Fractional pseudodifferential modeling through spatiotemporal duality applied to Geosciences

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Abstract. *Abstract potential theory and Dirichlet's priciple constitute the basic elements of the well-known classical theory of Markov processes and Dirichlet forms. In the spatial Gaussian framework, the duality condition allows the derivation of differential models whose solution is a Markov spatial process of finite-order. Gaussian fractional-order pseudodifferential models have been obtained by means of the duality condition in the papers by Ruiz-Medina, Angulo and Anh (2003) and Ruiz-Medina, Anh and Angulo (2004). The present paper extends these results in the spatiotemporal context using the theory of non-local Dirichlet forms, arising in the framework of symmetric Markov processes of pure jump type, allowing a spatiotemporal formulation of the duality condition. This condition leads, in particular, to the derivation in an abstract setting of a fractional-order pseudodifferential representation in space and time for Gaussian random fields, whose continuous spatiotemporal covariance kernel could be non-differentiable in the strong-sense. That is, new classes of spatiotemporal Gaussian processes can be introduced in this framework, including several classes of fractal spatiotemporal processes, as well as the class of spatiotemporal random fields, S/TRF-*ν/μ*, introduced in Christakos (1991), in connection with Stochastic Partial Differential Equations (SPDE).*

Keywords. *Distribution theory; Duality condition; Non-local Dirichlet forms; Tempered distributions; Weak-sense fractional-order differentiation.*

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1 Introduction

Increased attention has been paid in recent years to the functional modeling of spatiotemporal data, including the statistics literature, and the general scientific literature with applications in, e.g., environmental, ecological and health sciences. SPDE offers a suitable mathematical-probabilistic context to incorporate sample information from spatiotemporal referenced data (see Christakos, 2000). In this framework, the physical laws representing future changes that are currently unobservable are involved in the derivation of the spatiotemporal random field model, providing the ability to forecast new states arising in the evolution of the phenomena under study. In particular, the prediction and extrapolation problems are addressed in an accurate and flexible way, even in the presence of spatial heterogeneities and, in general, of non-stationary or non-linear behavior in space and/or time (see, for example, Ruiz-Medina and Angulo, 2007; Ruiz-Medina and Fernández-Pascual, 2010).

The spatiotemporal dependence structure is often interpreted as the temporal evolution of a spatial process described in the form of a dynamic model in discrete time (considered, for example, in the functional linear context by Gelfand, Banerjee and Gamerman, 2005; and in the autoregressive Hilbertian time series context by Bosq, 2000; Bosq and Blanke, 2007, and by Ruiz-Medina and Salmerón, 2010, among others). When continuous time is considered in the description of the temporal evolution of the spatial process of interest, SPDE and, in general, stochastic pseudodifferential evolution equations are considered (see Angulo *et al.*, 2005; Christakos, 1992; Christakos and Raghu, 1996; Kelbert *et al.*, 2005; Leonenko and Ruiz-Medina, 2006, among others). In the statistics literature, spatiotemporal covariance modeling in continuous time and space also constitutes an active area of research in the last few decades, since the work of Mardia and Goodall (1993) on separable space-time covariance functions to the more recent works on nonseparable space-time covariance modeling by Gneiting (2002); Gneiting, Kleiber and Schlather (2010); Porcu, Gregori and Mateu (2006); Porcu and Zastavnyi (2011); Stein (2005) and references therein.

The outline of the paper is as follows. The spatiotemporal duality condition is formulated for a given Reproducing Kernel Hilbert Space (RKHS) generated by a spatiotemporal covariance model. The covariance operator of the dual random field can be factorized in terms of a non-local symmetric Dirichlet form. The L^2 semigroup associated with this closed form defines the fundamental solution of the fractional-order pseudodifferential equation satisfied by the spatiotemporal zero-mean Gaussian random field, univocally determined by its RKHS (see, for example, Da Prato and Zabczyk, 2002). Finally, the asymptotic distributional properties of the introduced class of spatiotemporal Gaussian random fields are derived. Specifically, its asymptotic, in time, infinite-dimensional distribution admits a representation in terms of an infinite product of time-dependent Gaussian measures. For numerical illustration purposes certain simulation examples with mean quadratic local variation properties are studied.

2 Spatiotemporal duality condition

Let us consider the non-local symmetric Dirichlet form $(\mathcal{E},\mathcal{G})$ given by

$$\mathcal{E}(f,f)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f(\mathbf{x})-f(\mathbf{y}))^2 J(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y},\quad \forall f\in L^2(\mathbb{R}^d), \tag{1}$$

where $\mathcal{G}$ denotes the closure with respect to $\mathcal{E}_1=\mathcal{E}+\|\cdot\|^2_{L^2(\mathbb{R}^d)}$, with $\mathcal{E}_1(f,f)=\mathcal{E}(f,f)+\|f\|^2_{L^2(\mathbb{R}^d)}$, for all $f\in L^2(\mathbb{R}^d)$, of the set of continuous functions on $\mathbb{R}^d$ with compact support, and J denotes a kernel such that two positive constants K_1 and K_2 exist satisfying the condition

$$K_1\|\mathbf{x}-\mathbf{y}\|^{-d-\alpha} \leq J(\mathbf{x},\mathbf{y}) \leq K_2\|\mathbf{x}-\mathbf{y}\|^{-d-\beta},$$

for $0<\alpha<\beta<2$, and $\|\mathbf{x}-\mathbf{y}\|<1$.

