On excursion probabilities for a class of non-stationary random fields

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Abstract. *Asymptotic approximations of threshold exceedance probabilities for random fields have been established, under suitable conditions, in terms of connectivity properties defined by the expectation of Euler characteristic. In this paper, some extensions and related results concerning the order of approximation are investigated for the class of harmonizable random fields. In particular, the study is focused on transformations of stationary random fields by spatial deformations subject to appropriate regularity and boundedness assumptions. Other well-known significant cases of practical interest within this class are also addressed. Finally, several aspects of continuing research in this context are discussed.*

Keywords. *Euler characteristic; Excursion set; Harmonizable process; Non-stationary random field; Spatial deformation.*

1 Introduction

Random field models are involved in the representation and study of a wide range of real phenomena in Geophysics and Environmental Sciences, among other areas. There is a vast literature devoted to theoretical aspects and related statistical methodology, with particular emphasis in developments for the case of Gaussian and/or stationary random fields, although with increasing interest in more general scenarios (see, for example, [10], [12], [21] and [22]).

In real applications concerning the assessment on extremal behaviour, one of the most significant problems concerns the evaluation of different forms of excursion probabilities; in particular,

$$P\left\{\sup_{t\in T} X(t) \geq u\right\} \tag{1}$$

where, formally, X is a centered random field over the set $T \subseteq \mathbb{R}^d$, which is assumed to be compact under the usual metric, and u represents a given threshold. Probabilistic and statistical aspects of random field excursion sets and extrema are addressed in key references such as [1], [2], [9], [18], [20] and [23].

Geometrical characteristics of excursion sets defined by threshold exceedances are intrinsically related to excursion probabilities, and can be used in practice in the formulation of indicators for risk assessment. Derivation of formal asymptotic results involves a certain degree of complexity depending on model assumptions and probability specifications. In this context, Adler and Taylor [2] prove that, for a Gaussian random field and under appropriate conditions,

$$\left| P\left\{ \sup_{t\in T} X(t) \geq u \right\} - E\left[\varphi(A_u(X,T))\right] \right| < O\left(e^{\frac{-\alpha u^2}{2\sigma^2}} \right), \tag{2}$$

where φ represents the Euler characteristic, $A_u(X,T)$ is the excursion set of X over T at u-level, σ^2 is the variance of X (which is assumed to be constant) and $\alpha > 1$ is an identifiable constant. Some extensions of this important result for certain classes of non-Gaussian random fields are also developed in several subsequent papers (e.g. [3], [4]). The main aim of this approach is to find a suitable explicit error-bound for the approximation.

Spatial deformation has been used in different areas of application, such as image analysis or environmental studies, to represent certain forms of heterogeneity which can be explained by transformation of a reference stationary random field. See, for example, [19] in the context of sampling network design; [5], [6], [11], [17] concerning the estimation of deformed stationary or stationary-isotropic random fields; [16] on image warping and the evaluation of distortion by deformation; [13] regarding the joint estimation of spatial deformation and blurring under a generalized random field approach, with application to environmental data; [8] for the implementation of dynamic deformation in a spatio-temporal model, etc. Angulo and Madrid [7] study through simulation the effect of deformation on the asymptotic behavior of the Euler characteristic of threshold exceedance sets, considering different scenarios depending on local variability and long-range dependence properties of the underlying random field (see also [8] in the spatio-temporal case).

The main aim of this work is to study analytical aspects concerning the asymptotic approximation of threshold exceedance probabilities by the expectation of the Euler characteristic for a deformed random field. We first show that, under appropriate regularity and boundedness conditions, a suitable framework for generalization is provided by the class of harmonizable processes (see, for example, [14], [15], [21], [22]). In particular, we study the order of the error-bound approximation for stationary random fields subject to deformation. Extensions and related results for other well-known subclasses of harmonizable processes, as well as for other random field transformations, are also investigated.

2 Formal and Methodological aspects

This section introduces some basic elements and outlines the methodological approach followed in this research.

Given a random field X on $T \subseteq \mathbb{R}^d$ and a C^1-diffeomorphism $\Phi : T \longrightarrow D \subseteq \mathbb{R}^d$ with $|J_\Phi| > 0$, the deformed random field X_Φ is defined as

$$X_\Phi(s) = (X \circ \Phi^{-1})(s), \forall s \in D. \tag{3}$$

This formulation corresponds to a 'level'-type deformation, where the effect of the transformation by Φ is to reallocate in the spatial domain the random field variables, with state values remaining as originally. Another case of interest arises when the transformation also involves a local change of measure, corresponding to a 'flow'-type deformation, according to the following definition (see [7]):

$$X_{\tilde{\Phi}}(s) = (X \circ \Phi^{-1})(s)|J_{\Phi^{-1}}(s)|, \forall s \in D. \tag{4}$$

In both cases, stationarity and isotropy are not preserved, except for trivial transformations, in a deformed random field.

Under suitable conditions, the class of harmonizable processes provides a suitable framework for spectral analysis of random fields with local heterogeneities. A second-order random field X is said to be harmonizable ([21], [22]) if it can be represented as the Fourier-Stieltjes integral of a complex-valued process $Z(\lambda)$ as

$$X(s) = \int \exp(i\lambda s) dZ(\lambda), \tag{5}$$

satisfying that its spectral distribution function is of bounded variation, i.e.:

$$\int\int |dH(\lambda,\lambda')| < \infty \tag{6}$$

where $H(\lambda,\lambda') = \mathrm{Cov}(Z(\lambda), Z(\lambda'))$.

Methodological aspects Firstly, we investigate assumptions under which harmonizable random fields arise from spatial deformations. We study the bounded variation of the spectral distribution function of the transformed random field. Second, we address the problem of determining the order of the error-bound approximation. We investigate other subclasses of interest within the class of harmonizable processes, such as modulated stationary processes, output processes of linear systems, oscillatory processes, among others. In all the cases, we look into the asymptotic approximation of the excursion probabilities by the expectation of Euler characteristic of its excursion sets via tools in which the underlying basis is the Chern-Gauss-Bonnet theorem.

3 Final comments

In this paper, non-stationary random fields arising from deformation of stationary random fields are studied in the framework of harmonizable processes. Extensions of asymptotic results for approximation of threshold exceedance probabilities by the expected Euler characteristic of the excursion sets are investigated. The main focus is on the determination of the approximation error-bound order. Other cases within the class of harmonizable processes are also addressed under this approach.

Among other directions, continuing research involves extensions to other random field transformations such as blurring and its composition with deformation (see [7]), derivation of related results for other classes on non-stationary random fields, as well as formulation of dynamic extensions in the spatio-temporal context and consideration of certain generalized forms of excursion probabilities.

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