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EXPLICIT VERSIONS OF THE PRIME IDEAL THEOREM FOR DEDEKIND ZETA FUNCTIONS UNDER GRH

by

L. Grenié and G. Molteni

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EXPLICIT VERSIONS OF THE PRIME IDEAL THEOREM FOR DEDEKIND ZETA FUNCTIONS UNDER GRH

LOÏC GRENIÉ AND GIUSEPPE MOLTENI

Abstract. Let $\psi_{\mathbb{K}}$ be the Chebyshev function of a number field $\mathbb{K}$. Under GRH we prove an explicit upper bound for $|\psi_{\mathbb{K}}(x)-x|$ in terms of the degree and the discriminant of $\mathbb{K}$. The new bound improves significantly on previous known results.

1. Introduction

For a number field $\mathbb{K}$ we denote

$n_{\mathbb{K}}$ its dimension,
$\Delta_{\mathbb{K}}$ the absolute value of its discriminant,
r_1 the number of its real places,
r_2 the number of its imaginary places,
$d_{\mathbb{K}} := r_1+r_2-1$.

Moreover, throughout this paper $\mathfrak{p}$ denotes a nonzero prime ideal of the integer ring $\mathcal{O}_{\mathbb{K}}$ and $\mathrm{N}\mathfrak{p}$ its absolute norm. The von Mangoldt function $\Lambda_{\mathbb{K}}$ is defined on the set of ideals of $\mathcal{O}_{\mathbb{K}}$ as $\Lambda_{\mathbb{K}}(\mathfrak{I}) := \log \mathrm{N}\mathfrak{p}$ if $\mathfrak{I} = \mathfrak{p}^m$ for some $\mathfrak{p}$ and $m \in \mathbb{N}_{>0}$, and is zero otherwise. Moreover, the function $\pi_{\mathbb{K}}$ and the Chebyshev function $\psi_{\mathbb{K}}$ are defined as

$$\pi_{\mathbb{K}}(x) := \sharp\{\mathfrak{p}\colon \mathrm{N}\mathfrak{p} \leq x\}$$

and

$$\psi_{\mathbb{K}}(x) := \sum_{\substack{\mathfrak{I}\lhd\mathcal{O}_{\mathbb{K}}\\ 0<\mathrm{N}\mathfrak{I}\leq x}} \Lambda_{\mathbb{K}}(\mathfrak{I}) = \sum_{\substack{\mathfrak{p},\, m\\ \mathrm{N}\mathfrak{p}^m\leq x}} \log \mathrm{N}\mathfrak{p}.$$

The original prime number theorem states that

$$\pi_{\mathbb{Q}}(x) \sim \frac{x}{\log x} \qquad \text{as } x \to \infty$$

and was independently proved in 1896 by Hadamard and de la Vallée–Poussin, both following the ideas of Riemann. By the work of Chebyshev this claim is equivalent to

$$\psi_{\mathbb{Q}}(x) \sim x \qquad \text{as } x \to \infty.$$

The remainder in these asymptotic behaviors is strictly controlled by the distribution of the nontrivial zeros of the Riemann zeta function. This was first suggested by Riemann himself, and then confirmed by de la Vallée–Poussin in 1899, when he deduced the now standard estimate for the remainder from the classical zero free region for the Riemann zeta function. Actually, the Riemann Hypothesis

$$\zeta(s) \neq 0 \qquad \forall\, \mathrm{Re}(s) > 1/2$$

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is equivalent to the statements

$$\Big|\pi_{\mathbb{Q}}(x)-\int_2^x \frac{\mathrm{d}u}{\log u}\Big| \ll \sqrt{x}\log x$$

and

$$|\psi_{\mathbb{Q}}(x)-x| \ll \sqrt{x}\log^2 x,$$

as proved by von Koch in the first years of the twentieth century. A quantitative version of the von Koch result was proved by Schoenfeld [19] in 1976: as a consequence of his previous work in collaboration with Rosser [18] he showed that

$$|\psi_{\mathbb{Q}}(x)-x| \leq \frac{1}{8\pi}\sqrt{x}\log^2 x \qquad \forall\, x \geq 73.2. \tag{1.1}$$

The arguments of Hadamard and de la Vallée–Poussin were quickly adapted by Landau to prove analogous results for a generic number field $\mathbb{K}$, and in 1977 Lagarias and Odlyzko [5] modified the argument to explore the dependence of the remainder with respect to the parameters $\Delta_{\mathbb{K}}$ and $n_{\mathbb{K}}$. As a part of a more general result on Chebotarev's theorem, they proved that if $\zeta_{\mathbb{K}}$ satisfies the Generalized Riemann Hypothesis

$$\zeta_{\mathbb{K}}(s) \neq 0 \qquad \forall\, \mathrm{Re}(s) > 1/2,$$

then

$$|\psi_{\mathbb{K}}(x)-x| \ll \sqrt{x}[\log x \log \Delta_{\mathbb{K}}+n_{\mathbb{K}}\log^2 x],$$

where the implicit constant is independent of $\mathbb{K}$. Oesterlé repeated their argument, aiming to produce an explicit value of the absolute constants involved, and he proved that

$$|\psi_{\mathbb{K}}(x)-x| \leq \sqrt{x}\Big[\Big(\frac{\log x}{\pi}+2\Big)\log \Delta_{\mathbb{K}}+\Big(\frac{\log^2 x}{2\pi}+2\Big)n_{\mathbb{K}}\Big] \qquad \forall\, x \geq 1 \tag{1.2}$$

under GRH. This result was announced in [14], but unfortunately its proof has never appeared. Very recently Winckler [22, Th. 8.1] has also produced an explicit version of Lagarias and Odlyzko's work, and proved a result similar to (1.2), but with $\frac{23}{3}$ and $\frac{863}{31}$ as coefficients of logs in the $\log \Delta_{\mathbb{K}}$ and $n_{\mathbb{K}}$ parts, respectively.

In this paper we combine a new method to estimate convergent sums on zeros (see Lemma 3.1), a very recent result of Trudgian [21] on the number of zeros in the critical strip and up to $\pm T$, and an idea of Goldston [1], to deduce the following general result.

Theorem 1.1. *(GRH) For every $x \geq 3$ and $T \geq 5$ we have:*

$$|\psi_{\mathbb{K}}(x)-x| \leq F(x,T)\log \Delta_{\mathbb{K}}+G(x,T)n_{\mathbb{K}}+H(x,T) \tag{1.3}$$

with

$$F(x,T) = \frac{\sqrt{x}}{\pi}\Big[\log\Big(\frac{T}{2\pi}\Big)+6.01+\frac{5.84}{T}+\frac{5.52}{T^2}\Big]+1.02, \tag{1.4}$$

$$G(x,T) = \frac{\sqrt{x}}{\pi}\Big[\frac{1}{2}\log^2\Big(\frac{T}{2\pi}\Big)+\Big(2+\frac{5.84}{T}+\frac{5.52}{T^2}\Big)\log\Big(\frac{T}{2\pi}\Big)-1.41+\frac{29.04}{T}+\frac{31.46}{T^2}\Big]-2.10,$$

$$H(x,T) = \frac{x}{T}+\frac{\sqrt{x}}{\pi}\Big[25.57+\frac{25.97}{T}+\frac{28.57}{T^2}\Big]+\epsilon_{\mathbb{K}}(x,T)+8.35+1.22\frac{\delta_{n_{\mathbb{K}}\leq 2}}{x},$$

where $\epsilon_{\mathbb{K}}(x,T) := \max\big(0, d_{\mathbb{K}}\log x-1.44 n_{\mathbb{K}}\frac{\sqrt{x}}{T}\big)$ and δ is the Kronecker symbol.

Setting T to a constant one gets a bound of Chebyshev kind, with a main term independent of the parameters of the field; as a consequence the resulting bound is very strong when x is small with respect to the degree or the discriminant.
Setting $T = x/6$ one gets (1.2), for $x \geq 105$ for any non-rational field. By taking $T = 8$, the range can be extended for $x \in [20, 105]$: it follows immediately for $n_{\mathbb{K}} = 2$ and $\Delta_{\mathbb{K}} \geq 767842$, $n_{\mathbb{K}} = 3$ and $\Delta_{\mathbb{K}} \geq 5700$ or $n_{\mathbb{K}} \geq 4$; the remaining cases for quadratic and cubic fields can be checked by explicit computations.
Comparing the main increasing term $\sqrt{x}\log^2\left(\frac{T}{2\pi}\right)$ with the main decreasing term $\frac{x}{T}$ we are led to use $T(x) = c\frac{\sqrt{x}}{\log x}$ for suitable values of c. In fact, combining different choices for c we get the following result, which improves significantly on (1.2).

