { or } B<\sigma_{2}<\sigma_{1}<A\end{cases}
$$

or equivalently

$$
\begin{cases}2 \sigma_{1} \sigma_{2}-(A+B) & \sigma_{1}-(A+B) \sigma_{2}+2 A B<0 \\ \text { any }\left(\sigma_{1}, \sigma_{2}\right) & \text { if } B \leq \sigma_{2}<A, \sigma_{1} \geq A \text { or } B<\sigma_{1} \leq A, \sigma_{2} \leq B \\ & \text { if } \sigma_{1}>A, \sigma_{2}<B, \text { or } B<\sigma_{2}<\sigma_{1}<A\end{cases}
$$

This is equivalent to require $\left(\sigma_{1}, \sigma_{2}\right) \in \Theta_{1}(A, B)$ defined in $(14)_{1}$. If $\sigma_{2}>\sigma_{1}$, analogous steps, starting from (15), lead to require $\left(\sigma_{1}, \sigma_{2}\right) \in \Theta_{2}(A, B)$ defined in $(14)_{2}$. This concludes the first part of the Theorem. The second part of the Theorem follows straightforwardly.

Remark 1. We observe that $\Theta_{1}$ and $\Theta_{2}$ defined in (14) are limited by the line $\sigma_{1}=\sigma_{2}$ and by the hyperbola

$$
\left(\sigma_{1}-\frac{A+B}{2}\right)\left(\sigma_{2}-\frac{A+B}{2}\right)=\left(\frac{B-A}{2}\right)^{2}
$$



Fig. 3 Convergence set $\Theta(A, B)$ given by (14) (in blue) for the particular choice $A=$ $12, B=-4$.
see Figure 3. Since the latter curve depends on $m$ and $\boldsymbol{k}$, we have that also the regions $\Theta_{1}$ and $\Theta_{2}$ depend on $m$ and $\boldsymbol{k}$.

We also observe that $|\rho|=1$ for $\left(\sigma_{1}, \sigma_{2}\right) \in \partial \Theta(A, B) \backslash\{(A, A),(B, B)\}$ and for $\left(\sigma_{1}, \sigma_{2}\right) \rightarrow \infty$ inside $\Theta(A, B)$.

In the next result, we provide a better characterization of the regions $\Theta_{1}$ and $\Theta_{2}$.

Lemma 1. Fix $m$ and $\boldsymbol{k}$. If $0<\theta<1$, then the level sets

$$
\begin{equation*}
\left|\frac{\sigma_{1}-A}{\sigma_{2}-A} \frac{\sigma_{2}-B}{\sigma_{1}-B}\right|=\theta \tag{16}
\end{equation*}
$$

are the hyperbolae

$$
\begin{aligned}
& \left(\sigma_{1}-M-\frac{1+\theta}{1-\theta} D\right)\left(\sigma_{2}-M+\frac{1+\theta}{1-\theta} D\right)=-\frac{4 \theta}{(1-\theta)^{2}} D^{2}, \\
& \left(\sigma_{1}-M-\frac{1-\theta}{1+\theta} D\right)\left(\sigma_{2}-M+\frac{1-\theta}{1+\theta} D\right)=\frac{4 \theta}{(1+\theta)^{2}} D^{2}
\end{aligned}
$$

where

$$
M=\frac{A+B}{2}, D=\frac{A-B}{2} .
$$

For a given $\theta$, the above hyperbolae, restricted to $\Theta_{1}(A, B)$, delimit a region $\Theta_{1, \theta}(A, B)$, containing the point $(A, B)$, where

$$
\begin{equation*}
\left|\frac{\sigma_{1}-A}{\sigma_{2}-A} \frac{\sigma_{2}-B}{\sigma_{1}-B}\right| \leq \theta \tag{17}
\end{equation*}
$$

More precisely

$$
\Theta_{1, \theta}(A, B)=\Theta_{\theta}^{++}(A, B) \cup \Theta_{\theta}^{--}(A, B) \cup \Theta_{\theta}^{+-}(A, B) \cup \Theta_{\theta}^{-+}(A, B)
$$

where

$$
\begin{aligned}
\Theta_{\theta}^{++}(A, B)= & \left\{\left(\sigma_{1}, \sigma_{2}\right): \sigma_{1} \geq A, \sigma_{2} \geq B\right. \\
& \left.\left(\sigma_{1}-M-\frac{1-\theta}{1+\theta} D\right)\left(\sigma_{2}-M+\frac{1-\theta}{1+\theta} D\right) \leq \frac{4 \theta}{(1+\theta)^{2}} D^{2}\right\}, \\
\Theta_{\theta}^{--}(A, B)= & \left\{\left(\sigma_{1}, \sigma_{2}\right): \sigma_{1} \leq A, \sigma_{2} \leq B\right. \\
& \left.\left(\sigma_{1}-M-\frac{1-\theta}{1+\theta} D\right)\left(\sigma_{2}-M+\frac{1-\theta}{1+\theta} D\right) \leq \frac{4 \theta}{(1+\theta)^{2}} D^{2}\right\}
\end{aligned}
$$

$$
\Theta_{\theta}^{+-}(A, B)=\left\{\left(\sigma_{1}, \sigma_{2}\right): \sigma_{1} \geq A, \sigma_{2} \leq B\right.
$$

$$
\left.\left(\sigma_{1}-M-\frac{1+\theta}{1-\theta} D\right)\left(\sigma_{2}-M+\frac{1+\theta}{1-\theta} D\right) \geq-\frac{4 \theta}{(1-\theta)^{2}} D^{2}\right\}
$$

$$
\Theta_{\theta}^{-+}(A, B)=\left\{\left(\sigma_{1}, \sigma_{2}\right): \sigma_{1} \leq A, \sigma_{2} \geq B\right.
$$

$$
\left.\left(\sigma_{1}-M-\frac{1+\theta}{1-\theta} D\right)\left(\sigma_{2}-M+\frac{1+\theta}{1-\theta} D\right) \geq-\frac{4 \theta}{(1-\theta)^{2}} D^{2}\right\}
$$

Finally, $\Theta_{1, \theta}(A, B)$ contains the box $\Theta_{\theta}^{B}(A, B)$ with sides parallel to $\sigma_{2}=\sigma_{1}$ and $\sigma_{2}=-\sigma_{1}$ and containing the points

$$
\begin{aligned}
& E=\left(\frac{1+\sqrt{\theta}}{1-\sqrt{\theta}} D+M,-\frac{1+\sqrt{\theta}}{1-\sqrt{\theta}} D+M\right), \\
& F=\left(\frac{1-\sqrt{\theta}}{1+\sqrt{\theta}} D+M,-\frac{1-\sqrt{\theta}}{1+\sqrt{\theta}} D+M\right), \\
& G=\left(\frac{1+2 \sqrt{\theta}-\theta}{1+\theta} D+M, \frac{-1+2 \sqrt{\theta}+\theta}{1+\theta} D+M\right), \\
& H=\left(\frac{1-2 \sqrt{\theta}-\theta}{1+\theta} D+M, \frac{-1-2 \sqrt{\theta}+\theta}{1+\theta} D+M\right) .
\end{aligned}
$$

Proof This is an elementary exercise and the proof is left to the reader.
In Figure 4 we depicted the level sets (16) (left) and an example of regions $\Theta_{1, \theta}$ and $\Theta_{\theta}^{B}$ (right).


Fig. 4 Left: Level sets given by (16) for different values of $\theta$; Right: set $\Theta_{1, \theta}$ given by (17) and box $\Theta_{\theta}^{B}$ for the particular choice $\theta=0.3$. $A=12, B=-4$.

## 4 Estimates of optimized interface parameters

In general, the optimal symbols $\sigma_{1}^{o p t}=A(m, \boldsymbol{k})$ and $\sigma_{2}^{o p t}=B(m, \boldsymbol{k})$ are not effective in the practice since they lead to non-local interface conditions which are hardly implementable. For this reason, it is a common practice to look for the best symbols within a specific subset, for example the constants (Optimized Schwarz Method, see, e.g, $[14,16,17])$. Given $m$ and $\boldsymbol{k}$, we have in general that $A(m, \boldsymbol{k}) \neq-B(m, \boldsymbol{k})$, so that it does not make sense to look for the constant optimized value $p$ such that the reduction factor computed for $\sigma_{1}=p$ and $\sigma_{2}=-p$ is minimized.

In order to simplify our study, we assume however that $\sigma_{1}$ and $\sigma_{2}$ are related. In other words, rather than looking for the best possible point $\left(\sigma_{1}, \sigma_{2}\right)$ in $\mathbb{R}^{2}$, we will look for the best possible $\left(\sigma_{1}, \sigma_{2}\right)$ belonging to a properly chosen curve

$$
s(p)=\left\{\begin{array}{l}
\widetilde{\sigma}_{1}(p) \\
\widetilde{\sigma}_{2}(p)
\end{array} \quad p \in \mathbb{R}\right.
$$

so that in fact we obtain a minimization problem over the single scalar parameter $p$. In particular, given the curve $s(p)$, we consider the following

Problem 1. Find $\widehat{p} \in \Gamma$ which realizes

$$
\begin{gathered}
\max _{(m, \boldsymbol{k}) \in K}\left|\rho\left(m, \boldsymbol{k}, \widetilde{\sigma}_{1}(\widehat{p}), \widetilde{\sigma}_{2}(\widehat{p})\right)\right|=\max _{(m, \boldsymbol{k}) \in K}\left|\frac{\widetilde{\sigma}_{1}(\widehat{p})-A(m, \boldsymbol{k})}{\widetilde{\sigma}_{2}(\widehat{p})-A(m, \boldsymbol{k})} \cdot \frac{\widetilde{\sigma}_{2}(\widehat{p})-B(m, \boldsymbol{k})}{\widehat{\sigma}_{1}(\widehat{p})-B(m, \boldsymbol{k})}\right| \\
=\min _{p} \max _{(m, \boldsymbol{k}) \in K}\left|\frac{\widetilde{\sigma}_{1}(p)-A(m, \boldsymbol{k})}{\widetilde{\sigma}_{2}(p)-A(m, \boldsymbol{k})} \cdot \frac{\widetilde{\sigma}_{2}(p)-B(m, \boldsymbol{k})}{\widetilde{\sigma}_{1}(p)-B(m, \boldsymbol{k})}\right|,
\end{gathered}
$$

where $K$ is the set of the frequencies and $\Gamma \subset \mathbb{R}$ is the set where $s(p)$ is defined.

