Sampling from Max-Stable Processes Conditional on a Homogeneous Functional via an MCMC Algorithm

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Abstract. *For managing risks in climate or environmental fields, max-stable processes can be used as models for spatial and spatio-temporal extremes. When some information on the process of interest is available, conditional simulations provide probability distributions according to the information, allowing us to evaluate risks more precisely. Usually the information available is given by observed values of the process in some sites. Instead, in this work, we focus on the case that aggregated data are given. As condition, we consider a homogeneous functional like the integral or the maximum of the process. Due to the analytic intractability of the involved distributions, we propose a sampling algorithm based on MCMC techniques. The procedure consists of two steps where the second step is based on conditional sampling from a max-linear model. We illustrate the performance of the proposed algorithms in a simulation study and in an example of a real dataset of precipitation observations with a condition stemming from regional climate model outputs.*

Keywords. *Conditional Simulation; Max-Linear Model; Metropolis-Hastings Algorithm*

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1 Introduction

During the last years, max-stable processes have gained increasing attention as models for spatial and spatio-temporal extremes (cf. [5] and [6], for example). For instance, they are useful for managing risks in climate or environmental fields, and, thus, have frequently found application in these frameworks.

Sample-continuous max-stable processes on some compact (spatial or spatio-temporal) domain K are fully characterized by their spectral representation ([3],[8]): Let $\{U_i\}_{i\in\mathbb{N}}$ be the points of a Poisson point process on $(0,\infty)$ with intensity $u^{-2}\,\mathrm{d}u$. Independently from the Poisson point process, let W_i, $i \in \mathbb{N}$, be independent copies of some sample-continuous stochastic process $\{W(t), t \in K\}$ with $\mathbb{E}W(t) = 1$ for all

$t \in K$. Then,

$$X(t) = \max_{i \in \mathbb{N}} U_i W_i(t), \quad t \in K, \tag{1}$$

defines a max-stable process with standard Fréchet margins, i.e. $\mathbb{P}(X(t) \leq x) = \exp(-x^{-1})$, $x > 0$, $t \in K$.

Conditional simulations yield empirical probability distributions according to a model and some information available, usually observations in given locations. Thus, they provide useful information, especially in terms of risk assessment for the process of interest at locations where measurements are missing or vague. Recently, the question of conditional simulation of max-stable processes has become a focal point of interest. An exact and computationally efficient algorithm for conditional simulation of max-linear (i.e. spectrally discrete) models are developed by Wang and Stoev in [10]. In [1], Dombry and Eyi-Minko provide exact formulae for the conditional distribution of general max-stable processes in terms of its exponent measure. These formulae allow for an explicit calculation and implementation of a sampling algorithm in case of regular models (cf. [2], for example).

All the articles mentioned above deal with the distribution of the process X conditional on its value in some sites $t_1, \ldots, t_m \in K$. Thus, the proposed algorithms allow to investigate the probabilistic behaviour of X on the whole domain K given measurements or forecasts at some locations or instants of time, respectively. Here, we do not restrict ourselves to the value of the process at specific locations, but allow for a condition given by a more general functional. More precisely, we consider a homogeneous functional $\ell : C_+(K) \to [0, \infty)$, i.e. $\ell(af) = a\ell(f)$ for all $a \geq 0$ and $f \in C_+(K)$, where $C_+(K)$ denotes the space of all non-negative and continuous functions on K. Besides point evaluation, a functional like the integral or the maximum of the process satisfies this condition.

In the following, we aim to sample from the distribution of $X \mid \ell(X) = x$ for some $x > 0$. This might be of practical relevance in various situations, like in the case of local climate prediction according to outputs of regional climate models, given as an aggregated value on a grid cell. Due to the computational complexity, these models work on a coarse grid and thus provide forecasts for the average of the variable of interest over a rather large region, i.e. the integral of the corresponding random field.

Note that, in contrast to max-linear functionals, general homogeneous functionals do not allow for expressions in terms of the exponent measure of the underlying process. Thus, the techniques of [1] are not applicable in this general setting. Due to this lack of analytic tractability, we will choose an algorithmic approach for the sampling procedure based on MCMC methods. The procedure will consist of two steps where the second step is based on conditional sampling from a max-linear model.

2 Max-Linear Models

First, we consider a max-linear (i.e. spectrally discrete max-stable) model (cf. [10]):

$$X(t) = \max_{j=1,\ldots,n} a_j(t) Z_j, \qquad t \in K, \tag{2}$$

where Z_j are independently and identically standard Fréchet distributed random variables and $a_j \in C_+(K)$, $j = 1, \ldots, n$. Following [10], we will write $X = \mathbf{A} \odot \mathbf{Z}$ instead of (2), for short. Note that all the finite-dimensional marginal distributions of a max-stable process can be approximated arbitrarily well by such a model.

For simplicity, we assume that $\ell(X) > 0$ a.s. Then, on each ray $\{\mathbf{z} \in (0,\infty)^n : \mathbf{z} = c\mathbf{z}_0\}$, $\mathbf{z}_0 \in (0,\infty)^n$, there is one unique point $\mathbf{z}$ satisfying $\ell(\mathbf{A} \odot \mathbf{z}) = x$ for some given value $x > 0$, namely $\mathbf{z} = x \cdot \mathbf{z}_0 / \ell(\mathbf{A} \odot \mathbf{z}_0)$. Thus, instead of simulating $\mathbf{Z} = (Z_1, \ldots, Z_n)^\top$, we could equivalently sample $\mathbf{Y} = (Z_2/Z_1, \ldots, Z_n/Z_1)$ from the distribution of $\mathbf{Y} \mid \ell(X) = x$ which, in contrast to the distribution of $\mathbf{Z} \mid \ell(X) = x$, is absolutely continuous with respect to the Lebesgue measure with density $f_{\mathbf{Y}|\ell(X)=x}$.