Corollary 1.2. *(GRH) Suppose $x \geq 100$. Then*

$$|\psi_{\mathbb{K}}(x) - x| \leq \sqrt{x}\Big[\Big(\frac{\log x}{2\pi} + 2\Big)\log\Delta_{\mathbb{K}} + \Big(\frac{\log^2 x}{8\pi} + 2\Big)n_{\mathbb{K}}\Big]. \tag{1.5}$$

The range $x \geq 100$ can be extended for fields of large degree, in particular one has $x \geq 24$ when $n_{\mathbb{K}} \geq 8$, $x \geq 29$ for $n_{\mathbb{K}} = 7$, $x \geq 43$ for $n_{\mathbb{K}} = 6$ and $x \geq 72$ for $n_{\mathbb{K}} = 5$. Only small improvements are possible for cubic and quadratic fields with this method, and only at the cost of a very large quantity of numerical computations.

A different choice of c yields even better results for large x.

Corollary 1.3. *(GRH) For every $x \geq 3$, we have*

$$|\psi_{\mathbb{K}}(x) - x| \leq \sqrt{x}\Big[\Big(\frac{1}{2\pi}\log\Big(\frac{18.8\,x}{\log^2 x}\Big) + 2.3\Big)\log\Delta_{\mathbb{K}} + \Big(\frac{1}{8\pi}\log^2\Big(\frac{18.8\,x}{\log^2 x}\Big) + 1.3\Big)n_{\mathbb{K}} + 0.3\log x + 14.6\Big].$$

Moreover, if $x \geq 2000$, then

$$|\psi_{\mathbb{K}}(x) - x| \leq \sqrt{x}\Big[\Big(\frac{1}{2\pi}\log\Big(\frac{x}{\log^2 x}\Big) + 1.8\Big)\log\Delta_{\mathbb{K}} + \Big(\frac{1}{8\pi}\log^2\Big(\frac{x}{\log^2 x}\Big) + 1.1\Big)n_{\mathbb{K}} + 1.2\log x + 10.2\Big].$$

The first bound is stronger than (1.2) for $x \geq 1700$ if $\mathbb{K} \neq \mathbb{Q}$ (but $x \geq 280$ suffices when $n_{\mathbb{K}} \geq 3$ and $x \geq 115$ when $n_{\mathbb{K}} \geq 4$), and stronger than (1.5) for $x \geq 1.4\cdot 10^{16}$ (but $x \geq 5.6\cdot 10^{10}$ suffices when $n_{\mathbb{K}} \geq 3$ and $x \geq 2.2\cdot 10^{8}$ when $n_{\mathbb{K}} \geq 4$).
The second bound is always stronger than (1.2) when $\mathbb{K} \neq \mathbb{Q}$ and stronger than (1.5) for $x \geq 1.4\cdot 10^{32}$ (but $x \geq 9.3\cdot 10^{10}$ suffices when $n_{\mathbb{K}} \geq 3$ and $x \geq 6.3\cdot 10^{5}$ when $n_{\mathbb{K}} \geq 4$; the bad behavior for quadratic fields comes from the term $1.2\log x$). It is also stronger than (1.1), but only for extremely large x (actually $x \geq 3\cdot 10^{871}$). This is a consequence of the fact that our computations have not been optimized for $\mathbb{Q}$: actually this is possible in several steps and we believe that doing so the method should produce a better bound.

From Corollary 1.3 one quickly deduces the following explicit bound for the remainder of the $\pi_{\mathbb{K}}(x)$ function.

Corollary 1.4. *(GRH) For $x \geq \bar{x} \geq 3$ we have*

$$\begin{aligned}&\Big|\pi_{\mathbb{K}}(x) - \pi_{\mathbb{K}}(\bar{x}) - \int_{\bar{x}}^{x}\frac{\mathrm{d}u}{\log u}\Big|\\ &\qquad\leq \sqrt{x}\Big[\Big(\frac{1}{2\pi} - \frac{\log\log x}{\pi\log x} + \frac{5.8}{\log x}\Big)\log\Delta_{\mathbb{K}} + \Big(\frac{1}{8\pi} - \frac{\log\log x}{2\pi\log x} + \frac{3}{\log x}\Big)n_{\mathbb{K}}\log x + 0.3 + \frac{13.3}{\log x}\Big].\end{aligned}$$

We have made available at the address:
`http://users.mat.unimi.it/users/molteni/research/psi_GRH/psi_GRH_data.gp`
a file containing the PARI/GP [16] code we have used to compute the constants in this article.

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2. Preliminary inequalities

For $\mathrm{Re}(s) > 1$ we have

$$-\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \log(\mathrm{N}\mathfrak{p})(\mathrm{N}\mathfrak{p})^{-ms},$$

which in terms of standard Dirichlet series reads

$$-\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) = \sum_{n=1}^{\infty} \tilde{\Lambda}_{\mathbb{K}}(n) n^{-s}, \quad \text{with} \quad \tilde{\Lambda}_{\mathbb{K}}(n) := \begin{cases} \displaystyle\sum_{\mathfrak{p}|p,\, f_{\mathfrak{p}}|k} \log \mathrm{N}\mathfrak{p} & \text{if } n = p^k \\ 0 & \text{otherwise,} \end{cases}$$

where $f_{\mathfrak{p}}$ is the residual degree of $\mathfrak{p}$. The definition of $\tilde{\Lambda}_{\mathbb{K}}$ shows that $\tilde{\Lambda}_{\mathbb{K}}(n) \leq n_{\mathbb{K}}\Lambda(n)$ for every integer n.
Let

$$\Gamma_{\mathbb{K}}(s) := \Big[\pi^{-\frac{s+1}{2}}\Gamma\Big(\frac{s+1}{2}\Big)\Big]^{r_2} \Big[\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\Big]^{r_1+r_2} \tag{2.1}$$

and

$$\xi_{\mathbb{K}}(s) := s(s-1)\Delta_{\mathbb{K}}^{s/2}\Gamma_{\mathbb{K}}(s)\zeta_{\mathbb{K}}(s), \tag{2.2}$$

then the functional equation for $\zeta_{\mathbb{K}}$ reads

$$\xi_{\mathbb{K}}(1-s) = \xi_{\mathbb{K}}(s). \tag{2.3}$$

Moreover, since $\xi_{\mathbb{K}}(s)$ is an entire function of order 1 and does not vanish at $s = 0$, we have

$$\xi_{\mathbb{K}}(s) = e^{A_{\mathbb{K}}+B_{\mathbb{K}}s} \prod_{\rho} \Big(1-\frac{s}{\rho}\Big) e^{s/\rho} \tag{2.4}$$

for some constants $A_{\mathbb{K}}$ and $B_{\mathbb{K}}$, where ρ runs through all the zeros of $\xi_{\mathbb{K}}(s)$, which are precisely those zeros $\rho = \beta+i\gamma$ of $\zeta_{\mathbb{K}}(s)$ for which $0 < \beta < 1$ and are the so-called “nontrivial zeros” of $\zeta_{\mathbb{K}}(s)$. From now on ρ will denote a nontrivial zero of $\zeta_{\mathbb{K}}(s)$. We recall that the zeros are symmetric with respect to the real axis, as a consequence of the fact that $\zeta_{\mathbb{K}}(s)$ is real for $s \in \mathbb{R}$.
Differentiating (2.2) and (2.4) logarithmically we obtain the identity

$$\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) = B_{\mathbb{K}} + \sum_{\rho} \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) - \frac{1}{2}\log\Delta_{\mathbb{K}} - \Big[\frac{1}{s} + \frac{1}{s-1}\Big] - \frac{\Gamma'_{\mathbb{K}}}{\Gamma_{\mathbb{K}}}(s), \tag{2.5}$$

valid identically in the complex variable s.
Stark [20, Lemma 1] proved that the functional equation (2.3) implies that $B_{\mathbb{K}} = -\sum_{\rho} \rho^{-1}$

(see also [12] and [6, Ch. XVII, Th. 3.2]), and that once this information is available one can use (2.5) and the definition of the gamma factor in (2.1) to prove that the function $f_{\mathbb{K}}(s) := \mathrm{Re}\sum_\rho \frac{2}{s-\rho}$ can be exactly computed via the alternative representation

$$f_{\mathbb{K}}(s) = 2\mathrm{Re}\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s)+\log\frac{\Delta_{\mathbb{K}}}{\pi^{n_{\mathbb{K}}}}+\mathrm{Re}\Big(\frac{2}{s}+\frac{2}{s-1}\Big)+(r_1+r_2)\mathrm{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big)+r_2\mathrm{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{s+1}{2}\Big), \tag{2.6}$$

where $\sum_\rho \rho^{-1}$ and $\sum_\rho (s-\rho)^{-1}$ must be intended as symmetrical sums.

