UNIVERSITA' DEGLI STUDI DI BERGAMO
Department of Management, Economics and Quantitative Methods

Essays on Nonlinear Dynamics, Heterogeneous Agents and Evolutionary Games in Economics and Finance

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Acknowledgments

From mid-2010, coinciding with my enrollment as a Ph.D. student at the doctoral school in Economics, Applied Mathematics and Operational Research at the University of Bergamo, I started working on several research papers. This Thesis contains some of these works, the details of which are postponed to the following Chapters because I would like to use this short space dedicated to the acknowledgments to thank all people who, in one way or another, contributed to the writing of this Thesis.

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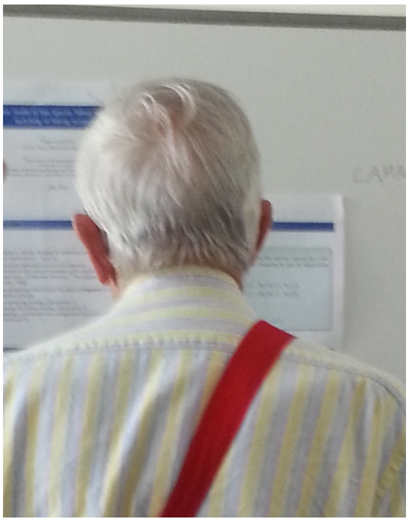
To my family



John Forbes Nash Jr., while reading the poster about the 3rd Chapter of this Thesis during the Fourth Congress of the Game Theory Society on July 22-26 2012 in Istanbul, Turkey.

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1 General Introduction

Many important features of economic and financial systems cannot be explained by means of theoretical models based on the paradigm of the rational and representative agent with unlimited computational ability and perfect information. In the last decades, with the intent of overcoming this drawback, a growing interest has emerged in alternative approaches which allow for factors such as bounded rationality and heterogeneity of agents, social interaction and learning, where agents' behavior is governed by adaptive evolutionary processes, based on simple "rules of thumb" (or "heuristics") or "trial and error" mechanisms. These new approaches have demonstrated how such evolutionary processes can lead to disequilibrium situations, path dependence, irreversibility, discontinuity as well as other nonlinear and complex phenomena observed in economics, finance and social sciences. Despite their complexity and less mathematical tractability, these types of models are becoming more and more popular for being able to explain the mechanisms and the reasons behind the complex patterns typical of economic and finance time series data and to offer a general and flexible modeling framework useful to tackle complicated economic issues, test possible solutions and derive economic policy implications.

In this Thesis three economic-financial models, based on the paradigm of bounded rationality, heterogeneity of agents and adaptive evolutionary selection of agents' behaviors, are introduced and studied by means of the qualitative theory of dynamical systems to shed some light on the effectiveness and the consequences of some regulations and "management practices" imposed by policy-makers and public authorities. Some of these regulations and management practices are proposed as possible solutions to real problems, others are already applied in reality and the here proposed theoretical models may be useful tools to test and discuss their validity. The models that we analyze are nonlinear dynamical systems evolving in continuous time, in discrete time or as mix of continuous and discrete time. The first is a two-population nonlinear evolutionary game which describes a possible application of financial options as a device to increase the quality of life in a city (A. Antoci, M. Galeotti and D. Radi "Financial Tools for the Abatement of Traffic Congestion: A Dynamical Analysis" Computational Economics (38:389-405) 2011). This model has been proposed to discuss the validity of a possible new financial mechanism to stimulate the "service providers" to increase the quality and the quantity of the public services. The second is a two species fishery model characterized by two logistic growth equations coupled with a replicator equation that regulates how fishermen select the species of fish to harvest (G. I. Bischi, F. Lamantia and D. Radi "Multispecies Exploitation with Evolutionary Switching of Harvesting Strategies" Natural Resource Modeling (26:546571) 2013). This model has been proposed to study a possible solution to the problems of overharvesting that plague the commercial exploitation of common-pool resources. The third is an asset pricing model with heterogeneous expectations and non-negative demand constraints that simulates a short selling regulation into force in some financial markets (F. Dercole and D. Radi "Does the "uptick rule" stabilize the stock market? Insights from rational equilibrium dynamics", submitted). The model has been studied to provide some indications about the validity of the short selling regulation in avoiding downward fluctuations in share prices. We prove that all these models exhibit complex dynamics for which their analysis is useful to draw policy implications for the specific problems.

This introduction has the following structure. In Section 1.1 we present a literature review on models of bounded rationality and heterogeneity of agents with the aim to provide the right framework for the works proposed in the three Chapters of this Thesis. In Section 1.2 we present the structure of the Thesis and a brief introduction to each of its three Chapters. In particular, we describe how the Chapters are related.

1.1 Bounded rationality, heterogeneity of agents and adaptive evolutionary process in economics and finance

In economic and financial models, the term "economic agents" has a general meaning and can refer to individuals (e.g. people), social groupings (e.g. firms) and/or more generally to decision makers. "Economic agent" (or "homo economicus") is often represented in "main stream" or "orthodox" literature as a fully-rational agent which has a complete knowledge of the economic system and is able to select the best course of actions among the set of those available in accordance to her/his own stable system of preferences. This form of rationality requires extreme assumptions concerning agents' information gathering and computing ability and it has been criticized on a number of grounds. Already in the 1950s, [95] underlined the necessity to replace the concept of a fully-rational agent with the concepts of "rational behavior". Agents adopt a rational behavior if they choose the best actions compatible with their limited set of information and their limited computational capacities.

Despite the concerns raised in [95], the fully-rational agent paradigm had been generally adopted in all economic models in the 1960s and in the 1970s, supported by the arguments made in [53] that non-rational agents if they exist will be sooner or later driven out of the market by rational agents, that will be able to take advantage of their superior rationality and earn high profits at the expense of the less rational agents. On the same line, [86] introduced the concept of *rational expectations*. This notion is based on the idea that the predictions of the economic theory cannot be substantially better than the predictions of the agents and that agents will not make systematic errors in their predictions because they can learn from their errors and adjust their predictions. In the 1970s, the validity of these arguments have been weakened by the discovery of deterministic chaos in simple non-linear systems by [80], see also [92] and [79] among others. In the presence of deterministic chaos, even forecasting rules consistent with the economic theory, and so based on a fully-rational assumption, can generate completely different (or wrong) predictions in the medium and long run due to very small round-errors made in measuring the current state of the system. Given the impossibility to measure the current state of a system with infinite precision, the risk of wrong predictions is very high. This phenomena is known as "butterfly effect" after [80] and it is related to the highly sensitive dependence on initial conditions of non-linear dynamical systems in their chaotic regime. These results shatter the Laplacian deterministic view of perfect predictability and restrict the validity of the arguments in support of rational expectations and the full rationality of agents in the context of a linear representation of the economic world. It is important to point out that, chaos and more general complex dynamics are not features of very complex dynamical systems but they can be observed in a very large class of simple nonlinear difference equations in one single variable, see, e.g., [79], and in some simple nonlinear dynamical systems of three differential equations, see, e.g., [92]. This makes it plausible to assume that complex economic systems can generate chaotic dynamics. The emergence of chaotic dynamics, i.e., dynamics that are neither periodic not quasi periodic, in a simple economic model was detected for the first time in [40]. He shows that, because of chaotic trajectories, the future behavior of a simple deterministic neoclassical growth model solution cannot be anticipated from its patterns in the past, i.e., unpredictability is observed in a deterministic economic model.

Inspired by these new results, economists revived the discussion about bounded-rationality and heterogeneity of agent assumption. In recent years, [75] criticizes the representative individual approach commonly used and underlines the importance of taking into account bounded rationality and heterogeneity of agents in economic models. In particular, he provides some example showing that the generally accepted idea *"even though the agents in an economy might be very heterogeneous, aggregate behavior could effectively be described by the behavior of a representative individual"* is not always true and sometimes can be misleading. In reality, agents can exhibit heterogeneous characteristics and have different expectations, interaction between which often lead to emergent of patterns that cannot be easily predicted at the population level by using representative agent models.

The validity of the bounded-rationality and heterogeneity of agents assumptions have been tested on an empirical ground as well. A lot of research has been done to find empirical evidence of bounded-rationality and heterogeneity of agents. In empirical finance, for example, [31] shows that different simple trading rules applied to real data can indeed yield positive returns, justifying their use and providing an empirical validation for theoretical asset pricing models with heterogeneous agents, see, e.g., [67] and [77] for recent surveys for these kinds of models.

The renewed interest for a more bumbling (awkward) kind of rationality in economics requires to model learning and adaptive selection processes. Already in the 1950s, [95] argued that the main difficulty to construct an economic theory around the concept of rational behavior is related to the lack of skills in modeling learning and choice processes. An important breakthrough in modeling these kinds of processes is represented by the introduction of the *evolutionary game theory* by John Maynard Smith in 1973, see [98] and [97], according to which the payoff (or the "fitness") of a strategy is a function of the current state of the game. The aim of the evolutionary games is to study the diffusion of different strategies in populations, where strategy has a general meaning and, concerning economics, it usually represents decision strategy, forecasting rules or any kind of economic agent's behavior. This notion of evolutionary selection connects the concepts of bounded rationality and heterogeneous agents. In particular, through the evolutionary process it is possible to define how different forms of rational behaviors spread across the populations.

Evolution is a concept strictly related to dynamics. In [101], for the first time, the evolutionary selection process was represented by a system of nonlinear first-order difference/differential equations with the introduction of the replicator dynamic[1]. Although being the best-known dynamic in evolutionary game theory, the replicator dynamic is not the only one. Another common evolutionary dynamic in economics is the logit dynamic. For recent general surveys on the topic see, e.g., [65] and [93].

The first class of games analyzed by means of evolutionary dynamics are the *linear games*, so called because the payoff functions are linear. It is worth to noting that, although the name of the games, the corresponding dynamical system is non-linear, see again [101]. There are two types of *linear games*, the symmetric games and the asymmetric games. The first describes the diffusion of strategies in one population

[1]This represents a turning points in economic modeling, for the first time the evolutionary theory, the heterogeneous agents framework and the nonlinear dynamic systems were put together to offer a powerful and flexible tool to tackle economic problems.

allowing for intra-population interaction. The second describes the diffusion of strategies in two different populations and only inter-population interaction is taken into account. A typical example from biology of a one-dimensional linear symmetric game is the so called "Hawk-Dove" game, where each individual from a single animal population has to choose between being either a "hawk", i.e., using an aggressive strategy, or a "dove", i.e., using a non aggressive strategy. Even though originally proposed in biology, this game has been found to be able to be applied in economics and finance. In finance, [38] uses a one-dimensional symmetric linear game to analyze a financial market where participants can either buy costly information ("hawk") or not ("doves"). Evolutionary linear asymmetric games are useful in economics and finance as well. For an application of an asymmetric linear game in economics, see, e.g., [50], which introduced and discussed the "Buyer-Seller" game where a buyer can either or not inspect and sellers can decide to report quality honestly or try to cheat. For a general overview of economic applications and discussions of linear evolutionary games see, e.g., [51].

In their more general setting, linear evolutionary games allow for both inter and intra population strategic interaction. In their work, [6] use this framework model to analyze the transfer of environmentally negative externalities between the northern hemisphere and the southern hemisphere in the global economy.

Linear specifications of evolutionary games are useful for simple illustrative examples but might be too restrictive to analyze real economic applications and non-linear payoff functions are usually required to obtain more realistic models. For example, [52] use fourth degree polynomials as a payoff functions for a one's country model of firms' choice between two models of internal organization. The work of [25] on wealth dynamics in financial markets provides another example of an evolutionary game with non-linear payoff functions. More sophisticated evolutionary games are based on non-linear payoff functions capturing both intra and inter population interaction. These kinds of games, in a two population setting, have been proposed in different forms by, for example, [15], [14], [11] and [20], to model environmental issues and to describe the diffusion of environmentally-friendly technologies. Using a similar set-up, in Chapter 2 of this Thesis a two-population evolutionary game has been proposed to model the use of financial options as a possible innovative device for raising funds aimed at improving the quality of the public services. The model describes the strategic interaction between two economic agents: city users that have to pay a fee to enter the city, but are in turn protected against the risk of low quality city life through self-insurance devices and service providers that can reduce costs by choosing to offer high quality public services. The model takes the form of a nonlinear dynamical system in continuous time. The global analysis underlines complex dynamics such as: multiple attractors, limit cycles and complex morphology of the basins of attraction. The aim of the work is to propose an interesting, flexible and innovative mechanism aimed at implementing and supporting environmental protection and social policies, to suggest an alternative use of financial instruments and to prove that the dynamics of the model can lead to a "welfare-improving attracting Nash equilibrium" in which all city users make use of the environmentally-friendly means of transportation and each service provider chooses to offer high quality services. Sufficient conditions to make this virtuous equilibrium a global attractor are also derived and different types of bifurcations are investigated and their socio-economic implications discussed.

More recent applications of the evolutionary game theory involves the use of adaptive evolutionary processes in nonlinear (economic) dynamical models. These models consist of a set of difference (or differential) equations describing the dynamics of some economic or financial variables combined with a set of difference (or differential) equations, usually $n-1$ where n is the number of available strategies (or agents' behaviors), describing the dynamics of the fraction of agents adopting a specific strategy. These models capture how the general economic environment affects the distribution of strategies among populations and how this distribution affects the general economic environment. Within this modeling framework, first [94] and later [110] integrate replicator and resource dynamics within a single framework to analyze common-dilemmas, better known as "the tragedy of the commons" after [62], in the context of a population of harvesters interacting in a common-pool resource game. In particular, [94] found that in the contest of a decentralized exploitation of common-property resources an equilibrium of cooperation behaviors guided by norms of restraint and punishment can be stable in an evolutionary sense against invasion by narrowly non-cooperative behaviors. By imposing some rules, [110] found that cooperative and non-cooperative behaviors can even coexist in a long-run equilibrium. More recently, [89] proposed a model of renewable resource exploitation with heterogeneous agents where the evolution of the distribution of strategies in a population is modeled through a replicator equation. In the model, agents (harvesters) choose between two predetermined harvesting strategies. The heterogeneity in strategies is expressed through different effort levels between two (sub)groups of harvesters. They show that under certain bio-economic conditions harvesting strategies characterized by different effort levels and harvesting costs can survive in the long-run as a result of the spatial and temporal features of the resource and the problem of imperfect information sharing. When applied to fisheries this model shows that cohabitation is possible between artisanal and industrial fisheries on the same fishing grounds as it has been observed from most existing cases.

All these works on common-pool resource games consider behavioral strategies which differ in the level of the resource exploitations. Taking a different stance in which behavioral strategies differ for the type of resources (species) targeted, in Chapter 3 of this Thesis a model with evolutionary switching of harvesting strategies has been proposed, see [23] and [24] as well. The model describes two fisheries where each fisherman can decide to harvest only in one of the two. The fishermen are assumed to have full-coordination intra-group capacity and set the harvesting rate at the Nash-Cournot equilibrium level continuously but they revise their decision about the species of fish to harvest at every fixed time interval according to an endogenous evolutionary mechanism based on past profits. This model replicates, in a stylized form, the operating mechanisms of a practice proposed by the Italian Fishery Authority and adopted in the Adriatic Sea to avoid overexploitation of several species of shellfish. The focus is on the sustainability of this way of fishing. This work contributes to enrich the literature on common-pool resource games considering an alternative policy strategy for fisheries and analyzing its effectiveness.

Evolutionary dynamics do not find application only in the realm of bio-economic writings. According to the paradigm of adaptive evolutionary models of bounded rationality and heterogeneity in price forecasting rules, [29] considered a cobweb commodity market model (traditionally proposed in the representative agent setting, see, e.g., [86]) with two types of producers characterized by different price expectations, rational vs. naive producers. This simple model with two kinds of expectations can explain both the coordination into rational expectations in a stable economy, that occurs for low evolutionary pressure, and the excess volatility with erratic price fluctuations observed in some unstable economies, which occurs for high evolutionary pressure. Along the same lines, [30] introduce heterogeneous expectations on share prices in a simple asset pricing model. The model is made of an asset pricing equation coupled with logit equations describing the diffusion of expectations, based either on simple forecasting rules or more sophisticated and costly techniques, among investors. The evolutionary process is based on the past profits generated by the different expectations. The model is able to replicate some stylized facts commonly observed in real financial time series, such as volatility clustering and fat tails in the returns, see, e.g., [57] and it has been validated through experimental designs and calibrations, for more details see, e.g., [68].

Using the same model with non-negativity constraints on the share demands in downward price movements, in Chapter 4 we try to understand the effect of a short selling regulation, the "uptick" rule, on price fluctuations. In this Chapter we attempt to learn some economic and financial implications of this policy measure in a context of complex dynamics, bifurcations and chaos. In particular, the analysis of the dynamics of the model is intended to underline the effectiveness of the regulation in achieving the objectives for which it has been proposed, i.e., to stabilize the prices of the stocks. These forms of contributions are becoming particularly important, since policy-makers are turning they attention to models of bounded rationality and heterogeneous agents generating complex phenomena in order to develop new approaches of policy analysis.

Heterogeneous agents and evolutionary dynamics naturally lead to highly *nonlinear* dynamical systems, because the fractions attached to the different rules are changing over time. Although all the models in this Thesis are nonlinear dynamical systems and even if nonlinear dynamical systems are an interesting field of research in economics and finance, in this short introduction we do not treat this argument. For a comprehensive treatment of this subject we refer to [54].

This short introduction aims to frame the models proposed in this Thesis in economic and finance literature and it does not pretend to provide a complete overview of the burgeoning literature on models of bounded rationality and heterogeneity of agents in economics and finance. There are many classes of behavioral models with bounded rational and heterogeneous agents that are not even considered here and that are becoming more and more important in economics and finance. Among them, the Agent-based Models (ABMs) are becoming very popular in the recent years due to the higher computation performances of modern computers, for a general introduction to this topic see, e.g., [102] and references therein.

In light of this brief introductory section, we can define the common features of the models proposed in this Thesis. They are economic models characterized by: (i) bounded rationality and heterogeneity of agents; (ii) agents that interact over time and choose the one to adopt among a finite set of feasible behaviors, strategies or decision rules; (iii) endogenous adaptive evolutionary processes that describe how agents revise their decisions about the behavior to adopt over time; (iv) adaptive evolutionary equations as functions of past-profits; (v) nonlinearity and complex dynamics.

The next Section provides indications about the structure of the Thesis.

1.2 Aim and Road map of the Thesis

This Thesis includes three Chapters, which have been written independently from each other. In each of the Chapters the dynamics of one particular nonlinear model characterized by heterogeneous agents endowed with bounded rationality is considered and evolutionary processes are investigated. Each Chapter is self-contained and can be read independently of the other Chapters. Indeed, each Chapter contains an introduction with a presentation to the problem, a description of the set-up of the model, a dynamic analysis, a discussion of the results, a conclusion and an appendix.

Here three issues in economics and finance are analyze by means of three different models, one for each Chapter. In each model, heterogeneous bounded rational agents select the decision strategies according to adaptive evolutionary processes. The models are characterized for being nonlinear dynamical systems with different forms of complex dynamics. For each model we will concentrate on the following topics:

(a) *Detect the dynamics of the model and through this figure out the complexity of the problem;*

(b) *Make a stability and bifurcation analysis with respect to the main economic parameters and discuss the economic consequences of the bifurcations;*

(c) *Develop a policy-analysis of the issues at stake and derive the policy implications in a context of complex dynamics.*

This work provides examples of how this class of models is useful to gain a deeper understanding of the mechanisms at the basis of economic and social systems, and to address real economic and financial issues. We conclude this introduction by briefly describing the contents of each of the Chapters.

Chapter 2 is based upon a joint work with A. Antoci[2] and M. Galeotti[3], and deals with a continuous time non-linear evolutionary game of *citizens* and *city service providers*. The model describes an innovative mechanism for raising funds to improve the quality and the quantity of public services in an urban area. Although the application discussed in the Chapter is specific, the proposed mechanism can be applied to tackle a wide class of problems in environmental economics. The Chapter contains the global analysis of the dynamics of the model with an analytical and numerical treatment of the possible

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bifurcations and the discussion of the related economic implications.

Chapter 3 is based upon a joint work with G. I. Bischi[4] and F. Lamantia[5], and deals with the bio-economic problems of a 2-species fishery in which each fisherman is allowed, at any instant of time, to exert his effort on one species only. The model is a hybrid system in which the dynamics of the fish species evolves in continuous time and fishermen revise their decision about the harvesting strategy to adopt only after a constant interval of time. The work is motivated by an actual application which was brought to our attention by C. Piccinetti, Professor of marine biology at the University of Bologna. The Chapter takes into consideration the issue of an alternative harvest strategy policy for fisheries.

Chapter 4 is the result of a combined work with F. Dercole[6], and deals with an evolutionary piecewise continuous nonlinear asset pricing model. The work represents an attempt to study the effects of a short selling regulation, the uptick rule, by means of a "simple" model. The Chapter includes a discussion about validity of the regulations on the basis of local and global analysis of the dynamics of the model developed using a combination of analytical and numerical tools.

Finally, we end this Thesis by some concluding remarks concerning the use of models of heterogeneous agents of bounded rationality to test the validity of regulatory policies in economics and finance.

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2 Financial tools for the abatement of traffic congestion: a dynamical analysis

In this Chapter we propose a nonlinear evolutionary model with application to environmental economics.

2.1 Introduction

Air pollution and the dangers for pedestrians and cyclists deriving from the widespread use of private cars in urban centers may further incentivize the use of private cars by city users. Indeed, the choice of using a car in these areas instead of going by bicycle or on foot has a self-enforcing nature: the greater the air pollution and traffic congestion levels in an urban area, the higher the incentive to go by car to reduce one's exposure to these problems. An increase in the use of private cars further increases air pollution and the dangers of urban traffic in turn, thus reinforcing the decision to go by car. As showed in the writings on environmental self-protection choices (see, e.g., [73], [7], [8], [10], [13], [27]), this mechanism may lead the urban community towards suboptimal Nash equilibria characterized by an excessive use of cars, unbearable levels of air pollution and traffic congestion.

Therefore, such a context calls for the definition of a system of sustainable mobility based on road pricing schemes (see, e.g., [59]), according to which the policy-maker increases the costs to enter into urban areas by car (parking taxes, congestion charges and so on) and collects revenues which can be used to finance private or public firms providing services aimed at the abatement of the negative effects of urban traffic and at the improvement of the quality of life in urban agglomerates (e.g. the management of public transport networks, of cycle and pedestrian lanes, of small-scale urban green areas etc.). These services can reduce the relative convenience of entering into urban areas with private cars and consequently can stop the undesirable self-enforcing process described above.

Road pricing instruments have been used to control traffic in several urban centers (e.g. Bergen, London, Milan, Singapore, Shanghai, Stockholm). The effectiveness of such policy instruments highly depends on the amount of the revenues raised via road pricing and on the quality of services provided to defend individuals from the effects due to traffic congestion. Service providers that furnish high quality services bear extra-costs and policy makers have to give monetary incentives which reduce unitary costs. In this context, an increase in the proportion of individuals using private cars generates an increase in traffic congestion but at the same time leads to an increase in the raised funds, which can be used as an incentive for providing high quality services.

All the proposals set forth to reduce traffic congestion are based on fixed taxes that individuals have to pay when entering into urban areas with private cars. In this Chapter we propose a simple mechanism aimed at implementing and supporting environmental protection policies in urban areas based on innovative financial instruments to be issued by a policy-maker (PM), which can be bought by two categories of involved agents, city users (CUs) and agencies providing the city services (SPs), that can be public or private. In particular, we consider the case of a city whose citizens and visitors (the city users) face the risk of a reduction in the quality of life, caused by urban traffic, on one hand, and by the poor quality of standard urban services, on the other.