The problem now is to choose an appropriate curve $s(p)$. Assume (as it will be the case in our cases) that

$$
\begin{equation*}
\bar{B}:=\max _{(m, \boldsymbol{k}) \in K} B(m, \boldsymbol{k}) \leq \bar{A}:=\min _{(m, \boldsymbol{k}) \in K} A(m, \boldsymbol{k}) \tag{18}
\end{equation*}
$$

Then, thanks to the definition of the set $\Theta_{1}$ in (14), we have that the stripe $\mathcal{S}(\bar{A}, \bar{B})=\left\{\left(\sigma_{1}, \sigma_{2}\right): \sigma_{1}>\sigma_{2}\right.$, and $\left.2 \bar{B}<\sigma_{1}+\sigma_{2}<2 \bar{A}\right\}$ is contained in $\Theta_{1}(A(m, \boldsymbol{k}), B(m, \boldsymbol{k}))$ for all $(m, \boldsymbol{k}) \in K$, so that for all points $\left(\sigma_{1}, \sigma_{2}\right)$ in $\mathcal{S}(\bar{A}, \bar{B})$ we have, owing to Theorem 1 ,

$$
\left|\frac{\sigma_{1}-\bar{A}}{\sigma_{2}-\bar{A}} \cdot \frac{\sigma_{2}-\bar{B}}{\sigma_{1}-\bar{B}}\right|<1
$$

We decided to look for the curve $s(p)$ within $\mathcal{S}(\bar{A}, \bar{B})$, thus guaranteeing $|\rho|<1$ for all $(m, \boldsymbol{k}) \in K$. The idea is to consider a curve which is far enough from the boundary of $\mathcal{S}(\bar{A}, \bar{B})$. This guarantees that $s(p)$ is far from all the boundaries $\partial \Theta_{1}(A(m, \boldsymbol{k}), B(m, \boldsymbol{k}))$, where $|\rho|=1$ for some $(m, \boldsymbol{k})$. To this aim, call

$$
\begin{equation*}
\bar{M}=\frac{1}{2}(\bar{A}+\bar{B}) \tag{19}
\end{equation*}
$$

and consider the line $s$

$$
\begin{equation*}
\sigma_{2}=-\sigma_{1}+2 \bar{M}, \sigma_{1} \geq \bar{M} \tag{20}
\end{equation*}
$$

which divides in two equal parts $\mathcal{S}(\bar{A}, \bar{B})$ and then is far from its boundary, see Figure 5.

Then, as long as the points belonging to $s$ are far from the line $\sigma_{1}=\sigma_{2}$ and from infinity, they are far from the boundary of the set $\Theta_{1}(A(m, \boldsymbol{k}), B(m, \boldsymbol{k}))$ for whatever $(m, \boldsymbol{k}) \in K$, and therefore give $|\rho|<1$ for whatever $(m, \boldsymbol{k}) \in K$. We look then for the best value of $p$ such that the reduction factor is minimized by taking $\sigma_{1}=p, \sigma_{2}=-p+2 \bar{M}$. This justifies the study of the following
Problem 2. Minimize the function
$p \mapsto \max _{(m, \boldsymbol{k}) \in K}|\widehat{\rho}(p, m, \boldsymbol{k})|:=\max _{(m, \boldsymbol{k}) \in K}\left|\frac{p-A(m, \boldsymbol{k})}{-p+2 \bar{M}-A(m, \boldsymbol{k})} \frac{-p+2 \bar{M}-B(m, \boldsymbol{k})}{p-B(m, \boldsymbol{k})}\right|$
for $p \geq \bar{M}$.
This optimization problem requires that we know exactly the functions $A(m, \boldsymbol{k})$ and $B(m, \boldsymbol{k})$. Nevertheless, one can obtain an interesting quantitative result even in the general case. Indeed, the following result holds.
Theorem 2. Assume that $A(m, \boldsymbol{k})$ and $B(m, \boldsymbol{k})$ are bounded on some set $K$, with $B<A$ for all $(m, \boldsymbol{k}) \in K$, and call
$D(m, \boldsymbol{k})=\frac{1}{2}(A(m, \boldsymbol{k})-B(m, \boldsymbol{k})), \quad M(m, \boldsymbol{k})=\frac{1}{2}(A(m, \boldsymbol{k})+B(m, \boldsymbol{k}))$,
$Q(m, \boldsymbol{k})=\frac{|M(m, \boldsymbol{k})-\bar{M}|}{D(m, \boldsymbol{k})}, \bar{Q}=\sup _{(m, \boldsymbol{k}) \in K} Q(m, \boldsymbol{k}), N=\frac{\inf _{(m, \boldsymbol{k}) \in K} D(m, \boldsymbol{k})}{\sup _{(m, \boldsymbol{k}) \in K} D(m, \boldsymbol{k})}$,


Fig. 5 Four possible scenarios with the stripe $S$ for the particular choices $m=0, k=0.6$ (top, left); $m=1, k=0.6$ (top, right); $m=0, k=12.5$ (bottom, left); $m=10, k=12.5$ (bottom, right). FSI problem, $A$ and $B$ given by (45)-(48).
with $\bar{M}$ given by (19). Assume that $\bar{A}$ and $\bar{B}$ defined by (18) are such that $\bar{B}<\bar{A}$ and let

$$
\begin{equation*}
\rho_{0}=\max \left\{\left(\frac{1-\sqrt{N}}{1+\sqrt{N}}\right)^{2} ;\left(\frac{1-\sqrt{1-\bar{Q}^{2}}}{\bar{Q}}\right)^{2}\right\} \tag{23}
\end{equation*}
$$

Then, for all $(m, \boldsymbol{k}) \in K$, we have

$$
\begin{equation*}
\widehat{\rho}(p, m, \boldsymbol{k})=\left|\frac{p-A(m, \boldsymbol{k})}{2 \bar{M}-p-A(m, \boldsymbol{k})} \frac{2 \bar{M}-p-B(m, \boldsymbol{k})}{p-B(m, \boldsymbol{k})}\right| \leq \rho_{0}, \tag{24}
\end{equation*}
$$

if and only if $p \in\left[p_{-}, p_{+}\right]$with

$$
\begin{align*}
& p_{-}=\bar{M} \\
& \quad+\sup _{(m, \boldsymbol{k}) \in K}\left\{\frac{1+\rho_{0}}{1-\rho_{0}} D(m, \boldsymbol{k})-\sqrt{(\bar{M}-M(m, \boldsymbol{k}))^{2}+\frac{4 \rho_{0}}{\left(1-\rho_{0}\right)^{2}}(D(m, \boldsymbol{k}))^{2}}\right\}, \\
& p_{+}=\bar{M} \\
& \quad+\inf _{(m, \boldsymbol{k}) \in K}\left\{\frac{1+\rho_{0}}{1-\rho_{0}} D(m, \boldsymbol{k})+\sqrt{(\bar{M}-M(m, \boldsymbol{k}))^{2}+\frac{4 \rho_{0}}{\left(1-\rho_{0}\right)^{2}}(D(m, \boldsymbol{k}))^{2}}\right\} . \tag{25}
\end{align*}
$$

Proof The proof is divided in two steps. In the first one, we look for the minimum value of $\theta$ which guarantees that the boxes $\Theta_{\theta}^{B}$ have a non-empty intersection as $(m, \boldsymbol{k})$ varies in $K$. This value will be precisely $\rho_{0}$ defined in (23), and the intersection will be a box which crosses the line $s$ in a segment. This means that all the points in this segment give $\widehat{\rho} \leq \rho_{0}$. Then, in the second part of the proof, we extend the endpoints of the box $\Theta_{\rho_{0}}^{B}$ lying on $s$ as long as it is still guaranteed that $\hat{\rho} \leq \rho_{0}$.

By hypothesis, $D(m, \boldsymbol{k}) \geq(\bar{A}-\bar{B}) / 2>0$, and this implies that $Q(m, \boldsymbol{k})$ is well defined and non negative. On the other hand,

$$
\begin{aligned}
2|M(m, \boldsymbol{k})-\bar{M}| & =|A(m, \boldsymbol{k})+B(m, \boldsymbol{k})-\bar{A}-\bar{B}| \\
& =|A(m, \boldsymbol{k})-\bar{A}-(\bar{B}-B(m, \boldsymbol{k}))| \\
& \leq A(m, \boldsymbol{k})-\bar{A}+(\bar{B}-B(m, \boldsymbol{k})) \\
& =2 D(m, \boldsymbol{k})-(\bar{A}-\bar{B}) .
\end{aligned}
$$

This implies

$$
\begin{aligned}
Q(m, \boldsymbol{k}) & \leq \frac{D(m, \boldsymbol{k})-(\bar{A}-\bar{B}) / 2}{D(m, \boldsymbol{k})}=1-\frac{1}{2} \frac{(\bar{A}-\bar{B})}{D(m, \boldsymbol{k})} \\
& \leq 1-\frac{1}{2} \frac{(\bar{A}-\bar{B})}{\sup _{(m, \boldsymbol{k}) \in K} D(m, \boldsymbol{k})}<1
\end{aligned}
$$

Next, take $\rho_{0}$ satisfying

$$
\begin{equation*}
\rho_{0} \geq\left(\frac{1-\sqrt{N}}{1+\sqrt{N}}\right)^{2} \tag{26}
\end{equation*}
$$