Using (2.2), (2.3) and (2.5) one sees that

$$\begin{aligned} \frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s) &= \frac{r_1+r_2-1}{s}+r_{\mathbb{K}}+O(s) \quad \text{as } s\to 0\\ \frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s) &= \frac{r_2}{s+1}+r_{\mathbb{K}}'+O(s+1) \qquad \text{as } s\to -1, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} r_{\mathbb{K}} &= B_{\mathbb{K}}+1-\frac{1}{2}\log\frac{\Delta_{\mathbb{K}}}{\pi^{n_{\mathbb{K}}}}-\frac{r_1+r_2}{2}\frac{\Gamma'}{\Gamma}(1)-\frac{r_2}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{2}\Big)\\ r_{\mathbb{K}}' &= -\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(2)-\log\frac{\Delta_{\mathbb{K}}}{\pi^{n_{\mathbb{K}}}}-\frac{n_{\mathbb{K}}}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{3}{2}\Big)-\frac{n_{\mathbb{K}}}{2}\frac{\Gamma'}{\Gamma}(1). \end{aligned}$$

In order to prove our result we need the following explicit bound for $r_{\mathbb{K}}$

$$|r_{\mathbb{K}}| \le 1.02\log\Delta_{\mathbb{K}}-2.10n_{\mathbb{K}}+8.35 \tag{2.8}$$

which is Lemma 3.2 in [2].

At last, we need two elementary lemmas. The first one is an optimized version of a lemma due to Littlewood [7].

Lemma 2.1. *If $x \ge -1$ and $1 \le \mathrm{Re}(\nu) \le 2$, then*

$$|(1+x)^\nu-1-\nu x| \le \Big(\frac{1}{2}+\Big(\frac{1}{\mathrm{Re}(\nu)}-\frac{1}{2}\Big)\max(0,-x)\Big)|\nu(\nu-1)x^2|.$$

Proof. The statement is obvious for $\nu = 1$, $\nu = 2$, $x = -1$ and $x = 0$, we thus suppose we are in another case. From the equality $f(x)-f(0)-f'(0)x = \int_0^x\int_0^u f''(v)\,\mathrm{d}v\,\mathrm{d}u$ one gets

$$\frac{(1+x)^\nu-1-\nu x}{\nu(\nu-1)} = \int_0^x\int_0^u (1+v)^{\nu-2}\,\mathrm{d}v\,\mathrm{d}u. \tag{2.9}$$

Let $x \ge 0$, then $|1+v|^{\mathrm{Re}(\nu)-2} \le 1$ thus

$$\frac{|(1+x)^\nu-1-\nu x|}{|\nu(\nu-1)|} \le \int_0^x\int_0^u \mathrm{d}v\,\mathrm{d}u = \frac{x^2}{2}.$$

Let $x \in (-1,0)$. Then

$$\frac{\mathrm{Re}(\nu)(\mathrm{Re}(\nu)-1)}{x^2}\Big|\int_0^x\Big|\int_0^u (1+v)^{\mathrm{Re}(\nu)-2}\,\mathrm{d}v\Big|\,\mathrm{d}u\Big| = \frac{(1+x)^{\mathrm{Re}(\nu)}-1-\mathrm{Re}(\nu)x}{x^2}. \tag{2.10}$$

The right-hand side may be written as $\sum_{k=0}^\infty \binom{\mathrm{Re}(\nu)}{k+2}x^k$ and its second derivative as $\sum_{k=0}^\infty \binom{\mathrm{Re}(\nu)}{k+4}(k+2)(k+1)x^k$. When $x \in (-1,0)$ each term of the series is positive; this proves that the right-hand side in (2.10) is convex in $(-1,0)$ so that its graph is below the line connecting its points with $x=-1$ and $x=0$. Said line has equation $y=(\frac{1}{2}+(\frac{1}{2}-\frac{1}{\mathrm{Re}(\nu)})x)\mathrm{Re}(\nu)(\mathrm{Re}(\nu)-1)$, thus (2.10) gives

$$\Big|\int_0^x \Big|\int_0^u (1+v)^{\mathrm{Re}(\nu)-2}\,\mathrm{d}v\Big|\,\mathrm{d}u\Big| \le \Big(\frac{1}{2}+\Big(\frac{1}{2}-\frac{1}{\mathrm{Re}(\nu)}\Big)x\Big)x^2$$

for $\mathrm{Re}(\nu)>1$ immediately and for $\mathrm{Re}(\nu)\ge 1$ by continuity. We get the claim comparing it with (2.9). ☐

Lemma 2.2. *Let*

$$f_1(x):=\sum_{r=1}^{\infty}\frac{x^{1-2r}}{2r(2r-1)},\qquad f_2(x):=\sum_{r=2}^{\infty}\frac{x^{2-2r}}{(2r-1)(2r-2)},$$

$$R_{r_1,r_2}(x):=-(r_1+r_2-1)(x\log x-x)+r_2(\log x+1)-(r_1+r_2)f_1(x)-r_2f_2(x).$$

Let $x\ge 3$, then

$$-(r_1+r_2-1)\log x\le R'_{r_1,r_2}(x)\le 1.22\frac{\delta_{n_{\mathbb{K}}\le 2}}{x}.$$

Proof. We have

$$f_1(x)=\frac{1}{2}\Big[x\log(1-x^{-2})+\log\Big(\frac{1+x^{-1}}{1-x^{-1}}\Big)\Big],\qquad f_2(x)=1-\frac{1}{2}\Big[\log(1-x^{-2})+x\log\Big(\frac{1+x^{-1}}{1-x^{-1}}\Big)\Big],$$

and

$$\begin{aligned}R'_{r_1,r_2}(x)&=-(r_1+r_2-1)\log x-\frac{1}{2}(r_1+r_2)\log(1-x^{-2})-\frac{1}{2}r_2\log\Big(\frac{1-x^{-1}}{1+x^{-1}}\Big)\\&=-(r_1+r_2-1)\log x-\frac{r_1}{2}\log(1-x^{-2})-r_2\log(1-x^{-1});\end{aligned}$$

this equality already proves the lower bound. The upper bound immediately follows for the cases where $r_1+r_2=1$. Suppose $r_1+r_2\ge 2$, writing $R'_{r_1,r_2}(x)$ as

$$R'_{r_1,r_2}(x)=\log x-\frac{r_1}{2}\log(x^2-1)-r_2\log(x-1),$$

then for $x>1$ one gets

$$R'_{r_1,r_2}(x)\le\log x-(r_1+r_2)\log(x-1)\le\log x-2\log(x-1)\le 0$$

where the last inequality is true for $x\ge\frac{3+\sqrt{5}}{2}=2.61\dots$ ☐

3. Upper bounds

For the proof of the theorem we need bounds for three sums on nontrivial zeros, namely for

$$\sum_{|\gamma|\le T}1,\qquad \sum_{|\gamma|\ge T}\frac{1}{|\rho|^2}\quad\text{and}\quad\sum_{|\gamma|\le T}\frac{1}{|\rho|}.$$

The first sum is simply the number $N_{\mathbb{K}}(T)$ of nontrivial zeros in the rectangle $0<\mathrm{Re}(s)<1$, $|\mathrm{Im}(s)|\le T$. It has been explicitly estimated by Trudgian [21] in a work improving Kadiri–Ng's paper [4]. We estimate the second sum by partial summation using this result. For the last one a simple partial summation is not possible since both Kadiri–Ng's and Trudgian's results are proved only for $T\ge 1$ and improve when the range is further restricted to $T\ge T_0$ with a $T_0\ge 1$. As a consequence we bound the part of the third sum coming from the zeros far enough of the real axis by partial summation, and the remaining with a different technique.

In fact, in [2] we have shown a new method to bound converging sums on zeros under GRH. The method works very well but depends on several parameters whose values are fixed via a trial and error approach. Thus, in order to apply it we need to fix a value for T_0, and the final result will only be valid in the range $T \geq T_0$. After several tests the choice $T_0 = 5$ seemed to represent a good compromise between the need of having a large T_0 (to take advantage of the better estimante in Trudgian's result) and a small T_0 (to make the final theorem valid in a larger range). Our result is as follows.