According to the proposed mechanism, each city user has to choose (ex-ante) whether to use a private car (choice (a)) or to use a more environmentally-friendly transportation means: walking, bicycles, buses, trolleys or trams (choice (b)). According to their choice the PM requires them to buy two different *tickets*, including *cash-or-nothing call options*[1], called A (at a price p_a) and B (at a price $p_b < p_a$). The ticket prices, p_a and p_b, are fixed by the PM. On the other hand, each SP has to choose (ex-ante) whether to improve (choice (c)) or not (choice (d)) the services it furnishes. Accordingly the PM requires them to subscribe two different contracts, similar to *cash-or-nothing put options*, called C and D.

The tickets A and B imply a cost for CUs if the value of a properly defined index $\mathbf{Q}$ of the *city's quality of life*, measured by an independent agency at the end of any fixed period, is above a fixed threshold value $\mathbf{Q} \geq \mathbf{Q}^*$, but offer a reimbursement in the case $\mathbf{Q} < \mathbf{Q}^*$. When $\mathbf{Q} < \mathbf{Q}^*$, the CUs owning the *ticket* A receive a reimbursement equal to $\beta_a p_a$, while the CUs owning the *ticket* B receive a reimbursement equal to $\beta_b p_b$, where β_a and β_b are two parameters satisfying the condition $0 < \beta_a \leq \beta_b \leq 1$ ($\beta_a = \beta_b = 1$ means that both amounts p_a and p_b are totally reimbursed). If $\mathbf{Q} \geq \mathbf{Q}^*$ the CUs do not receive any refund.

[1] A (call or put) cash-or-nothing option is a type of option whose payoff is either some fixed amount of cash or nothing at all (see, e.g., [74])

On the side of the SPs, those that select choice (c), i.e. improving the services they furnish, will bear a unitary extra-cost but, if the quality target $\mathbf{Q}^*$ is achieved, they will be rewarded.

Contract C is very similar to the contract in which the "Environmental Policy Bond" regime, introduced by [70], [71] and [72][2], is based. Environmental Policy Bonds are auctioned by the Public Administration on the open market, but, unlike ordinary bonds, can be redeemed at face value only if a specified environmental objective has been achieved. They do not bear any interest, and the yield investors can gain depends on the difference between the auctioned price and the face value in the case of redemption. Economic agents involved in the environmental objective, *either polluters or not*, once in possession of the bonds, have a strong interest to operate in such a way that the objective itself is quickly achieved, so to cash in the expected gains as soon as possible. Differently from the Environmental Policy Bond regime, contract C considered in our model can be only bought by the SPs which intend to provide high quality services (the quality of services is assumed to be observable).

Tickets A and B, bought by the CUs, can be regarded as the joint implementation of a fixed environmental tax and a potential refund. The prospective of a refund, in case $\mathbf{Q} < \mathbf{Q}^*$, makes these policy instruments more acceptable to public opinion; in fact, they can be considered as self-insurance products[3] whose purchase can offer protection (or mitigation) from some environmental risk. By relying on the citizens' aversion to environmental risks, these self-insurance instruments can be a partial alternative to new taxes or forms of public indebtedness.

[2]Even before these works, different types of financial instruments have been proposed to achieve social and environmental goals (see, e.g., [90], [39] and [103]).

[3]These devices or policy instruments affect the size of the losses due to a risk of environmental and urban degradation. Thus, they can be considered a measure of self-insurance, see [46], and can be labeled self-insurance products.

According to the mechanism we propose, virtuous service providers choosing to offer high quality services can reduce the costs by subscribing the environmental contracts C. City users have to pay to enter into the city, but can protect themselves against a low quality of life in a city by using a self-insurance device. The policy-maker can achieve the goal of improving the city quality of life at a low cost, since the costs taken on by city users compensate, at least partially, the financial aids to virtuous service providers and so do not imply any aggravation of the public budget. In such a context, an increase in the number of city users choosing to enter into the city with private cars has a negative effect on the value of the quality index **Q**; however, since they have to pay a higher ticket ($p_a > p_b$), they contribute to increase the funds used to offer an incentive for virtuous behavior by service providers, which, in turn, contributes to increase the value of **Q**. Therefore, as a consequence of the mechanism described above, a strong interdependency between city users and service providers' behavior occurs. The aim of this Chapter is to study the dynamics that may arise in such a context. To this purpose, the choice processes of city users and service providers are modeled via a two-population evolutionary game, where the population of city users strategically interacts with that of service providers. Specifically these processes are modeled by the so-called replicator dynamics (e.g., see [108]), according to which a given choice spreads among the population as long as its expected payoff is greater than the average one. As it has emerged from the model, such a dynamics may lead to a "welfare-improving attracting Nash equilibrium", in which all city users choose to use environmentally-friendly means of transportation and all service providers choose to offer high quality services. The basin of attraction of this equilibrium expands as the reimbursement due to the visitors increases.

The model presented in this Chapter is a generalization of the model analyzed in [12]. In [9], [11], [15] a similar fund raising mechanism has been analyzed in a context in which firms have to decide whether to adopt environmentally-friendly technology.

The structure of the remainder of the Chapter is as follows. Section 2.2 develops the game theoretic model. In Sections 2.3 and 2.4 the model is analyzed. Section 2.5 concludes. Appendix 2.6 contains the proofs of the results.

2.2 The model

We assume that, at each time t, city users (CUs) and agencies providing the city services (SPs) play a one shot population game (i.e. all CUs and all SPs play the game simultaneously). Each city user has to choose (ex-ante) whether to use a private car (choice (a)) or not (choice (b)). The policy-maker requires them to buy two different *tickets*, including *cash-or-nothing call options*, called A (at a price p_a) and B (at a price p_b), according to their choices. Each SP has to opt (ex-ante) between choice (c) and choice (d), i.e., whether to improve or not the services it furnishes. The policy-maker requires them to subscribe two different contracts, similar to *cash-or-nothing put options*, called C and D, according to their choices.

We assume the two populations to be constant over time and normalize to 1 the number of both CUs and SPs. Let the variable $x(t)$ denote the proportion of CUs adopting choice (a) at time t ($0 \leq x(t) \leq 1$) and $1 - x(t)$ the proportion of CUs adopting choice (b). Analogously, let $y(t)$ denote the proportion of SPs adopting choice (c) at the time t ($0 \leq y\,(t) \leq 1$), and $1 - y(t)$ the proportion of SPs adopting choice (d).

The ticket prices, $p_a > p_b$, are fixed by the policy-maker (PM).

The index $\mathbf{Q}$ is a measure of the *city's quality of life*, whose target is fixed by the PM at a sufficiently high value $\mathbf{Q}^*$. We imagine that, at the end of any fixed period, an independent agency measures $\mathbf{Q}$.

At the end of a period in which $\mathbf{Q} < \mathbf{Q}^*$, the CUs owning the *ticket* A receive a reimbursement equal to $\beta_a p_a$, while the CUs owning the *ticket* B receive a reimbursement equal to $\beta_b p_b$, where β_a and β_b are two parameters satisfying the condition $0 < \beta_a \leq \beta_b \leq 1$ ($\beta_a = \beta_b = 1$ means that both amounts, p_a and p_b, are totally reimbursed). If $\mathbf{Q} \geq \mathbf{Q}^*$ the CUs do not receive any refund (this means that the value of the *cash-or-nothing call options* is zero). Hence the values of the *cash-or-nothing call options*, or *tickets* A and B, depend on the index $\mathbf{Q}$, which is their *underlying*.

It is also reasonable to assume that each CU's payoff is affected not only by the price of the *tickets* and the amount of the reimbursements, but by the *city's quality of life,* depending negatively on the use of private cars and positively on better urban services.

Consequently we assume the payoff of a CU buying ticket A to be given by:

- $\pi_a^1 = \gamma_1 y - \delta x - \rho p_a$ if $\mathbf{Q} \geq \mathbf{Q}^*$
- $\pi_a^2 = \gamma_2 y - \delta x - \rho (1 - \beta_a) p_a$ if $\mathbf{Q} < \mathbf{Q}^*$

while the payoff of a CU buying ticket B will be given by:

- $\pi_b^1 = \varepsilon_1 y - \eta x - \rho p_b$ if $\mathbf{Q} \geq \mathbf{Q}^*$
- $\pi_b^2 = \varepsilon_2 y - \eta x - \rho (1 - \beta_b) p_b$ if $\mathbf{Q} < \mathbf{Q}^*$

All the parameters are positive. Moreover, $\gamma_1 > \gamma_2$, $\varepsilon_1 > \varepsilon_2$, $\varepsilon_i > \gamma_i$, $\varepsilon_1 - \varepsilon_2 > \gamma_1 - \gamma_2$, implying, in particular, that the citizens not using private cars derive more advantages from increases in service quality. All the citizen payoffs are negatively correlated to the number of citizens using private cars by the two parameters δ and η (we assume $\eta \geq \delta$). Notice that both p_a and p_b are multiplied by ρ. This parameter can be thought as a measure of the citizen willingness to pay for urban services. For the sake of simplicity, we assume that all the citizens have the same willingness.

On the side of the SPs, those adopting choice (c), i.e. improving the services they furnish, will bear a unitary extra-cost θ, $\theta > 0$, but, if the quality target is achieved, will get a reward $\lambda + \mu x - \nu y$, where $\lambda - \nu > \theta$, $\mu, \nu > 0$. The parameter θ represents the cost of increasing the level of services, while λ is a fixed amount of money; $\mu > 0$ means that the reward is positively related to the number of CUs (x) who decide to use a private car. The reason is obvious: if we fix $p_a > p_b$, the more are the citizens choosing option A, the larger are the financial resources available to pay the SPs' reward. On the other hand, an increase in y implies that more SPs will be entitled to the financial aid, thus reducing the reimbursement available to each one. Therefore, the payoff of a SP adopting choice (c) is given by:

- $\pi_c^1 = \lambda + \mu x - \nu y - \theta$ if $\mathbf{Q} \geq \mathbf{Q}^*$
- $\pi_c^2 = -\theta$ if $\mathbf{Q} < \mathbf{Q}^*$

Without loss of generality, we can normalize to zero the payoff of a SP adopting choice (d) (i.e. who decide not to provide high quality services).

Finally we assume:

$$\mathbf{P}\left(\mathbf{Q} \geq \mathbf{Q}^{*}\right) = \sigma\left(1-x\right) + \left(1-\sigma\right) y,\ 0 < \sigma < 1$$

where $\mathbf{P}$ denotes a probability. This is equivalent to saying that, if all SPs improve the services they provide and all CUs renounce to private cars, then $\mathbf{Q}$ will be *almost surely* above the threshold level $\mathbf{Q}^{*}$ ($\mathbf{P}\left(\mathbf{Q} \geq \mathbf{Q}^{*}\right) = 1$). The parameter σ represents the weight of CUs renouncing to private cars on the probability that $\mathbf{Q} \geq \mathbf{Q}^{*}$.

Finally we obtain the expected payoffs:

- $E\pi_a = \left(\gamma_1 y - \delta x - \rho p_a\right) \mathbf{P}\left(\mathbf{Q} \geq \mathbf{Q}^{*}\right) + \left[\gamma_2 y - \delta x - \rho\left(1-\beta_a\right) p_a\right] \mathbf{P}\left(\mathbf{Q} < \mathbf{Q}^{*}\right)$

- $E\pi_b = \left(\varepsilon_1 y - \eta x - \rho p_b\right) \mathbf{P}\left(\mathbf{Q} \geq \mathbf{Q}^{*}\right) + \left[\varepsilon_2 y - \eta x - \rho\left(1-\beta_b\right) p_b\right] \mathbf{P}\left(\mathbf{Q} < \mathbf{Q}^{*}\right)$

- $E\pi_c = \left(\lambda + \mu x - \upsilon y - \theta\right) \mathbf{P}\left(\mathbf{Q} \geq \mathbf{Q}^{*}\right) - \theta \mathbf{P}\left(\mathbf{Q} < \mathbf{Q}^{*}\right)$

- $E\pi_d = 0$

The process of adopting strategies is modeled by the so called replicator dynamics (see, e.g., [108]), according to which the strategies whose expected payoffs are greater than the average payoff spread within the population at the expense of the others. In our case:

$$\begin{aligned} \dot{x} &= x\left(E\pi_a - \overline{E\pi}_{CU}\right) \qquad (2.1) \\ \dot{y} &= y\left(E\pi_c - \overline{E\pi}_{SP}\right) \end{aligned}$$

where

$$\begin{aligned}\overline{E\pi}_{CU} &= x \cdot E\pi_a + (1-x) \cdot E\pi_b \\ \overline{E\pi}_{SP} &= y \cdot E\pi_c + (1-y) \cdot E\pi_d\end{aligned}$$

are the average payoffs, respectively, of the two populations of CUs and SPs.

We assume that, in our context, replicator dynamics is generated by the following mechanism of expectations' formation. At the end of each period t (whose length, in a continuous time framework, is reduced to zero), the values of x and y become common knowledge to the agents (e.g., one can imagine that these values are frequently reported and updated on the web page of the Public Administration and on the local media). On the basis of such values, agents form their expectations about the relative performance of the available strategies in the next period (in other words, the current values of x an y are used as a proxy for the values of these variables in the close future).

The replicator system (2.1) in the square $[0,1]^2$ can be written as:

$$\begin{aligned}\dot{x} &= x(1-x)(E\pi_a - E\pi_b) = x(1-x)F(x,y) \\ \dot{y} &= y(1-y)(E\pi_c - E\pi_d) = y(1-y)G(x,y)\end{aligned} \tag{2.2}$$

Setting

$$
\begin{aligned}
\varphi &= (\varepsilon_1 - \gamma_1) - (\varepsilon_2 - \gamma_2) \\
\psi &= \varepsilon_2 - \gamma_2 \\
\zeta &= \eta - \delta \\
\tau &= \rho\,(\beta_b p_b - \beta_a p_a) \\
\omega &= \rho\,[(1-\beta_a)\,p_a - (1-\beta_b)\,p_b]
\end{aligned}
$$

we obtain:

$$
\begin{aligned}
F(x,y) &= (-\varphi y + \tau)\,[\sigma(1-x) + (1-\sigma)\,y] + \zeta x - \psi y - \omega \\
G(x,y) &= (\lambda + \mu x - \nu y)\,[\sigma(1-x) + (1-\sigma)\,y] - \theta
\end{aligned}
$$

where:

$$
\varphi, \psi > 0,\ \zeta \geq 0,\ \omega > \max(0,\tau);\ 0 < \sigma < 1;\ \lambda, \mu, \nu, \theta > 0,\ \lambda - \nu > \theta \qquad (2.3)
$$

2.3 Fixed points and stability

The analysis of fixed points of (2.2) and their stability is summarized by the following Propositions.

Proposition 2.1 *The boundary fixed points are at least four (the vertices of* $[0,1]^2$*) and at the most nine (the maximum that can be attained). At least one and at most four* *of them are attractors: in particular, the vertex* $(0,1)$ *is always attracting and the four vertices are all attracting if and only if the boundary fixed points are eight.*

Proposition 2.2 *The minimum number of interior fixed points (i.e. belonging to* $(0,1)^2$*) is zero and the maximum (which can be attained) is three. At most one of them is an attractor and at most one of them is a saddle. When the interior fixed points are three, two of them are repellors and one is a saddle.*

According to Propositions 2.1 and 2.2, the desirable equilibrium $(x,y) = (0,0)$, where all CUs choose an environmentally-friendly transportation means and all service agencies provide high quality services, is always a local attractor under evolutionary dynamics. However, other fixed points can be simultaneously locally attractive. In such a context, the equilibrium selection depends on the initial distributions $(x(0), y(0))$ of strategies in the two interacting populations.

Figure (2.1) illustrates the case when the boundary fixed points are eight and thus all the four vertices are attractors. The two vertices $(0,1)$ and $(1,0)$ are two stable equilibria in which the choice of the CUs is consistent with the one of the SPs and vice versa. In the first one, the CUs use the public services and the SPs increase the quality of these services. The CUs enjoy a good level of quality of public services and the SPs get a reward for their efforts made in increasing the quality of these services. In the second one, the CUs do not use the public services and the SPs do not increase the quality of these public services. One choice is consistent with the other. Indeed, if the CUs do not choose to use public services it does not make sense to increase the service quality because nobody will benefit from this improvement.

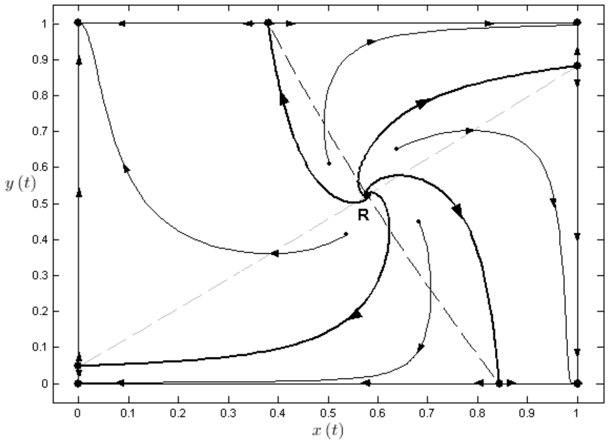

Figure 2.1: All the four vertices of the square are attractors. Parameters: $\lambda = 460$, $\mu = 120$, $\nu = 40$, $\psi = 48$, $\varphi = 25$, $\omega = 320$, $\theta = 240$, $\tau = 315$, $\sigma = 0.5$, $\zeta = 350$.

It is interesting to note that in the situation depicted in Figure (2.1), the equilibrium $(1, 1)$, in which SPs increase the quality of the public services but CUs choose to use their private car, is also stable. This may sound like a contradiction. However, this equilibrium is stable due to the fact that the payoff of the CUs who choose to use the public services is strongly negatively affected by the CUs who choose to use the private car, i.e. $\zeta = \eta - \delta$ is very large. This makes the CUs prefer the use of private cars even if the quality of the public services is high at least in a neighborhood of equilibrium $(1, 1)$. At the same time, the amount of money that SPs receive because the CUs choose to use the private car, which is measured by parameter μ, is too large, making the choice of increasing the quality of the services quite convenient even if the CUs do not choose to use these serveces. In this case the policy-maker should reduce this specific component of the reward. Moreover, the equilibrium $(0, 0)$ is also stable. This is an equilibrium in which the CUs use the public services but the SPs do not choose to increase the quality of the services they provide. Clearly a sub-optimal situation that is related to the fact that the expected profits of the SPs who do not choose to increase the quality of the services they provide is larger in a neighborhood of the equilibrium $(0, 0)$ than the expected profits related to the opposite strategy. This is due to the fact that the fixed component of the reward λ due to the SPs if $\mathbf{Q} \geq \mathbf{Q}^*$ and the weight σ of the CUs who choose to use the public services in increasing the value of the index $\mathbf{Q}$ are not large enough. Assuming the inability to affect the value of σ, in this case the policy-maker should increase the value of the fixed component of the reward for the SPs. This is a clear example that a wrong incentive scheme can lead to undesired situations.

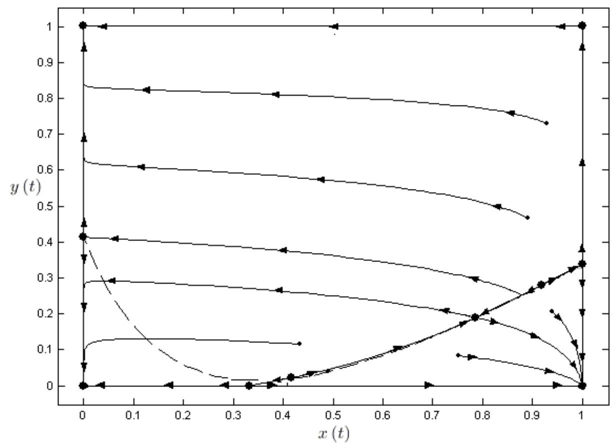

Figure 2.2: Three interior fixed points. Parameters: $\lambda = 11.45$, $\mu = 36$, $\nu = 0.25$, $\psi = 0.25$, $\varphi = 210$, $\omega = 10.0833$, $\theta = 8$, $\tau = 10$, $\sigma = 0.5$, $\zeta = 20.50$.

Figure (2.2) shows an example where three interior fixed points exists.

2.4 Limit cycles and bifurcations

The main result of this Section is the proof that system (2.2), defined in the square $[0,1]^2$, can possess two limit cycles, an attracting one surrounded by a repelling one. This number may not be the maximum. However the result already shows the complexity of the configurations that our apparently simple system can exhibit. Moreover we show how the two limit cycles can reduce to one, either repelling or attracting, through, respectively, a Hopf or a saddle-connection bifurcation, or can disappear through a *collision* (then the bifurcating cycle is interiorly attracting and externally repelling). In the latter case the vertex $(0,1)$ becomes the only *attracting set* of the system. Finally we provide necessary and sufficient conditions for $(0,1)$ to attract all the trajectories lying in $(0,1)^2$.

We start with the following results.

Proposition 2.3 *If system* (2.2) *exhibits some limit cycles in* $(0,1)^2$*, they all surround the same interior fixed point.*

Theorem 2.4 *System* (2.2) *exhibits, for suitable values of the parameters, two limit cycles in* $(0,1)^2$*, an attracting one surrounded by a repelling one.*

Figure (2.3) shows the case illustrated in Theorem 2.4: there is only one interior fixed point, a repellor, surrounded by two limit cycles, respectively attracting and repelling.

Now we want to show how the phase picture of system (2.2) can evolve, through bifurcations, starting from the one described in the above Theorem 2.4: that is, from two

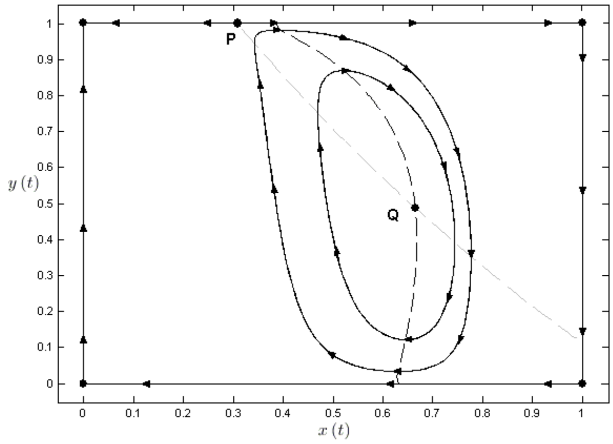

Figure 2.3: Two limit cycles surrounding the only interior (*repelling*) fixed point Q. Parameters: $\lambda = 520$, $\mu = 20$, $\nu = 340$, $\psi = 0.9$, $\varphi = 7.5$, $\omega = 96$, $\theta = 140$, $\tau = 94.5$, $\sigma = 0.7$, $\zeta = 93$.

limit cycles to one (repelling or attracting) to no one. In the latter case we will see that $(0, 1)$ becomes an attractor for all the trajectories lying in $(0, 1)^2$ (except the one at the repelling fixed point), which can be considered a desirable outcome of the model.

Let us summarize the situation described in Theorem 2.4, when two limit cycles appear. The following conditions hold:

- $\frac{\partial F}{\partial x} > 0$ along $\{F = 0\} \cap [0, 1]^2$ and $\frac{\partial G}{\partial x} < 0$ along $\{G = 0\} \cap [0, 1]^2$.

- $F = 0$ intersects the boundary of $[0, 1]^2$ at the two (open) edges $x = 1$ and $y = 1$, while $G = 0$ intersects the boundary of $[0, 1]^2$ at the two (open) edges $y = 0$ and $y = 1$.

- At the point $P = (\overline{x}, 1)$, where $F(\overline{x}, 1) = 0$, $G(\overline{x}, 1) > 0$.

- Consequently there is only one interior fixed point $Q = (\widetilde{x}, \widetilde{y})$ and we assume $\frac{\partial G}{\partial y}(\widetilde{x}, \widetilde{y}) < 0$.