This implies

$$
\frac{1-\sqrt{\rho_{0}}}{1+\sqrt{\rho_{0}}} \leq \sqrt{N}
$$

so that by the definition of $N$ we have
$\widehat{p}_{-}:=\sup _{(m, \boldsymbol{k}) \in K} D(m, \boldsymbol{k}) \frac{1-\sqrt{\rho_{0}}}{1+\sqrt{\rho_{0}}}+\bar{M} \leq \widehat{p}_{+}:=\inf _{(m, \boldsymbol{k}) \in K} D(m, \boldsymbol{k}) \frac{1+\sqrt{\rho_{0}}}{1-\sqrt{\rho_{0}}}+\bar{M}$.
By noticing that $p_{-} \leq \widehat{p}_{-}$and $p_{+} \geq \widehat{p}_{+}$, we have from the previous inequality that the interval of $p$ defined by (25) is non empty. Now, we observe that for
any $p \in\left[\widehat{p}_{-}, \widehat{p}_{+}\right]$, and for any $(m, \boldsymbol{k}) \in K$, the points $(p, 2 \bar{M}-p)$ belong to the box $\Theta_{\rho_{0}}^{B}(A(m, \boldsymbol{k}), B(m, \boldsymbol{k}))$. This is more easily seen after a rotation of $\pi / 4$ (and a dilation of $1 / \sqrt{2}$ ), given by

$$
\zeta\left[\begin{array}{l}
x \\
y
\end{array}\right]=\frac{1}{2}\left[\begin{array}{l}
x-y \\
x+y
\end{array}\right]
$$

Thus we have to show that for any $p \in\left[\widehat{p}_{-}, \widehat{p}_{+}\right]$and for any $(m, \boldsymbol{k}) \in K$, the points $(p-\bar{M}, \bar{M})$ belong to the axes-parallel box $\zeta \Theta_{\rho_{0}}^{B}(A(m, \boldsymbol{k}), B(m, \boldsymbol{k}))$ with sides containing the points (see Lemma 1)

$$
\begin{array}{rlrl}
\zeta E & =\left(\frac{1+\sqrt{\rho_{0}}}{1-\sqrt{\rho_{0}}} D, M\right), & \zeta F & =\left(\frac{1-\sqrt{\rho_{0}}}{1+\sqrt{\rho_{0}}} D, M\right), \\
\zeta G & =\left(\frac{1-\rho_{0}}{1+\rho_{0}} D, \frac{2 \sqrt{\rho_{0}}}{1+\rho_{0}} D+M\right), \zeta H & =\left(\frac{1-\rho_{0}}{1+\rho_{0}} D, \frac{-2 \sqrt{\rho_{0}}}{1+\rho_{0}} D+M\right),
\end{array}
$$

or equivalently that

$$
\frac{1-\sqrt{\rho_{0}}}{1+\sqrt{\rho_{0}}} D \leq p-\bar{M} \leq \frac{1+\sqrt{\rho_{0}}}{1-\sqrt{\rho_{0}}} D
$$

and

$$
\frac{-2 \sqrt{\rho_{0}}}{1+\rho_{0}} D+M \leq \bar{M} \leq \frac{2 \sqrt{\rho_{0}}}{1+\rho_{0}} D+M .
$$

The first condition follows immediately from the definition of $\widehat{p}_{-}$and $\widehat{p}_{+}$, see (27), while the second reduces to

$$
Q(m, \boldsymbol{k})=\frac{|\bar{M}-M(m, \boldsymbol{k})|}{D(m, \boldsymbol{k})} \leq \frac{2 \sqrt{\rho_{0}}}{1+\rho_{0}} .
$$

The latter inequality holds true if

$$
\bar{Q} \leq \frac{2 \sqrt{\rho_{0}}}{1+\rho_{0}}
$$

that is for

$$
\sqrt{\rho_{0}} \geq \frac{1-\sqrt{1-\bar{Q}^{2}}}{\bar{Q}}
$$

The latter condition, together with (26), are satisfied under hypothesis (23) and this concludes the first part of the proof.

The condition $p \in\left[\widehat{p}_{-}, \widehat{p}_{+}\right]$provides a sufficient condition for the satisfaction of (24). We want now to extend such a range so to obtain also a necessary condition. To this aim, observing that the box $\Theta_{\rho_{0}}^{B}(A(m, \boldsymbol{k}), B(m, \boldsymbol{k}))$ is contained in $\Theta_{1, \rho_{0}}(A(m, \boldsymbol{k}), B(m, \boldsymbol{k}))$ (see Lemma 1), we are sure to satisfy (24) until the line $s$ does not intersect the boundary of $\Theta_{1, \rho_{0}}(A(m, \boldsymbol{k}), B(m, \boldsymbol{k}))$, defined by the two branches of the hyperbola

$$
\left(\sigma_{1}-M-\frac{1+\rho_{0}}{1-\rho_{0}} D\right)\left(\sigma_{2}-M+\frac{1+\rho_{0}}{1-\rho_{0}} D\right)=-\frac{4 \rho_{0}}{\left(1-\rho_{0}\right)^{2}} D^{2}
$$

see Lemma 1. Thus, replacing $\sigma_{2}=2 \bar{M}-\sigma_{1}$ in the above equation gives

$$
\begin{gathered}
\left(\sigma_{1}-M-\frac{1+\rho_{0}}{1-\rho_{0}} D\right)\left(-\sigma_{1}+2 \bar{M}-M+\frac{1+\rho_{0}}{1-\rho_{0}} D\right)=-\frac{4 \rho_{0}}{\left(1-\rho_{0}\right)^{2}} D^{2} \\
\sigma_{1}^{2}-2\left(\bar{M}+\frac{1+\rho_{0}}{1-\rho_{0}} D\right) \sigma_{1}+2 M \bar{M}-M^{2}+2 \frac{1+\rho_{0}}{1-\rho_{0}} D \bar{M}+D^{2}=0
\end{gathered}
$$

with solutions

$$
\sigma_{1}=\bar{M}+\frac{1+\rho_{0}}{1-\rho_{0}} D(m, \boldsymbol{k}) \pm \sqrt{(\bar{M}-M(m, \boldsymbol{k}))^{2}+\frac{4 \rho_{0}}{\left(1-\rho_{0}\right)^{2}}(D(m, \boldsymbol{k}))^{2}}
$$

Since we want that the points $(p, 2 \bar{M}-p) \in \Theta_{1, \rho_{0}}(A(m, \boldsymbol{k}), B(m, \boldsymbol{k}))$ for all $(m, \boldsymbol{k}) \in K$, then it is necessary and sufficient that $p \in\left[p_{-}, p_{+}\right]$defined in (25).

In Figure 6 we reported four possible sets $\Theta_{1, \rho_{0}}$ for different values of ( $m, \boldsymbol{k}$ ). In particular, this figure as well as Figure 5 are related to the fluid-structure interaction problem described below, where $A$ and $B$ are given by (45)-(48), and the other parameters are defined in Section 6.2. We notice that the points $\left(p_{-}, s\left(p_{-}\right)\right)$and $\left(p_{+}, s\left(p_{+}\right)\right)$are always in such sets.
Remark 2. One could obtain a sharper result in the previous Theorem by replacing $\bar{M}=\frac{1}{2}(\bar{A}+\bar{B})$ with the number that minimizes the quantity

$$
\sup _{(m, \boldsymbol{k}) \in K} \frac{|M(m, \boldsymbol{k})-\bar{M}|}{D(m, \boldsymbol{k})} .
$$

Of course such a choice can only be done if the functions $M(m, \boldsymbol{k})$ and $D(m, \boldsymbol{k})$ are known.

## 5 Examples of possible applications

In this section we present three possible applications of the general results reported above. In particular in Sections 5.1 and 5.2 we present two problems considered so far in the literature, whereas in Section 5.3 we derive a completely new analysis and optimization for the fluid-structure interaction (FSI) problem.
5.1 The diffusion-reaction problem with a flat interface

This problem has been considered and analyzed in [14]. In particular, we have $n=2, d=1, a=-\infty, b=+\infty, R=0$, and problem (8) with $\mathcal{L}_{1}=\mathcal{L}_{2}=$ $-\triangle+\xi, \xi>0, \Omega=\mathbb{R}^{2}, \Omega_{1}=\left\{(x, y) \in \mathbb{R}^{2}: x<0, y \in \mathbb{R}\right\}, \Omega_{2}=\{(x, y) \in$ $\left.\mathbb{R}^{2}: x>0, y \in \mathbb{R}\right\}, \Sigma:=\Sigma_{0}=\left\{(x, y) \in \mathbb{R}^{2}: x=0, y \in \mathbb{R}\right\}, \gamma_{1}=\gamma_{2}=+\infty$


Fig. 6 Four possible sets $\Theta_{1, \rho_{0}}\left(\rho_{0}=0.32\right)$ for the particular choices $m=0, k=0.6$ (top, left); $m=1, k=0.6$ (top, right); $m=0, k=12.5$ (bottom, left); $m=10, k=12.5$ (bottom, right). FSI problem, $A$ and $B$ given by (45)-(48).
and $\delta=1$. The only frequency coordinate is $k$, that is the one related to the longitudinal direction $y$. In this case, the solutions (10) are given by

$$
{\widehat{u_{1}}}^{j}(x, k)=X_{1}^{j}(k) e^{x \sqrt{k^{2}+\xi}}, \quad{\widehat{u_{2}}}^{j}(x, k)=X_{2}^{j}(k) e^{-x \sqrt{k^{2}+\xi}}
$$

for some functions $X_{1}^{j}$ and $X_{2}^{j}$ [14]. Then, the expressions in (11) become

$$
A(k)=-B(k)=\sqrt{k^{2}+\xi}
$$

and the reduction factor (12) reads

$$
\begin{equation*}
\rho(k)=\frac{\sigma_{1}(k)-\sqrt{k^{2}+\xi}}{\sigma_{2}(k)-\sqrt{k^{2}+\xi}} \cdot \frac{\sigma_{2}(k)+\sqrt{k^{2}+\xi}}{\sigma_{1}(k)+\sqrt{k^{2}+\xi}}, \tag{28}
\end{equation*}
$$

see also [14].