Let $\{X(t,\mathbf{z}),\ t\in\mathbb{R}_+,\ \mathbf{z}\in D\subset\mathbb{R}^d\}$ be a spatiotemporal zero-mean Gaussian Random Field (RF) on a Dirichlet regular compact domain D. Assume that the spatiotemporal convariance kernel $r_X(t,s,\mathbf{z},\mathbf{y})$ $= E[X(t,\mathbf{z})X(s,\mathbf{y})]$, defining the integral covariance operator R_X of X satisfies $r_X(t,s,\mathbf{u},\mathbf{v})=0$, for all $\mathbf{u},\mathbf{v}\in\partial D$, with ∂D denoting the boundary of D with null Legesgue measure. For simplicity in what follows we also assume that D is a convex regular domain.

Definition 1 *Let* $\left\{\int_D \widetilde{X}(t,\mathbf{y})\varphi(t,\mathbf{y})dtd\mathbf{y},\ \varphi(t,\cdot)\in\overline{H}^{\gamma}(D),\ t\in\mathbb{R}_+\right\}$ *be a zero-mean Gaussian Generalized Random Field (GRF). Here,* $\overline{H}^{\gamma}(D)$ *denotes the fractional Sobolev space of order* $\gamma\geq\max\{\alpha,\beta\}$ *constituted of functions* ψ *with compact support contained in D such that* $\int_D\left|(-\Delta)^{\gamma/2}(\psi)(\mathbf{z})\right|^2 d\mathbf{z}<\infty$, *where* $(-\Delta)^{\gamma/2}$ *denotes the inverse of the Riesz potential of order* γ *(Triebel, 1978). The GRF* $\widetilde{X}$ *defines the dual RF of X iff satisfies the following conditions:*

(i) $E\left[\int_{\mathbb{R}_+\times D}\widetilde{X}(t,\mathbf{y})\varphi(t,\mathbf{y})dtd\mathbf{y}\right]=0,$ *for all* $\varphi(t,\cdot)\in\overline{H}^{\gamma}(D)$, *and for each* $t\in\mathbb{R}_+$.

(ii) $\int_{\mathbb{R}_+\times\mathbb{R}_+}\int_{D\times D}E\left[\widetilde{X}(t,\mathbf{y})X(s,\mathbf{z})\right]\varphi(t,\mathbf{y})\phi(s,\mathbf{z})dtdsd\mathbf{y}d\mathbf{z}=\int_{\mathbb{R}_+\times D}\varphi(t,\mathbf{y})\phi(t,\mathbf{y})dtd\mathbf{y}$, *for all* $\varphi(t,\cdot),\phi(t,\cdot)\in\overline{H}^{\gamma}(D)$, *and for each* $t\in\mathbb{R}_+$.

3 Fractional-order pseudodifferential representation

Let X be a spatiotemporal zero-mean Gaussian RF satisfying conditions (i)-(ii) in Definition 1. Then, X satisfies, in the weak-sense, the following fractional-order pseudodifferential evolution equation:

$$\int_D \psi(\mathbf{x})\frac{\partial}{\partial t}X(t,\mathbf{x})d\mathbf{x}=\mathcal{E}(X(t,\cdot),\psi)+\int_D\psi(\mathbf{x})\varepsilon(t,\mathbf{x})d\mathbf{x}, \tag{2}$$

for all $\psi\in\overline{H}^{\gamma}(D)$, where ε denotes spatiotemporal Gaussian white noise. Equation (2) holds in the strong-sense, i.e., pointwise, for $\gamma>d/2$. The proof of the weak-sense identity (2) follows from Definition 1, in particular, from condition (ii), the covariance kernel of $\widetilde{X}$ defines a non-local symmetric Dirichlet form as in (1), and, for all $\varphi\in R_X(L^2(\mathbb{R}_+\times H^{\gamma}(D))$,

$$\begin{aligned}
R_X^{-1}(\varphi)(\varphi) &= E\left[\int_{\mathbb{R}_+\times\mathbb{R}_+}\int_{D\times D}\widetilde{X}(t,\mathbf{x})\widetilde{X}(s,\mathbf{y})\varphi(t,\mathbf{x})\varphi(s,\mathbf{y})dtdsd\mathbf{x}d\mathbf{y}\right]\\
&=\int_{\mathbb{R}_+\times\mathbb{R}_+}\left\langle\mathcal{F}\left(\varphi(t,\cdot)-\varphi(s,\cdot)\right)*\mathcal{F}\left(\varphi(t,\cdot)-\varphi(s,\cdot)\right),\mathcal{H}(\mathcal{F}(J),t,s)\right\rangle_{L^2}dtds,\\
\mathcal{H}(\mathcal{F}(J),t,s) &= \left[\frac{\exp\left(-(t-s)\mathcal{F}(J)\right)-\exp\left(-(t+s)\mathcal{F}(J)\right)}{\mathcal{F}(J)}\right]^{-1},\quad\forall t,s\in\mathbb{R}_+,
\end{aligned}\tag{3}$$

where $\langle\cdot,\cdot\rangle_{L^2}$ represents the inner product in L^2, with $\mathcal{F}(J)$ denoting the Fourier transform of J in equation (1), and $\mathcal{F}\left(\varphi(t,\cdot)-\varphi(s,\cdot)\right)*\mathcal{F}\left(\varphi(t,\cdot)-\varphi(s,\cdot)\right)$ being the convolution in the spatial domain.

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