Lemma 3.1. *(GRH) One has*

$$\sum_{|\gamma|\leq 5} \frac{1}{|\rho|} \leq 1.02\log\Delta_{\mathbb{K}} - 1.63 n_{\mathbb{K}} + 7.04.$$

Proof. We apply the same technique we have already used for Lemma 4.1 in [2]. Thus, let $f(s,\gamma) := 4(2s-1)/((2s-1)^2+4\gamma^2)$, so that $f_{\mathbb{K}}(s) = \sum_\gamma f(s,\gamma)$, and let $g(\gamma) := 2(1+4\gamma^2)^{-1/2}\chi_{[-5,5]}(\gamma)$, so that $\sum_{|\gamma|\leq 5}|\rho|^{-1} = \sum_\gamma g(\gamma)$. We look for a finite linear combination of $f(s,\gamma)$ at suitable points s_j such that

$$g(\gamma) \leq F(\gamma) := \sum_j a_j f(s_j,\gamma) \qquad \forall \gamma\in\mathbb{R}, \tag{3.1}$$

so that

$$\sum_{|\gamma|\leq 5} \frac{1}{|\rho|} \leq \sum_j a_j f_{\mathbb{K}}(s_j); \tag{3.2}$$

once (3.2) is proved, we recover a bound for the sum on zeros recalling the identity (2.6). According to this approach the final coefficient of $\log\Delta_{\mathbb{K}}$ will be the sum of all a_j, thus we are interested into linear combinations for which this sum is as small as possible. We set $s_j = 1+j/2$ with $j = 1,\dots,2q+3$ for a suitable integer q. Let $\Upsilon \subset (0,\infty)$ be a set with q numbers. We require:

(1) $F(\gamma) = g(\gamma)$ for all $\gamma\in\Upsilon\cup\{0,5\}$,
(2) $F'(\gamma) = g'(\gamma)$ for all $\gamma\in\Upsilon$,
(3) $\lim_{\gamma\to\infty}\gamma^2F(\gamma) = \lim_{\gamma\to\infty}\gamma^2 g(\gamma) = 0$.

This produces a set of $2q+3$ linear equations for the $2q+3$ constants a_j, and we hope that these satisfy (3.1) for every γ. We choose $q := 22$ and $\Upsilon := \{0.6, 1, 1.9, 2.9, 3.9, 10, 13, 14, 15, 16, 17, 18, 19, 20, 30, 40, 50, 100, 10^3, 10^4, 10^5, 10^6\}$. Finally, with an abuse of notation we take for a_j the solution of the system, rounded above to 10^{-7}: this produces the numbers in Table 4. Then, using Sturm's algorithm, we prove that the values found actually give an upper bound for g, so that (3.2) holds with such a_j's. These constants verify

$$\begin{aligned} &\sum_j a_j = 1.011\dots, && \sum_j a_j\Big(\frac{2}{s_j}+\frac{2}{s_j-1}\Big) \leq 7.04,\\ &\sum_j a_j\frac{\Gamma'}{\Gamma}\Big(\frac{s_j}{2}\Big) \leq -1.13, && \sum_j a_j\frac{\Gamma'}{\Gamma}\Big(\frac{s_j+1}{2}\Big) \leq -0.31. \end{aligned} \tag{3.3}$$

We write $\sum_j a_j\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s_j)$ as

$$-\sum_n \tilde\Lambda_{\mathbb{K}}(n)S(n) \quad\text{with}\quad S(n) := \sum_j \frac{a_j}{n^{s_j}}.$$

We check numerically that $S(n) > 0$ for $n \le 60975$ and that it is negative for $60975 < n \le 128000$. Then, since the sign of a_j alternates, we can easily prove that each pair $\frac{a_1}{n^{s_1}}+\frac{a_2}{n^{s_2}}$, ..., $\frac{a_{2q+1}}{n^{s_{2q+1}}}+\frac{a_{2q+2}}{n^{s_{2q+2}}}$ and the last term $\frac{a_{2q+3}}{n^{s_{2q+3}}}$ are negative for every $n \ge 128000$, thus

$$\begin{aligned}\sum_j a_j \frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s_j) &= -\sum_n \tilde\Lambda_{\mathbb{K}}(n)S(n) \le -n_{\mathbb{K}} \sum_{n>60975} \Lambda(n)S(n) \\ &= -n_{\mathbb{K}}\Big[\sum_{n=1}^{\infty} \Lambda(n)S(n) - \sum_{n\le 60975} \Lambda(n)S(n)\Big] \\ &= n_{\mathbb{K}}\Big[\sum_j a_j \frac{\zeta'}{\zeta}(s_j) + \sum_{n\le 60975} \Lambda(n)S(n)\Big] \le 0.12 n_{\mathbb{K}}.\end{aligned} \tag{3.4}$$

The result now follows from (2.6), and (3.2–3.4). □

Now we can bound the sums.

First sum. Trudgian [21] has proved that

$$\Big|N_{\mathbb{K}}(T) - \frac{T}{\pi}\log\Big(\Big(\frac{T}{2\pi e}\Big)^{n_{\mathbb{K}}}\Delta_{\mathbb{K}}\Big)\Big| \le \frac{1}{\pi}(c_1(\eta)W_{\mathbb{K}}(T) + c_2(\eta)n_{\mathbb{K}} + c_3(\eta)) \qquad \forall T \ge T_0 \ge 1, \tag{3.5}$$

where $W_{\mathbb{K}}(T) := \log\Delta_{\mathbb{K}} + n_{\mathbb{K}}\log(T/2\pi)$, $c_1(\eta) = \pi D_1$, $c_2(\eta) = \pi(D_2 + D_1\log 2\pi)$ and $c_3(\eta) = \pi D_3$ and the D_j are Trudgian's constants which depend on T_0, $\eta \in (0, \frac{1}{2}]$ and on two other parameters p and r. We thus have

$$N_{\mathbb{K}}(T) \le \frac{T}{\pi}\Big(1+\frac{c_1(\eta)}{T}\Big)W_{\mathbb{K}}(T) - \frac{T}{\pi}\Big(1-\frac{c_2(\eta)}{T}\Big)n_{\mathbb{K}} + \frac{c_3(\eta)}{\pi} \qquad \forall T \ge T_0.$$

We fix $\eta = \frac{1}{2}$, $p = -\eta = -\frac{1}{2}$ (this choice differs from the one in [21]) and $r = \frac{1+\eta-p}{1/2+\eta}$ (as in [21]), so that actually $r = 2$; recall that $T_0 = 5$. Following Trudjan's argument we find $D_1 = 0.459\ldots$, $D_2 = 1.996\ldots$, $D_3 = 2.754\ldots$, hence

$$N_{\mathbb{K}}(T) \le \frac{T}{\pi}\Big(1+\frac{1.45}{T}\Big)W_{\mathbb{K}}(T) - \frac{T}{\pi}\Big(1-\frac{8.93}{T}\Big)n_{\mathbb{K}} + \frac{8.66}{\pi} \qquad \forall T \ge 5. \tag{3.6}$$

Second sum. We proceed by partial summation. Let Formula (3.5) for $N_{\mathbb{K}}(T)$ be written as $A(T)+R(T)$, respectively the asymptotic and the remainder term. Then

$$\begin{aligned}\sum_{|\gamma|\ge T}\frac{1}{|\rho|^2} = \sum_{|\gamma|\ge T}\frac{1}{1/4+\gamma^2} &\le \int_T^{+\infty}\frac{\mathrm{d}A(\gamma)}{1/4+\gamma^2} + \frac{R(T)}{1/4+T^2} + \int_T^{+\infty}\frac{2\gamma R(\gamma)\,\mathrm{d}\gamma}{(1/4+\gamma^2)^2} \\ &= \int_T^{+\infty}\frac{\mathrm{d}A(\gamma)}{1/4+\gamma^2} + \frac{2R(T)}{1/4+T^2} + \int_T^{+\infty}\frac{R'(\gamma)\,\mathrm{d}\gamma}{1/4+\gamma^2} \\ &= \int_T^{+\infty}\frac{\mathrm{d}A(\gamma)}{1/4+\gamma^2} + \frac{2R(T)}{1/4+T^2} + \frac{c_1(\eta)}{\pi}n_{\mathbb{K}}\int_T^{+\infty}\frac{\gamma^{-1}\,\mathrm{d}\gamma}{1/4+\gamma^2} \\ &\le \int_T^{+\infty}\frac{\mathrm{d}A(\gamma)}{1/4+\gamma^2} + \frac{2R(T)}{T^2} + \frac{c_1(\eta)}{2\pi T^2}n_{\mathbb{K}}.\end{aligned}$$

Using

$$\int_T^{+\infty}\frac{\mathrm{d}\gamma}{1/4+\gamma^2} = 2\,\mathrm{atan}\Big(\frac{1}{2T}\Big) \quad \text{which is} \le \frac{1}{T}, \text{ and } \ge \frac{1}{T} - \frac{1/12}{T^3}$$

$$\int_T^{+\infty} \frac{\log\gamma}{1/4+\gamma^2}\,\mathrm{d}\gamma \leq \int_T^{+\infty} \gamma^{-2}\log\gamma\,\mathrm{d}\gamma = \frac{\log(eT)}{T},$$

one has

$$\int_T^{+\infty} \frac{\mathrm{d}A(\gamma)}{1/4+\gamma^2} \leq \frac{W_{\mathbb{K}}(T)}{\pi T}+\Big(1+\frac{\log 2\pi}{12T^2}\Big)\frac{n_{\mathbb{K}}}{\pi T}.$$