Now, by noticing that $A>B$ for all $k$, we can can apply Theorem 1 , and we have from (14) that the exact convergence set is given by $\Theta=\Theta_{1} \cup \Theta_{2}$, where

$$
\begin{aligned}
& \Theta_{1}=\left\{\left(\sigma_{1}, \sigma_{2}\right): \sigma_{2}<\sigma_{1} \text { and } \sigma_{1} \sigma_{2}<k^{2}+\xi\right\}, \\
& \Theta_{2}=\left\{\left(\sigma_{1}, \sigma_{2}\right): \sigma_{2}>\sigma_{1} \text { and } \sigma_{1} \sigma_{2}>k^{2}+\xi\right\} .
\end{aligned}
$$

Regarding the optimization procedure, we have from (19) that $\bar{M}=0$, so that in this case it makes sense to look for the same optimal value, see also [14]. In particular, assuming $k \in\left[k_{\min }, k_{\max }\right]$, Problem 2 becomes
Problem 3. Minimize the function

$$
p \mapsto \max _{k \in\left[k_{\min }, k_{\max }\right]}\left|\frac{p-\sqrt{k^{2}+\xi}}{p+\sqrt{k^{2}+\xi}}\right|^{2},
$$

for $p \geq 0$.
From (21) and (22) we have $D(k)=\sqrt{k^{2}+\xi}, M(k)=Q(k)=\bar{Q}=0$ and $N=\sqrt{\frac{k_{\text {min }}^{2}+\xi}{k_{\text {max }}^{2}+\xi}}$. Then, by noticing that $\lim _{x \rightarrow 0} \frac{1-\sqrt{1-x^{2}}}{x}=0$, we have from

$$
\begin{equation*}
\rho_{0}=\left(\frac{1-\sqrt[4]{\frac{k_{\min }^{2}+\xi}{k_{\max }+\xi}}}{1+\sqrt[4]{\frac{k_{\min }^{2}+\xi}{k_{\max }+\xi}}}\right)^{2} \tag{23}
\end{equation*}
$$

Then, since the hypotheses of Theorem 2 hold true, by its application we have

$$
|\widehat{\rho}|=\left|\frac{p-\sqrt{k^{2}+\xi}}{p+\sqrt{k^{2}+\xi}}\right|^{2} \leq\left(\frac{1-\sqrt[4]{\frac{k_{\min }^{2}+\xi}{k_{\max }^{2}+\xi}}}{1+\sqrt[4]{\frac{k_{\min }^{2}+\xi}{k_{\max }^{2}+\xi}}}\right)^{2}
$$

for all $k \in\left[k_{\min }, k_{\max }\right]$, provided that $p$ belongs to the range defined by (25). Moreover, we have the following characterization of such a range:

$$
\begin{equation*}
\sqrt{k_{\min }^{2}+\xi} \leq p \leq \sqrt{k_{\max }^{2}+\xi} \tag{29}
\end{equation*}
$$

We observe that Problem 3 has been exhaustively studied in [14]. In particular, the following optimized value has been found:

$$
\begin{equation*}
p_{o p t}=\left(\left(k_{\min }^{2}+\xi\right)\left(k_{\max }^{2}+\xi\right)\right)^{1 / 4} \tag{30}
\end{equation*}
$$

leading to the best reduction factor $\rho_{\text {opt }}=\rho\left(k_{\min }\right)=\rho\left(k_{\max }\right)$. We observe that the above value of $p_{\text {opt }}$ falls in (29).

Just to provide a quantitative result, referring to the numerical simulations reported in [14], Table 6.3, we consider the diffusion-reaction problem solved in the unit square with $\xi=100$ and $h=1 / 50$. We then have $k_{\text {min }}=\pi / H=\pi$, with $H$ the dimension of the domain, and $k_{\max }=\pi / h=50 \pi$, with $h$ the space
discretization parameter. Then our estimates, using the range defined by (29), tell us that the reduction factor satisfies

$$
\begin{equation*}
|\widehat{\rho}|=\left|\frac{p-\sqrt{k^{2}+\xi}}{p+\sqrt{k^{2}+\xi}}\right|^{2} \leq 0.35 \tag{31}
\end{equation*}
$$

for all $k \in[\pi, 50 \pi]$, provided that

$$
10.5 \leq p \leq 157.4
$$

Moreover, from (30) we have $p_{\text {opt }}=40.6$ with $\rho_{\text {opt }}=0.35$. This highlights the optimality of estimate (31).

Remark 3. Observe that equation (3) does not depend on $n$ but only on $d$. This means that the same analysis and optimization of above is obtained by considering the diffusion-reaction problem with a flat interface in 3D. Accordingly, in [17] the convergence factor (28) has been obtained also in the 3D case, provided that $k$ is substituted by $\left(k_{1}, k_{2}\right)$ with $k_{1}$ and $k_{2}$ the frequency coordinates related to the variables $y_{1}$ and $y_{2}$.
5.2 The diffusion-reaction problem with a cylindrical interface

This problem has been introduced and studied in [17] to consider those situations where the interface is not flat but of cylindrical type, see Figure 2, right. In particular, we have $n=3, d=2, a=0, b=+\infty$, and problem (8) with $\mathcal{L}_{1}=\mathcal{L}_{2}=-\triangle+\xi, \Omega=\mathbb{R}^{3}, \Omega_{1}:=\left\{\left(x_{1}, x_{2}, y\right) \in \mathbb{R}^{3}: x_{1}^{2}+x_{2}^{2}<R^{2}\right\}, \Omega_{2}:=$ $\left\{\left(x_{1}, x_{2}, y\right) \in \mathbb{R}^{3}: x_{1}^{2}+x_{2}^{2}>R^{2}\right\}, \Sigma_{R}=\left\{\left(x_{1}, x_{2}, y\right) \in \mathbb{R}^{3}: x_{1}^{2}+x_{2}^{2}=\right.$ $\left.R^{2}\right\}, \gamma_{2}=+\infty, \delta=1$, and condition (7) holds. The frequency coordinates are $k \in \mathbb{R}$, related to the longitudinal direction $y$, and $m \in \mathbb{Z}$, related to the one-dimensional torus $S^{1}=\left\{x_{1}^{2}+x_{2}^{2}=1\right\}$. In this case, the solutions (10) are given by

$$
{\widehat{u_{1}}}^{j}(r, m, k)=X_{1}^{j}(k) I_{m}(\alpha r), \quad{\widehat{u_{2}}}^{j}(r, m, k)=X_{2}^{j}(k) K_{m}(\alpha r),
$$

for some functions $X_{1}^{j}$ and $X_{2}^{j}$, where $\alpha=\sqrt{k^{2}+\xi}, I_{m}$ and $K_{m}$ are the modified Bessel functions, see [21], and $r=\sqrt{x_{1}^{2}+x_{2}^{2}}$, as usual. Then, the expressions in (11) become

$$
\begin{equation*}
A(m, k)=-\alpha \frac{K_{m}^{\prime}(\alpha R)}{K_{m}(\alpha R)}, \quad B(m, k)=-\alpha \frac{I_{m}^{\prime}(\alpha R)}{I_{m}(\alpha R)} \tag{32}
\end{equation*}
$$

see also [17]. Notice that owing to the properties of the modified Bessel functions, we have $A(m, k)>0, \forall k, m$, and $B(m, k)<0, \forall k, m$. Then, the reduction factor (12) reads

$$
\rho(m, k)=\left|\frac{\sigma_{1}(m, k)+\alpha \frac{K_{m}^{\prime}(\alpha R)}{K_{m}(\alpha R)}}{\sigma_{2}(m, k)+\alpha \frac{K_{m}^{\prime}(\alpha R)}{K_{m}(\alpha R)}} \cdot \frac{\sigma_{2}(m, k)+\alpha \frac{I_{m}^{\prime}(\alpha R)}{I_{m}(\alpha R)}}{\sigma_{1}(m, k)+\alpha \frac{I_{m}^{\prime}(\alpha R)}{I_{m}(\alpha R)}}\right|,
$$

see also [17].
Now, we can apply again Theorem 1, and we have from (14) that the exact convergence set is given by $\Theta=\Theta_{1} \cup \Theta_{2}$, where

$$
\begin{aligned}
& \Theta_{1}=\left\{\left(\sigma_{1}, \sigma_{2}\right): \sigma_{2}<\sigma_{1}\right. \text { and } \\
& \left(\sigma_{1}+\frac{\alpha}{2}\left(\frac{K_{m}^{\prime}(\alpha R)}{K_{m}(\alpha R)}+\frac{I_{m}^{\prime}(\alpha R)}{I_{m}(\alpha R)}\right)\right)\left(\sigma_{2}+\frac{\alpha}{2}\left(\frac{K_{m}^{\prime}(\alpha R)}{K_{m}(\alpha R)}+\frac{I_{m}^{\prime}(\alpha R)}{I_{m}(\alpha R)}\right)\right)< \\
& \\
& \left.\left(\frac{\alpha}{2}\left(\frac{K_{m}^{\prime}(\alpha R)}{K_{m}(\alpha R)}-\frac{I_{m}^{\prime}(\alpha R)}{I_{m}(\alpha R)}\right)\right)^{2}\right\}
\end{aligned}
$$

$\Theta_{2}=\left\{\left(\sigma_{1}, \sigma_{2}\right): \sigma_{2}>\sigma_{1}\right.$ and

$$
\begin{aligned}
&\left(\sigma_{1}+\frac{\alpha}{2}\left(\frac{K_{m}^{\prime}(\alpha R)}{K_{m}(\alpha R)}+\frac{I_{m}^{\prime}(\alpha R)}{I_{m}(\alpha R)}\right)\right)\left(\sigma_{2}+\frac{\alpha}{2}\left(\frac{K_{m}^{\prime}(\alpha R)}{K_{m}(\alpha R)}+\frac{I_{m}^{\prime}(\alpha R)}{I_{m}(\alpha R)}\right)\right)> \\
&\left.\left(\frac{\alpha}{2}\left(\frac{K_{m}^{\prime}(\alpha R)}{K_{m}(\alpha R)}-\frac{I_{m}^{\prime}(\alpha R)}{I_{m}(\alpha R)}\right)\right)^{2}\right\} .
\end{aligned}
$$