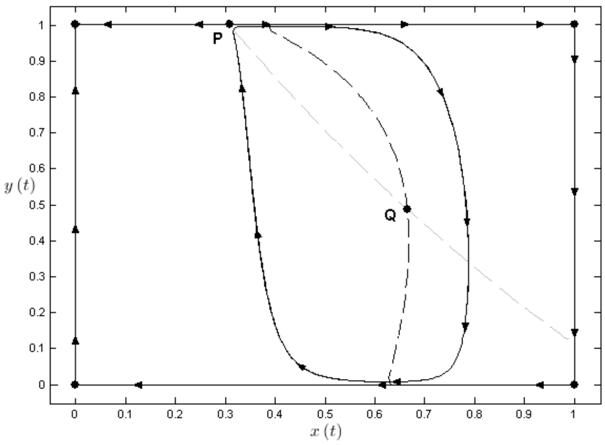

Figure 2.4: One repelling limit cycle surrounding the only interior (*attracting*) fixed point Q. Parameters: $\lambda = 520$, $\mu = 20$, $\nu = 340$, $\psi = 0.663$, $\varphi = 5.5252$, $\omega = 70.7232$, $\theta = 140$, $\tau = 69.6182$, $\sigma = 0.7$, $\zeta = 68.5131$.

It follows that the fixed points of the system in $[0, 1]^2$ are six: the attractor $(0, 1)$, the saddles $(0, 0)$, $(1, 0)$, $(1, 1)$, $(\overline{x}, 1)$ and the interior fixed point $Q = (\widetilde{x}, \widetilde{y})$ which can be either an attractor or a repellor (observe that $\overline{x} < \widetilde{x}$). In fact, multiplying F by a suitable $h > 0$, we can make $traceJ(Q) > 0$, so that Q is a repellor. In such a case there may exist, as we have seen, two limit cycles surrounding Q. Suppose we have precisely this phase picture. Now, if we multiply F by a positive $h < 1$, the attracting limit cycle shrinks, until it disappears when, at $h = \overline{h}$, $traceJ(Q) = 0$. For smaller values of h the system exhibits only one repelling limit cycle, surrounding Q (see Figure (2.4)). This is the effect of a supercritical Hopf bifurcation. Let us see, instead, how only the attracting limit cycle can remain. By the usual notations, let us modify the coefficients ζ and ω of F, leaving all the others unvaried. Precisely, we replace ζ and ω by $\zeta' = \zeta + \rho$ and $\omega' = \omega + \rho\widetilde{x}$, $\rho > 0$. At the same time we multiply G by some $k > 1$ in such a way that $traceJ(Q)$ remains positive but does not increase (or suitably decreases). Then, when ρ, starting from 0, increases, the fixed point $P = (\overline{x}, 1)$ moves rightward, while the repelling limit cycle expands vertically, until, for a suitable pair $\left(\overline{\rho}, \overline{k}\right)$, $F(\overline{x}, 1) = G(\overline{x}, 1) = 0$. At this stage P is a *saddle-node* and the repelling limit cycle has become a loop through P (i.e. a *saddle-connection*). For higher values of ρ there is only one attracting limit cycle surrounding Q, while P has become a repellor and there exists a new interior fixed point, a saddle $S = (\widehat{x}, \widehat{y})$, with $\widehat{x} < \widetilde{x}$ and $\widehat{y} > \widetilde{y}$. The basin of the attracting cycle is bounded, precisely, by the stable manifold of S, whose *arcs* originate in P (see Figure (2.5)). We observe, however, that, by just letting the curve $F = 0$ approach the curve $G = 0$ on the edge $y = 1$ of the square, without adjusting at the same time $traceJ(Q)$, the resulting phase picture might have been different: that is, when the boundary fixed point, say $\widetilde{P}$, becomes a repellor and an interior saddle S appears, the two limit cycles may persist (see Figure (2.6)).

Finally we show that both limit cycles can disappear through a *collision.* In fact, let us start again from the two limit cycles *configuration.* Then, by multiplying F by some $h > 1$, $traceJ(Q)$ increases, while the attracting limit cycle around Q expands and the repelling one shrinks, until, at a suitable value $\overline{\overline{h}}$, they coincide (i.e. *collide).* The resulting *compound* limit cycle is interiorly attracting and externally repelling (see Figure (2.7)). Then, for $h > \overline{\overline{h}}$, no cycle exists and all the trajectories starting in $(0, 1)^2$ (except, of course, the one at Q) converge to the *virtuous* point $(0, 1)$.

From the analysis contained in this section it emerges that the basin of attraction of the desirable equilibrium $(x, y) = (0, 1)$ may have a rather complex morphology, even

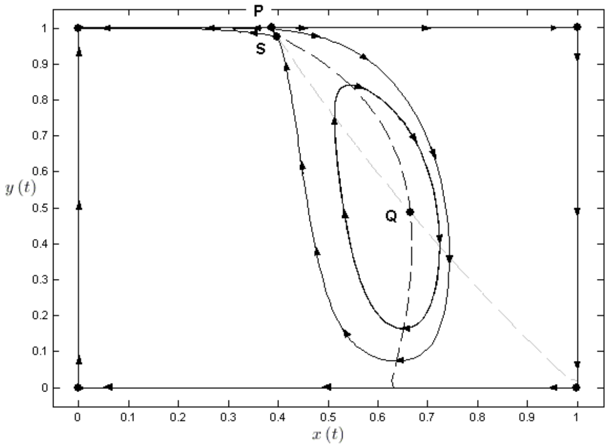

Figure 2.5: One attracting limit cycle. Parameters: $\lambda = 676$, $\mu = 26$, $\nu = 442$, $\psi = 0.9$, $\varphi = 7.5$, $\omega = 101.9827$, $\theta = 182$, $\tau = 94.5$, $\sigma = 0.7$, $\zeta = 102$.

when $(0,1)$ is the unique attractive fixed point, as it is shown in Figure (2.3), where the repelling cycle separates the basin of attraction of the attractive one from that of the vertex $(0,1)$.

The path-dependence of evolutionary dynamics could be viewed as a shortcoming of the proposed fund-raising mechanism. Nevertheless, we can give necessary and sufficient conditions for the *global attractiveness* of $(0,1)$ relatively to the open square $(0,1)^2$, namely:

Proposition 2.5 *The vertex* $(0,1)$ *attracts all the trajectories lying in the open square* $(0,1)^2$ *if and only if the following conditions hold:*

1. $\sigma\lambda \geq \theta$;

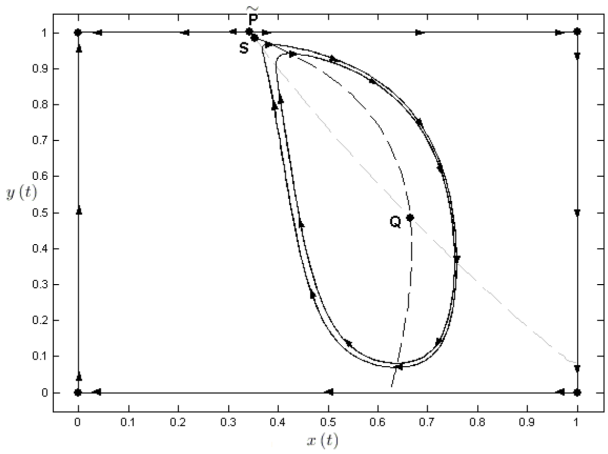

Figure 2.6: Two limit cycles with two interior fixed points. Parameters: $\lambda = 508.0347$, $\mu = 38$, $\nu = 340$, $\psi = 0.9$, $\varphi = 7.5$, $\omega = 98.3266$, $\theta = 140$, $\tau = 94.5$, $\sigma = 0.7$, $\zeta = 96.5$.

2. $\omega \geq \zeta$;

3. there exists $h > 0$ *such that* $hF(1,1) + G(1,1) \leq 0$;

4. $F(x,y) < 0$ *along* $\{G(x,y) = 0\} \cap (0,1)^2$

We omit the proof of the above Proposition 2.5, which can be drawn through straightforward steps. Let us observe that the first three conditions guarantee that the vertices, in the order, $(0,0)$, $(0,1)$ and $(1,1)$ are not attractors, while the fourth condition, together with the other three, guarantees that no interior fixed point exists and no other boundary fixed point, except $(0,1)$, attracts trajectories lying in $(0,1)^2$.

We can also give a sufficient condition, which allows an easier economical interpretation,

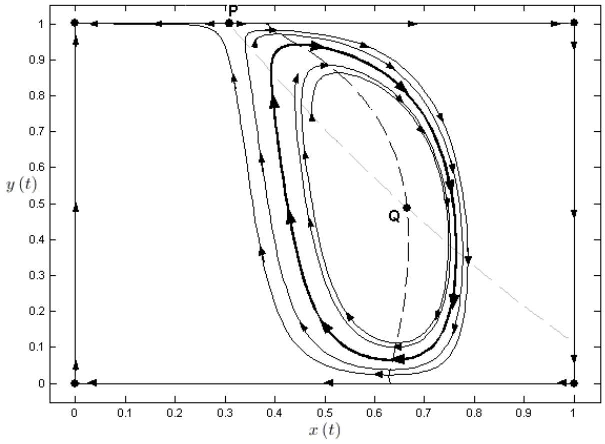

Figure 2.7: One *compound* limit cycle (*internally attracting and externally repelling*). Parameters: $\lambda = 520$, $\mu = 20$, $\nu = 340$, $\psi = 0.9483$, $\varphi = 7.9024$, $\omega = 101.1504$, $\theta = 140$, $\tau = 99.5699$, $\sigma = 0.7$, $\zeta = 97.9894$.

for the vertex $(0, 1)$ to attract all the trajectories lying in $(0, 1)^2$. This condition is obtained observing that $\dot{x} < 0$ holds for every values of x and y in the open square $(0, 1)^2$ if:

$$\min\left\{(1-\beta_a)\,p_a - (1-\beta_b)\,p_b,\; p_a - p_b + \sigma\left(\beta_b p_b - \beta_a p_a\right)\right\} \geq \frac{\eta-\delta}{\rho} \tag{2.4}$$

Under condition (2.4) all the trajectories in $(0, 1)^2$ approach the edge $x = 0$ of the square. So, if both (2.4) and $\sigma\lambda \geq \theta$ are satisfied:

$$\sigma\lambda \geq \theta \text{ and } \min\left\{(1-\beta_a)\,p_a - (1-\beta_b)\,p_b,\; p_a - p_b + \sigma\left(\beta_b p_b - \beta_a p_a\right)\right\} \geq \frac{\eta-\delta}{\rho} \tag{2.5}$$

then the vertex $(0,1)$ attracts all the trajectories lying in the open square $(0,1)^2$. The condition (2.5) requires (coeteris paribus):

(1) a sufficiently large difference $p_a - p_b$ between the prices of the tickets A and B bought by the city users;

(2) a sufficiently large difference $\beta_b - \beta_a$ between the reimbursement rates of the tickets B and A, respectively;

(3) a sufficiently large fixed amount of money λ (i.e. the component of the reward $\lambda + \mu x - \nu y$ not depending on the shares x and y) received, when $\mathbf{Q} \geq \mathbf{Q}^*$, by each service provider deciding to provide high quality services;

(4) a high enough sensitivity (measured by the parameter σ) of the probability $\mathbf{P}\left(\mathbf{Q} \geq \mathbf{Q}^*\right) = \sigma\left(1-x\right) + \left(1-\sigma\right) y$ with respect to variations in x, the share of city users using private cars.

Notice that, in (2.5), the threshold value $\frac{\eta-\delta}{\rho}$ increases (coeteris paribus) if the difference $\eta - \delta$ increases, where η (respectively δ) measures the negative impact due to an increase in x on city users adopting choice (b) (respectively (a)). In fact, in a context characterized by a high value of $\eta - \delta$, the decision to use a private car has a strong self-enforcing nature that favors the emergence of undesirable outcomes characterized by a widespread use of private cars.

Furthermore, the numerical simulation showed in Figure (2.8) suggests the possibility of expanding the basin of attraction of $(0,1)$ by increasing the reimbursement rate β_b due to city users renouncing private cars (choice (b)). This implies that if the policy-maker levies a simple entrance ticket on the city users with no chance of being reimbursed, this could minimize the basin of attraction of the virtuous equilibrium $(0, 1)$, increasing the initial critical mass of $1 - x$ and y that are needed to approach it. Increasing the reimbursement share, therefore, might paradoxically lower the costs of the financial mechanism for the policy maker: if the system converges to $(0, 1)$, no reimbursement will be paid by the policy-maker to city users and the *extra* entries obtained from the *call option* component of the *tickets* can be used by the policy-maker to finance service providers for their virtuous behavior.

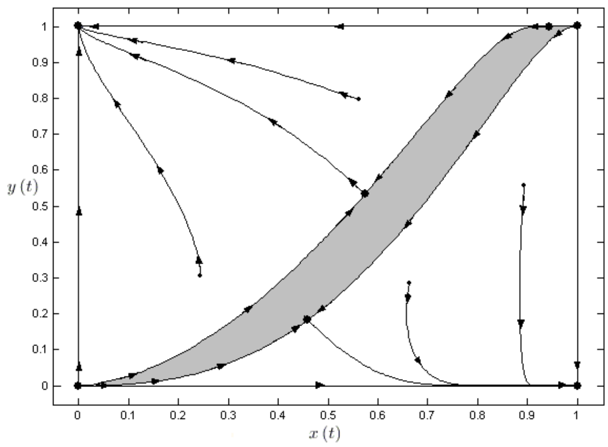

Figure 2.8: Attractors $(0, 1)$ and $(1, 0)$. Increasing the reimbursement rate β_b (hence increasing ω and τ) the basin of attraction of $(0, 1)$ expands. Parameters: $\lambda = 8$, $\mu = 6$, $\nu = 0.5$, $\psi = 1$, $\varphi = 4$, $\omega = 0.4 \left(\omega' = 1.2\right)$, $\theta = 5$, $\tau = 0.2 \left(\tau' = 1.4\right)$, $\sigma = 0.8$, $\zeta = 3$.

2.5 Conclusions

In this Chapter we have analyzed the effects on economic agents' behavior generated by the introduction of a simple mechanism aimed at implementing and supporting environmental protection policies in urban areas. According to the proposed mechanism, service providers choosing to offer high quality services can reduce their costs; while city users have to pay for entering into urban areas, but can protect themselves against a low quality of life in these areas through a self-insurance device. The dynamics arising from the interaction between these categories of economic agents are modeled by a two-population evolutionary game according to which the population of city users strategically interacts with that of service providers. In turn the policy-maker can achieve the goal of improving the city users' quality of life at a low cost, since the costs borne by city users compensate, at least partially, the financial aid for virtuous service providers and so do not imply any aggravation of the public budget. However we did not introduce any explicit budget constraint, whose possible satisfaction can affect the dynamics of the replication system, due, in particular, to the value of the parameter ρ, expressing the *citizen non-willingness to pay for urban services* (see Section 2.2). At any rate for the always attracting equilibrium $(0, 1)$ the budget constraint reduces to $p_b \geq \lambda - \nu$.

From the analysis of the model it emerges that the welfare-improving equilibrium $(x, y) = (0, 1)$, where all city users choose an environmentally-friendly transportation means and all services agencies provide high quality services, is always a local attractor under evolutionary dynamics. However, the basin of attraction of such an equilibrium may have a rather complex morphology. In fact, in Sections 2.3 and 2.4 we show that other attracting fixed points can coexist with $(0, 1)$ and, furthermore, that two limit cycles may arise, an attracting one surrounded by a repelling one. In our construction the latter separates the basin of attraction of the former from the basin of attraction of the *virtuous vertex*

$(0,1)$. Therefore, evolutionary dynamics is strongly path-dependent. This feature of dynamics may be viewed as a shortcoming of the proposed mechanism. Nevertheless we have given a sufficient condition (see (2.5)) assuring the global attractiveness of the equilibrium $(0,1)$.

The present analysis could be extended in several directions. In particular, in an optimal control framework in which the policy-maker aims at maximizing its own objective function by choosing the values of the control variables p_a, p_b, β_a and β_b, it would be interesting to compare the costs for the policy-maker of the two alternative regimes described above (with and without reimbursement), taking the budget constraint explicitly into account.

2.6 Appendix

Proof of Proposition 2.1. The conditions (2.3) imply, as it can easily be checked, that the vertex $(0,1)$ is attracting. Moreover the hyperbola $F(x,y)=0$, having asymptotes $y=\frac{\sigma\tau-\zeta}{\sigma\varphi}$ and $-\sigma x+(1-\sigma)\,y=c$, for a suitable c, has at most one intersection with each horizontal edge of the square $[0,1]^2$. On the other hand $G(x,y)=0$ has one intersection with $x=0$ at some point $(0,\widehat{y})$ with $\widehat{y}>1$. It follows that $G(x,y)=0$ has at most one intersection with the vertical edge $x=0$ of the square and at most two intersections with the vertical edge $x=1$. Hence the boundary fixed points are at most nine. Moreover, as in $(0,0)$ and $(0,1)$ $F\,(x,y)<0$, the possible fixed points on the open horizontal edges cannot be attracting. Analogously, since $G\,(x,y)>0$ in $(0,1)$ and $G\,(x,y)<0$ in $(1,1)$, the possible fixed point on the open vertical edge $x=0$ and one of the possible fixed points on the open vertical edge $x=1$ cannot be attracting either. In fact, it follows that, when the boundary fixed points are eight, the four vertices are all attracting. In case of nine boundary fixed points, instead, the vertices $(0,0)$, $(0,1)$ and $(1,0)$ are still attracting, while $(1,1)$ is a saddle and there exists a fourth attractor on the open edge $x=1$. In all the other cases the attractors are less than four. ■

Proof of Proposition 2.2. The two hyperbolas $F(x,y)=0$ and $G(x,y)=0$ have a common asymptotic direction (that of the line $-\sigma x+(1-\sigma)\,y=0$). Hence their intersections in the Euclidean plane are at most three. It is easily seen that only one branch of $G(x,y)=0$ intersects $(0,1)^2$. On the other hand it can be checked that $F(x,y)=0$ does not intersect the vertical edge $x=0$ of the square and, in case $\sigma\tau-\zeta\geq 0$ (i.e. $\frac{\partial F}{\partial x}<0$ in $(0,1)^2$), it does not intersect the horizontal edge $y=0$ either. It follows that the possible intersection of $F(x,y)=0$ with $(0,1)^2$ is the graph of a function $x=f(y)$ defined in an interval $(y',y'')\subseteq(0,1)$. Therefore the intersections between the two hyperbolas in $(0,1)^2$ are the solutions of the equation $G(f(y),y)=0$ with $y\in(y',y'')$, hence the possible zeros in (y',y'') of a third degree polynomial $P(y)$. By possibly exchanging $P(y)$ with $-P(y)$, it is easily observed that, if $(\overline{x},\overline{y})$ is a fixed point of the system in $(0,1)^2$, then $sign\det J\,(\overline{x},\overline{y})=signP'(\overline{y})$, where J denotes the Jacobian matrix. As a consequence, if two consecutive simple zeros of $P(y)$ correspond to interior fixed points of system (2.2), one of them is a saddle and the other one is generically either an attractor or a repellor. Vice-versa a double zero of $P(y)$, giving rise to an intersection between $F=0$ and $G=0$ in $(0,1)^2$, corresponds to a saddle-node of the system, while, as we will see, a triple zero of $P(y)$, if it corresponds to an interior fixed point of the system, gives rise to an *improper repellor*. Let us consider now the case of three interior fixed points (see Figure 2). From the previous considerations, through a careful investigation, one can check that in this case $\frac{\partial F}{\partial x}>0$ and $\frac{\partial F}{\partial y}<0$ along $\{F=0\}\cap(0,1)^2$ while $\frac{\partial G}{\partial y}>0$ along $\{G=0\}\cap(0,1)^2$. Hence the function $x=f\,(y)$, above defined, is increasing, with $0<y<\overline{y}\leq 1$, and no one of the interior fixed points, say $P_i=(x_i,y_i)$, $i=1,2,3$, is attracting (since $traceJ(P_i) > 0$). Next we want to show that P_3, the one with the highest coordinates, is a repellor. In fact, it is easily seen that the arc $x = f(y)$, $y_3 < y < \overline{y}$, lies in the region $\{G > 0\} \cap (0,1)^2$. Consider then a *curvilinear triangle* having two *edges* on $F = 0$ and $G = 0$ and vertices $P_3 = (x_3, y_3)$, $Q' = (f(y^*), y^*)$, $Q'' = (g(y^*), y^*)$, where $y_3 < y^* < \overline{y}$, $G(g(y^*), y^*) = 0$ and $(g(y^*), y^*) \in (0,1)^2$. Then the backward (i.e. negative) trajectory of Q' cannot leave the *triangle* and thus converges to P_3. It follows that P_3, being a non-degenerate fixed point with a parabolic repelling sector (see footnote 5), is a repellor. From what precedes, we conclude that P_1 is also a repellor and P_2 is a saddle. ■

Proof of Proposition 2.3. It is well-known that the sum of the indexes of the fixed points surrounded by a limit cycle must be +1[4]. As we have shown, if there are two non-degenerate interior fixed points, one of them is a saddle. Vice-versa, in the case the interior fixed points are three, accounting for multiplicity, the proof of Proposition 2.2 shows that the one of them with the highest coordinates cannot lie in the interior of a cycle. This proves the present Proposition. ■

Proof of Theorem 2.4. Let us first consider a system Σ_0 satisfying, besides (2.3), the following conditions:

1. $G_0(0,0) > 0 > G_0(1,1)$

2. $F_0(1,1) > 0 > F_0(1,0)$

3. There exists a fixed point $P_0 = (x^*, 1)$, $0 < x^* < 1$, such that $F_0(x^*, 1) = 0 <$

[4]For a non-degenerate fixed point the index is +1 if it is not a saddle, −1 if it is a saddle, while a saddle-node has index 0 etc. (see, e.g., [78]).

$G_0(x^*, 1)$

4. There exists a fixed point $Q_0 = (1, y^*)$, $0 < y^* < 1$, such that $F_0(1, y^*) = G_0(1, y^*) = \frac{\partial G_0}{\partial y}(1, y^*) = 0$

Conditions (1)-(4), in addition to the previous ones, are easily seen to imply:

- $\frac{\partial F_0}{\partial x} > 0$ for $y \in [y^*, 1]$
- $\frac{\partial G_0}{\partial x} < 0$ along $\{G_0(x, y) = 0\} \cap [0, 1]^2$
- $\frac{\partial G_0}{\partial y} > 0$ along $\{G_0(x, y) = 0\} \cap [0, 1] \times [0, y^*)$
- $\frac{\partial G_0}{\partial y} < 0$ along $\{G_0(x, y) = 0\} \cap [0, 1] \times (y^*, 1]$

It follows that Σ_0 has exactly six fixed points in $[0, 1]^2$, all lying on the boundary: the attractor $(0, 1)$, the saddles $(0, 0)$, $(1, 0)$, $(1, 1)$, $(x^*, 1)$ and the degenerate fixed point $Q_0 = (1, y^*)$, whose Jacobian matrix has the form $J = \begin{pmatrix} 0 & 0 \\ -e & 0 \end{pmatrix}$ with $e > 0$. Actually it can be checked that Q_0 possesses an elliptic sector and a parabolic sector lying in $(0, 1)^2$, plus a hyperbolic sector lying outside the square[5]. More precisely, the elliptic sector is bounded by a curve (a *polycycle*) constituted by the stable manifold, in $(0, 1)^2$,

[5] Let P be an isolated fixed point of a smooth planar system. Consider an open neighborhood D of P (e.g. a disc). A region $U \subseteq D - P$ is said (see, e.g., [4]):

- an elliptic sector of P if it is constituted by trajectories having P both as α and ω limit-set;
- a parabolic sector of P if it is constituted by trajectories having P either as α or as ω limit-set;
- a hyperbolic sector of P if it is constituted by trajectories having P neither as α nor as ω limit-set.

For the definitions of α and ω limit-sets see also [91] pag.146. For a graphic representation of elliptic, parabolic and hyperbolic sectors, see, e.g., [5] pag. 86.

of $P_0 = (x^*, 1)$ and by the segments $\{x^* \leq x \leq 1,\, y = 1\}$ and $\{x = 1,\, y^* \leq y \leq 1\}$; while the parabolic sector is constituted by trajectories having α-limit in Q_0 and ω-limit in $(0, 1)$.