Thus

$$\begin{aligned}\sum_{|\gamma|\geq T}\frac{\pi}{|\rho|^2} &\leq \frac{W_{\mathbb{K}}(T)}{T}+\Big(1+\frac{\log 2\pi}{12T^2}\Big)\frac{n_{\mathbb{K}}}{T}+\frac{2}{T^2}\Big[c_1(\eta)\log\Delta_{\mathbb{K}}+\Big(c_1(\eta)\log\Big(\frac{T}{2\pi}\Big)+c_2(\eta)+\frac{c_1(\eta)}{4}\Big)n_{\mathbb{K}}+c_3(\eta)\Big]\\ &=\Big(1+\frac{2c_1(\eta)}{T}\Big)\frac{W_{\mathbb{K}}(T)}{T}+\Big(1+\frac{\log 2\pi}{12T^2}\Big)\frac{n_{\mathbb{K}}}{T}+\Big(2c_2(\eta)+\frac{c_1(\eta)}{2}\Big)\frac{n_{\mathbb{K}}}{T^2}+\frac{2c_3(\eta)}{T^2}\end{aligned}$$

hence

$$\sum_{|\gamma|\geq T}\frac{\pi}{|\rho|^2} \leq \Big(1+\frac{2.89}{T}\Big)\frac{W_{\mathbb{K}}(T)}{T}+\Big(1+\frac{18.61}{T}\Big)\frac{n_{\mathbb{K}}}{T}+\frac{17.31}{T^2} \qquad \forall T\geq 5. \tag{3.7}$$

Third sum. We proceed again by partial summation, plus the contribution of Lemma 3.1 to bound the part of the sum coming from low-lying zeros. We have

$$\begin{aligned}\sum_{|\gamma|\leq T}\frac{1}{|\rho|} &= \sum_{|\gamma|\leq T}\frac{1}{(1/4+\gamma^2)^{1/2}} \leq \sum_{|\gamma|\leq 5}\frac{1}{|\rho|}+\sum_{5\leq|\gamma|\leq T}\frac{1}{(1/4+\gamma^2)^{1/2}}\\ &\leq \sum_{|\gamma|\leq 5}\frac{1}{|\rho|}+\int_5^T\frac{\mathrm{d}A(\gamma)}{(1/4+\gamma^2)^{1/2}}+\frac{2R(5)}{\sqrt{101}}+\frac{R(T)}{(1/4+T^2)^{1/2}}+\int_5^T\frac{\gamma R(\gamma)\,\mathrm{d}\gamma}{(1/4+\gamma^2)^{3/2}}\\ &= \sum_{|\gamma|\leq 5}\frac{1}{|\rho|}+\int_5^T\frac{\mathrm{d}A(\gamma)}{(1/4+\gamma^2)^{1/2}}+\frac{4R(5)}{\sqrt{101}}+\int_5^T\frac{R'(\gamma)\,\mathrm{d}\gamma}{(1/4+\gamma^2)^{1/2}}\\ &= \sum_{|\gamma|\leq 5}\frac{1}{|\rho|}+\frac{4R(5)}{\sqrt{101}}+\int_5^T\frac{\mathrm{d}A(\gamma)}{(1/4+\gamma^2)^{1/2}}+\frac{c_1(\eta)}{\pi}n_{\mathbb{K}}\int_5^T\frac{\gamma^{-1}\,\mathrm{d}\gamma}{(1/4+\gamma^2)^{1/2}}\\ &\leq \sum_{|\gamma|\leq 5}\frac{1}{|\rho|}+\frac{4R(5)}{\sqrt{101}}+0.2\frac{c_1(\eta)}{\pi}n_{\mathbb{K}}+\int_5^T\frac{\mathrm{d}A(\gamma)}{(1/4+\gamma^2)^{1/2}}.\end{aligned}$$

Using

$$\int_5^T\frac{\mathrm{d}\gamma}{(1/4+\gamma^2)^{1/2}} = \log\Big(\frac{2T+\sqrt{4T^2+1}}{10+\sqrt{101}}\Big)$$

which is $\leq \log T - \log 5$ and $\geq \log T - 1.62$ for $T\geq 5$, and

$$\int_5^T\frac{\log\gamma\,\mathrm{d}\gamma}{(1/4+\gamma^2)^{1/2}} \leq \int_5^T\frac{\log\gamma}{\gamma}\,\mathrm{d}\gamma = \frac{\log^2 T}{2}-\frac{\log^2 5}{2},$$

one has

$$\int_5^T\frac{\mathrm{d}A(\gamma)}{(1/4+\gamma^2)^{1/2}} \leq \Big(\log\Big(\frac{T}{2\pi}\Big)+0.23\Big)\frac{\log\Delta_{\mathbb{K}}}{\pi}+\Big(\log^2\Big(\frac{T}{2\pi}\Big)-0.01\Big)\frac{n_{\mathbb{K}}}{2\pi}$$

thus recalling Lemma 3.1 we get

$$\sum_{|\gamma|\leq T}\frac{\pi}{|\rho|} \leq\Big(\log\Big(\frac{T}{2\pi}\Big)+0.23\Big)\log\Delta_{\mathbb{K}}+\frac{n_{\mathbb{K}}}{2}\Big(\log^2\Big(\frac{T}{2\pi}\Big)-0.01\Big)+\sum_{|\gamma|\leq 5}\frac{\pi}{|\rho|}+\frac{4\pi R(5)}{\sqrt{101}}+0.2c_1(\eta)n_{\mathbb{K}}$$

$$\le\Big(\log\Big(\frac{T}{2\pi}\Big)+0.23\Big)\log\Delta_{\mathbb{K}}+\frac{n_{\mathbb{K}}}{2}\Big(\log^2\Big(\frac{T}{2\pi}\Big)-0.01\Big)+\pi(1.02\log\Delta_{\mathbb{K}}-1.63n_{\mathbb{K}}+7.04)$$
$$+\frac{4}{\sqrt{101}}\Big(c_1(\eta)(\log\Delta_{\mathbb{K}}+n_{\mathbb{K}}\log\big(\frac{5}{2\pi}\big))+c_2(\eta)n_{\mathbb{K}}+c_3(\eta)\Big)+0.2c_1(\eta)n_{\mathbb{K}}$$
$$=\Big(\log\Big(\frac{T}{2\pi}\Big)+0.23+1.02\pi+\frac{4}{\sqrt{101}}c_1(\eta)\Big)\log\Delta_{\mathbb{K}}+7.04\pi+\frac{4}{\sqrt{101}}c_3(\eta)$$
$$+\Big(\frac{1}{2}\log^2\Big(\frac{T}{2\pi}\Big)-\frac{1}{2}0.01-1.63\pi+\frac{4\log(5/2\pi)}{\sqrt{101}}c_1(\eta)+\frac{4}{\sqrt{101}}c_2(\eta)+0.2c_1(\eta)\Big)n_{\mathbb{K}}$$

hence

$$\sum_{\substack{\rho\\|\gamma|\le T}}\frac{\pi}{|\rho|}\le\Big(\log\Big(\frac{T}{2\pi}\Big)+4.01\Big)\log\Delta_{\mathbb{K}}+\Big(\frac{1}{2}\log^2\Big(\frac{T}{2\pi}\Big)-1.41\Big)n_{\mathbb{K}}+25.57\qquad\forall T\ge 5. \tag{3.8}$$

4. Proofs

Proof of Theorem 1.1. Let

$$\psi^{(1)}_{\mathbb{K}}(x):=\int_0^x\psi_{\mathbb{K}}(t)\,\mathrm{d}t.$$

As observed by Goldston [1], since $\psi_{\mathbb{K}}(x)\ge 0$, one has the double inequality

$$\begin{aligned}\psi_{\mathbb{K}}(x)&\le\frac{\psi^{(1)}_{\mathbb{K}}(x+h)-\psi^{(1)}_{\mathbb{K}}(x)}{h}&&\text{if } h>0,\\ \psi_{\mathbb{K}}(x)&\ge\frac{\psi^{(1)}_{\mathbb{K}}(x+h)-\psi^{(1)}_{\mathbb{K}}(x)}{h}&&\text{if } -x<h<0.\end{aligned}\tag{4.1}$$

As in [3, Ch. IV Sec. 4, p. 73] and [5, Sec. 5], considering the integral representation

$$\psi^{(1)}_{\mathbb{K}}(x)=-\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s)\frac{x^{s+1}}{s(s+1)}\,\mathrm{d}s$$

one gets for every $x>1$ the identity

$$\psi^{(1)}_{\mathbb{K}}(x)=\frac{x^2}{2}-\sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)}-xr_{\mathbb{K}}+r'_{\mathbb{K}}+R_{r_1,r_2}(x)$$

where $R_{r_1,r_2}(x)$ is defined in Lemma 2.2 and $r_{\mathbb{K}}$ and $r'_{\mathbb{K}}$ are defined in (2.7). Thus

$$\frac{\psi^{(1)}_{\mathbb{K}}(x+h)-\psi^{(1)}_{\mathbb{K}}(x)}{h}=x+\frac{h}{2}-\sum_{\rho}\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}-r_{\mathbb{K}}+R'_{r_1,r_2}(\eta)$$

for a suitable η in the interval between x and $x+h$. Hence, for every $x\ge 3$ and $h\ne 0$ such that $x+h>1$, Lemma 2.2 gives

$$-d_{\mathbb{K}}\log x\le\frac{\psi^{(1)}_{\mathbb{K}}(x+h)-\psi^{(1)}_{\mathbb{K}}(x)}{h}-\Big(x+\frac{h}{2}-\sum_{\rho}\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}-r_{\mathbb{K}}\Big)\le 1.22\frac{\delta_{n_{\mathbb{K}}\le 2}}{x}. \tag{4.2}$$

We will now split the sum on the zeros in two parts: above and below T. The technique is the same for $h>0$ and $h<0$ but the constants are slightly different, we thus proceed separately for the two cases.