Regarding the optimization procedure, first of all we notice that the function $A$ is increasing both in $k$ and in $m$, whereas $B$ is decreasing both in $k$ and in $m$, see [17]. Then, assuming $k \in\left[k_{\min }, k_{\max }\right]$ and $m \in\left[m_{\min }, m_{\max }\right]$, from (18) we have $\bar{A}=-\alpha_{\min } \frac{K_{m_{\min }}^{\prime}\left(\alpha_{\text {min }} R\right)}{K_{m_{\min }}\left(\alpha_{\text {min }} R\right)}$ and $\bar{B}=-\alpha_{\text {min }} \frac{I_{m_{\min }}^{\prime}\left(\alpha_{\text {min }} R\right)}{I_{m_{\text {min }}}\left(\alpha_{\text {min }} R\right)}$, where we have set $\alpha_{\text {min }}=\sqrt{k_{\text {min }}^{2}+\xi}$. Then, (19) gives

$$
\begin{equation*}
\bar{M}=-\frac{\alpha_{\min }}{2}\left(\frac{K_{m_{\min }}^{\prime}\left(\alpha_{\min } R\right)}{K_{m_{\min }}\left(\alpha_{\min } R\right)}+\frac{I_{m_{\min }}^{\prime}\left(\alpha_{\min } R\right)}{I_{m_{\min }}\left(\alpha_{\min } R\right)}\right) \tag{33}
\end{equation*}
$$

so that Problem 2 becomes
Problem 4. Minimize the function

$$
p \mapsto \max _{\substack{m \in\left[m_{\text {minin }}, m_{\text {max }}\right] \\ k \in\left[k_{\text {min }}, k_{\text {max }}\right]}}\left|\frac{p+\alpha \frac{K_{m}^{\prime}(\alpha R)}{K_{m}(\alpha R)}}{-p+2 \bar{M}+\alpha \frac{K_{m}^{\prime}(\alpha R)}{K_{m}(\alpha R)}} \frac{-p+2 \bar{M}+\alpha \frac{I_{m}^{\prime}(\alpha R)}{I_{m}(\alpha R)}}{p+\alpha \frac{I_{m}^{\prime}(\alpha R)}{I_{m}(\alpha R)}}\right|,
$$

for $p \geq \bar{M}$ and with $\bar{M}$ given by (33).
We can then compute numerically from (22) the values of $\bar{Q}$ and $N$ and apply again Theorem 2 obtaining a quantitative convergence result.

We observe that Problem 4 has been studied in [17] under the assumption $\bar{M}=0$. Indeed, it has been there noticed that $A \simeq-B$ for general frequencies, apart for small values of $m, k$ and $\xi$ at the same time. In particular, the following optimized value has been found:

$$
\begin{equation*}
p_{o p t}=\sqrt{-\frac{A_{+} B_{+}\left(A_{-}-B_{-}\right)+A_{-} B_{-}\left(A_{+}-B_{+}\right)}{A_{+}-B_{+}-A_{-}+B_{-}}} \tag{34}
\end{equation*}
$$

where $A_{-}:=A\left(m_{\min }, k_{\min }\right), B_{-}:=B\left(m_{\min }, k_{\min }\right), A_{+}:=A\left(m_{\max }, k_{\max }\right)$, $B_{+}:=B\left(m_{\max }, k_{\max }\right)$, and $A, B$ given by (32).

Referring to the numerical results shown in [17], we report here again a quantitative example to illustrate the application of our results. Take a cylinder whose length is 5 cm and radius 1 cm and where the interface is located at $R=0.5$. For $\xi=1, k_{\min }=0, k_{\max }=62, m_{\min }=0, m_{\max }=20$, we obtain $\bar{A}=1.79, \bar{B}=-0.24, \bar{M}=0.77$ so that our estimate based on (27) gives that

$$
|\widehat{\rho}|=\left|\frac{p+\alpha \frac{K_{m}^{\prime}(\alpha R)}{K_{m}(\alpha R)}}{-p+1.54+\alpha \frac{K_{m}^{\prime}(\alpha R)}{K_{m}(\alpha R)}} \frac{-p+1.54+\alpha \frac{I_{m}^{\prime}(\alpha R)}{I_{m}(\alpha R)}}{p+\alpha \frac{I_{m}^{\prime}(\alpha R)}{I_{m}(\alpha R)}}\right| \leq 0.62
$$

for all $k \in[0,62]$ and $m \in[0,20]$, provided that

$$
9.437 \leq p \leq 9.439
$$

Moreover, from (34) we have $p_{\text {opt }}=8.70$ with $\rho_{\text {opt }}=0.62$. Observe that in this case $p_{\text {opt }}$ does not fall in the range estimated by our result. This is not surprising, since the two optimization procedures have been performed with different values of $\bar{M}$. However, we observe that $\rho_{\text {opt }}$ is precisely equal to $\rho_{0}$.
5.3 The fluid-structure interaction problem with a cylindrical interface

### 5.3.1 Problem setting

We are in the case of a cylindrical interface, that is $n=3, d=2$. We consider the problem arising from the interaction between an incompressible, inviscid and linear fluid occupying the domain $\Omega_{f}:=\left\{\left(x_{1}, x_{2}, y\right) \in \mathbb{R}^{3}: x_{1}^{2}+x_{2}^{2}<R^{2}\right\}$, and a linear elastic structure modeled with the wave equation occupying the domain $\Omega_{s}:=\left\{\left(x_{1}, x_{2}, y\right) \in \mathbb{R}^{3}: R^{2}<x_{1}^{2}+x_{2}^{2}<(R+H)^{2}\right\}$. The two subproblems interact at the common interface $\Sigma_{R}=\left\{\left(x_{1}, x_{2}, y\right) \in \mathbb{R}^{3}: x_{1}^{2}+\right.$ $\left.x_{2}^{2}=R^{2}\right\}$. In particular, after a time discretization (for the sake of simplicity we consider here a BDF1 scheme for both subproblems, see [18]), the coupled problem at time $t^{n+1}:=(n+1) \Delta t, \Delta t$ being the time discretization parameter,
reads

$$
\begin{cases}\rho_{f} \delta_{t} \boldsymbol{u}+\nabla p=\mathbf{0} & \text { in } \Omega_{f},  \tag{35}\\ \nabla \cdot \boldsymbol{u}=0 & \text { in } \Omega_{f}, \\ \int_{-\infty}^{\infty} \int_{S^{1}}\left|p\left(r \boldsymbol{x}^{\prime}, \boldsymbol{y}\right)\right| d \sigma\left(\boldsymbol{x}^{\prime}\right) d \boldsymbol{y} & \text { bounded as } r \rightarrow 0^{+}, \\ \boldsymbol{u} \cdot \boldsymbol{n}=\delta_{t} \boldsymbol{\eta} \cdot \boldsymbol{n} & \text { on } \Sigma_{R}, \\ -p \boldsymbol{n}=\lambda \nabla \boldsymbol{\eta} \boldsymbol{n} & \text { on } \Sigma_{R}, \\ \boldsymbol{\eta} \times \boldsymbol{n}=\mathbf{0} & \text { on } \Sigma_{R} \\ \rho_{s} \delta_{t t} \boldsymbol{\eta}-\lambda \triangle \boldsymbol{\eta}=\mathbf{0} & \text { in } \Omega_{s}, \\ \gamma_{S T} \boldsymbol{\eta}+\lambda \nabla \boldsymbol{\eta} \boldsymbol{n}=P_{\text {ext }} \boldsymbol{n} & \text { on } \Sigma_{o u t},\end{cases}
$$

where, as usual, $r=\sqrt{x_{1}^{2}+x_{2}^{2}}, \rho_{f}$ and $\rho_{s}$ are the fluid and structure densities, $\lambda$ the square of the wave propagation velocity, $\delta_{t} w:=\frac{w-w^{n}}{\Delta t}, \delta_{t t} w:=\frac{\delta_{t} w-\delta_{t} w^{n}}{\Delta t}$, $\Sigma_{\text {out }}=\Sigma_{R+H}=\left\{\left(x_{1}, x_{2}, y\right) \in \mathbb{R}^{3}: x_{1}^{2}+x_{2}^{2}=(R+H)^{2}\right\}$ is the external surface of the structure domain, $\boldsymbol{n}$ is the unit vector orthogonal to the interface $\Sigma_{R}$ or $\Sigma_{R+H}$ defined by $\boldsymbol{n}=\frac{\left(x_{1}, x_{2}, 0\right)}{\sqrt{x_{1}^{2}+x_{2}^{2}}}$, and we have omitted the time index $n+1$. Problem (35) $)_{1-3}$ is the fluid problem, problem (35) $)_{7-8}$ is the structure problem equipped with a Robin condition at the external surface to account for the effect of an elastic surrounding tissue with elasticity modulus $\gamma_{S T}$ [24], $P_{e x t}$ is the external pressure, whereas $(35)_{4-6}$ are the coupling conditions at the FS interface. The fluid and the structure problems have to be completed with initial and boundary conditions along the $y$ direction, the latter reducing to the assumption of decay to zero for $|y| \rightarrow \infty$. We also observe that the coupling at the interface is allowed only in the normal direction.

By combining linearly $(35)_{4}$ and $(35)_{5}$ we obtain two generalized Robin boundary conditions. Observe that in the fluid problem the viscous terms have been neglected so that the fluid Cauchy stress tensor reduces to the only pressure. In particular, setting $u_{r}=\boldsymbol{u} \cdot \boldsymbol{n}$ and $\eta_{r}=\boldsymbol{\eta} \cdot \boldsymbol{n}$ and introducing the operator $\mathcal{S}_{f}$, we obtain

$$
\mathcal{S}_{f} u_{r}-p=\mathcal{S}_{f} \delta_{t} \eta_{r}+\lambda \partial_{r} \eta_{r},
$$

that is

$$
\mathcal{S}_{f} \Delta t \delta_{t} u_{r}-p=\mathcal{S}_{f} \delta_{t} \eta_{r}+\lambda \partial_{r} \eta_{r}-S_{f} u_{r}^{n}
$$

Then, the transmission condition for the fluid problem can be rearranged as

$$
\begin{equation*}
\mathcal{S}_{f} \Delta t \delta_{t} u_{r}-p=\frac{\mathcal{S}_{f}}{\Delta t} \eta_{r}+\lambda \partial_{r} \eta_{r}+F_{1}\left(u_{r}^{n}, \eta_{r}^{n}\right) \tag{36}
\end{equation*}
$$

where $F_{1}$ accounts for terms at previous time steps. Analogously, by introducing the operator $\mathcal{S}_{s}$, we obtain the following interface condition for the
structure problem

$$
\begin{equation*}
\frac{\mathcal{S}_{s}}{\Delta t} \eta_{r}+\lambda \partial_{r} \eta_{r}=\mathcal{S}_{s} \Delta t \delta_{t} u_{r}-p+F_{2}\left(u_{r}^{n}, \eta_{r}^{n}\right) \tag{37}
\end{equation*}
$$

where again $F_{2}$ accounts for terms at previous time steps. Then, at time $t^{n+1}$, the corresponding iterative generalized Schwarz method reads:

Given $\boldsymbol{u}^{0}, p^{0}, \eta^{0}$, solve for $j \geq 0$ until convergence

1. Fluid problem

$$
\begin{cases}\rho_{f} \delta_{t} \boldsymbol{u}^{j+1}+\nabla p^{j+1}=\mathbf{0} & \text { in } \Omega_{f},  \tag{38}\\ \nabla \cdot \boldsymbol{u}^{j+1}=0 & \text { in } \Omega_{f}, \\ \int_{-\infty}^{\infty} \int_{S^{1}}\left|p\left(r \boldsymbol{x}^{\prime}, \boldsymbol{y}\right)\right| d \sigma\left(\boldsymbol{x}^{\prime}\right) d \boldsymbol{y} & \text { bounded as } r \rightarrow 0^{+}, \\ \mathcal{S}_{f} \Delta t \delta_{t} u_{r}^{j+1}-p^{j+1}=\frac{\mathcal{S}_{f}}{\Delta t} \eta_{r}^{j}+\lambda \partial_{r} \eta_{r}^{j}+F_{1}\left(u_{r}^{n}, \eta_{r}^{n}\right) \text { on } \Sigma_{R} ;\end{cases}
$$

2. Structure problem

$$
\begin{cases}\rho_{s} \delta_{t t} \boldsymbol{\eta}^{j+1}-\lambda \triangle \boldsymbol{\eta}^{j+1}=\mathbf{0} & \text { in } \Omega_{s}  \tag{39}\\ \gamma_{S T} \boldsymbol{\eta}^{j+1}+\lambda \nabla \boldsymbol{\eta}^{j+1} \boldsymbol{n}=P_{e x t} \boldsymbol{n} & \text { on } \Sigma_{o u t} \\ \frac{\mathcal{S}_{s}}{\Delta t} \eta_{r}^{j+1}+\lambda \partial_{r} \eta_{r}^{j+1}=\mathcal{S}_{s} \Delta t \delta_{t} u_{r}^{j+1}-p^{j+1}+F_{2}\left(u_{r}^{n}, \eta_{r}^{n}\right) & \text { on } \Sigma_{R} \\ \boldsymbol{\eta}^{j+1} \times \boldsymbol{n}=\mathbf{0} & \text { on } \Sigma_{R}\end{cases}
$$

### 5.3.2 Convergence analysis

In order to perform a convergence analysis of the generalized Schwarz method (38)-(39), we need to write the coupled problem (35) in a different manner, such that it falls in the general framework of problem (8). To this aim, we first notice that the divergence free condition on $\boldsymbol{u}(35)_{2}$ allows us to rewrite the fluid problem (35) $)_{1-3}$ only in the unknown pressure

$$
\begin{cases}\Delta p=0 & \text { in } \Omega_{f},  \tag{40}\\ \int_{-\infty}^{\infty} \int_{S^{1}}\left|p\left(r \boldsymbol{x}^{\prime}, \boldsymbol{y}\right)\right| d \sigma\left(\boldsymbol{x}^{\prime}\right) d \boldsymbol{y} & \text { bounded as } r \rightarrow 0^{+} .\end{cases}
$$

Then, we notice that structure problem $(35)_{7-8}$ along the $r$ direction reads as follows

$$
\left\{\begin{array}{l}
\left(\frac{\rho_{s}}{\Delta t^{2}}-\lambda \triangle\right) \eta_{r}=0 \text { in } \Omega_{s}  \tag{41}\\
\gamma_{S T} \eta_{r}+\lambda \partial_{r} \eta_{r}=0 \text { on } \Sigma_{\text {out }},
\end{array}\right.
$$

where we have set to zero the forcing term $P_{\text {ext }}$ and the quantities at the previous time steps, since we analyze the convergence to the zero solution.

Following [16], thanks to the relation

$$
\begin{equation*}
\frac{\partial p}{\partial r}=-\rho_{f} \delta_{t} u_{r}=-\frac{\rho_{f}}{\Delta t}\left(u_{r}-u_{r}^{n}\right) \quad \text { on } \Sigma_{R} \tag{42}
\end{equation*}
$$

obtained by restricting the first equation of the fluid problem $(35)_{1}$ on the FS interface, it is possible to rewrite the interface conditions $(35)_{4-5}$ along the normal direction in terms of the pressure solely as follows

$$
\begin{array}{ll}
p=-\lambda \frac{\partial \eta_{r}}{\partial r} & \text { on } \Sigma_{R} \\
\frac{\partial p}{\partial r}=-\frac{\rho_{f}}{\Delta t^{2}} \eta_{r} & \text { on } \Sigma_{R} \tag{43}
\end{array}
$$

where we have set to zero the terms at time $n$ since we analyze the convergence to the zero solution. The previous interface conditions are of type $(6)_{3,4}$ with $u_{1}=p, u_{2}=\eta_{r}, \delta=0, \kappa_{D}=-1, \kappa_{N}=-\frac{\rho_{f}}{\Delta t^{2}}$. Then, the FSI problem written in terms of the fluid pressure and structure displacement given by (40), (41), (43) falls in the general framework of problem (6), where $\mathcal{L}_{1}=-\triangle, \mathcal{L}_{2}=$ $-\lambda \Delta+\frac{\rho_{s}}{\Delta t^{2}}, \Omega_{1}=\Omega_{f}, \Omega_{2}=\Omega_{s}, \gamma_{2}=\gamma_{S T}$ for $r=R+H$. Analogously, owing to (42), conditions (36) and (37) could be rewritten as follows

$$
\begin{align*}
& \frac{\rho_{f}}{\Delta t} \mathcal{S}_{f}^{-1} p+\frac{\partial p}{\partial r}=-\frac{\rho_{f}}{\Delta t^{2}} \eta_{r}-\frac{\rho_{f} \lambda}{\Delta t} \mathcal{S}_{f}^{-1} \frac{\partial \eta_{r}}{\partial r} \\
& \frac{\rho_{f}}{\Delta t^{2}} \eta_{r}+\frac{\lambda \rho_{f}}{\Delta t} \mathcal{S}_{s}-1 \frac{\partial \eta_{r}}{\partial r}=-\frac{\rho_{f}}{\Delta t} \mathcal{S}_{s}^{-1} p-\frac{\partial p}{\partial r} \tag{44}
\end{align*}
$$

where we have set $F_{1}=F_{2}=0$ since we analyze the convergence to the zero solution. Then, it is easy to check that the FSI problem given by (40), (41) and (44) falls in the general framework of problem (8) where $\mathcal{S}_{1}=\frac{\rho_{f}}{\Delta t} \mathcal{S}_{f}{ }^{-1}, \mathcal{S}_{2}=$ $\frac{\rho_{f}}{\Delta t} \mathcal{S}_{s}{ }^{-1}$.

We have the following
Proposition 2. Set

$$
\begin{equation*}
A(m, k)=-\frac{\lambda \Delta t \beta\left(K_{m}^{\prime}(\beta R)-\chi I_{m}^{\prime}(\beta R)\right)}{K_{m}(\beta R)-\chi I_{m}(\beta R)}, \quad B(m, k)=-\frac{\rho_{f} I_{m}(k R)}{\Delta t k I_{m}^{\prime}(k R)} \tag{45}
\end{equation*}
$$

Then, the reduction factor of iterations (38)-(39) is given by

$$
\begin{equation*}
\rho^{j}(m, k)=\rho(m, k)=\left|\frac{\sigma_{f}(m, k)-A(m, \boldsymbol{k})}{\sigma_{s}(m, k)-A(m, \boldsymbol{k})} \cdot \frac{\sigma_{s}(m, k)-B(m, \boldsymbol{k})}{\sigma_{f}(m, k)-B(m, \boldsymbol{k})}\right|, \tag{46}
\end{equation*}
$$

where $\sigma_{f}$ and $\sigma_{s}$ are the symbols of $\mathcal{S}_{f}$ and $\mathcal{S}_{s}$, respectively, and where we have set

$$
\begin{equation*}
\beta(k):=\sqrt{k^{2}+\frac{\rho_{s}}{\lambda \Delta t^{2}}}, \tag{47}
\end{equation*}
$$

and

$$
\begin{equation*}
\chi(m, k):=\frac{\gamma_{S T} K_{m}(\beta(R+H))+\lambda \beta K_{m}^{\prime}(\beta(R+H))}{\gamma_{S T} I_{m}(\beta(R+H))+\lambda \beta I_{m}^{\prime}(\beta(R+H))} . \tag{48}
\end{equation*}
$$

Moreover, the exact convergence set is given by (14) with $\sigma_{1}=\sigma_{f}, \sigma_{2}=\sigma_{s}$ and $A$ and $B$ given by (45).