Let us now perturb Σ_0 into Σ, by replacing F_0 and G_0 with:

$$\begin{aligned} F(x,y) &= F_0(x,y) + \alpha\varepsilon^2 \\ G(x,y) &= G_0(x,y) - \varepsilon \end{aligned} \tag{2.6}$$

where $\varepsilon > 0$ is arbitrarily small and α is suitably chosen. As a consequence, the fixed points of Σ are the four vertices of $[0,1]^2$ and a point P on $y = 1$ near P_0 with the above characteristics, plus, in the place of Q_0, a point Q in $(0,1)^2$, which, generically, is either an attractor or a repellor.

In order to better understand the situation, let us consider the coordinates $u = 1 - x$ and $z = y - y^*$. In such coordinates F and G can be written as:

$$\begin{aligned} F(u,z) &= -u(a + bz) + wz - dz^2 + \alpha\varepsilon^2 \\ G(u,z) &= u(k - lz) - mu^2 - nz^2 - \varepsilon \end{aligned} \tag{2.7}$$

where $a, b, w, d, k, l, m, n > 0$, while the coordinates of $Q = (\widetilde{u}, \widetilde{z})$ satisfy: $\widetilde{u} = \frac{\varepsilon}{k} + o(\varepsilon)$, $\widetilde{z} = \frac{a\varepsilon}{wk} + o(\varepsilon)$. It follows that, by multiplying F by a suitable $h > 0$, we can set:

$$a = \left(l + 2n\frac{a}{w}\right) y^*(1 - y^*)$$

which implies

$$traceJ(Q) = \beta\varepsilon^2 + o\left(\varepsilon^2\right) \tag{2.8}$$

Suppose, now, Q is an attractor for Σ. Hence, since Σ has no repelling fixed point in $[0,1]^2$, it follows from Poincaré-Bendixson Theorem (see [64]) that there must exist a repelling limit cycle surrounding Q.

We want to show that this cycle is not originated by a Hopf bifurcation. In fact, in such a case, its diameter should be, because of (2.8), of the order ε^2. But we can show the existence of a *winding inward* trajectory starting at a point whose distance from Q is $\overline{q}\varepsilon$, with $\overline{q} > 0$ independent of ε. Hence this trajectory is contained in the region bounded by the repelling cycle and, consequently, the latter cannot be originated by a Hopf bifurcation. Therefore the Hopf bifurcation at Q, which generically takes place when α crosses a suitable value $\overline{\alpha}$, is supercritical, i.e. an attracting limit cycle arises around the fixed point Q when it becomes a repellor. Thus, in this case, two (at least) limit cycles exist, an attracting one surrounded by a repelling one.

So, we have to show the existence of a *winding inward* trajectory starting at a distance from Q of the order ε. We will utilize the coordinates (u, z), but referring to the system $\Sigma = \left(\dot{x}, \dot{y}\right)$. Let us first consider the line $r)$ $\frac{\partial G}{\partial z} = \frac{\partial G}{\partial y} = 0$. Then $r)$ $lu + 2nz = 0$ intersects $G = 0$ at a point $(\widehat{u}, \widehat{z})$, where $\widehat{u} = \frac{\varepsilon}{k} + o(\varepsilon)$, $\widehat{z} = -\frac{l\widehat{u}}{2n}$. By straightforward computations one can check that the *backward* (i.e. negative) trajectory starting at $H_0 = (u_0, z_0)$, with $z_0 = -\frac{l\varepsilon}{4nk}$, $G\left(u_0, z_0\right) = 0$, intersects $F(u, z) = 0$ at a point $H_1 = (u_1, z_1)$ without intersecting $r)$. Moreover one can see that $G\left(u_1, z_1\right) = o(\varepsilon)$.[6] Consider

[6]Suppose, by contradiction, $|G(u_1, z_1)| \geq p\varepsilon$, with p independent of ε. Hence there would exist $(u', z') \in$

now the backward trajectory of H_1 and its intersection $H_2 = (u_0, z_2)$ with $u = u_0$. Let $\overline{z}$ satisfy $F(u_0, \overline{z}) = 0$. Then, from the above considerations and recalling $y^* > \frac{1}{2}$, one can show that $z_2 - \overline{z} - (\overline{z} - z_0) = z_2 + z_0 - 2\overline{z} > \gamma\varepsilon$, where $\gamma > 0$ is independent of ε for ε sufficiently small. Take now the first intersections with $F = 0$ of the *forward* (i.e. positive) trajectory of H_0 and the *backward* trajectory of H_2, and denote them, respectively, as $H_3 = (u_3, z_3)$ and $H_4 = (u_4, z_4)$. If it were $u_3 \geq u_4$ (and thus $x_3 \leq x_4$), then there should be a value $\overline{u}$, $u_0 < \overline{u} < u_4$, where, denoting by $(\overline{u}, z')$, $(\overline{u}, \overline{z})$, $(\overline{u}, z'')$ the intersections of $u = \overline{u}$ with, respectively, the arc (H_0, H_3), the curve $F = 0$ and the arc (H_2, H_4), $z'' - \overline{z} = \overline{z} - z' = \delta > 0$. Moreover one can show that the above intersections should lie in the region $G > 0$. However, since $\delta^2 << \delta$ (δ being of order at most ε), it is easily seen that, in absolute value, the *speed* at $(\overline{u}, z')$ would be higher than the *speed* at $(\overline{u}, z'')$, i.e.

$$\frac{(z' + y^*)(1 - z' - y^*)G(\overline{u}, z')}{\overline{u}(1 - \overline{u})|F(\overline{u}, z')|} > \frac{(z'' + y^*)(1 - z'' - y^*)G(\overline{u}, z'')}{\overline{u}(1 - \overline{u})F(\overline{u}, z'')} \tag{2.9}$$

which implies $u_3 < u_4$, leading to a contradiction. Therefore it is $u_3 < u_4$, i.e. $x_3 < x_4$, which means that the trajectory through H_0 winds inward. This concludes the proof.

■

(H_1, H_0) such that $|G(u', z')| = \frac{p\varepsilon}{2}$. But then, as it is easily calculated, $\frac{p\varepsilon}{2} \leq \int_{z'}^{z_1} \frac{d|G(u(z),z)|}{dz} dz = \int_{z'}^{z_1} \left[\frac{(k-lz-2mu)u(1-u)|F(u,z)|}{z(1-z)|G(u,z)|} + lu + 2nz\right] dz < \underline{q}\varepsilon^2$ for a suitable $\underline{q}$ independent of ε, which produces a contradiction when ε is sufficiently small.

3 Multi-species exploitation with evolutionary switching of harvesting strategies

In this Chapter we propose a nonlinear evolutionary model for the exploitation of common-pool resources where the harvesting at each time is based on Cournot competition.

3.1 Introduction

In order to avoid the overexploitation of some fisheries, management institutions usually enforce forms of regulation, either by imposing harvesting restrictions, such as constant efforts, individual fishing quotas, taxation, etc., or by limiting the kinds of fish to be caught or the regions where exploitation is allowed (see e.g. [35], [3] [48], [49], [19] and [22]). Usually, optimal policies are established by solving suitable long-run optimization problems, for instance estimating the maximum sustainable yield (MSY) for fisheries and assessing the relative social impact. Any miscomputation of the total catch can easily lead the resource to the verge of collapse. Moreover, fishers have to accept the adopted fishing restriction and there is the problem of controlling compliance to the prescribed catch. To make things more complicated, the strategic interaction between fishers usually gives an incentive to free-ride and overexploit to single agents, according to the well-known problem of the "tragedy of the commons" (see [62]), as documented by empirical fisheries data in [81].

However, even economic externalities may have an indirect impact on harvesting pressure and can be employed in regulations. For instance, increasing harvesting (and thus quantity of the resource on the market) usually leads to price reductions and so to lower profits. Similarly cost externalities come into play, as stock depletion leads to increments of landing costs and so again to lower profits. Some experiments on endogenous regulatory policies of common pool resources have been recently performed on the basis of these self-regulating economic externalities. In particular, fishing institutions only establish general rules, and then fishers are allowed to decide fishing strategies on their own. Along these lines, it is more reasonable to assume that exploiters choose their catch in order to maximize their short-term profit instead of solving optimal control problems. In fact, the long-run sustainability of exploitation is more an objective for the farsighted regulator, whereas it is more likely that fishers behave myopically.

For example, a recent law proposed in Italy to regulate the harvesting of two non-interacting shellfish (*Venerupis aurea* and *Callista chione*) in the Adriatic Sea, requires that each agent can harvest only one species in any three-year period, possibly revising his/her choice in predefined successive periods, but no limits on individual quotas are set. In other words, instead of imposing a difficult-to-control policy (e.g. imposed effort, total allowable catch, etc.), the fishing institution only establishes that each vessel can harvest just one species in each period and has to stick to this choice for a given time interval. In the revision of their strategy, agents compare their average profits with the ones obtained by the agents who made a different choice over the last fishing period. These average profits are taken as a proxy of the fitness of a strategy, according to the paradigms of evolutionary game theory (see [108], [65]).

The aim of the Chapter is to use analytical and numerical methods to analyze the economic consequences of this kind of self-regulating fishery, as well as to shed some light on the sustainability of this form of exploitation in comparison to other policies. Indeed, our analysis gives evidence of possible advantages of profit-driven self-regulated harvesting strategy choices over other practices, both from the point of view of biomass levels (i.e., biological sustainability) and wealth (economic profitability). Moreover, the simulation results suggest that this kind of myopic evolutionary regulation in certain cases can ensure a virtuous trade-off between profit maximization and resource conservation.

We develop a standard model for each of two species, then allow fishers to switch between the two fisheries at prespecified time periods. We consider four management strategies: 1) unrestricted harvesting; 2) splitting the fishers between the two fisheries equally; 3) allowing the fishers to choose continuously; and 4) only allowing switching at prespecified periods. Case 4) gives rise to a hybrid dynamic model, which is the nearest to a real-world application but quite difficult to study analytically, so the other cases mainly serve as benchmarks. Even if far from the real system we want to describe, particularly cases 2) and 3) can give useful suggestions about the directions of investigation of the more realistic hybrid system, as well as some intuitive interpretations of the properties observed through numerical simulations.

The structure of the Chapter is as follows. The bioeconomic model is introduced in

Section 3.2, where agents' harvesting functions are defined under various assumptions about fishing restrictions. Section 3.3 defines the switching mechanism exploiters employ to decide the species to harvest from period to period. The main properties of the model with switching in continuous time are also studied in this Section. Some numerical simulations are proposed in Section 3.4 in order to understand peculiar features of the proposed hybrid system. Section 3.5 concludes that allowing switching between separate fisheries may have a long-term positive effect on stocks and profits under certain conditions. Appendix 3.6 contains the algebra to obtain the agents' cost functions and the proofs of the Propositions.

3.2 The bioeconomic models

Let us consider a simple marine ecosystem with two non-interacting species, indexed by 1 and 2, each with its own habitat, and biomass (or density) measures X_1 and X_2 respectively, both subject to commercial harvesting. We assume that their time evolution is described by a dynamical system of the form:

$$\begin{aligned} \dot{X}_1 &= X_1 G_1(X_1) - H_1(X_1, X_2) \\ \dot{X}_2 &= X_2 G_2(X_1) - H_2(X_1, X_2) \end{aligned} \tag{3.1}$$

where $\dot{X}_i$ denotes the derivatives of biomass with respect to time, G_i specifies the natural growth function and H_i represents the instantaneous harvesting of species $i = 1, 2$.

According to [82], we assume that the two populations of (shell)fish follow a logistic natural growth of the form:

$$G_i = \rho_i \left(1 - \frac{X_i}{k_i}\right) \quad ; i = 1, 2 \tag{3.2}$$

where ρ_i and k_i are, respectively, the intrinsic rate of growth and the carrying capacity of species i.

These two species are harvested by N agents. We assume that H_i, the current total harvesting of species i, is wholly supplied to the market, and prices are determined according to the following horizontal differentiated linear inverse demand system (see [96] and [60]):

$$\begin{aligned} p_1 &= f_1(H_1, H_2) = a_1 - b_1(H_1 + \sigma H_2) \\ p_2 &= f_2(H_1, H_2) = a_2 - b_2(\sigma H_1 + H_2) \end{aligned} \tag{3.3}$$

where a_i is the reservation price for species i, b_i represents the slope of the demand for fish i and $\sigma \in [0, 1]$ is the symmetric degree of substitutability between the two fish varieties. In particular, if $\sigma = 0$ then the two varieties are independent in demand. On the other hand, for $\sigma = 1$ they are perfect substitutes[1]. In addition, we also assume quadratic harvesting costs for both species, i.e., for harvesting h_i units of species i an agent incurs a cost given by

$$C_i(X_i, h_i) = \gamma_i \frac{h_i^2}{X_i} \tag{3.4}$$

where γ_i is the cost parameter for catching species i. This cost function is obtained and employed in [35] and [99] and used by several authors, see, e.g., [37]. As shown in the Appendix 3.6, equation (3.4) can be derived from a Cobb-Douglas type "production

[1] In the context we are considering, we disregard the case $\sigma < 0$, related to varieties that are demand complementary.

function" with fishing effort (labor) and fish biomass (capital) as production inputs. This production function exhibits decreasing marginal returns to both input factors: for the biomass they are a consequence of gear saturation, which occurs whenever the fishing nets have a maximum capacity, whereas decreasing catch-per-unit-effort (CPUE) captures the problem of congestion among fishing vessels. In particular, the Cobb-Douglas production function is based on the assumption that gear saturation and congestion reduce the mortality rate of one unit of biomass and the CPUE in a smooth manner. Other mathematical forms of the production function can capture similar effects, as suggested in [35]. For other cost functions in fishing see, e.g., [32].

3.2.1 Unrestricted Harvesting

Although rarely observed in real-world examples of fisheries, the case of unrestricted harvesting developed in this Section serves as a benchmark case for comparison purposes. By unrestricted harvesting we mean that a generic fisher has no constraints on the quantity and kind of fish to harvest. The current profit of a generic agent that harvests the quantities h_1^F and h_2^F of species 1 and 2 reads:

$$\pi^F = p_1 h_1^F + p_2 h_2^F - \gamma_1 \frac{\left(h_1^F\right)^2}{X_1} - \gamma_2 \frac{\left(h_2^F\right)^2}{X_2}$$

If fisher q, $q = 1, ..., N$ is allowed to catch without constraints and tries to maximize current profits, his/her problem is given by $\max_{h_{1,q}^F; h_{2,q}^F} \pi_q^F$, where

$$\pi_q^F = \left\{ a_1 - b_1 \left[h_{1,q}^F + \sigma h_{2,q}^F + \sum_{u=1; u\neq q}^{N} \left(h_{1,u}^F + \sigma h_{2,u}^F \right) \right] \right\} h_{1,q}^F +$$
$$+ \left\{ a_2 - b_2 \left[h_{2,q}^F + \sigma h_{1,q}^F + \sum_{u=1; u\neq q}^{N} \left(h_{2,u}^F + \sigma h_{1,u}^F \right) \right] \right\} h_{2,q} +$$
$$- \gamma_1 \frac{\left(h_{1,q}^F \right)^2}{X_1} - \gamma_2 \frac{\left(h_{2,q}^F \right)^2}{X_2}$$

and $h_{i,q}^F$, $i = 1, 2$; $q = 1, ..., N$, denotes the harvesting of species i by fisher q in case of unrestricted harvesting.

Instantaneous optimal harvesting can be obtained by solving the system of first-order conditions[2] $\frac{\partial \pi_q^F}{\partial h_{1,q}} = 0$ and $\frac{\partial \pi_q^F}{\partial h_{2,q}} = 0$. Note that, since all agents face the same optimization problem, we can solve the system of first-order conditions by letting $h_{i,q}^F = h_{i,u}^F, i = 1, 2;\ q, u = 1, ..., N$. Hence, in the case of unrestricted harvesting, the equilibrium harvesting quantities $h_i^{F,*}$ by a representative player for catching species i reads

$$h_i^{F,*}(X_i, X_j) = \frac{a_j(b_j + Nb_i)X_iX_j\sigma - a_iX_i(b_j(1+N)X_j + 2\gamma_j)}{(b_i + Nb_j)(b_j + Nb_i)X_iX_j\sigma^2 - (b_i(1+N)X_i + 2\gamma_i)(b_j(1+N)X_j + 2\gamma_j)};$$
$$i, j = 1, 2; i \neq j$$

In the particular case $b_1 = b_2 = b = 0$, i.e., perfectly elastic demands for both species, the individual optimal harvesting of species i and the resulting total instantaneous profit

[2]Specific conditions can be given for the sufficiency of these conditions, i.e. for the concavity of profit π_q^F with respect to $h_{1,q}^F$ and $h_{2,q}^F$. For instance, in the case $b_1 = b_2$ (assumption that will be considered in the following) it is easy to prove that profits are strictly concave. However, we can assume by continuity that the same holds for $b_1 \simeq b_2$.

can be written in the following simplified form:

$$h_i^{F,*} = \frac{a_i X_i}{2\gamma_i}, \qquad \pi_i^{F,*} = \sum_{i=1}^{2} \frac{a_i}{2} h_i^{F,*}; \qquad i = 1, 2. \tag{3.5}$$

In fisheries models, prices are often assumed to be constant as fish is considered a staple food for the majority of consumers, there are many substitutes for each species and many fish are internationally traded, see [35] and [37].

3.2.2 Restricted harvesting

In this Subsection, we obtain the harvesting function under the assumption that an authority restricts each agent to catch only one species at a time. Let us assume that, in a given time period, agents are partitioned into two groups, with $m_1 = m$ agents in group 1 (harvesting species 1 only) and $m_2 = N - m$ agents in group 2 (harvesting species 2 only).

Given the specifications of cost functions and prices as above, the profit of fisher q in group $i\,(=1,2)$ when harvesting $h_{i,q}$ reads

$$\pi_{i,q} = p_i h_{i,q} - \gamma_i \frac{h_{i,q}^2}{X_i}, \; i = 1, 2 \tag{3.6}$$

Therefore, in deciding his/her instantaneous harvesting of species i, the representative fisher q in group i solves the problem $\max\limits_{h_{i,q}} \pi_{i,q}$, where

$$\pi_{i,q} = \left\{ a_i - b_i \left[h_{i,q} + \sum_{u \in m_i; u \neq q} h_{i,u} + \sigma \sum_{u \in m_j} h_{j,u} \right] \right\} h_{i,q} - \gamma_i \frac{h_{i,q}^2}{X_i}; \; i,j = 1,2, i \neq j$$

By taking the first-order conditions and employing the symmetry property that players within each group are homogeneous (i.e. $h_{i,q} = h_{i,u}, i = 1,2;\ q,u \in m_i$), we obtain the following harvesting quantities at a Nash equilibrium

$$h_i^*(X_i, X_j) = \frac{a_i X_i (b_j X_j [1 + N r_j] + 2\gamma_j) - a_j b_i N r_j X_i X_j \sigma}{(b_i X_i (1 + Nr) + 2\gamma_i)(b_j X_j (1 + N(1-r)) + 2\gamma_j) - b_i b_j N^2 (1-r) r X_i X_j \sigma^2}; i,j = 1,2, i \neq j \quad (3.7)$$

where $r_1 = r = \frac{m_1}{N}$ and $r_2 = (1-r) = \frac{m_2}{N}$ represent, respectively, the fractions of agents in group 1 and 2.

By inserting (3.7) into (3.6), we get optimal individual profits

$$\pi_i^* = \left(b_i + \frac{\gamma_i}{X_i} \right) (h_i^*)^2 \quad (3.8)$$

which shows that profits are non-negative. Of course, if profits are positive (or at least non-negative), then also optimal harvesting (3.7) is positive.

The assumption $b_1 = b_2 = b = 0$ allows us to obtain a simpler expression for individual optimal harvesting profit, which constitutes a useful benchmark in the following:

$$h_i^* = \frac{a_i X_i}{2\gamma_i}, \qquad \pi_i^* = \frac{a_i}{2} h_i^*; \qquad i = 1,2. \quad (3.9)$$

Although harvesting expressions in (3.5) and (3.9) are the same because there is no interaction via the demand curve, in the case of restricted harvesting each agent cannot access both stocks. In fact, profits for each fisher in (3.5) are the sum of profits from both species, whereas in (3.9) profits to each agent come from the only species caught.

3.3 Switching mechanism

In this Section we explain the basic dynamic mechanism that regulates how the fraction $r(t)$ of exploiters of species 1 (or, equivalently, the fraction $1-r(t)$ of exploiters of species 2) is updated over time in the case of restricted harvesting. In this case, recall that an authority imposes that fishers have to stick to the decided strategy for a given period of time $s > 0$, after which they can reconsider their decisions on the basis of observed profits. This period-by-period adaptive mechanism can be described by an endogenous evolutionary dynamics, for instance through a replicator equation in discrete time (see [108], [65] and [21]). More specifically, let us assume that at the end of each time period of length s, a representative agent in group i assesses his/her average profit $\overline{\pi}_i^*$ over that period, given by

$$\overline{\pi}_i^*(t) = \frac{\int_{t-s}^{t} \pi_i^*(\tau)\, d\tau}{s} \quad ; i = 1, 2 \tag{3.10}$$

If the magnitude of $\overline{\pi}_i^*(t)$ can be estimated by all agents, i.e. it is a common knowledge, it can be employed as a shared fitness measure for playing strategy i. This leads to the following dynamic model, expressed by continuous time growth and harvesting of the fish species and discrete (or pulse) fishing strategy switching (a discrete decision-driven time)[3]

$$\begin{cases} \dot{X}_1(t) = X_1(t)G_1(X_1(t)) - Nr(t)h_1^*(X_1(t), X_2(t)) \\ \dot{X}_2(t) = X_2(t)G_2(X_2(t)) - N(1-r(t))h_2^*(X_1(t), X_2(t)) \\ r(t) = \begin{cases} r(t-s)\frac{\overline{\pi}_1^*(t)}{r(t-s)\overline{\pi}_1^*(t)+[1-r(t-s)]\overline{\pi}_2^*(t)} & \text{if } \frac{t}{s} = \lfloor \frac{t}{s} \rfloor \\ r\left(\lfloor \frac{t}{s} \rfloor s\right) & \text{otherwise} \end{cases} \end{cases} \tag{3.11}$$

where $\lfloor x \rfloor$ is the largest integer not greater than x (i.e. the floor of x), and $h_i^*(X_1(t), X_2(t))$, $\overline{\pi}_i^*(t)$, $i = 1, 2$ are given, respectively, in (3.7) and (3.10). The third equation states that at each switching time, each representative fisher is assumed to know the average profits during the previous period both for fishers of the same group as well as for fishers of the other group. If we observe $\overline{\pi}_1^*(t) > \overline{\pi}_2^*(t)$ then $r(t)$ increases, i.e., a fraction of fishers harvesting species 2 switches to harvest species 1, otherwise $r(t)$ decreases.

Given $X_1(0)$, $X_2(0)$ and $r(0)$, for each $t \geq 0$ the time evolution of $X_i(t)$, $i = 1, 2$, and $r(t)$ is thus regulated by the hybrid dynamical system (3.11). The term hybrid indicates that $X_1(t)$ and $X_2(t)$ evolve in continuous time, whereas $r(t)$ is updated according to a discrete time scale.