Suppose first $h>0$. Under GRH we have

$$\Big|\sum_{|\gamma|\geq T}\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big|\leq\sum_{|\gamma|\geq T}x^{\frac{3}{2}}\frac{\big(1+\frac{h}{x}\big)^{\frac{3}{2}}+1}{h|\rho(\rho+1)|}\leq A\frac{x^{\frac{3}{2}}}{h}\sum_{|\gamma|\geq T}\frac{1}{|\rho^2|},$$

with $A:=1+\big(1+\frac{h}{x}\big)^{\frac{3}{2}}$, thus from (3.7) and for $T\geq 5$ we get

$$\Big|\sum_{|\gamma|\geq T}\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big|\leq\frac{Ax^{\frac{3}{2}}}{\pi Th}\Big(\Big(1+\frac{2.89}{T}\Big)W_{\mathbb{K}}(T)+\Big(1+\frac{18.61}{T}\Big)n_{\mathbb{K}}+\frac{17.31}{T}\Big).$$

We rewrite

$$\begin{aligned}\sum_{|\gamma|<T}\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}&=\sum_{|\gamma|<T}\frac{x^\rho}{\rho}+\sum_{|\gamma|<T}\frac{(x+h)^{\rho+1}-x^{\rho+1}-h(\rho+1)x^\rho}{h\rho(\rho+1)}\\&=\sum_{|\gamma|<T}\frac{x^\rho}{\rho}+hx^{-1/2}\sum_{|\gamma|<T}w_\rho x^{i\gamma}\end{aligned}$$

with

$$w_\rho:=\frac{\big(1+\frac{h}{x}\big)^{\rho+1}-1-(\rho+1)\frac{h}{x}}{\rho(\rho+1)\big(\frac{h}{x}\big)^2}.$$

From Lemma 2.1 we know that $|w_\rho|\leq\frac{1}{2}$ so that from (3.6) we deduce

$$\Big|\sum_{|\gamma|<T}w_\rho x^{i\gamma}\Big|\leq\frac{1}{2\pi}\Big[(T+1.45)W_{\mathbb{K}}(T)-(T-8.93)n_{\mathbb{K}}+8.66\Big]$$

for every $T\geq 5$, giving

$$\begin{aligned}\Big|\sum_{\rho}\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}-\sum_{|\gamma|<T}\frac{x^\rho}{\rho}\Big|\leq{}&\frac{Ax\sqrt{x}}{\pi Th}\Big[\Big(1+\frac{2.89}{T}\Big)W_{\mathbb{K}}(T)+\Big(1+\frac{18.61}{T}\Big)n_{\mathbb{K}}+\frac{17.31}{T}\Big]\\&+\frac{Th}{2\pi\sqrt{x}}\Big[\Big(1+\frac{1.45}{T}\Big)W_{\mathbb{K}}(T)-\Big(1-\frac{8.93}{T}\Big)n_{\mathbb{K}}+\frac{8.66}{T}\Big].\end{aligned}$$

The comparison of the main terms suggests taking $h=\frac{2x}{T}$; this brings $A=1+\big(1+\frac{2}{T}\big)^{3/2}\leq 2+\frac{3}{T}+\frac{3}{2T^2}$ and

$$\begin{aligned}\frac{\pi}{\sqrt{x}}\Big|\sum_{\rho}\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}-\sum_{|\gamma|<T}\frac{x^\rho}{\rho}\Big|\leq{}&\Big(1+\frac{3}{2T}+\frac{3}{4T^2}\Big)\Big[\Big(1+\frac{2.89}{T}\Big)W_{\mathbb{K}}(T)+\Big(1+\frac{18.61}{T}\Big)n_{\mathbb{K}}+\frac{17.31}{T}\Big]\\&+\Big[\Big(1+\frac{1.45}{T}\Big)W_{\mathbb{K}}(T)-\Big(1-\frac{8.93}{T}\Big)n_{\mathbb{K}}+\frac{8.66}{T}\Big].\end{aligned}$$

After some simplifications we thus have for $T\geq 5$

$$\begin{aligned}\frac{\pi}{\sqrt{x}}\Big|\sum_{\rho}\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}-\sum_{|\gamma|<T}\frac{x^\rho}{\rho}\Big|\leq{}&\Big[2+\frac{5.84}{T}+\frac{5.52}{T^2}\Big]W_{\mathbb{K}}(T)\\&+\Big[\frac{29.04}{T}+\frac{31.46}{T^2}\Big]n_{\mathbb{K}}+\frac{25.97}{T}+\frac{28.57}{T^2}.\end{aligned}\tag{4.3}$$

For $h<0$ the computation is similar with only a few differences. We now have $A \leq 2$ and $|w_\rho| \leq \frac{1}{2}+\frac{|h|}{6x}$ from Lemma 2.1, thus

$$\Big|\sum_{\rho}\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}-\sum_{|\gamma|<T}\frac{x^\rho}{\rho}\Big| \leq \frac{2x\sqrt{x}}{\pi T|h|}\Big[\Big(1+\frac{2.89}{T}\Big)W_{\mathbb{K}}(T)+\Big(1+\frac{18.61}{T}\Big)n_{\mathbb{K}}+\frac{17.31}{T}\Big] + \frac{T|h|}{2\pi\sqrt{x}}\Big(1+\frac{|h|}{3x}\Big)\Big[\Big(1+\frac{1.45}{T}\Big)W_{\mathbb{K}}(T)-\Big(1-\frac{8.93}{T}\Big)n_{\mathbb{K}}+\frac{8.66}{T}\Big].$$

The situation is the same, thus we similarly take $h=-\frac{2x}{T}$ (we then have $x+h>1$ if $T\geq 5$), producing

$$\frac{\pi}{\sqrt{x}}\Big|\sum_{\rho}\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}-\sum_{|\gamma|<T}\frac{x^\rho}{\rho}\Big| \leq \Big[\Big(1+\frac{2.89}{T}\Big)W_{\mathbb{K}}(T)+\Big(1+\frac{18.61}{T}\Big)n_{\mathbb{K}}+\frac{17.31}{T}\Big] + \Big(1+\frac{2}{3T}\Big)\Big[\Big(1+\frac{1.45}{T}\Big)W_{\mathbb{K}}(T)-\Big(1-\frac{8.93}{T}\Big)n_{\mathbb{K}}+\frac{8.66}{T}\Big]$$

and after some simplifications we get

$$\frac{\pi}{\sqrt{x}}\Big|\sum_{\rho}\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}-\sum_{|\gamma|<T}\frac{x^\rho}{\rho}\Big| \leq \Big[2+\frac{5.01}{T}+\frac{0.97}{T^2}\Big]W_{\mathbb{K}}(T)+\Big[\frac{26.88}{T}+\frac{5.96}{T^2}\Big]n_{\mathbb{K}}+\frac{25.97}{T}+\frac{5.78}{T^2}. \tag{4.4}$$

Let $M_{W,\pm}(T)$, $M_{n,\pm}(T)$ and $M_{c,\pm}(T)$ be the functions of T such that the right-hand side of (4.3) and (4.4) respectively are

$$M_{W,+}(T)W_{\mathbb{K}}(T)+M_{n,+}(T)n_{\mathbb{K}}+M_{c,+}(T)$$
$$M_{W,-}(T)W_{\mathbb{K}}(T)+M_{n,-}(T)n_{\mathbb{K}}+M_{c,-}(T),$$

and their differences let be denoted as

$$D_W(T) := M_{W,+}(T)-M_{W,-}(T) = \frac{0.83}{T}+\frac{4.55}{T^2}$$
$$D_n(T) := M_{n,+}(T)-M_{n,-}(T) = \frac{2.16}{T}+\frac{25.50}{T^2}$$
$$D_c(T) := M_{c,+}(T)-M_{c,-}(T) = \frac{22.79}{T^2}.$$