Proof We need to determine the solutions of problems (40) and (41) at iteration $j$. Regarding the fluid problem (40), from (4) with $d=2$ we obtain $\widehat{p}^{j+1}(r, m, k)=X_{f, 1}^{j+1}(m, k) I_{m}(k r)+X_{f, 2}^{j+1}(m, k) K_{m}(k r)$, for suitable functions $X_{f, 1}^{j}$ and $X_{f, 2}^{j}$, where the dependence on $l$ vanished since for $d=2$ the multiplicity of the eigenvalues related to the spherical harmonics is constant $\left(k_{m}=2\right)$. The boundedness assumption $(40)_{2}$ on the pressure together with the properties of the modified Bessel functions entail $X_{f, 2}^{j+1}(m, k)=0, \forall j$, thus

$$
\widehat{p}^{j+1}(r, m, k)=X_{f}^{j+1}(m, k) I_{m}(k r),
$$

where, for the sake of simplicity, we have set $X_{f, 1}^{j+1}(m, k)=X_{f}^{j+1}(m, k)$.
Regarding the structure problem (41) ${ }_{1}$, from (4) with $d=2$ we obtain $\widehat{\eta}_{r}^{j}(r, m, k)=X_{s, 1}^{j}(m, k) I_{m}(\beta r)+X_{s, 2}^{j}(m, k) K_{m}(\beta r)$ for suitable functions $X_{s, 1}^{j}$ and $X_{s, 2}^{j}$ and with $\beta$ given by (47). Now, we impose condition (41) ${ }_{2}$, obtaining

$$
\gamma_{S T} \widehat{\eta}_{r}^{j}+\lambda \partial_{r} \widehat{\eta}_{r}^{j}=0 \quad \text { on } \widehat{\Sigma}_{\text {out }}:=\{r=R+H, m \in \mathbb{N}, k \in \mathbb{R}\} .
$$

This leads to

$$
\begin{aligned}
\gamma_{S T}\left(X_{s, 1}^{j} I_{m}(\beta r)+X_{s, 2}^{j}\right. & \left.K_{m}(\beta r)\right) \\
& +\left.\lambda \beta\left(X_{s, 1}^{j} I_{m}^{\prime}(\beta r)+X_{s, 2}^{j} K_{m}^{\prime}(\beta r)\right)\right|_{r=R+H}=0,
\end{aligned}
$$

and thus to $X_{s, 1}^{j}=-\chi X_{s, 2}^{j}$, where $\chi$ is given by (48). Therefore, the structure solution is

$$
\widehat{\eta}_{r}^{j}(r, m, k)=X_{s}^{j}(m, k)\left[K_{m}(\beta r)-\chi(m, k) I_{m}(\beta r)\right]
$$

where, for the sake of simplicity, we have set $X_{s, 2}^{j}(m, k)=X_{s}^{j}(m, k)$.
Now, the direct application of Proposition 1 with $g_{1}(m, k)=I_{m}(k r)$ and $g_{2}(m, k)=K_{m}(\beta r)-\chi(m, k) I_{m}(\beta r)$ leads to the first part of thesis.

The second part of the thesis is a straightforward application of Theorem 1.

Remark 4. In the case $\sigma_{f} \rightarrow \infty, \sigma_{s}=0$ we obtain the Dirichlet-Neumann scheme, which is known to be characterized by poor convergence properties when the fluid and structure densities are similar, as happens in haemodynamics (added mass effect, see $[5,13])$ This is confirmed by our analysis which leads for the Dirichlet-Neumann scheme to the following reduction factor:

$$
\rho^{D N}(m, k)=\frac{\rho_{f} I_{m}(k R)\left(K_{m}(\beta R)-\chi I_{m}(\beta R)\right)}{\lambda \Delta t \beta\left(K_{m}^{\prime}(\beta R)-\chi I_{m}^{\prime}(\beta R)\right) \Delta t k I_{m}^{\prime}(k R)}
$$

which increases for big values of the ratio $\rho_{f} / \beta=\rho_{f} / \sqrt{\rho_{s} /\left(\lambda \Delta t^{2}\right)+k^{2}}$, that is when the fluid and structure densities are similar.

### 5.3.3 Optimization

The optimal symbols which guarantee that the reduction factor (46) annihilates are $\sigma_{f}^{o p t}(m, k)=A(m, k)$ and $\sigma_{s}^{o p t}(m, k)=B(m, k)$, where $A$ and $B$ are given by (45). Again, these quantities lead to non-implementable interface conditions, so that we apply the theory developed in Section 4, allowing to obtain an optimization problem with respect to a scalar variable solely. We observe that in this case the determination of the maximum and of the minimum of $A(m, k)$ in (45) is not trivial, so that it is not possible anymore to write an explicit expression for $\bar{M}$, which needs to be computed numerically, see the next section.

## 6 Numerical results

In this section we present some numerical results to highlight the effectiveness of the theoretical findings reported in the previous sections for the FSI problem.

### 6.1 Generalities

We considered the coupling between the incompressible Navier-Stokes equations written in the Arbitrary Lagrangian-Eulerian formulation [7] and the linear infinitesimal elasticity, see for example [25]. In particular, we studied the effectiveness of the estimates reported in Section 5.3 and obtained for the simplified models, when applied to complete fluid and structure models. To do this, we compared their performance with the one related to the optimized values of $\sigma_{f}$ and $\sigma_{s}$ reported in [16], where the linear/non-viscous fluid (35) $1_{1-2}$ has been coupled with the independent rings model for a membrane $[11,26]$, and where the 2 D convergence analysis and optimization have been performed in the case of a flat interface. In particular, the following optimized values have been found

$$
\sigma_{f}^{f l a t}=\left(\frac{\rho_{s}}{\Delta t}+\varphi \Delta t\right) H_{s}, \quad \sigma_{s}^{f l a t}=\frac{2 \rho_{f}}{\Delta t k_{\max }}
$$

where $\varphi=\frac{E H_{s}}{\left(1-\nu^{2}\right) R^{2}}$, with $H_{s}$ the structure thickness, $E$ the Young modulus, $\nu$ the Poisson modulus, and $R$ the fluid domain radius. As noticed in [12], when the surrounding tissue is considered, the membrane models need to be rewritten by incorporating the surrounding elasticity coefficient $\gamma_{S T}$ in the membrane elastic coefficient, that is by substituting $\varphi$ with $\varphi+\gamma_{S T}$. Then, the optimized values related to the 2D/flat analysis become

$$
\begin{equation*}
\sigma_{f}^{f l a t}=\left(\frac{\rho_{s}}{\Delta t}+\left(\varphi+\gamma_{S T}\right) \Delta t\right) H_{s}, \quad \sigma_{s}^{f l a t}=\frac{2 \rho_{f}}{\Delta t k_{\max }} . \tag{49}
\end{equation*}
$$

In all the numerical experiments, we used the BDF scheme of order 1 for both the subproblems with a semi-implicit treatment of the fluid convective term. Moreover, we used the following data: fluid viscosity $\mu=0.035 d y n e / \mathrm{cm}^{2}$, fluid density $\rho_{f}=1 \mathrm{~g} / \mathrm{cm}^{3}$, structure density $\rho_{s}=1.1 \mathrm{~g} / \mathrm{cm}^{3}$, Poisson ratio $\nu=0.49$, Young modulus $E=3 \cdot 10^{6}$ dyne $/ \mathrm{cm}^{2}$. All these data are inspired from haemodynamic applications where the computational domains are often characterized by a cylindrical shape, see, e.g, [11]. We observe that the simplified structure model $(35)_{7}$ considered in the analysis is characterized by only two parameters, $\rho_{s}$ and $\lambda$, whereas the linear infinitesimal elasticity considered in the numerical experiments by three parameters, $\rho_{s}$ and the Lamé constants $\lambda_{1}=E /(2(1+\nu))$ and $\lambda_{2}$. Here, to compute $A(m, k)$ in (45) and the other quantities needed to build the estimates reported in Theorem 2, we assumed that the value of $\lambda$ could be approximated by $G \lambda_{1}$, with $G=\pi^{2} / 12$ the Timoshenko correction factor.

We prescribed in all the numerical experiments the following pressure $P_{\text {in }}$ at the inlet

$$
P_{\text {in }}= \begin{cases}1000 \text { dyne } / \mathrm{cm}^{2} & t \leq 0.08 \mathrm{~s} \\ 0 & 0.08 \mathrm{~s}<t \leq T\end{cases}
$$

where $T=0.20 s$, and absorbing resistance conditions at the outlets $[25,26]$.
In all the cases the optimized interface symbols are constant so that the Optimized Schwarz Method coincides with the Robin-Robin algorithm, introduced in the FSI context in [2] and then also considered in [1,3,27]. The fluid domain has been treated explicitly (semi-implicit approach, see [4, 9, 25]).

For the numerical discretization, we used $P 1 b u b b l e-P 1$ finite elements for the fluid subproblem and $P 1$ finite elements for the structure subproblem, and a time discretization parameter $\Delta t=0.001 \mathrm{~s}$

All the numerical results have been obtained with the parallel Finite Element library LIFEV developed at MOX - Politecnico di Milano, INRIA - Paris, CMCS - EPF of Lausanne and Emory University - Atlanta.

### 6.2 The case of a straight cylinder

In the first set of numerical experiments, we considered a cylinder with length $L=5 \mathrm{~cm}$, partitioned in two non-overlapping subdomains, an inner cylinder for the fluid problem with radius $R=0.5 \mathrm{~cm}, 4680$ tetrahedra and 1050 vertices (corresponding to 7830 degrees of freedom for the velocity and 1050 for the pressure), and an external cylindrical crown for the structure with thickness $H_{s}$ and 1260 vertices (corresponding to 3780 degrees of freedom for the structure displacement). We studied the performance of the Optimized Schwarz Method when the thickness structure is $H_{s}=0.1 \mathrm{~cm}$ and $H_{s}=0.5 \mathrm{~cm}$ and the surrounding tissue parameter is $\gamma_{S T}=1.5 \cdot 10^{6}$ dyne $/ \mathrm{cm}^{3}$ and $\gamma_{S T}=3 \cdot 10^{6}$ dyne $/ \mathrm{cm}^{3}$. The space discretization parameter is $h=0.25 \mathrm{~cm}$, and the frequencies vary in the ranges $m=0, \ldots, 10$ and $0,6 \leq k \leq 12,5$.

The estimates in Theorem 2 provide the following results:

1. $\boldsymbol{H}_{\boldsymbol{s}}=\mathbf{0 . 1}, \gamma_{\boldsymbol{S T}}=\mathbf{1 . 5} \cdot \mathbf{1 0}^{\mathbf{6}}: \bar{M}=793$ and $\rho_{0}=0.42$, with $p \in[1123,6383]$;
2. $\boldsymbol{H}_{\boldsymbol{s}}=\mathbf{0 . 1}, \gamma_{\boldsymbol{S T}}=\mathbf{3 . 0} \cdot \mathbf{1 0}^{\mathbf{6}}: \bar{M}=1323$ and $\rho_{0}=0.32$, with $p \in[1983,7521]$;
3. $\boldsymbol{H}_{\boldsymbol{s}}=\mathbf{0 . 5}, \gamma_{S T}=\mathbf{1 . 5} \cdot \mathbf{1 0}^{\mathbf{6}}: \bar{M}=864$ and $\rho_{0}=0.41$, with $p \in[1260,7209]$;
4. $\boldsymbol{H}_{s}=\mathbf{0 . 5}, \gamma_{S T}=3.0 \cdot \mathbf{1 0}^{6}: \bar{M}=1008$ and $\rho_{0}=0.38$, with $p \in[1459,7354]$.