In the limiting case $s \to 0$, i.e., with fishers changing their strategy continuously (i.e., the species to harvest), $\overline{\pi}_i^*(t) = \pi_i^*(t)$ and the last equation in (3.11) can be replaced with

$$\dot{r}(t) = r(t)\left[\pi_1^*(t) - (r(t)\pi_1^*(t) + (1-r(t))\pi_2^*(t))\right] = r(t)(1-r(t))\left[\pi_1^*(t) - \pi_2^*(t)\right] \tag{3.12}$$

which is the well-known replicator equation in continuous time (see again [108] and [65]), stating that $\dot{r}(t) > 0$ if $\pi_1^*(t) > \pi_2^*(t)$. In this case, the model assumes the simpler form of a nonlinear 3-dimensional system of ordinary differential equations (ODEs). This

[3]Notice that in the hybrid system (3.11) the time index t has to be expressed in order to ensure synchronism between discrete and continuous dynamical variables.

simpler specification constitutes a useful benchmark. In fact, an equilibrium point for the system with continuous replicator (3.12) is also a fixed point for the hybrid system (3.11), although the converse is not necessarily true. This follows from the fact that the first and the second dynamic equations in the two specifications are identical, and the replicator dynamics in discrete time have the same equilibrium conditions: $r(t) = r(t-s)$ for $r = 0$, $r = 1$ or $\pi_1^*(t) = \pi_2^*(t)$. In fact, if instantaneous profits are identical in equilibrium, then the average profits of the two strategies over the non-switching time interval of length s are also identical. Nonetheless, we can have an equilibrium point such that the average profits of the two strategies over the interval s are equal, even though instantaneous profits are not equal over the interval. As we shall see, in the case (3.11), $r(t)$ becomes a piecewise-constant function, like an endogenously driven bang-bang parameter whose discontinuous jumps occur at discrete times and lead to sudden switching among different dynamic scenarios, which is typical behavior of hybrid systems, see, e.g., [18], [58] and [61].

3.3.1 Equilibria and stability analysis with continuous switching

In order to obtain analytical results, let us consider the system (3.1) with replicator dynamics in continuous time (3.12) and constant prices (i.e., $b_1 = b_2 = 0$). The dynamical model assumes the form of the following system of ODEs:

$$\begin{cases} \dot{X}_1 = X_1\rho_1\left(1 - \frac{X_1}{k_1}\right) - Nr\frac{a_1 X_1}{2\gamma_1} \\ \dot{X}_2 = X_2\rho_2\left(1 - \frac{X_2}{k_2}\right) - N\left(1 - r\right)\frac{a_2 X_2}{2\gamma_2} \\ \dot{r} = r(1-r)\left[\frac{a_1^2 X_1}{4\gamma_1} - \frac{a_2^2 X_2}{4\gamma_2}\right] \end{cases} \tag{3.13}$$

where we omitted the dependence on t, as no confusion arises. Note that in the invariant subspaces defined by $r = 0$ and $r = 1$, the first two differential equations in (3.13) are uncoupled. The following Propositions (proved in Appendix 3.6, see at the end of this Chapter) describe the steady states of the model and their local stability properties. To keep the notation short, in these Propositions it is useful to define the aggregate parameters

$$\alpha_1(r) = \rho_1 - \frac{a_1 N r}{2\gamma_1} \text{ and } \alpha_2(r) = \rho_2 - \frac{a_2 N(1-r)}{2\gamma_2} \tag{3.14}$$

Proposition 3.1 (Boundary equilibria and their stability) *For the system of ODEs (3.13) the following statements hold:*

- *the total extinction fixed points $E_r^0 = (0, 0, r)$, where $r = [0, 1]$, are unstable non-hyperbolic nodes provided that $\alpha_i(r) > 0$, $i = 1, 2$;*

- *the equilibria with harvesting of only one species are given by:*

  - *$E_1^0 = (k_1, 0, 0)$ and $E_2^0 = (0, k_2, 1)$ [extinction of the harvested species], which are saddle points;*

  - *$E_1^1 = \left(0, k_2\left(1 - N\frac{a_2}{2\gamma_2\rho_2}\right), 0\right)$, if $Na_2 < 2\gamma_2\rho_2$, and $E_2^1 = \left(k_1\left(1 - N\frac{a_1}{2\gamma_1\rho_1}\right), 0, 1\right)$, if $Na_1 < 2\gamma_1\rho_1$ [extinction of the non-harvested species], which are saddle points;*

  - *$E_1^2 = \left(k_1, k_2\left(1 - N\frac{a_2}{2\gamma_2\rho_2}\right), 0\right)$ if $Na_2 < 2\gamma_2\rho_2$ [both viable species with no harvesting of species 1], which is a stable node provided that*

$$a_1 < \sqrt{\frac{a_2^2 k_2 \gamma_1 (2\gamma_2\rho_2 - Na_2)}{2k_1\gamma_2^2\rho_2}} \tag{3.15}$$

    and a saddle point if the reverse inequality in (3.15) holds;

– $E_2^2 = \left(k_1\left(1 - N\frac{a_1}{2\gamma_1\rho_1}\right), k_2, 1\right)$ *if* $Na_1 < 2\gamma_1\rho_1$ *[both viable species with no harvesting of species 2] which is a stable node provided that*

$$a_2 < \sqrt{\frac{a_1^2 k_1 \gamma_2 (2\gamma_1\rho_1 - Na_1)}{2k_2\gamma_1^2\rho_1}} \tag{3.16}$$

and a saddle point if the reverse inequality in (3.16) holds.

The next Proposition characterizes an equilibrium with $r^* \in (0,1)$, which is the case in which each species is always harvested by some fishers, of course with the restriction that each agent is allowed to fish only one species.

Proposition 3.2 (Inner equilibrium and its stability) *For the system of ODEs (3.13) the following statements hold:*

- *There exists a unique inner equilibrium* $E^* = (X_1^*, X_2^*, r^*)$ *with* $r^* \in (0,1)$*, where*

$$X_i^* = \frac{a_j^2 k_1 k_2 \gamma_i (2a_2\gamma_1\rho_1 + 2a_1\gamma_2\rho_2 - a_1 a_2 N)}{2\left(a_2^3 k_2 \gamma_1^2 \rho_1 + a_1^3 k_1 \gamma_2^2 \rho_2\right)}, i = 1, 2; i \neq j \tag{3.17}$$
$$r^* = \frac{\gamma_1\rho_1\left(a_2^3 k_2 N\gamma_1 - 2a_2^2 k_2 \gamma_1\gamma_2\rho_2 + 2a_1^2 k_1 \gamma_2^2 \rho_2\right)}{N\left(a_2^3 k_2 \gamma_1^2 \rho_1 + a_1^3 k_1 \gamma_2^2 \rho_2\right)}$$

- *Equilibrium biomass levels* $X_i^* > 0, i = 1, 2$ *iff* $\alpha_i(r^*) > 0$*, with* $0 < r^* < 1$*; this occurs in the following cases:*

  – *case 1: If* $\alpha_1(1) > 0$ *and* $\alpha_2(0) > 0$ *then the carrying capacity* k_1 *must satisfy*

$$\widehat{k}_1 = \frac{a_2^2 k_2 \gamma_1 (2\gamma_2\rho_2 - Na_2)}{2a_1^2\gamma_2^2\rho_2} < k_1 < \frac{2a_2^2 k_2^2 \gamma_1^2 \rho_1}{2a_1^2\gamma_1\gamma_2\rho_1 - Na_1^3\gamma_2} = \overline{k}_1;$$

– *case 2: If* $\alpha_1(1) > 0$ *and* $\alpha_2(0) < 0$ *then the carrying capacity* k_1 *must satisfy* $0 < k_1 < \overline{k}_1$*;*

– *case 3: If* $\alpha_1(1) < 0$ *and* $\alpha_2(0) > 0$ *then the carrying capacity* k_1 *must satisfy* $k_1 > \widehat{k}_1$*;*

– *case 4: If* $\alpha_1(1) < 0$ *and* $\alpha_2(0) < 0$ *then it must be* $\frac{2\gamma_1\rho_1}{N} < a_1 < \frac{2a_2\gamma_1\rho_1}{Na_2-2\gamma_2\rho_2}$*, and at* $a_1 = \frac{2a_2\gamma_1\rho_1}{Na_2-2\gamma_2\rho_2}$ *it is* $X_1^* = X_2^* = 0$*;*

– *finally, if* $k_1 = \widehat{k}_1$ *then it is* $r^* = 0$ *and* $E^* = E_1^2 = \left(k_1, k_2\left(1 - N\frac{a_2}{2\gamma_2\rho_2}\right), 0\right)$ *whereas if* $k_1 = \overline{k}_1$ *then it is* $r^* = 1$ *and* $E^* = E_2^2 = \left(k_1\left(1 - N\frac{a_1}{2\gamma_1\rho_1}\right), k_2, 1\right)$*;*

- *If the equilibrium* $E^* = (X_1^*, X_2^*, r^*)$ *involves positive biomasses, then it is stable under the replicator dynamics in continuous time.*

The cases discussed in the previous Proposition help to understand through which contacts (with border equilibria) the inner equilibrium appears or disappears. In particular, the last two cases indicate that the inner equilibrium can have a contact with extinction equilibrium E_r^0 or with the border equilibria E_1^2 and E_2^2.

It is also interesting to notice that, by Proposition 3.1, a single species, say species 1, does not become extinct provided that $\frac{2\gamma_1\rho_1}{a_1N} > r$, whereas by Proposition 3.2, the condition $r^* < 1$ leads to $\frac{2\gamma_1\rho_1}{a_1N} < 1 + \frac{2a_2^2k_2\gamma_1^2\rho_1}{Na_1^3k_1\gamma_2}$. Therefore, if

$$\frac{2\gamma_1\rho_1}{a_1N} \in \left(r, 1 + \frac{2a_2^2k_2\gamma_1^2\rho_1}{Na_1^3k_1\gamma_2}\right) \tag{3.18}$$

then species 1 will survive both with an exogenous fixed r or with a r^* to which the continuous time switching mechanism converges. However, if the fixed $r = \overline{r} \in (0,1)$ is such that

$$\frac{2\gamma_1\rho_1}{a_1 N} \in (0, r) \tag{3.19}$$

i.e., too much harvesting pressure is imposed on species 1, then this fixed $\overline{r}$ will lead the resource to extinction, whereas an endogenous r could avoid the occurrence of extinction for species 1 (the same reasoning applies, of course, to species 2).

For the sake of comparison, the analytical results on the coexistence of both species can be synthesized as follows:

Corollary 1: *If $k_1, k_2 > 0$ and $\frac{2\rho_1\gamma_1}{Na_1} + \frac{2\rho_2\gamma_2}{Na_2} > 1$, the model with continuous replicator dynamics (3.13) converges to one of the following fixed points with coexistence of the two species:*

- E_1^2 *if* $\frac{2\gamma_2\rho_2}{a_2 N} \in \left(1 + \frac{2a_1^2 k_1 \gamma_2^2 \rho_2}{N a_2^3 k_2 \gamma_1}, +\infty\right)$;
- E_2^2 *if* $\frac{2\gamma_1\rho_1}{a_1 N} \in \left(1 + \frac{2a_2^2 k_2 \gamma_1^2 \rho_1}{N a_1^3 k_1 \gamma_2}, +\infty\right)$;
- E^* *whenever* $\frac{2\gamma_1\rho_1}{a_1 N} < 1 + \frac{2a_2^2 k_2 \gamma_1^2 \rho_1}{N a_1^3 k_1 \gamma_2}$ *and* $\frac{2\gamma_2\rho_2}{a_2 N} < 1 + \frac{2a_1^2 k_1 \gamma_2^2 \rho_2}{N a_2^3 k_2 \gamma_1}$.

In the model (3.13) with the last differential equation dropped and the fraction r exogenously fixed to $\overline{r}$, if $k_1, k_2 > 0$ and $\frac{2\rho_1\gamma_1}{Na_1} + \frac{2\rho_2\gamma_2}{Na_2} > 1$, any $\overline{r} \in \left(1 - \frac{2\rho_2\gamma_2}{Na_2}, \frac{2\rho_1\gamma_1}{Na_1}\right)$ ensures the coexistence of both species.

In short, if there is a coexistence equilibrium for the model (3.13) then there is at least an $\overline{r}$ such that also the model with r exogenously fixed converges to a coexistence equilibrium. On the contrary, if there exists an $\overline{r}$ such that the model with r exogenous converges to a coexistence equilibrium then also the model with continuous replicator dynamics converges to a coexistence equilibrium.

3.4 Numerical simulations

Numerical simulations are important for shedding some light on the dynamics of the more realistic model of discrete time switching of fishing strategy. This Section is mainly devoted to investigating cyclical or more complex behaviors dictated by the hybrid structure of the model, which are impossible to observe in the benchmark case of continuous switching. In fact, assuming discrete time strategy switching is more realistic than continuous time adjustments, but analytical results can be obtained under continuous adjustments (as in the previous Section) and so comparisons between continuous and discrete switching are insightful. In particular, here we compare the dynamics of the system with discrete and continuous replicator equations and these cases with two simpler management strategies, namely unrestricted harvesting and splitting the fishers between the two fisheries equally. Moreover, we investigate the role played by s (the switching time) as well as the effects of non-constant prices, i.e. demand functions (3.3) with slope $b_i \neq 0$, $i = 1, 2$.

Before describing and discussing the simulations, we recall that, in general, the set of fixed points of the system with a continuous time replicator is a subset of the set of fixed points of the hybrid model. Moreover, even though a fixed point under a continuous time replicator is also a fixed point in the hybrid system, its stability properties can be different, as clearly shown below.

Let us begin the numerical investigation with a complete symmetric setting of the parameter values except for the instantaneous growth rates of the two species. For illustrative purposes only, the values are chosen at the following level:

$$\begin{gathered} \rho_1 = 90; \rho_2 = 140; k_1 = k_2 = 80; a_1 = a_2 = 50; \gamma_1 = \gamma_2 = 9; \\ b_1 = b_2 = 0; N = 40; \sigma = 0.5, s = 3. \end{gathered} \tag{3.20}$$

Both species are assumed to have the same carrying capacity, the same (constant) price in the market and the same cost to catch; the different intrinsic growth rates satisfy the relations $\alpha_1(1) < 0$, $\alpha_2(0) > 0$ and $k_1 > \widehat{k}_1$, so that an inner equilibrium with harvesting of both species exists and it is stable in the case of continuous time switching according to the Proposition in the previous Section. All the numerical simulations are obtained starting from the initial condition $X_1(0) = X_2(0) = 10$ and $r(0) = 0.5$ (which remains the same in the cases without evolutionary switching). Of course, as also shown in the first row of Figure (3.1), in the absence of harvesting the two non-interacting species always settle on the respective carrying capacities in the long run. Under this parameter constellation, if unrestricted oligopolistic harvesting takes place, then the first species becomes extinct (see Figure (3.1), row 2). On the other hand, if the exploiters are split equally in the two groups ($r = 0.5$), the trajectory converges to an equilibrium, say $E^+\left(X_1^+, X_2^+\right)$, with $X_2^+ > X_1^+$ (Figure (3.1), row 3). Instead, in the case of continuous and discrete replicator dynamics the trajectories converge to the unique (globally stable) inner equilibrium $E^*\left(X_1^*, X_2^*, r^*\right)$ with $X_1^* = X_2^*$ and $r^* = 0.3913$ (Figure (3.1), rows 4 and 5). In the case of continuous time replicator dynamics, this confirms the analytical results of the previous Section. Moreover, the numerical simulation shows that the same asymptotic dynamics occur in the case of impulsive adjustment in discrete time as well, even if a difference can be seen in the initial transient part of the trajectory. Since by assumption it is $\alpha_1(1) < 0$, it is not sustainable to let all agents harvest species 1, so that $E_2^2 = \left(k_1\left(1 - N\frac{a_1}{2\gamma_1\rho_1}\right), k_2, 1\right)$ is not a feasible equilibrium of the model under replicator dynamics. On the other hand, all the other border equilibria described in the Proposition "boundary equilibria and their stability" exist and are unstable, as shown in Figure (3.1). This means that under replicator dynamics the system is able to adjust endogenously the two fractions of fishers that harvest species one or two, putting less fishing pressure on the species with lower growth rate (species one in this specific example). This avoids the overexploitation of the species with respect to the other. In other words, this is a clear example of an autonomous self-regulating system. Moreover, even from an economic point of view, in this case the evolutionary mechanism represents a good solution ensuring a higher level of average profits, as shown in the second column of Figure (3.1) depicting the corresponding profits versus time. As well, the distribution of profits between fishers in the two groups also appears fairer in the case of evolutionary strategy switching (in both continuous and discrete time) than in the other cases.

It is interesting to observe, from the analytical expression of the inner equilibrium X_i^* in (3.17), that only asymmetries in the values of the economic parameters a_i and γ_i, $i = 1, 2$, can create differences in the long-run levels of biomass of the two species. For this reason, Figure (3.2) is obtained under the same parameters of Figure (3.1) but with a decreased value of the cost parameter $\gamma_1 = 5$, i.e., catching fish 1 becomes less expensive. The trajectory of the dynamical system with continuous replicator converges to the inner equilibrium $E^* = (X_1^*, X_2^*, r^*)$ for which $X_1^* < X_2^*$, while the trajectory of the dynamical system with a discrete replicator converges to a closed invariant orbit surrounding the equilibrium E^*, suggesting its instability under discrete switching, see last row of Figure (3.2). Here the dynamics are cyclic around the unstable fixed point, and this is due to the hybrid nature of the dynamical system with discrete replicator

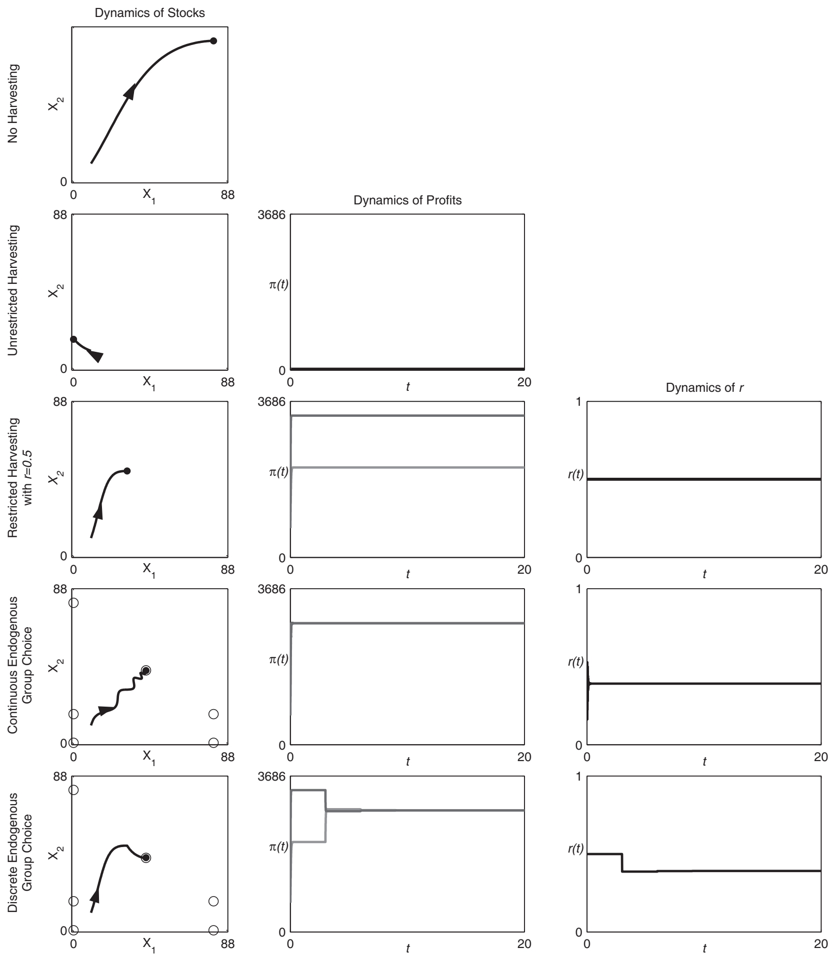

Figure 3.1: With parameters $\rho_1 = 90; \rho_2 = 140$; $k_1 = k_2 = 80$; $a_1 = a_2 = 50$; $\gamma_1 = 9; \gamma_2 = 9$; $b_1 = b_2 = 0$; $N = 40$; $r(0) = 0.5$; $\sigma = 0.5$; $s = 3$ and initial condition $X_1(0) = 10$, $X_2(0) = 10$ the trajectories in the space (X_1, X_2) are represented in the first column, profits $\pi(t)$ (black line for $\pi_1(t)$ and gray line for $\pi_2(t)$) in the second column and the fraction $r(t)$ of fishers that harvest species 1 third column. Different rows represent different policies for harvesting constraints. Row (1): biological independent species with logistic growth and without harvesting. Row (2): unrestricted oligopolistic harvesting. Row (3): two groups of fishers each harvesting only one species with imposed fraction $r = 0.5$. Row (4): endogenously adjusting $r(t)$ according to a continuous time replicator dynamics. Row (5): hybrid model with $r(t)$ evolving according to discrete time replicator dynamics.

equations and cannot be observed in the other dynamical models here considered.

Concerning the profits for the different harvesting strategies, the system with (continuous or discrete) replicator dynamic is able to ensure higher income for operators than in the other cases. In fact, with unrestricted oligopolistic harvesting, the extinction of one species and the depletion of the other occur due to overfishing, which sharply reduces total average profits, see again Figure (3.2), second column. Note that fixing the fraction of exploiters (recall that here we assumed $r = 0.5$) can even lead to higher profits, but at the cost of the extinction of the first species, so that half of the fishers (the ones who are exogenously assigned to harvest species one only) are forced to abandon their activity because there is no longer stock to harvest for them (see Figure (3.2), row 3). It follows that, despite the high level of total profits generated with the fixed fraction, this is not a desirable situation indeed.

For the sake of comparison between the cases in Figures (3.1) and (3.2), it is interesting to investigate the reasons why the extinction of species 1 occurs only in the case of Figure (3.2), when r is exogenously determined. From the analytical results, every time the evolutionary mechanism settles endogenously to the level r^* ensuring the coexistence of the two species, it is possible to fix exogenously an $\overline{r}$ that ensures the coexistence and vice versa. However, to fix exogenously an $\overline{r}$ requires correctly estimating the parameters of the model and their possible changes over time and to adjust r accordingly and immediately in case of relevant changes in these values. On the contrary, the evolutionary mechanism is able to react endogenously to changes in the economical and biological parameters without requiring any external intervention, thus ensuring the coexistence of the two species (whenever the conditions stated in the previous Section hold). This represents an important advantage of the evolutionary model that justifies its use. This aspect can be better appreciated by a cross simulation analysis. Starting from the

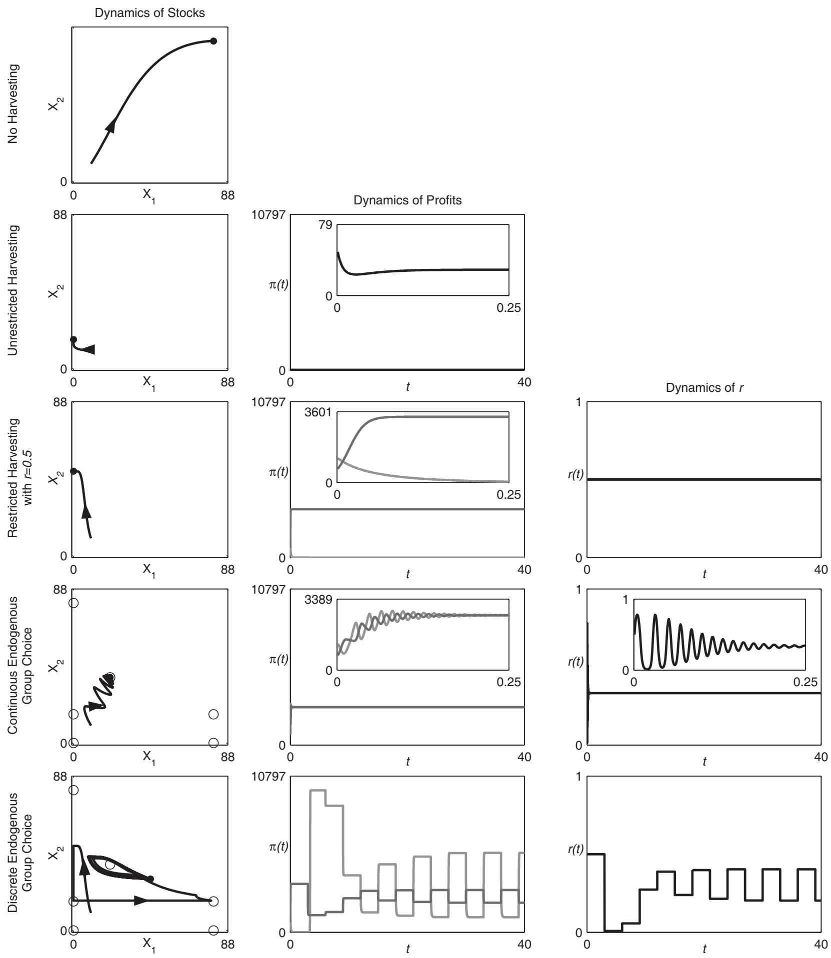

Figure 3.2: Initial condition and parameters as in Figure (3.1), except parameter $\gamma_1 = 5$. The explanation of the panels is the same as in Figure (3.1).

simulation represented in Figure (3.1), fixing exogenously $r = 0.5$ is enough to ensure the coexistence of the two species, as condition $\frac{2\gamma_1\rho_1}{a_1 N} = 0.81 \in (0.5, 1.81)$ is fulfilled, see (3.18). However, if the cost parameter γ_1 decreases from 9 to 5 as in Figure (3.2) (e.g., because a new fishing technique has been introduced), then it is $\frac{2\gamma_1\rho_1}{a_1 N} = 0.45 \in (0, 0.5)$ so that species 1 goes extinct, see (3.19). Therefore, in the case of a fixed r, if this value is not reduced exogenously by the authority, the risk of extinction of one of the two shellfish species is high, as happens in Figure (3.2) second row. In this specific case, the fixed value of r should be in the range $(0, 0.45)$ in order to avoid the extinction of species 1. As it is clear from this example, this requires continuous monitoring of the system (biological and economical parameters). On the contrary, the evolutionary mechanism is able to adjust r autonomously avoiding the risk of extinction of species.