By (4.1–4.4) we have

$$\Big|\psi_{\mathbb{K}}(x)-x-\sum_{|\gamma|<T}\frac{x^\rho}{\rho}\Big| \leq \frac{\sqrt{x}}{\pi}\big(M_{W,+}(T)W_{\mathbb{K}}(T)+M_{n,+}(T)n_{\mathbb{K}}+M_{c,+}(T)\big) + \frac{x}{T}+|r_{\mathbb{K}}|+1.22\frac{\delta_{n_{\mathbb{K}}\leq 2}}{x}+\max\Big(0, d_{\mathbb{K}}\log x-\frac{\sqrt{x}}{\pi}\big(D_W(T)W_{\mathbb{K}}(T)+D_n(T)n_{\mathbb{K}}+D_c(T)\big)\Big). \tag{4.5}$$

The last term is bounded by $\epsilon_{\mathbb{K}}(x,T)$, since $D_c(T)$ is positive and $\frac{1}{n_{\mathbb{K}}}D_W(T)W_{\mathbb{K}}(T)+D_n(T) \geq 1.44\pi/T$ when $T\geq 5$. Moreover, by (3.8) we have

$$\Big|\sum_{\substack{\rho\\|\gamma|<T}}\frac{x^\rho}{\rho}\Big| \leq \frac{\sqrt{x}}{\pi}\Big[\Big(\log\Big(\frac{T}{2\pi}\Big)+4.01\Big)\log\Delta_{\mathbb{K}}+\Big(\frac{1}{2}\log^2\Big(\frac{T}{2\pi}\Big)-1.41\Big)n_{\mathbb{K}}+25.57\Big] \tag{4.6}$$

for $T\geq 5$, thus the claim follows from (4.5), (4.6) and the upper bound for $|r_{\mathbb{K}}|$ in (2.8). ☐

Proof of Corollary 1.2. When $\mathbb{K}=\mathbb{Q}$ the claim is weaker than (1.1), thus from now on we can assume that $n_{\mathbb{K}}\geq 2$. Let the claim in the corollary be written as

$$|\psi_{\mathbb{K}}(x)-x|\leq F_{c,\Delta}(x)\log\Delta_{\mathbb{K}}+G_{c,n}(x)n_{\mathbb{K}}.$$

We prove that for every field the bound coming from the theorem is smaller than the one in the corollary, i.e. that

$$(F_{c,\Delta}(x)-F(x,T))\log\Delta_{\mathbb{K}}+(G_{c,n}(x)-G(x,T))n_{\mathbb{K}}-H(x,T)\geq 0 \tag{4.7}$$

for every $x\geq 100$. In order to prove it we need a choice for $T=T(x)$: we set $T(x)=\frac{c\sqrt{x}}{\log x}$ with $c\in[4.8,8]$ (in this way $T\geq 5$ for every $x\geq 36$). An elementary argument proves that the left-hand side in (4.7) is $\sqrt{x}$ times a function which increases in x when $x\geq 100$.

Proof. Dividing the left-hand side of (4.7) by $\frac{\sqrt{x}}{\pi}$ we get

$$\Big[\frac{1}{2}\log x-\Big[\log\Big(\frac{T}{2\pi}\Big)+6.01+\frac{5.84}{T}+\frac{5.52}{T^2}\Big]+\frac{0.98\pi}{\sqrt{x}}\Big]\log\Delta_{\mathbb{K}}$$
$$+\Big[\frac{1}{8}\log^2 x-\Big[\frac{1}{2}\log^2\Big(\frac{T}{2\pi}\Big)+\Big(2+\frac{5.84}{T}+\frac{5.52}{T^2}\Big)\log\Big(\frac{T}{2\pi}\Big)-1.41+\frac{29.04}{T}+\frac{31.46}{T^2}\Big]+\frac{4.10\pi}{\sqrt{x}}-\frac{\pi}{n_{\mathbb{K}}\sqrt{x}}H(x,T)\Big]n_{\mathbb{K}}$$

whose derivative is

$$\Big[\frac{1}{2x}-\Big[\log\Big(\frac{T}{2\pi}\Big)+6.01+\frac{5.84}{T}+\frac{5.52}{T^2}\Big]'-\frac{0.49\pi}{x\sqrt{x}}\Big]\log\Delta_{\mathbb{K}}$$
$$+\Big[\frac{\log x}{4x}-\Big[\frac{1}{2}\log^2\Big(\frac{T}{2\pi}\Big)+\Big(2+\frac{5.84}{T}+\frac{5.52}{T^2}\Big)\log\Big(\frac{T}{2\pi}\Big)-1.41+\frac{29.04}{T}+\frac{31.46}{T^2}\Big]'-\frac{2.05\pi}{x\sqrt{x}}-\Big(\frac{\pi}{n_{\mathbb{K}}}\frac{H(x,T)}{\sqrt{x}}\Big)'\Big]n_{\mathbb{K}}.$$

Since $T'>0$ for $x\geq e^2$, the function $-\frac{\log(T/2\pi)}{T^2}$ is increasing for $\frac{\sqrt{x}}{\log x}\geq\frac{2\pi\sqrt{e}}{c}$, and since $c\in[4.8,8]$, it is satisfied for every $x\geq 100$. Moreover,

$$-\pi\frac{\epsilon_{\mathbb{K}}(x,T)}{n_{\mathbb{K}}\sqrt{x}}=-\pi\max\Big(0,\frac{d_{\mathbb{K}}}{n_{\mathbb{K}}}-\frac{1.44}{c}\Big)\frac{\log x}{\sqrt{x}}$$

increases for $x\geq e^2$ for every combination of $d_{\mathbb{K}},n_{\mathbb{K}}$ and c. Thus, removing some increasing terms it is sufficient to prove that

$$\Big[\frac{1}{2x}-\Big(\log\Big(\frac{T}{2\pi}\Big)\Big)'-\Big(\frac{5.84}{T}\Big)'-\frac{0.49\pi}{x\sqrt{x}}\Big]\log\Delta_{\mathbb{K}}+\Big[\frac{\log x}{4x}-\frac{1}{2}\Big(\log^2\Big(\frac{e^2T}{2\pi}\Big)\Big)'-\Big(\frac{\pi}{n_{\mathbb{K}}}\frac{\sqrt{x}}{T}\Big)'$$
$$-5.84\Big(\frac{1}{T}\log\Big(\frac{T}{2\pi}\Big)\Big)'-\Big(\frac{29.04}{T}\Big)'-\frac{2.05\pi}{x\sqrt{x}}\Big]n_{\mathbb{K}}\geq 0$$

which after some computations becomes

$$\Big[\frac{2}{\log x}+\frac{5.84}{c}\frac{\log x-2}{\sqrt{x}}-\frac{0.98\pi}{\sqrt{x}}\Big]\log\Delta_{\mathbb{K}}$$
$$+\Big[1-\frac{2\pi}{cn_{\mathbb{K}}}-\log\Big(\frac{ce^2}{2\pi\log x}\Big)\Big(1-\frac{2}{\log x}\Big)+\frac{5.84}{c\sqrt{x}}\log\Big(\frac{c\sqrt{x}}{2\pi e\log x}\Big)(\log x-2)+\frac{29.04}{c}\frac{\log x-2}{\sqrt{x}}-\frac{4.10\pi}{\sqrt{x}}\Big]n_{\mathbb{K}}\geq 0.$$

Recalling the restriction $c\in[4.8,8]$, one proves that both the coefficient of $\log\Delta_{\mathbb{K}}$ and of $n_{\mathbb{K}}$ are positive for all $x\geq 12$. ☐

We further notice that the coefficients of $\log\Delta_{\mathbb{K}}$ and of $n_{\mathbb{K}}$ in (4.7) are positive when $x \geq 100$: in fact they can be written as $\sqrt{x}$ times a monotonous function of x (repeating the previous argument, this time without the contribution of $H(x,T)$), and their value in $x = 100$ is positive for every $c \in [4.8, 8]$. Now we split the argument according to the value of $n_{\mathbb{K}}$.

$n_{\mathbb{K}} \geq 8$. We are assuming GRH, so $\log\Delta_{\mathbb{K}} \geq n_{\mathbb{K}}\log(11.916) - 5.8507$ (see [8–11, 13] and entry $b = 1.6$ of Table 3 in [9]). Thus we can prove the claim by proving that

$$(F_{c,\Delta}(x) - F(x,T))(n_{\mathbb{K}}\log(11.916) - 5.8507) + (G_{c,n}(x) - G(x,T))n_{\mathbb{K}} - H(x,T) \geq 0,$$

and since the coefficient of $n_{\mathbb{K}}$ is positive, it is sufficient to prove it for $n_{\mathbb{K}} = 8$. We set $c = 8$. We have verified that the left-hand side is $\sqrt{x}$ times an increasing function (for $x \geq 100$), thus the inequality can be proved for every $x \geq 100$ simply by testing its value in $x = 100$.