In all the cases, we ran the simulations with different values of $p$ within the estimated ranges, in order to find the best one. In Figure 9 we reported the fluid pressure in the deformed domain at 4 different instants.


Fig. 7 Fluid pressure wave traveling along the deformed computational domain. From left to right, we have $t=0.001 \mathrm{~s}, t=0.004 \mathrm{~s}, t=0.009 \mathrm{~s}, t=0.013 \mathrm{~s}$.

In Table 1 we reported the average number of iterations per time step to reach convergence for the values of $p$ within the estimated ranges which guaranteed the best convergence properties, and for (49). Observe from this

|  | $2 \mathrm{D} /$ flat |  | $3 \mathrm{D} / \mathrm{cyl}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| $H_{s}-\gamma_{S T}$ | $\sigma_{f}-\sigma_{s}$ | \# iter | $\sigma_{f}-\sigma_{s}$ | \# iter |
| $0.1-1.5 \cdot 10^{6}$ | $3189-160$ | 5.7 | $2250-664$ | 4.9 |
| $0.1-3.0 \cdot 10^{6}$ | $4689-160$ | 6.7 | $3375-729$ | 4.6 |
| $0.5-1.5 \cdot 10^{6}$ | $9946-160$ | 24.6 | $3250-1522$ | 6.3 |
| $0.5-3.0 \cdot 10^{6}$ | $11446-160$ | 21.5 | $3750-1734$ | 6.4 |

Table 1 Values of the optimized interface parameters and of (49), and average number of iterations per time step. Cylindrical simulation.
result the robustness of the optimized values estimated by Theorem 2. Indeed, the average number of iterations per time step seems to be independent of the parameters. On the contrary, the optimized interface parameters estimated with the 2D/flat analysis worked very well for small values of the structure thickness $H_{s}$ (obtaining however worst performance with respect to the ones obtained by our analysis), whereas they did not work as well for a greater value of $H_{s}$. This suggests that the cylindrical analysis and optimization could in general improve the efficiency of the Robin-Robin scheme for the FSI problem in haemodynamics.

### 6.3 Carotid simulation

In this section we report the numerical results obtained in a real geometry, namely a human carotid, see Figure 8. The fluid mesh is composed by 9655


Fig. 8 Carotid fluid (left) and structure (right) computational domains.
vertices and 51173 tetrahedra, corresponding to 80138 degrees of freedom for the velocity and 9655 for the pressure, whereas the structure mesh is composed by 11052 vertices corresponding to 33156 degrees of freedom for the structure displacement.

We set $\gamma_{S T}=3 \cdot 10^{6}$ dyne $/ \mathrm{cm}^{3}$, the frequencies vary in the ranges $m=$ $0, \ldots, 19$ and $0,7 \leq k \leq 42$, the structure thickness is $H_{s}=0.06 \mathrm{~cm}$ and the fluid domain radius at the inlet is $R=0.24 \mathrm{~cm}$. In this case we do not have a uniform value of $R$ along the computational domain, so that we decided to use the value at the inlet to compute (49) and the estimates provided by Theorem 2. In particular, we have $\bar{M}=1544, \rho_{0}=0.56$, with $p \in[1981,15034]$.

We ran the simulations with different values of $p$ within the estimated range. In Figure 9 we reported the fluid pressure in the deformed domain at 4 different instants. The best performance has been obtained for $\sigma_{f}=4375$ and $\sigma_{s}=-1287$ which allowed to obtain convergence in 12.5 iterations (in average) per time step. These results confirmed the suitability of the RobinRobin scheme in real haemodynamic application and the effectiveness of the estimates provided by Theorem 2.


Fig. 9 Fluid pressure wave traveling along the deformed carotid domain. From left to right, we have $t=0.001 \mathrm{~s}, t=0.005 \mathrm{~s}, t=0.010 \mathrm{~s}, t=0.015 \mathrm{~s}$.

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## References

1. M. Astorino, F. Chouly, and M. Fernández. Robin based semi-implicit coupling in fluidstructure interaction: stability analysis and numerics. SIAM J. Sci. Comp., 31(6):40414065, 2009.
2. S. Badia, F. Nobile, and C. Vergara. Fluid-structure partitioned procedures based on Robin transmission conditions. J. Comput. Physics, 227:7027-7051, 2008.
3. S. Badia, F. Nobile, and C. Vergara. Robin-Robin preconditioned Krylov methods for fluid-structure interaction problems. Comput. Methods Appl. Mech. Engrg., 198(33-36):2768-2784, 2009.
4. S. Badia, A. Quaini, and A. Quarteroni. Splitting methods based on algebraic factorization for fluid-structure interaction. SIAM J. Sci. Comp., 30(4):1778-1805, 2008.
5. P. Causin, J.F. Gerbeau, and F. Nobile. Added-mass effect in the design of partitioned algorithms for fluid-structure problems. Comput. Methods Appl. Mech. Engrg., 194(42-44):4506-4527, 2005.
6. V. Dolean, M.J. Gander, and L. Gerardo Giorda. Optimized Schwarz Methods for Maxwell's equations. SIAM J. Sci. Comp., 31(3):2193-2213, 2009.
7. J. Donea. An arbitrary Lagrangian-Eulerian finite element method for transient dynamic fluid-structure interaction. Comput. Methods Appl. Mech. Engrg., 33:689-723, 1982.
8. O. Dubois. Optimized Schwarz methods with Robin conditions for the advection- diffusion equation. In O. B. Widlund and D. E. Keyes, editors, Domain Decomposition Methods in Science and Engineering XVI, pages 181-188. Springer-Verlag, 2006.
9. M.A. Fernández, J.F. Gerbeau, and C. Grandmont. A projection semi-implicit scheme for the coupling of an elastic structure with an incompressible fluid. Int. J. Num. Methods Engrg., 69(4):794-821, 2007.
10. G.B. Folland. Introduction to partial differential equations. Princeton University press, 1995.
11. L. Formaggia, A. Quarteroni, and A. Veneziani (Eds.). Cardiovascular Mathematics Modeling and simulation of the circulatory system. Springer, 2009.
12. L. Formaggia, A. Quarteroni, and C. Vergara. On the physical consistency between three-dimensional and one-dimensional models in haemodynamics. J. Comput. Physics, 244:97-112, 2013.
13. C. Forster, W. Wall, and E. Ramm. Artificial added mass instabilities in sequential staggered coupling of nonlinear structures and incompressible viscous flow. Comput. Methods Appl. Mech. Engrg., 196(7):1278-1293, 2007.
14. M.J. Gander. Optimized Schwarz Methods. SIAM J. Numer. Anal., 44(2):699-731, 2006.
15. M.J. Gander, F. Magoulès, and F.Nataf. Optimized Schwarz methods without overlap for the Helmholtz equation. SIAM J. Sci. Comp., 24:38-60, 2002.
16. L. Gerardo Giorda, F. Nobile, and C. Vergara. Analysis and optimization of RobinRobin partitioned procedures in fluid-structure interaction problems. SIAM J. Numer. Anal., 48(6):2091-2116, 2010.
17. G. Gigante, M. Pozzoli, and C. Vergara. Optimized Schwarz Methods for the diffusionreaction problem with cylindrical interfaces. SIAM J. Numer. Anal., in press.
18. E. Hairer, S.P. Nørsett, and G. Wanner. Solving ordinary differential equations: Nonstiff problems. Springer Series in Comput. Math. Springer, 1993.
19. C. Japhet. Optimized Krylov-Ventcell method. Application to convection-diffusion problems. In P.E. Bjorstad, M.S. Espedal, and D.E. Keyes, editors, Proceedings of the Ninth International Conference on Domain Decomposition Methods, pages 382-389, 1998.
20. C. Japhet, N. Nataf, and F. Rogier. The optimized order 2 method. Application to convection-diffusion problems. Fut Gen COmput Syst, 18:17-30, 2001.
21. N. Lebedev. Special Functions and Their Applications. Courier Dover Publications, 1972.
22. P.L. Lions. On the Schwartz alternating method III. In T. Chan, R. Glowinki, J. Periaux, and O.B. Widlund, editors, Proceedings of the Third International Symposium on Domain Decomposition Methods for PDE's, pages 202-223. Siam, Philadelphia, 1990.
23. F. Magoulès, P. Ivanyi, and B.H.V. Topping. Non-overlapping Schwarz method with optimized transmission conditions for the Helmholtz equation. Comput. Methods Appl. Mech. Engrg., 193:4797-4818, 2004.
24. P. Moireau, N. Xiao, M. Astorino, C. A. Figueroa, D. Chapelle, C. A. Taylor, and J.-F. Gerbeau. External tissue support and fluidstructure simulation in blood flows. Biomechanics and Modeling in Mechanobiology, 11(1-2):1-18, 2012.
25. F. Nobile, M. Pozzoli, and C. Vergara. Time accurate partitioned algorithms for the solution of fluid-structure interaction problems in haemodynamics. Computer \& Fluids, 86:470-482, 2013
26. F. Nobile and C. Vergara. An effective fluid-structure interaction formulation for vascular dynamics by generalized Robin conditions. SIAM J. Sci. Comp., 30(2):731-763, 2008.
27. F. Nobile and C. Vergara. Partitioned algorithms for fluid-structure interaction problems in haemodynamics. Milan Journal of Mathematics, 80(2):443-467, 2012.
28. A. Qaddouria, L. Laayounib, S. Loiselc, J Cotea, and M.J. Gander. Optimized Schwarz methods with an overset grid for the shallow-water equations: preliminary results. Appl. Num. Math, 58:459-471, 2008.
29. B. Stupfel. Improved transmission conditions for a one-dimensional domain decomposition method applied to the solution of the Helmhotz equation. J. Comput. Physics, 229:851-874, 2010.

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