Another key aspect that deserves to be delved into is the different dynamics of the models with continuous and discrete replicator dynamics, i.e., the effects of s on the dynamics of the model. It is worth noticing that s, the time interval after which the fishers can choose to change their fishing strategy, influences the amplitude of the oscillations. When $s \to 0$ the amplitude tends to zero, and the hybrid dynamical system has a behavior similar to the one obtained with a continuous replicator dynamic. However, when s increases, the presence of cycles of greater amplitude can be detected, see Figure (3.3). From the two pictures in Figure (3.3), it is easy to see that the orbits surrounding the inner equilibrium are characterized by two switching times, i.e., they are of period $2s$. Along these orbits the biomass levels of the two species always move in opposite directions, one increases and one decreases, this opposite relationship of growth changes at each switching time according to $r(t)$, which takes two values $\{\underline{r}_s, \overline{r}_s\}$ along the orbits depending on s.

Let us now consider the same values for parameters as in Figures (3.2) and (3.3) changing the level of the carrying capacity of species 2 only, namely $k_2 = 10$. For the dynamical

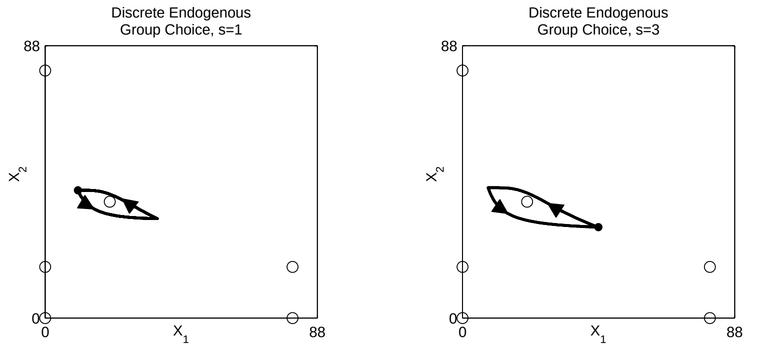

Figure 3.3: Two trajectories in the space (X_1, X_2) with initial condition and parameters as in Figure (3.2) for the dynamical model with discrete replicator equation. The picture gives evidence of how the amplitude of the closed orbits changes by changing the switching time interval s, with $s = 1$ in the left panel and $s = 3$ in the right panel.

system with continuous replicator dynamics, the inner equilibrium $E^* = (X_1^*, X_2^*, r^*)$ is positive, hence it is stable according to Proposition 3.2 in the previous Section. More precisely, this is the situation described in case 3 of Proposition 3.2. Regarding the system with discrete replicator equations, Figure (3.4) shows some dynamical behaviors of the hybrid model for different values of the switching time s. Numerical evidence shows that the inner equilibrium is unstable under the adaptive discrete dynamics. For $s = 1$ the trajectory passes very close to the inner equilibrium and draws a quite erratic path around it. For $s = 3$ and $s = 10$ more regular orbits can be observed. The time series of the individual and total profits are quite irregular as well, see the last column of Figure (3.4). It is worth noticing that profits arising from fishing species 1 (black lines) are quite regular along their time series. Instead, the profits arising from fishing species 2 (gray lines) exhibit a quite irregular pattern with long periods characterized by high profits and short periods characterized by low profits.

Numerical simulations are also useful for obtaining some insights on how decreasing

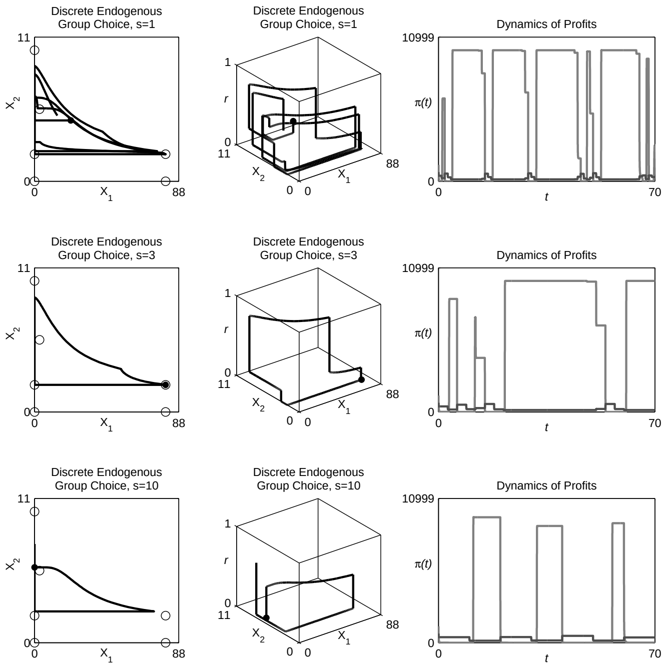

Figure 3.4: For the hybrid dynamic model with initial condition and parameters as in Figure (3.2) but $k_2 = 10$. In the first column the trajectories are projected in the space (X_1, X_2); in the second column the same trajectories are represented in the phase space (X_1, X_2, r) and in the third columns the versus-time representation of profits along the trajectories are represented for both fishers harvesting species 1 (black line) and 2 (gray line) respectively. The different rows are obtained for different values of the discrete switching time s, given by $s = 1$, $s = 3$ and $s = 10$ respectively.

inverse demand functions can influence the dynamic behaviors of the models studied here. If we repeat all the numerical simulations performed in the previous examples with $b_i \neq 0$, $i = 1, 2$, we see that the total harvesting and profits decrease, but in general the positive effects of the switching mechanism on reducing overexploitation of the two fish species can still be appreciated. In order to give an idea of the difference between the dynamics of the models with negative slopes of the inverse demand functions (3.3), i.e., decreased prices with increased total harvesting, Figure (3.5) shows the dynamics of the models with $b_i = 0.05$, $i = 1, 2$ and the other parameters values as in Figure (3.1). As for the case with zero slope demand, the model with unrestricted harvesting leads to the extinction of species 1. On the other hand, when r is defined exogenously or endogenously by fishers under a profit-driven adaptive process, it is possible to prevent species one from extinction and increase the general level of profits, see Figure (3.5).

The positive effect of the switching mechanism can be appreciated even for larger values of b_i, $i = 1, 2$. In order to avoid too much harvesting reduction due to demand effects, in Figure (3.6) we modify the parameter values as follows:

$$\rho_1 = 80; \rho_2 = 140; k_1 = 50; k_2 = 80; a_1 = 220; a_2 = 200; \gamma_1 = \gamma_2 = 9;$$
$$b_1 = b_2 = 0.1; N = 40; \sigma = 0.5; s = 3.$$

With respect to the other examples, the reservation prices have been increased and the growth rate and the carrying capacity for species one have been decreased. Having a smaller carrying capacity, species 1 is more rare in nature than species 2 and thus we assume it has a higher reservation price. The numerical simulations in Figure (3.5) gives evidence of overexploitation in the cases of unrestricted harvesting and restricted harvesting with a fixed proportion of exploiters. In both cases, the level of harvesting is not sustainable over time and in the long run the species with the lower intrinsic

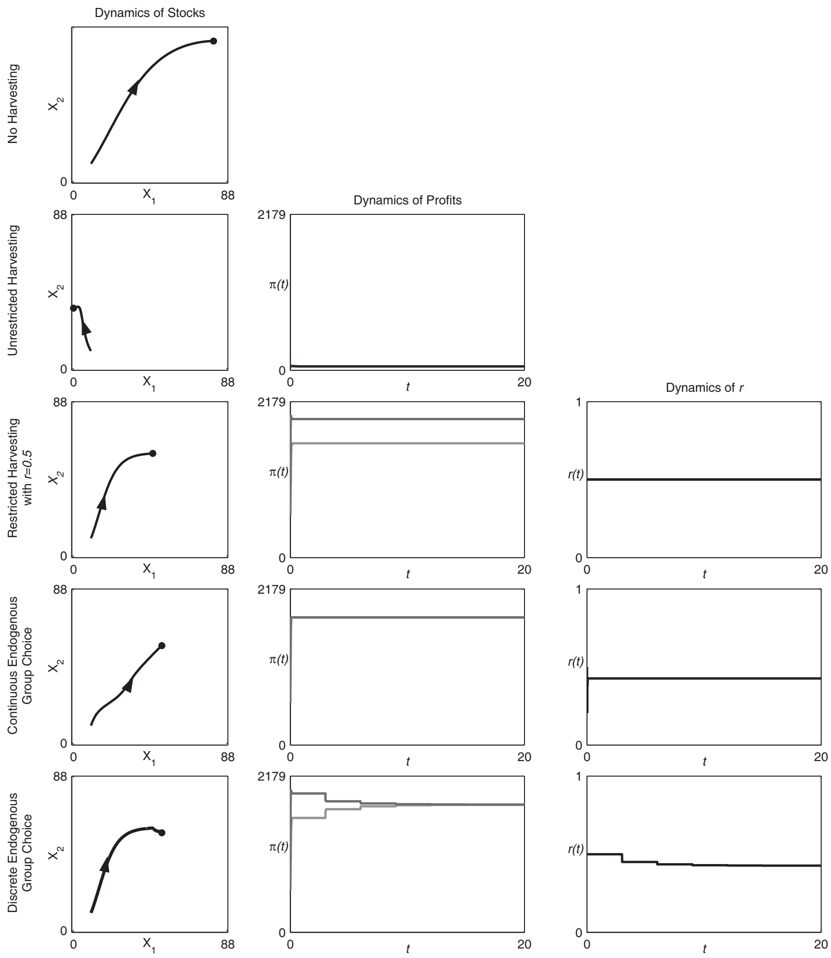

Figure 3.5: Same parameters values and initial condition as in Figure (3.1) but $b_1 = b_2 = 0.05$. The meaning of the panels is the same as in Figures (3.1) and (3.2).

growth rate will go extinct. However, when fishers can adjust their strategy myopically according to past profits, the extinction problem for species with lower growth rate could be avoided. It is worth noticing that, with these parameter values, the discrete switching mechanism performs even better than the instantaneous one. In fact, the first mechanism ensures a level of profits at least as high as the second one and the biological equilibrium has higher level of biomass for both species, so that there is a higher probability of surviving in the long run even in presence of exogenous shocks, which may temporarily reduce the natural rate of growth of the two (shell)fish species.

3.5 Conclusions

This Chapter proposes a dynamical system to model a fishery where two non-interacting fish species are harvested by a population of fishers, each allowed to catch just one species at a time and with the possibility of changing their fishing choice at specific times, according to a profit-driven replicator dynamic. The dynamic model is hybrid, since the growth of the fish species as well as harvesting activities occur in continuous time, whereas decisions about the species to catch take place in discrete time.

The analytical and numerical results show that this type of evolutionary mechanism may lead to a good compromise between total profit maximization, profit distribution among fishers and resource conservation, thanks to evolutionary self-regulation mainly based on cost externalities. In fact, the reduction of biomass of one species leads to its increasing landing cost and, consequently, favors the endogenous switching to the more abundant species. Moreover, severe overfishing of one species causes decreasing prices and consequently decreasing profits. Of course, in cases where both fisheries

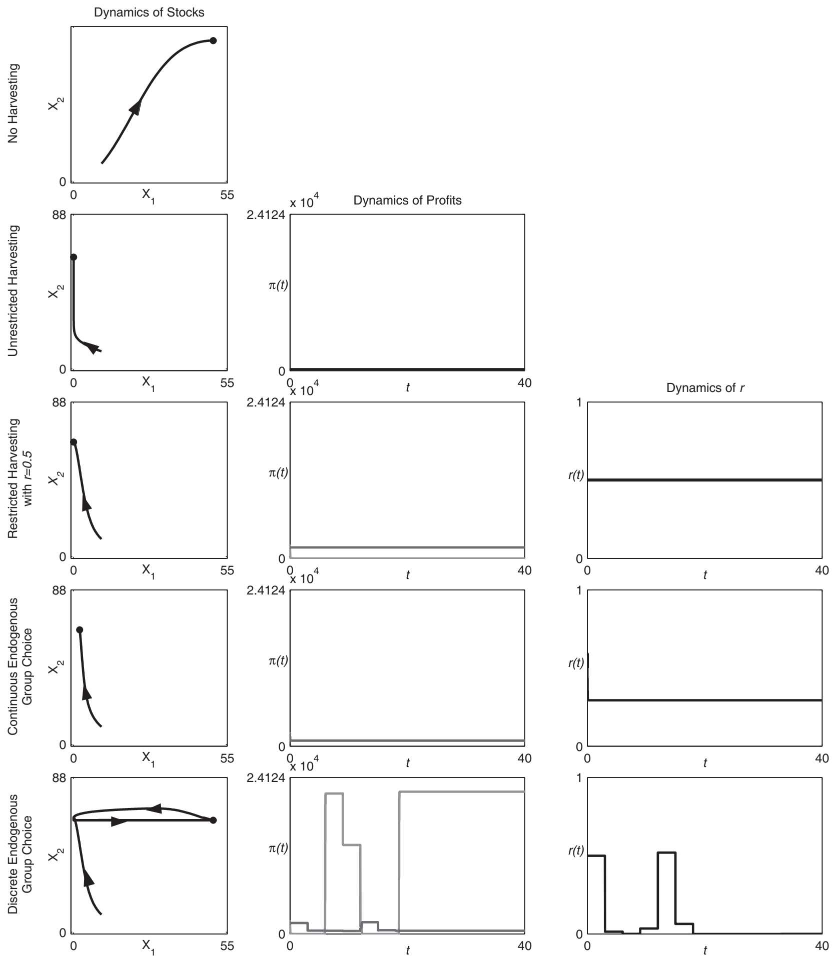

Figure 3.6: Same initial conditions as in Figure (3.2) and parameters $\rho_1 = 80; \rho_2 = 140$; $k_1 = 50$; $k_2 = 80$; $a_1 = 220$; $a_2 = 200$; $\gamma_1 = \gamma_2 = 9$; $b_1 = b_2 = 0.1$; $N = 40$; $r(0) = 0.5$; $\sigma = 0.5$; $s = 3$. The meaning of the panels is the same as in Figures (3.1) and (3.2).

are declining in terms of both stocks and profits, the evolutionary switching method proposed can at most allow fishers to move to the least-bad fishery option, with the lone result of slackening fishery decline.

Some simpler benchmark cases, with fixed prices and/or continuous time switching, have also been developed here. These benchmarks constitute a useful guide, even a sort of basic foundation, on which the (mainly numerical) analysis of the more realistic model can be performed, namely with variable market prices and discrete strategy switching.

The model studied in this Chapter offers a glimpse into the interesting properties of myopic and adaptive harvesting mechanisms driven by endogenous evolutionary processes. However, this is just a starting point for further and deeper analysis. There are several aspects of the model that deserve to be explored more deeply. For example, the variable r, i.e., the fraction of fishers harvesting a given fish stock, is assumed to take any value in the interval $[0, 1]$, where 0 and 1 are always equilibria. Indeed, when r converges to 0 or 1, one of the two species is no longer harvested and consequently it is not available in the market. This could be a reasonable practice only if the two species are perfect substitutes in consumer tastes (corresponding to the case $\sigma = 1$ in our model). Otherwise consumers may be heavily penalized by such an outcome. This issue will be addressed in future work, for example by introducing constraints on the dynamics of r or by assuming that the fractions of fishers harvesting one of the two species have a fixed component and a time varying portion, so that the non-switching portion ensures that both fish species are always available in the market. The endogenously switching components, on the other hand, help to regulate the fishing pressure so that the more abundant species is more harvested due to lower costs. The research can be extended in other different directions as well, for example it would be interesting to compare the results obtained here for the endogenous and myopic adaptive switching process with those obtained in models where an optimal fraction r is computed according to an optimal control problem, in which a regulator maximizes a social welfare function over a planning horizon. Another interesting extension is to formulate the problem in terms of choosing effort and then include a constraint on total effort in the fishers' profit maximization problems of restricted and unrestricted harvesting. The same arguments may be applied to the choice of an optimal length s of the switching interval, since it seems to be an important parameter in our numerical experiments. Moreover, the stability analysis for the model with continuous evolutionary switching mechanisms may be extended by using more sophisticated mathematical tools to provide qualitative indications on the behavior of the hybrid dynamical in the long run. Finally, the model can also be extended to the case of interacting species. For example, a similar model has been proposed by G. I. Bischi, Fabio Lamantia and myself, in [23] for the simulation of a fishery where a predator-prey system is exploited in the presence of the same endogenous evolutionary self-regulating method. Even if it is quite difficult to harvest only a single species when two fish populations interact in the same environment, the simulation of such a situation can provide useful theoretical information on the understanding of the trade-off between species interactions and endogenous evolutionary processes based on economic forces.

3.6 Appendix

Cost Function. Following [99], [35] and [37], we obtain here the cost function (3.4) employed in the profit maximization problems. Let us assume that current harvesting h is obtained through a Cobb-Douglas production function of the stock X and fishing effort E (see e.g. [35], pages 222-223.) with total factor productivity ρ

$$h(X, E) = \rho X^{\alpha} E^{\beta}$$

from which $E = \rho^{-1/\beta} X^{-\alpha/\beta} h^{1/\beta}$. Moreover, assuming that the "production function" $h(X, E)$ is an homogeneous function of degree 1 with $\alpha = \beta = \frac{1}{2}$ and that total cost of fishing is proportional to exerted effort, i.e. $C = \delta E$, then it is

$$C = \delta \rho^{-2} X^{-1} h^2 = \gamma \frac{h^2}{X}$$

Without loss of generality, we assume that $\rho = 1$, so that γ can be interpreted as a cost parameter. ■

Proof of Proposition 3.1. Any steady state of the dynamical system (3.13) must satisfy the algebraic system:

$$\begin{cases} X_1 \left(\rho_1 \left(1 - \frac{X_1}{k_1} \right) - N r \frac{a_1}{2\gamma_1} \right) = 0 \\ X_2 \left(\rho_2 \left(1 - \frac{X_2}{k_2} \right) - N \left(1 - r \right) \frac{a_2}{2\gamma_2} \right) = 0 \\ r(1-r) \left[\frac{a_1^2 X_1}{4\gamma_1} - \frac{a_2^2 X_2}{4\gamma_2} \right] = 0 \end{cases}$$

from which we get the equilibria E_j^k, $k = 0, 1, 2$ and $j = r, 1, 2$ listed in the Proposition 3.1. The Jacobian matrix for the dynamical system (3.13) is given by:

$$J\left(X_1, X_2, r\right) = \begin{bmatrix} \alpha_1(r) - \frac{2\rho_1 X_1}{k_1} & 0 & -N \frac{a_1 X_1}{2\gamma_1} \\ 0 & \alpha_2(r) - \frac{2\rho_2 X_2}{k_2} & N \frac{a_2 X_2}{2\gamma_2} \\ r(1-r) \frac{a_1^2}{4\gamma_1} & -r(1-r) \frac{a_2^2}{4\gamma_2} & (1-2r) \left[\frac{a_1^2 X_1}{4\gamma_1} - \frac{a_2^2 X_2}{4\gamma_2} \right] \end{bmatrix}$$

Evaluated at the equilibria with extinction of both species $E_r^0 = (0, 0, r)$, the Jacobian is the following triangular matrix:

$$J\left(E_r^0\right)=\begin{bmatrix}\alpha_1(r) & 0 & 0\\ 0 & \alpha_2(r) & 0\\ r(1-r)\frac{a_1^2}{4\gamma_1} & -r(1-r)\frac{a_2^2}{4\gamma_2} & 0\end{bmatrix}$$

from which it follows that the eigenvalues are the entries in the main diagonal, so we get the statement for the non hyperbolic stability given in the Proposition.