To this end, we propose an algorithm of Metropolis-Hastings type (cf. [9], for example). Given a current state $\mathbf{y}^{(k)}$, we propose a new state $\mathbf{y}^*$, sampled from the unconditional density $f_{\mathbf{Y}}$ of $\mathbf{Y}$. The proposal is accepted with probability

$$a(\mathbf{y}^{(k)}, \mathbf{y}^*) = \min\left\{1, \frac{f_{\mathbf{Y}}(\mathbf{y}^{(k)}) f_{\mathbf{Y}|\ell(X)=x}(\mathbf{y}^*)}{f_{\mathbf{Y}}(\mathbf{y}^*) f_{\mathbf{Y}|\ell(X)=x}(\mathbf{y}^{(k)})}\right\},$$

which can be calculated explicitly. It can be shown that the distribution of the resulting Markov chain converges to the desired conditional distribution of $\mathbf{Y} \mid \ell(X) = x$. Based on a realization $\mathbf{y}$ from this distribution, a sample from the distribution of the process X conditional on $\ell(X) = x$ is obtained by

$$\frac{x}{\ell(\mathbf{A} \odot (1, \mathbf{y})^\top)} \cdot \left(\mathbf{A} \odot \begin{pmatrix} 1 \\ \mathbf{y} \end{pmatrix}\right).$$

3 General Max-Stable Processes

We now consider a general max-stable process. In view of its spectral representation (1), it has a similar structure as a max-linear process where the Poisson points $\{U_i\}_{i\in\mathbb{N}}$ correspond to the Fréchet random variables Z_j and the spectral functions W_i take the role of the coefficient functions a_j. Thus, in the general case, the coefficient functions are random and their number is infinite.

To cope with these problems, we use a general construction principle for Poisson point processes via exponentially distributed random variables and rewrite (1) in the form

$$X(t) =_d \max_{k\in\mathbb{N}} \left(\textstyle\sum_{i=1}^k E_i\right)^{-1} W_k(t), \quad t \in K, \tag{3}$$

where E_i, $i \in \mathbb{N}$, are independent standard exponentially distributed random variables.

Further, we note that the choice of the spectral functions W_k is not unique. In particular, according to Corollary 9.4.5 in [4], they can be chosen such that $\sup_{t\in K} W(t) \leq C$ a.s. for some $C > 0$. By this observation, only a finite random number $N(\mathbf{W}, \mathbf{E})$ of functions, defined by

$$N(\mathbf{W},\mathbf{E}) = \min\left\{N \in \mathbb{N} : \left(\textstyle\sum_{i=1}^N E_i)\right)^{-1} C \leq \inf_{t\in K} \max_{k=1,\ldots,N} \left(\textstyle\sum_{i=1}^k E_i\right)^{-1} W_k(t)\right\},$$

may contribute to the process X (cf. [7]). Thus, we have

$$X(t) =_d \max_{k\leq N(\mathbf{E},\mathbf{W})} \left(\textstyle\sum_{i=1}^k E_i\right)^{-1} W_k(t), \quad t \in K,$$

which can be interpreted as a model of max-linear type with a random number $N(\mathbf{W}, \mathbf{E})$ of random coefficient functions $\mathbf{W} = (W_k)_{k=1}^{N(\mathbf{W},\mathbf{E})}$.

Thus, based on our results for the max-linear case, we propose a two-step procedure:

1. Draw a random number N and a vector $\mathbf{w} = (w_k(t))_{k=1}^N$ of spectral functions from the joint distribution of $N(\mathbf{E},\mathbf{W}),\mathbf{W} \mid \ell(X) = x$ via a Metropolis-Hastings algorithm. Here, for an arbitrary proposal distribution for $N(\mathbf{E},\mathbf{W})$ and the unconditional distribution of $\mathbf{W}$ as proposal distribution for the spectral functions, the corresponding acceptance probability can by calculated explicitly.

2. Simulate a vector $\mathbf{e} = (e_k)_{k=1}^N$ from the distribution of $\mathbf{E} \mid N(\mathbf{E},\mathbf{W}) = N$, $\mathbf{W} = \mathbf{w}$, $\ell(X) = x$ via a Metropolis-Hastings algorithm. The algorithm can be designed in a similar way as for the case of a max-linear model.

Then, a realization of the max-stable process $X \mid \ell(X) = x$ is given by

$$X(t) = \max_{k=1,\dots,N} \left(\sum_{i=1}^k e_i\right)^{-1} w_k(t), \quad t \in K.$$

Finally, we analyze the performance of the MCMC algorithms and their convergence, both in the max-linear and in the general max-stable setting, in a simulation study. To this end, we compare their results with the exact conditional distribution which is known in the case of ℓ being a point evaluation (cf. [10, 1]). Further, the performance of the algorithms is illustrated in a real dataset example: a max-stable model is fitted to extreme precipitation in the French region of Cevennes, and simulations of the fall pointwise maxima are performed conditionally on the outputs of a regional climate model.

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