$n_{\mathbb{K}} = 5$, 6 and 7. We repeat the previous argument, but now with the minimal discriminants which are 1609, 9747 and 184607, respectively (see [13, Table 1]).

$2 \leq n_{\mathbb{K}} \leq 4$. For every such degree one checks that (4.7) holds true when $\Delta_{\mathbb{K}} > \overline{\Delta_{\mathbb{K}}}$ where $\overline{\Delta_{\mathbb{K}}}$ is in Table 1 (by monotonicity in x it is sufficient to check the claim for $x = 100$); we adjust the parameter c to get a smaller $\overline{\Delta_{\mathbb{K}}}$.

TABLE 1. Minimal discriminants $\overline{\Delta_{\mathbb{K}}}$ for (4.7).

$r_2 \backslash n_{\mathbb{K}}$	2 ($c = 4.8$)	3 ($c = 5.1$)	4 ($c = 6$)
0	172921407	1350275	10311
1	103995324	369421	2584
2			648

This proves the claim for all fields but those with $n_{\mathbb{K}} \leq 4$ and $\Delta_{\mathbb{K}} \leq \overline{\Delta_{\mathbb{K}}}$. Actually, all fields with small degree and small discriminants are known [15] (for quadratic fields we use the fundamental discriminants below $\overline{\Delta_{\mathbb{K}}}$), and the number of these exceptions is in Table 2.

TABLE 2. Number of exceptional fields for (4.7).

$r_2 \backslash n_{\mathbb{K}}$	2	3	4
0	52561764	74747	54
1	31610787	65708	73
2			22

For each exceptional field we come back to (4.7) and prove it for every $x \geq \bar{x}$ in Table 3 (using again the monotonicity in x); we adjust the parameter c to get a smaller $\bar{x}$.

TABLE 3. Minimal x for the exceptional fields for (4.7); the minimal discriminants come from [13, Table 1]; $\bar{x}$ is the one associated with the smallest discriminant.

$n_{\mathbb{K}}$	2 ($c = 4.8$)	3 ($c = 5$)	4 ($c = 5$)
minimal $\Delta_{\mathbb{K}}$	3	23	117
$\bar{x}$	1566020	980	184

At last we test the claim for the exceptional fields in the exceptional range in Table 3 by computing $|\psi_{\mathbb{K}}(x)-x|$ (with PARI/GP [16]) and by checking that the difference with the bound is at least 1: in this way we only need to check the integers x in the range. This idea works for the fields in our list of degree 3 and 4. For quadratic fields both the number of fields and $\bar{x}$ are much larger. Luckily, the value of $\bar{x}$ drops down quickly when the discriminant increases, and for discriminants larger that 100 it is already only 5040, which can be checked very fast. Therefore the really long computations are only those for quadratic fields with discriminants below 100. The entire check can be made in approximately 40 hours on a 2011 personal computer. □

Proof of Corollary 1.3. In (1.4) we make the choice $T = \frac{10}{e}\frac{\sqrt{x}}{\log x}$, for which the condition $T \geq 5$ is satisfied for every $x \geq 3$. The term $\epsilon_{\mathbb{K}}(x,T)$ in Theorem 1.1 is $\leq 0.61 d_{\mathbb{K}} \log x$, and

$$\begin{aligned}
F(x,T) &\leq \frac{\sqrt{x}}{\pi}\Big[\frac{1}{2}\log\Big(x\frac{25e^2}{\pi^2}\frac{e^{\frac{11.68}{T}+\frac{11.04}{T^2}}}{\log^2 x}\Big)+4.01\Big]+1.02,\\
G(x,T) &\leq \frac{\sqrt{x}}{\pi}\Big[\frac{1}{8}\log^2\Big(x\frac{25e^2}{\pi^2}\frac{e^{\frac{11.68}{T}+\frac{11.04}{T^2}}}{\log^2 x}\Big)-3.41+\frac{17.36}{T}+\frac{3.37}{T^2}-\frac{32.23}{T^3}-\frac{15.23}{T^4}\Big]-2.10,\\
H(x,T) &\leq \frac{e}{10}\sqrt{x}\log x+25.57\frac{\sqrt{x}}{\pi}+0.61 d_{\mathbb{K}}\log x+2.75\log x+8.76.
\end{aligned}$$

The first claim in Corollary 1.3 follows plugging these bounds in (1.3), after some simplifications. For the second inequality we set $T = \frac{2\pi}{e^2}\frac{\sqrt{x}}{\log x}$; in this case the term $\epsilon_{\mathbb{K}}(x,T)$ in Theorem 1.1 is 0, the condition $T \geq 5$ requires $x \geq 2000$, and the claim follows as the previous one. □

Proof of Corollary 1.4. Let

$$\vartheta_{\mathbb{K}}(x) := \sum_{\substack{\mathfrak{p} \\ \mathrm{N}\mathfrak{p}\leq x}} \log \mathrm{N}\mathfrak{p}.$$

Then one has

$$\pi_{\mathbb{K}}(x)-\pi_{\mathbb{K}}(\bar{x})-\int_{\bar{x}}^{x}\frac{\mathrm{d}u}{\log u} = \int_{\bar{x}}^{x}\frac{\mathrm{d}(\vartheta_{\mathbb{K}}(u)-u)}{\log u},$$

which by partial integration gives

$$\Big|\pi_{\mathbb{K}}(x)-\pi_{\mathbb{K}}(\bar{x})-\int_{\bar{x}}^{x}\frac{\mathrm{d}u}{\log u}\Big| \leq \int_{\bar{x}}^{x}\frac{\mathrm{d}|\vartheta_{\mathbb{K}}(u)-u|}{\log u} \leq \frac{|\vartheta_{\mathbb{K}}(x)-x|}{\log x}+\int_{\bar{x}}^{x}\frac{|\vartheta_{\mathbb{K}}(u)-u|\,\mathrm{d}u}{u\log^2 u}. \tag{4.8}$$

Moreover, there are at most $n_{\mathbb{K}}$ ideals of the form $\mathfrak{p}^m$ ($\mathfrak{p}$ prime) of a given norm in $\mathbb{K}$, so

$$|\psi_{\mathbb{K}}(x)-\vartheta_{\mathbb{K}}(x)| \leq n_{\mathbb{K}}|\psi_{\mathbb{Q}}(x)-\vartheta_{\mathbb{Q}}(x)| \leq 1.43\, n_{\mathbb{K}}\sqrt{x},$$

where the last inequality is Theorem 13 in [17]. This shows that $\vartheta_{\mathbb{K}}(x)$ satisfies the same bound of $\psi_{\mathbb{K}}(x)$, at the cost of adding $1.43 n_{\mathbb{K}}\sqrt{x}$. Substituting this bound and the first inequality in Corollary 1.3 into (4.8) and after some numerical approximations one gets the corollary. □

TABLE 4. Constants for $\sum_{|\gamma|\leq 5}|\rho|^{-1}$ in Lemma 3.1.

j	$a_j{\cdot}10^7$	j	$a_j{\cdot}10^7$
1	−324328089	25	−52154912212245427675107284117
2	115693093357	26	72227309752304735434420743120
3	−10579381239203	27	−91546659026910381192366828396
4	495540769876127	28	106117853961289012764032450733
5	−14528281352885983	29	−112369546004525999862866475251
6	296347058332550155	30	108533470948598920563558219043
7	−4498154499661073603	31	−95431698456287244651252772381
8	53248447239339829090	32	76206788473674179730998288621
9	−508947342104081739447	33	−55105812322315804526845019881
10	4033084416071505510477	34	35955970546002972861665837368
11	−27051470635668143949707	35	−21079935102298710141936369413
12	156121546937577920978167	36	11047616237574616067334355219
13	−785529078417852387859619	37	−5143709248575449263188160534
14	3482495472267374521416188	38	2111566552644017238627810350
15	−13720533216155265613103988	39	−757162365842762640320305866
16	48375037637788872322025183	40	234379624034767935847527151
17	−153492067547835461489301521	41	−61692234538384117080736694
18	440289327629182231371781424	42	13534020670767148307863583
19	−1145934878685670756527108765	43	−2407266538638620726296042
20	2713965041058219158192688004	44	333452115133845423979326
21	−5861973594145453618923885659	45	−33740880236473501034280
22	11566694720865120123031709900	46	2218003445878553284287
23	−20874589384842483010331503670	47	−71076474624305025203
24	34482298986730410055952580804	—	—

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(L. Grenié) Dipartimento di Ingegneria, Università di Bergamo, viale Marconi 5, I-24044 Dalmine Italy

E-mail address: loic.grenie@unibg.it

(G. Molteni) Dipartimento di Matematica, Università di Milano, via Saldini 50, I-20133 Milano, Italy

E-mail address: giuseppe.molteni1@unimi.it