Without loss of generality, in the rest of the proof we assume that only the second species is harvested ($r=0$), as with $r=1$ one has just to swap indexes in the first two coordinates of the equilibria and the stability analysis is the same. Let us consider the equilibrium where species 1 is at the carrying capacity and species 2 vanishes, i.e. $E_1^0=(k_1,0,0)$

$$J\left(E_1^0\right)=\begin{bmatrix}-\rho_1 & 0 & -N\frac{a_1k_1}{2\gamma_1}\\ 0 & \alpha_2(0) & 0\\ 0 & 0 & \frac{a_1^2k_1}{4\gamma_1}\end{bmatrix}$$

The Jacobian matrix assumes again a triangular structure, with eigenvalues $-\rho_1<0$ and $\frac{a_1^2k_1}{4\gamma_1}>0$ so E_1^0 is always a saddle point. At the fixed point $E_1^1=\left(0,k_2\left(1-N\frac{a_2}{2\gamma_2\rho_2}\right),0\right)$ the Jacobian matrix becomes

$$J\left(E_1^1\right)=\begin{bmatrix}\rho_1 & 0 & 0\\ 0 & -\alpha_2(0) & N\frac{a_2k_2}{2\gamma_2\rho_2}\alpha_2(0)\\ 0 & 0 & -\frac{a_2^2k_2^2}{4\gamma_2\rho_2}\alpha_2(0)\end{bmatrix}$$

whose eigenvalues are $\rho_1>0$, whereas the other two are negative provided that $\alpha_2(0)>0$, i.e. $Na_2<2\gamma_2\rho_2$, whereas if the reverse inequality holds the second component of the equilibrium becomes negative. Finally, at $E_1^2=\left(k_1,k_2\left(1-N\frac{a_2}{2\gamma_2\rho_2}\right),0\right)$ we have

$$J\left(E_1^2\right) = \begin{bmatrix} -\rho_1 & 0 & -N\frac{a_1 k_1}{2\gamma_1} \\ 0 & -\alpha_2(0) & \frac{a_2 N}{2\gamma_2}\frac{k_2}{\rho_2}\alpha_2(0) \\ 0 & 0 & \frac{a_1^2 k_1}{4\gamma_1} - \frac{a_2^2}{4\gamma_2}\frac{k_2}{\rho_2}\alpha_2(0) \end{bmatrix}$$

By the previous discussion, the first two eigenvalues are negative provided that $\alpha_2(0) > 0$, i.e. $Na_2 < 2\gamma_2\rho_2$. In this case the third eigenvalue is also negative whenever $\alpha_2(0) > \rho_2 \frac{a_1^2 k_1 \gamma_2}{a_2^2 k_2 \gamma_1}$ (> 0), which is equivalent to condition (3.15). ■

Proof of Proposition 3.2. From the definition of equilibrium we have that $X_i^* = k_i\left(1 - Nr_i^* \frac{a_i}{2\gamma_i\rho_i}\right) = \frac{k_i}{\rho_i}\alpha_i(r_i^*)$, where $r_1^* = r^*$ and $r_2^* = 1 - r^*$, i.e. $X_i^* > 0 \Leftrightarrow \alpha_i(r_i^*) > 0$. By solving inequalities $\alpha_i(r_i^*) > 0$, $i = 1, 2$ with the condition $1 > r^* > 0$, we get the different cases described in the Proposition. Concerning the stability, the Jacobian matrix evaluated at the inner equilibrium $E^* = (X_1^*, X_2^*, r^*)$ can be rewritten as:

$$J\left(E^*\right) = \begin{bmatrix} -\alpha_1(r^*) & 0 & -N\frac{a_1 X_1^*}{2\gamma_1} \\ 0 & -\alpha_2(r^*) & N\frac{a_2 X_2^*}{2\gamma_2} \\ r^*(1-r^*)\frac{a_1^2}{4\gamma_1} & -r^*(1-r^*)\frac{a_2^2}{4\gamma_2} & 0 \end{bmatrix}$$

Thus, at the equilibrium the Jacobian matrix has the structure

$$J\left(E^*\right) = \begin{bmatrix} J_{11} & 0 & J_{13} \\ 0 & J_{22} & J_{23} \\ J_{31} & J_{32} & 0 \end{bmatrix}$$

where the elements J_{11}, J_{22}, J_{13} and J_{32} are negative and J_{23} and J_{31} are positive. Therefore, for the characteristic polynomial

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

with

$$a_1 = J_{11} + J_{22}; \quad a_2 = -J_{23}J_{32} - J_{31}J_{13} + J_{11}J_{22}\ ;\ a_3 = J_{13}J_{22}J_{31} + J_{11}J_{23}J_{32}$$

satisfies the Routh-Hurwitz criterion, as:

$$a_1 > 0;\ a_2 > 0;\ a_3 > 0$$

and

$$a_1a_2 - a_3 = -J_{11}J_{22}\left(J_{11} + J_{22}\right) + J_{11}J_{13}J_{31} + J_{23}J_{32}\left(2J_{11} + J_{22}\right) > 0$$

Therefore whenever the equilibrium $E^* = (X_1^*, X_2^*, r^*)$ is feasible (i.e. it involves positive biomasses), it is also stable. ■

4 Does the “uptick rule” stabilize the stock market? Insights from Adaptive Rational Equilibrium Dynamics

In this Chapter we propose a nonlinear evolutionary model with application to finance.

4.1 Introduction

Short selling is the practice of selling financial instruments that have been borrowed, typically from a broker-dealer or an institutional investor, with the intent to buy the same class of financial instruments in a future period and return them back at the maturity of the loan.

By short selling, investors open a so-called “short position”, that is technically equivalent to holding a negative amount of shares of the traded asset, with the expectation that the asset will recede in value in the next period. At the closing time specified in the short selling contract, the debt is compounded with interest which occurred during the period of the financial operation, for this reason short sellers prefer to close the short position and reopen a new one with the same features, rather than extending the position over the closing time (see, e.g., [74]).

A short position is the counterpart of the (more conventional) "long position", i.e. buying a security such as a stock, commodity, or currency, with the expectation that the asset will rise in value.

Short selling is considered the father of the modern derivatives and, as such, it has a double function: it can be used as an insurance device, by hedging the risk of long positions in related stocks thus allowing risky financial operations, or for speculative purposes, to profit from an expected downward price movement. Moreover, financial speculators can sell short stocks in an effort to drive down the related price by creating an imbalance of sell-side interest, the so called "bear raid" action. This feedback may lead to the market collapse, and has indeed been observed during the financial crises of 1937 and 2007, see, e.g., [84].

Many national authorities have developed different kinds of short selling restrictions to avoid the negative effect of this financial practice[1]. Most of the regulations are based on "price tests", i.e., short selling is allowed or restricted depending on some tests based on recent price movements. The best known and most widely applied of such regulations is the so-called "*uptick rule*", or rule 10a-1, imposed in 1938 by the U.S. Securities and Exchange Commission[2] (hereafter SEC) to protect investors and was in force until 2007.

[1] It is worth mentioning that short sale restrictions are nearly as old as organized exchanges. The first short selling regulation was enacted in 1610 in the Amsterdam stock exchange. For a review of the history of short sale restrictions, see Short History of the Bear, Edward Chancellor, October 29, 2001, copyright David W. Tice and Co.

[2] The rule was originally introduced under the Securities Exchange Act of 1934.

This rule regulated short selling into all U.S. stock markets and in the Toronto Stock Exchange. Other financial markets, like the London Stock Exchange and the Tokyo Stock Exchange, have different or no short selling restrictions (for a summary of short sale regulations in approximately 50 different countries see [28]).

The uptick rule originally stated that short sales are allowed only on an uptick, i.e., at a price higher than the last reported transaction price. The rule was later relaxed to allow short sales to take place on a zero-plus-tick as well, i.e., at a price that is equal to the last sale price but only if the most recent price movement has been positive. Conversely, short sales are not permitted on minus- or zero-minus-ticks, subject to narrow exceptions[3].

In adopting the uptick rule, the SEC sought to achieve three objectives[4]:

(i) *allowing relatively unrestricted short selling in an advancing market;*

(ii) *preventing short selling at successively lower prices, thus eliminating short selling as a tool for driving the market down; and*

(iii) *preventing short sellers from accelerating a declining market by exhausting all remaining bids at one price level, causing successively lower prices to be established by long sellers.*

The last two objectives have been partially confirmed by the empirical analysis (see, e.g., [1] and reference therein). Instead, the regulation does not seem to be effective

[3] In the Canadian stock markets, the tick test was introduced under rule 3.1 of UMIR (Universal Market Integrity Rules). It prevents short sales at a price that is less than the last sale price of the security.

[4] Quoted from the Securities Exchange Act Release No. 13091 (December 21, 1976), 41 FR 56530 (1976 Release).

in producing the first desired effect. The observed number of executed short sales is indeed lower under uptick rule than in the unconstrained case, during phases with an upward market trend, see again [1]. This is due to the asynchrony between placement and execution of a short-sell order, since the rising of the price in between these two operations can make the trade not feasible under the uptick rule.

Moreover, empirical evidence provides uniform support of the idea that short selling restrictions often cause share prices to rise. From a theoretical point of view, there is no clear argument for explaining this mispricing effect of the uptick rule. According to [83] this is due to a reduction in stock supply owing to the short sale restriction. More generally, theoretical models with heterogeneous agents and differences in trading strategies support the idea that share values become overvalued under short selling restrictions due to the fact that "pessimistic" and "bear" traders (expecting negative price movements) are ruled out of the market (see, e.g., [63]). In contrast, theoretical models based on the assumption that all agents have rational expectations suggest that short selling restrictions do not change on the average the stock prices (see, e.g., [43]).

However, given the complexity of the phenomena, and the impossibility of isolating the effects of a regulation from other concomitant changes in the economic scenario, the effectiveness of the uptick rule in meeting the three above objectives, and its possible side effects on shares' prices, are still far from being completely clarified.

Guided by the aim to provide further insight on the argument, this Chapter studies the effects on share prices in an artificial market of a short selling restriction based on a tick test similar to the one imposed by the uptick rule in real financial markets. Using an artificial asset pricing model makes it easier in assessing the effects of the uptick rule in isolation from other exogenous shocks, though artificial modeling necessarily trades realism for mathematical tractability.

We consider an asset pricing model of adaptive rational equilibrium dynamics (A.R.E.D.), where heterogeneous beliefs on the future prices of a risky asset, together with traders' adaptability based on past performances, have shown to endogenously sustain price fluctuations. Asset pricing models of A.R.E.D. (hereafter referred to simply as ARED asset pricing models) are discrete-time dynamical systems based on the empirical evidence that investors with different trading strategies coexist in the financial market (see, e.g., [100]). These simple models provide a theoretical justification for many "stylized facts" observed in the real financial time series, such as, financial bubbles and volatility clustering (see [56], and [57]). Stochastic models based on the same assumptions are even used to study exchange rate volatility and the implication of some specific financial policies (see, e.g., [109]).

We extend, in particular, the deterministic model introduced in [30], where, in the simplest case, agents choose between two predictors of future prices of a risky asset, i.e. a fundamental predictor and a non-fundamental predictor. Agents that adopt the fundamental predictor are called *fundametalists*, while agents that adopt the non-fundamental predictor are called *noise traders* or non-fundamental traders. Fundamentalists believe that the price of a financial asset is determined by its fundamental value (as given by the present discounted value of the stream of future dividends, see [66]) and any deviation from this value is only temporary. Non-fundamental traders, sometimes called *chartists* or *technical analysts*, believe that the future price of a risky asset is not completely determined by fundamentals and it can be predicted by simple technical trading rules (see, e.g., [47], [85], and [88]).

In the model, agents revise their "beliefs", prediction to be adopted, according to an evolutionary mechanism based on the past realized profits. As a result, the fundamental value is a fixed point of the price dynamics, as, once there, both fundamentalists and non-fundamental traders predict the fundamental price. As long as the sensitivity of traders in switching to the best performing predictor is relatively low, the fundamental equilibrium is stable, but the fundamental stability is typically lost at higher intensities of the traders' choice across the predictors, making room for financial bubbles.

It is worth to remember that [30] investigated the peculiar case of zero supply of outside shares. Under this assumption each bought share is sold short. We therefore consider a positive supply of outside share, that is essential to ensure financial transactions when short selling is forbidden[5]. Moreover, we pair the fundamental predictor with first a technical linear predictor and then with a technical nonlinear predictor and compare the results obtained with and without the uptick rule.

As linear predictor, we consider the chartist predictor introduced in [30]. This facilitates the comparison of our results with those in [30] and related papers. As nonlinear predictor, we introduce a new predictor, "Smoothed Price Rate Of Change" or S-ROC predictor, that extrapolates future prices by applying the rate of change averaged on past prices with a confidence mechanism smoothing out extreme unrealistic rates (for an overview of this class of predictors, see [47]).

For what concerns the implementation of the regulation, we implement the uptick rule as it was in its original formulation, i.e., short selling is allowed only on an uptick. Note, however, that in an artificial asset pricing model a zero-tick is possible only at equilibrium, so that allowing or forbidding short sales on zero-plus-ticks makes basically

[5]We consider a positive supply of outside shares for the asset pricing model under Walrasian market clearing at each period. A similar model under the market maker scenario has been considered by [69].

no change in the observed price dynamics. In fact, with a positive supply of shares, traders take long positions at the fundamental equilibrium, so only the non-fundamental equilibria at which one type of trader is prohibited to go short are affected by the rule behavior on zero-plus-ticks (moreover, such equilibria are irrelevant to study the global price dynamics, as will be explained in Section 4.2.3).

From the mathematical point of view, the uptick rule makes the asset pricing model a piecewise-smooth dynamical system[6], namely a system in which different mathematical rules can be applied to determine the next price, and the rule to be applied depends on the current state of the system, that is, on the fact that trader types are interested in going short and whether short selling is allowed or not. Non-smooth dynamical systems are certainly more problematic to analyze, both analytically and numerically (though non-smooth dynamics is a very active topic in current research, see [42], and [36], and references therein) so we will limit the analytical treatment to stationary solutions.

Piecewise-smooth dynamical systems have already been used as models in finance. [107] proposed a one-dimensional piecewise-linear asset pricing model, where traders adopt different buying and selling strategies in response to different market movements. Other examples can be found in [105], [106], and [104]. Two ARED piecewise-smooth systems modeling short selling restrictions have been also proposed. Modifying the model in [30], [16] restricted short selling by allowing limited short positions at each trading period, whereas [41] investigated the complete ban on short selling. Thus both contributions implement short selling restrictions that are not based on price tests.

The results of our theoretical analyses are in line with the empirical evidence. The sale price that is established in our model when one trader type is prohibited from going

[6]To be precise, the model is a piecewise-continuous dynamical system. However, the class of piecewise-smooth dynamical systems contains the class of piecewise-continuous dynamical systems.

short is indeed systematically higher than the unconstrained price. Thus, constrained downward movements below the fundamental value are less pronounced, whereas constrained upward movements above the fundamental value can be larger. We provide a more complete explanation for this effect, suggesting that it is due to the combination of two mechanisms: on one side, the short selling restriction reduces the possibility for pessimistic or bear traders to bet on downward movements below the fundamental value, avoiding excessive underpricing, but at the same time, when prices are above the fundamental value, the restriction reduces the possibility for fundamentalists to drive down the prices back to the fundamental value by opening short positions. This is in agreement with the last two goals established by the SEC (see above). The first stated objective of the uptick is always realized in our model, since the market clearing is assumed to be synchronous among all traders.

When non-fundamental traders adopt the S-ROC predictor, we observe that the overpricing due to the uptick rule disappears due to the smoothness of the predictor that makes non-fundamental traders not confident with extreme price deviations from the fundamental value. Indeed, the expectations of large price deviations produced by the uptick rule force the non-fundamental trader to believe in the fundamental value with the effect of reducing, instead of increasing, the price deviations. The stabilizing effect however vanishes when traders become highly sensitive in switching to the strategy with best recent performance.

The Chapter is organized as follows. Section 4.2.1 briefly reviews the unconstrained asset pricing model, summarizing from [30] and setting the notation and most of the modeling equations that will be used in next Sections. Section 4.2.2 is also preliminary and recaps the concept of fundamental equilibrium, including its stability analysis and some new results. Section 4.2.3 formulates the piecewise-smooth model constrained by the uptick rule, and discusses the existence and stability of fundamental and non-fundamental equilibria. So far, no explicit price predictors is introduced, whereas Section 4.2.4 presents the price predictors for which the unconstrained and constrained models will be studied and compared in Sections 4.3 and 4.4. Section 4.3 presents the analytical results concerning the existence and stability of fixed points. Some of the results concerning the unconstrained model are new and interesting per se. Section 4.4 presents a series of numerical tests, confirming the analytical results and investigating non-stationary (periodic, quasi-periodic, and chaotic) regimes. In Section 4.4.3 we discuss in detail our economic findings. Section 4.5 concludes and lists a series of related interesting topics for further research. All the analytical results presented in Sections 4.2 and 4.3 are proved in Appendix 4.6.

4.2 The ARED asset pricing model with and without the uptick rule

We consider the asset pricing model with heterogeneous beliefs and adaptive traders introduced by [30]. While in the original model a zero supply of outside shares was considered, making short selling essential to ensure the exchanges, we consider the case of positive supply, so that short selling will no longer be necessary and a constraint on it can be imposed. In this generalized version of the original model, we introduce a negative demand constraint according to the uptick rule, in order to study the effects of this regulation on price fluctuations.

4.2.1 The unconstrained ARED asset pricing model

Consider a financial market where traders invest either in a single risky asset, supplied in S shares [7] of (ex-dividend) price p_t at period t, or in a risk free asset perfectly elastically supplied at gross return R (where $R = 1 + r$, with $r \in (0, 1)$). The risky asset pays random dividend $\tilde{y}_t$ in period t, where the divided process $\tilde{y}_t$ is IID (Identically Independently Distributed) with $E_t [\tilde{y}_{t+1}] = \bar{y}$ constant. Thus, denoting by $W_{h,t}$ the economic wealth of a generic trader of type h at the beginning of period t, and by $z_{h,t}$ the number of shares held by the trader in period t, we have the following wealth equation (or individual intertemporal budget constraint):

$$\tilde{W}_{h,t+1} = R(W_{h,t} - p_t z_{h,t}) + \tilde{p}_{t+1} z_{h,t} + \tilde{y}_{t+1} z_{h,t} = RW_{h,t} + (\tilde{p}_{t+1} + \tilde{y}_{t+1} - Rp_t) z_{h,t}, \quad (4.1)$$

where tilde denotes random variables, $W_{h,t} - p_t z_{h,t}$ is the amount of money invested in the risk free asset in period t and $\tilde{R}_{t+1} = \tilde{p}_{t+1} + \tilde{y}_{t+1} - Rp_t$ is the excess return per share realized at the end of the period.

Let $E_{h,t}, V_{h,t}$ denote the "beliefs" of investor of type h about the conditional expectation and conditional variance of wealth. They are assumed to be functions of past prices and dividends. We assume that each investor type is a myopic mean variance maximizer, so for type h the demand for shares $z_{h,t}$ solves

$$\max_{z_{h,t}} \left\{ E_{h,t} \left(\tilde{W}_{t+1} \right) - \frac{a}{2} V_{h,t} \left(\tilde{W}_{t+1} \right) \right\}$$

i.e.,

$$z_{h,t}(p_t) = \frac{E_{h,t}[\tilde{R}_{t+1}]}{aV_{h,t}[\tilde{R}_{t+1}]} = \frac{E_{h,t}[\tilde{p}_{t+1} + \tilde{y}_{t+1}] - Rp_t}{aV_{h,t}[\tilde{R}_{t+1}]},$$

[7] S is in fact the supply of traded assets in each period. Obviously when short selling is allowed assets are borrowed outside the pool of this S shares making the total supply higher than S.

where a is the risk aversion coefficient and p_t is to be determined by the market clearing between all demands and the supply S of shares. For simplicity (as done in [30], see [55], for an extension), we assume that traders have common and constant beliefs about the variance, i.e. $V_{h,t}\left[\tilde{R}_{t+1}\right] = \sigma^2$, $\forall h$, and common and correct beliefs about the dividend, i.e. $E_{h,t}\left[\tilde{y}_{t+1}\right] = E_t\left[\tilde{y}_{t+1}\right] = \bar{y}$, $\forall h$. Moreover, the number N of traders and S of supplied shares in each period (not considering the extra supply of shares due to short sales) are kept constant. Let H be the number of available "beliefs" or price predictors $E_{h,t}[\tilde{p}_{t+1} + \tilde{y}_{t+1}]$, $h = 1, \ldots, H$, each obtained at a cost C_h, and denote by $n_{h,t}$ the fraction of traders adopting predictor h in period t, the market clearing imposes

$$N \sum_{h=1}^{H} n_{h,t} z_{h,t}(p_t) = S, \quad z_{h,t}(p_t) = \frac{E_{h,t}[\tilde{p}_{t+1} + \tilde{y}_{t+1}] - Rp_t}{a\sigma^2}, \tag{4.2}$$

which is solved for p_t, thus obtaining

$$p_t = \frac{1}{R}\left(\sum_{h=1}^{H} n_{h,t} E_{h,t}[\tilde{p}_{t+1} + \tilde{y}_{t+1}] - a\sigma^2 \frac{S}{N}\right), \tag{4.3}$$

Let us substitute the expression of p_t in $z_{h,t}\left(p_t\right)$, $\forall h \in H$, to obtain the actual demands

$$z_{h,t} = \frac{1}{a\sigma^2}\left(E_{h,t}[\tilde{p}_{t+1} + \tilde{y}_{t+1}] - \sum_{k=1}^{H} n_{k,t} E_{k,t}[\tilde{p}_{t+1} + \tilde{y}_{t+1}]\right) + \frac{S}{N} \tag{4.4}$$

(no longer functions of the price p_t), and let us use p_t to calculate the net profits $R_t z_{h,t-1} - C_h$, $h = 1, \ldots, H$, realized in period t.

Eq. (4.4) gives the number of shares held by a trader of type h in period t. If negative, the trader is in a short position. If positive, the trader is in a long position.

At this point, the fractions $n_{h,t+1}$, $h = 1, ..., H$, for the next period are determined as functions of the positions of the traders and of the last available net profits. In particular, the following discrete choice model is used:

$$n_{h,t+1} = \frac{\exp\left(\beta(R_t z_{h,t-1} - C_h)\right)}{\sum_{k=1}^{H} \exp\left(\beta(R_t z_{k,t-1} - C_k)\right)}, h = 1, ..., H-1, \tag{4.5}$$

where β measures the intensity of traders' choice across predictors (traders' adaptability). The above procedure can then be iterated to compute the next price p_{t+1}.

If all agents have common beliefs on the future prices, i.e. $E_{h,t} = E_t \; \forall h$, the pricing equation (4.3) reduces to

$$Rp_t = E_t[\tilde{p}_{t+1} + \tilde{y}_{t+1}] - a\sigma^2 \frac{S}{N}.$$

This equation admits a unique solution $\tilde{p}_t^* \equiv \bar{p}$, where

$$\bar{p} = \frac{\bar{y} - a\sigma^2 S/N}{R-1}, \tag{4.6}$$

that satisfies the "no bubbles" condition $\lim_{t\to\infty}\left(E\tilde{p}_t^*/R^t\right) = 0$. This price, given as the discounted sum of expected future dividends, would prevail in a perfectly rational world and is called the *fundamental price* (see, e.g., [66, 69]). Of course, we assume $\bar{p} > 0$, i.e., sufficiently high dividend $\bar{y}$ or limited supply of ouside shares per investor S/N. Given the assumptions about the dividend process and the fundamental price and focusing only on the deterministic skeleton of the model, i.e. $\tilde{y}_t = \bar{y} \; \forall t$, we have that $E_{h,t}[\tilde{p}_{t+1} + \tilde{y}_{t+1}] = E_h[p_{t+1}] + \bar{y}$, where the price predictors $E_h[p_{t+1}]$, $h = 1, \ldots, H$, are deterministic functions of L known past prices $\{p_{t-1}, p_{t-2}, \ldots, p_{t-L}\}$, $L \geq 1$.

It is useful to rewrite the model in terms of price deviations from a benchmark price $\bar{p}$. In the following, let $s = S/N$ and denote by x_t the price deviation from the fundamental value, i.e., $x_t = p_t - \bar{p}$. Defining the traders' beliefs on the next deviation x_{t+1} as

$f_h(\mathbf{x}_t) = E_h[p_{t+1}] - \bar{p}$, with $\mathbf{x}_t = (x_{t-1}, x_{t-2}, \ldots, x_{t-L})$ being the vector of the last L available deviations, the demand functions to be used in the market clearing in Eq. (4.2) become

$$z_{h,t}(x_t) = \frac{f_h(\mathbf{x}_t) - Rx_t}{a\sigma^2} + s, \tag{4.7}$$

while the pricing equation (4.3) and the actual demands (4.4) can be written in deviations as

$$x_t = \frac{1}{R}\sum_{h=1}^{H} n_{h,t} f_h(\mathbf{x}_t) \quad \text{and} \quad z_{h,t} = \frac{1}{a\sigma^2}\left(f_h(\mathbf{x}_t) - \sum_{k=1}^{H} n_{k,t} f_k(\mathbf{x}_t)\right) + s, \tag{4.8}$$

and the excess of return in (4.5) can be expressed in deviations as

$$R_t = x_t - Rx_{t-1} + \delta_t + a\sigma^2 s. \tag{4.9}$$

where $\delta_t = y_t - \bar{y}$ is a shock due to the dividend realization. As mentioned above, we focus on the deterministic skeleton of the model, i.e., we fix $\delta_t = 0\ \forall t$.

Substituting Eqs. (4.7) and (4.9) into (4.5) and coupling the pricing equation in (4.8) with (4.5), the ARED model can be rewritten as

$$x_t = \frac{1}{R}\sum_{h=1}^{H} n_{h,t} f_h(\mathbf{x}_t), \tag{4.10a}$$

$$n_{h,t+1} = \frac{\exp\left(\beta\left((x_t - Rx_{t-1} + a\sigma^2 s)\left(\frac{f_h(\mathbf{x}_{t-1}) - Rx_{t-1}}{a\sigma^2} + s\right) - C_h\right)\right)}{\sum_{k=1}^{H}\exp\left(\beta\left((x_t - Rx_{t-1} + a\sigma^2 s)\left(\frac{f_k(\mathbf{x}_{t-1}) - Rx_{t-1}}{a\sigma^2} + s\right) - C_k\right)\right)},\ h = 1, \ldots, H-1 \tag{4.10b}$$

(recall that $\sum_{h=1}^{H} n_{h,t} = 1$). Given the current composition $n_{h,t}$ of the traders' population, the first equation computes the price deviation for period t, while the second updates the traders' fractions for the next period. The past deviations $(x_{t-1}, x_{t-2}, \ldots, x_{t-(L+1)})$ appearing in vectors $\mathbf{x}_t$ and $\mathbf{x}_{t-1}$, together with the fractions $n_{h,t}$, $h = 1, \ldots, H-1$, constitute the state of the system[8].

The initial condition is composed of the opening price deviation x_0 and of the traders' fractions $n_{h,1}$, $h = 1, ..., H$, to be used in the first period. In fact, assuming that each price predictor can be customized to the case when the number of past available prices is less then L, then Eq. (4.10a) can be applied at $t = 1$ (to determine the price deviation x_1 in period 1), whereas Eq.s (4.10b) can only be applied at $t = 2$, so that $n_{h,2} = n_{h,1}$ is used. For $t > L$ the price predictors in Eqs. (4.10) can be regularly applied.

Note that Eq. (4.10a) guarantees a positive price for any period t (i.e., $x_t > -\bar{p}$), provided all price predictions are such ($f_h(\mathbf{x}_t) > -\bar{p}$ for all $h = 1, \ldots, H$).

When there are only two types of traders, $H = 2$, it is convenient to express the fractions $n_{1,t}$ and $n_{2,t}$ as a function of $m_t = n_{1,t} - n_{2,t} \in (-1, 1)$, i.e.,

$$n_{1,t} = \frac{1 + m_t}{2} \quad \text{and} \quad n_{2,t} = \frac{1 - m_t}{2}. \tag{4.11}$$

In this specific case, model (4.10) can be rewritten as

$$x_t = \frac{1}{2R}\left((1 + m_t) f_1(\mathbf{x}_t) + (1 - m_t) f_2(\mathbf{x}_t)\right), \tag{4.12a}$$

$$m_{t+1} = \tanh\left(\frac{\beta}{2}\left((x_t - Rx_{t-1} + a\sigma^2 s)\frac{f_1(\mathbf{x}_{t-1}) - f_2(\mathbf{x}_{t-1})}{a\sigma^2} - (C_1 - C_2)\right)\right), \tag{4.12b}$$

where $(x_{t-1}, x_{t-2}, \ldots, x_{t-(L+1)}, m_t)$ is the system's state and (x_0, m_1) identifies the initial condition.

[8]Note that, by writing Eq. (4.10b) for $n_{h,t}$ and substituting it into Eq. (4.10a), one can write x_t as a recursion on the last $L + 2$ deviations. This gives a more compact and homogeneous system's state ($L + 2$ price deviations instead of $L + 1$ deviations and $H - 1$ traders' fractions), however, the formulation (4.10) is physically more appropriate and easier to initialize.

4.2.2 The fundamental equilibrium

The following lemma gives the condition under which the fundamental price is an equilibrium of model (4.10) (or model (4.12) when $H = 2$):

Lemma 4.1 *If all predictors satisfy $f_h(\mathbf{0}) = 0$, $h = 1, \ldots, H$, with $\mathbf{0}$ the vector of L zeros, then $(\bar{x}^{(0)}, \bar{n}_h^{(0)})$ with*

$$\bar{x}^{(0)} = 0 \quad \textit{and} \quad \bar{n}_h^{(0)} = \frac{\exp\left(-\beta C_h\right)}{\sum_{k=1}^{H} \exp\left(-\beta C_k\right)}$$

[or $(\bar{x}^{(0)}, \bar{m}^{(0)})$ with $\bar{m}^{(0)} = \tanh\left(-\beta/2\left(C_1 - C_2\right)\right)$ if $H = 2$] is a fixed point of model (4.10) *[*(4.12)*], at which all strategies equally demand $\bar{z}_h^{(0)} = s$. We call this steady state* fundamental equilibrium*. H of the associated eigenvalues are zero and the remaining L ones are the roots of the characteristic equation*

$$\lambda^L - \gamma_1 \lambda^{L-1} + \cdots - \gamma_L = 0, \quad \gamma_i = \frac{1}{R} \sum_{h=1}^{H} \bar{n}_h \frac{\partial}{\partial x_{t-i}} f_h(\mathbf{x}_t) \bigg|_{\mathbf{x}_t = \mathbf{0}}, \quad i = 1, \ldots, L.$$

Lemma 4.1 also reveals that the price dynamics is not reversible, at least locally to the fundamental equilibrium (due to the presence of zero eigenvalues), so that prices cannot be reconstructed backward in time.

We now state a simple condition that rules out the possibility of other equilibria:

Lemma 4.2 *If $f_h(\bar{x}\mathbf{1})/\bar{x} < R$ [or if $f_h(\bar{x}\mathbf{1})/\bar{x} > R$] for all $h = 1, \ldots, H$ and $\bar{x} \neq 0$, with* $\mathbf{1}$ *the vector of L ones, then the fundamental equilibrium is the only fixed point of model* (4.10)*.*

As we will recall in Section 4.2.4, traders with price predictors such that $|f_h(\bar{x}\mathbf{1})/\bar{x}| < 1$ believe that tomorrow's price will revert to its fundamental value ($x_t \to 0$), whereas, at an equilibrium, trend followers obviously extrapolate the equilibrium price, so their price predictors are such that $f_h(\bar{x}\mathbf{1})/\bar{x} = 1$. Lemma 4.2 therefore shows that non-fundamental equilibria are possible only in the presence of at least one of the two mentioned types of traders and traders that believe that nonzero price deviations will amplify in the short run, even if they have been recently constant.

4.2.3 The ARED asset pricing model constrained by the uptick rule

When trading restrictions imposed by the uptick-rule are introduced, we must distinguish between two situations: if prices are rising, e.g. we simply look at the last movement available $x_{t-1} - x_{t-2}$, then short selling is allowed and the unconstrained model (4.10) still applies. In contrast, in a downward (or stationary) movement ($x_{t-1} \leq x_{t-2}$), traders' demands are forced to be non-negative, i.e., the demand functions to be used in the market clearing in Eq. (4.2) are

$$z_{h,t}(x_t) = \max\left\{0, \frac{f_h(\mathbf{x}_t) - Rx_t}{a\sigma^2} + s\right\}. \tag{4.13}$$

Note that the forward dynamics remain uniquely defined. In fact, given the past price deviations in $\mathbf{x}_t$ and the traders' fractions $n_{h,t}$, the per capita demand $d(x_t) = \sum_{h=1}^{H} n_{h,t} z_{h,t}(x_t)$ is a piecewise-linear, continuous function of the deviation x_t, that is decreasing up to the deviation at which it vanishes together with the highest of the single agents' demand curves, and $d(x_t) = 0$ for larger deviations (see Figure (4.1)). There is therefore a unique deviation x_t at which the market clears, i.e., $d(x_t) = s > 0$.

Also note that, given the same traders' fractions, the constrained price[9] is higher than the unconstrained price[10] ($x_t > x_{U,t}$), so that positive predictions ($f_h(\mathbf{x}_t) > -\bar{p}$ for all $h = 1, \ldots, H$) still yield positive prices ($x_t > -\bar{p}$).

When solving Eq. (4.2) for x_t with the constrained demands (4.13), $2^H - 1$ cases must be further distinguished, depending on which of the optimal demands in (4.7) are forced to zero by (4.13) (obviously not all demands can vanish). The uniqueness of forward dynamics guarantees that only one of the cases clears the market.

For simplicity, hereafter we will only consider the case with two types of traders ($H = 2$), so one of the following three cases is realized at each period:

0: no trader is prohibited from going short (equivalently, both types of traders hold nonnegative amounts of shares in period t), i.e.,
(4.7) implies $z_{1,t} \geq 0$ and $z_{2,t} \geq 0$, with $x_t = \dfrac{1}{R}\left(n_{1,t}\, f_1(\mathbf{x}_t) + n_{2,t}\, f_2(\mathbf{x}_t)\right)$,

1: traders of type 1 are prohibited from going short (only traders of type 2 hold shares in period t), i.e.,
(4.7) implies $z_{1,t} < 0$ and $z_{2,t} > 0$, with $x_t = \dfrac{1}{R}\left(f_2(\mathbf{x}_t) - a\sigma^2 s \dfrac{n_{1,t}}{n_{2,t}}\right)$,

[9] In this part of the Chapter we introduce the notation $x_{U,t}$ for the unconstrained price determined by model in Section 4.2.1 to distinguish it from the constrained price x_t. Since there is no risk of confusion, this distinction is not made in other parts of the Chapter for the sake of avoiding cumbersome notations.

[10] It is clear that all the traders' demand functions are always characterized by the same slope. However, the intercept of the demands with the $x = 0$ axis, i.e. $s + \frac{f_h(\mathbf{x}_t)}{a\sigma^2}$, changes over time and it depends on the past share prices. For example if the predictor of a trader is based on L past prices of the share, i.e. $\mathbf{x}_t = (x_{t-1}, ..., x_{t-L})$, the intercept of its demand with the $x = 0$ axis depends on all of these prices. Thus, to prove that the constrained prices are always higher than the unconstrained one given the same past prices, it does not necessarily mean a price dynamic characterized by larger fluctuations for the constrained model. The situation can be the opposite when we consider predictors based on a large number of past deviations and especially when they are non-linear. In other words, simple static considerations on the shape of the constrained demands do not help us to understand entirely the effect of the uptick rule on the price dynamics.

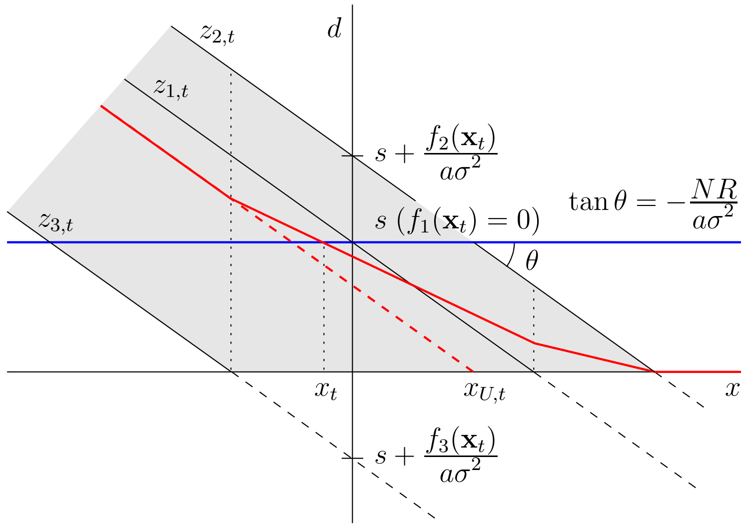

Figure 4.1: Per-capita demand (d, red) and supply (s, blue) curves as functions of the price deviation x to be realized in period t. A case with $H = 3$ types of traders is sketched, where $z_{1,t}$ is the demand of ”fundamentalists” (see Sect. 4.2.4), while $z_{2,t}$ and $z_{3,t}$ are the demands of ”non-fundamentalists” ($f_2(\mathbf{x}_t) > 0$, $f_3(\mathbf{x}_t) < 0$, and negative (dashed) demands are obtained with (4.7)). The resulting per capita demand curve is piecewise-linear, continuous, and decreasing. Depending on the traders' fractions ($n_{1,t}$, $n_{2,t}$, $n_{3,t}$), it can take on different configurations in the shaded area (the most negative of which corresponds to $z_{3,t}$ when $n_{3,t} = 1$; the case shown corresponds to $n_{1,t} = n_{2,t} = n_{3,t} = 1/3$). At the unconstrained price $x_{U,t}$ traders of type 3 are in a short position in period t.

2: traders of type 2 are prohibited from going short (only traders of type 1 hold shares in period t), i.e.,

(4.7) implies $z_{1,t} > 0$ and $z_{2,t} < 0$, with $x_t = \frac{1}{R}\left(f_1(\mathbf{x}_t) - a\sigma^2 s \frac{n_{2,t}}{n_{1,t}}\right)$.

Then, m_{t+1} must be computed (see (4.11)) and, again, there are three cases, depending on the signs of the optimal demands at period $(t-1)$ (see Eq. (4.5)). In order to simplify the model formulation, we prefer to enlarge the system's state, by including the traders' demands $z_{h,t-1}$ realized in period $(t-1)$, $h = 1, 2$, in lieu of the farthest price deviation

$x_{t-(L+1)}$. The state variables therefore are

$$(x_{t-1}, x_{t-2}, \ldots, x_{t-L},\ z_{1,t-1},\ z_{2,t-1},\ m_t) \quad \text{if} \quad L \geq 2, \tag{4.14a}$$

$$(x_{t-1}, x_{t-2},\ z_{1,t-1},\ z_{2,t-1},\ m_t) \quad \text{if} \quad L = 1, \tag{4.14b}$$

as we need x_{t-2} to apply the uptick rule, and we can update the traders' fractions by simply replacing (4.12b) with

$$m_{t+1} = \tanh\left(\frac{\beta}{2}\left((x_t - Rx_{t-1} + a\sigma^2 s)(z_{1,t-1} - z_{2,t-1}) - (C_1 - C_2)\right)\right). \tag{4.15}$$

The uptick rule makes the ARED model piecewise smooth, namely the space of the state variables is partitioned into three regions associated with different equations for updating the system's state (see [42] and references therein). By defining the regions

$$\begin{aligned} U\ &:\ x_{t-1} > x_{t-2}, \\ Z_0 &:\ x_{t-1} \leq x_{t-2}, \quad z_{1,t} \geq 0, \quad z_{2,t} \geq 0, \\ Z_1 &:\ x_{t-1} \leq x_{t-2}, \quad z_{1,t} < 0, \quad z_{2,t} > 0, \\ Z_2 &:\ x_{t-1} \leq x_{t-2}, \quad z_{1,t} > 0, \quad z_{2,t} < 0, \end{aligned} \tag{4.16a}$$

where

$$z_{1,t} = \frac{1 - m_t}{2}\,\frac{f_1(\mathbf{x}_t) - f_2(\mathbf{x}_t)}{a\sigma^2} + s \quad \text{and} \quad z_{2,t} = \frac{1 + m_t}{2}\,\frac{f_2(\mathbf{x}_t) - f_1(\mathbf{x}_t)}{a\sigma^2} + s \tag{4.16b}$$

are the optimal demands from (4.8), we can write the forward dynamics as follows:

$$x_t = \begin{cases} \frac{1}{2R}\left((1+m_t)f_1(\mathbf{x}_t)+(1-m_t)f_2(\mathbf{x}_t)\right) & \text{if} \quad (\mathbf{x}_t, m_t) \in U \cup Z_0, \\ \frac{1}{R}\left(f_2(\mathbf{x}_t) - a\sigma^2 s \frac{1+m_t}{1-m_t}\right) & \text{if} \quad (\mathbf{x}_t, m_t) \in Z_1, \\ \frac{1}{R}\left(f_1(\mathbf{x}_t) - a\sigma^2 s \frac{1-m_t}{1+m_t}\right) & \text{if} \quad (\mathbf{x}_t, m_t) \in Z_2, \end{cases} \tag{4.17a}$$

$$z_{1,t} = \begin{cases} \frac{1-m_t}{2}\frac{f_1(\mathbf{x}_t)-f_2(\mathbf{x}_t)}{a\sigma^2}+s & \text{if} \quad (\mathbf{x}_t, m_t) \in U \cup Z_0, \\ 0 & \text{if} \quad (\mathbf{x}_t, m_t) \in Z_1, \\ \frac{2s}{1+m_t} & \text{if} \quad (\mathbf{x}_t, m_t) \in Z_2, \end{cases} \tag{4.17b}$$

$$z_{2,t} = \begin{cases} \frac{1+m_t}{2}\frac{f_2(\mathbf{x}_t)-f_1(\mathbf{x}_t)}{a\sigma^2}+s & \text{if} \quad (\mathbf{x}_t, m_t) \in U \cup Z_0, \\ \frac{2s}{1-m_t} & \text{if} \quad (\mathbf{x}_t, m_t) \in Z_1, \\ 0 & \text{if} \quad (\mathbf{x}_t, m_t) \in Z_2, \end{cases} \tag{4.17c}$$

$$m_{t+1} = \tanh\left(\frac{\beta}{2}\left((x_t - Rx_{t-1} + a\sigma^2 s)(z_{1,t-1} - z_{2,t-1}) - (C_1 - C_2)\right)\right). \tag{4.17d}$$

The same model can be rewritten in compact notations as follows:

$$(x_t, z_{1,t}, z_{2,t}, m_{t+1}) = \begin{cases} G_1\left(\mathbf{x}_t, z_{1,t-1}, z_{2,t-1}, m_t\right) & \text{if} \quad (\mathbf{x}_t, m_t) \in U \cup Z_0, \\ G_2\left(\mathbf{x}_t, z_{1,t-1}, z_{2,t-1}, m_t\right) & \text{if} \quad (\mathbf{x}_t, m_t) \in Z_1, \\ G_3\left(\mathbf{x}_t, z_{1,t-1}, z_{2,t-1}, m_t\right) & \text{if} \quad (\mathbf{x}_t, m_t) \in Z_2, \end{cases} \tag{4.18}$$

where G_1, G_2 and G_3 are three systems that define the asset pricing model with uptick

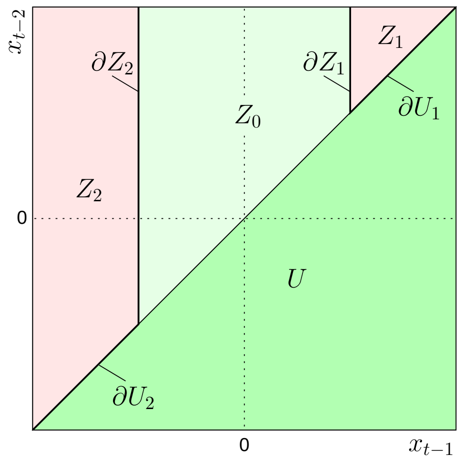

Figure 4.2: Partition of the state space into the three regions $U \cup Z_0$ (green) and Z_h, $h = 1, 2$ (pink): the projection on the (x_{t-1}, x_{t-2}) space in the case where traders use the fundamental and chartist predictors (see Section 4.2.4).

rule.

Region $U \cup Z_0$ is separated from region Z_h, $h = 1, 2$, by the two boundaries

$$\partial U_h : \; x_{t-1} = x_{t-2}, \; z_{h,t} \leq 0 \quad \text{and} \quad \partial Z_h : \; x_{t-1} \leq x_{t-2}, \; z_{h,t} = 0 \tag{4.19}$$

(see Figure (4.2), where a projection on the (x_{t-1}, x_{t-2}) space is shown). Across boundary ∂U_k the system is discontinuous, i.e., the corresponding expressions on the right-hand sides of (4.17a–c) assume different values on ∂U_k. In contrast, the system is continuous (but not differentiable) at the boundaries ∂Z_h.

Similarly to the unconstrained model (4.10), the initial condition of model (4.17) is set by the opening price deviation x_0 and the traders' initial composition m_1. However, Eq. (4.17d) also requires the traders' demand $z_{1,0}$ and $z_{2,0}$ which can be conventionally set at s.

Finally, let's discuss the fixed points of model (4.17) (or equivalently (4.18)), which necessarily lie on the boundary of region U and are still denoted with the pair $(\bar{x}, \bar{m})$ (the equilibrium demands, which also characterize the equilibria of model (4.17), can be obtained from Eqs. (4.17b,c)). Each of the three systems defining model (4.17), system G_1 (equivalent to model (4.10) adding the demands as state variables) which defines the dynamics of the model in region $U \cup Z_0$ and the two systems, G_1 and G_2, which define the dynamics of the model in regions Z_1 and Z_2, respectively, have their own fixed points, which we call either *admissible* or *virtual* according to the region of the state space in which they are. In this Chapter, a generic fixed point or equilibrium of system G_1 is called admissible if it lies in region $U \cup Z_0$ and virtual otherwise, a generic fixed point of system G_2 is called admissible if it lies in region Z_1 and virtual otherwise, and a generic fixed point of system G_3 is called admissible if it lies in region Z_2 and virtual otherwise. Virtual fixed points are not equilibria of model (4.17), but tracking their position is useful in the analysis.

The fundamental equilibrium is always an admissible fixed point of system G_1, also called the unconstrained system because equivalent to model (4.10) adding the demands as state variables. Indeed, it lies on the boundary between regions U and Z_0, with positive demands (equal to s), i.e., it is always an interior point of region $U \cup Z_0$. Its local stability is therefore ruled by Lemma 4.1 (the storage of the previous demands in lieu of the farthest past deviation brings the number of zero eigenvalues to $2H - 1$, $2H$ if $L = 1$, see (4.14b)), while the existence of (admissible or virtual) non-fundamental equilibria of the unconstrained system G_1 is ruled by Lemma 4.2.

The fixed points of the other two systems, G_2 and G_3, are of little interest. They lie on the boundaries ∂U_2 and ∂U_1, respectively, across which model (4.17) is discontinuous. Hence, there are arbitrarily small perturbations from the fixed point entering region U, for which the unconstrained system G_1 will map the system's state far from the fixed point. The fixed points of the two systems G_2 and G_3 are therefore (highly) unstable and will not be considered in the analysis.

4.2.4 Classical price predictors

In this Section we briefly introduce the price predictors used in this Chapter (see, e.g., [47], [33], and [34], for a more complete survey of the most classical types of price predictors used in the literature). The first one, $f_1(\mathbf{x}_t)$, called fundamental predictor, will be paired with each of the others, the non-fundamental predictors, in the analysis of Sections 4.3 and 4.4. For this reason, all non-fundamental predictors will be denoted by $f_2(\mathbf{x}_t)$.

Fundamental predictor

Fundamental traders, or *fundamentalists*, believe that prices return to their fundamental value. The simplest fundamental prediction is the fundamental price for period $t+1$, irrespectively of the recent trend:

$$E_1[p_{t+1}] = \bar{p}, \quad f_1(\mathbf{x}_t) = 0. \tag{4.20}$$

More generally, fundamentalists believe that prices will revert to the fundamental value by a factor v at each period:

$$E_1[p_{t+1}] = \bar{p} + v(p_{t-1} - \bar{p}), \quad f_1(\mathbf{x}_t) = v x_{t-1}, \quad 0 \leq v < 1, \tag{4.21}$$

the smaller is v, the highest is the expected speed of convergence to the fundamental price.

As in [30], we assume that "training" costs must be borne to obtain enough "understanding" of how markets work in order to believe in the fundamental price, so fundamentalists incur into a cost $C_1 > 0$ at each prediction.

Chartist predictor

The second type of simple trader that we consider is called *chartist* or *trend chaser*. This type of trader believes that any mispricing will continue, i.e. the chartist predictor is formally equivalent to predictor (4.21):

$$E_2[p_{t+1}] = \bar{p} + g(p_{t-1} - \bar{p}), \quad f_2(\mathbf{x}_t) = g x_{t-1}, \quad g > 1, \tag{4.22}$$

but amplifies, instead of damping, nonzero price deviations from the fundamental.

The chartist prediction is not costly.

Rate of change (ROC) predictor

The third type of simple trader that we consider is called "*nonlinear technical analyst*" or "*ROC trader*".

The ROC (*"Price Rate Of Change"*) is a nonlinear prediction which applies the price rate of change averaged over the last $L-1$ periods,

$$\mathrm{ROC} = \left(\frac{p_{t-1}}{p_{t-L}}\right)^{\frac{1}{L-1}} = \left(\frac{\bar{p}+x_{t-1}}{\bar{p}+x_{t-L}}\right)^{\frac{1}{L-1}}, \quad L \geq 2, \tag{4.23a}$$

to p_{t-1} twice to extrapolate p_{t+1}:

$$E_2[p_{t+1}] = p_{t-1}\mathrm{ROC}^2, \quad f_2(\mathbf{x}_t) = (\bar{p}+x_{t-1})\mathrm{ROC}^2 - \bar{p}. \tag{4.23b}$$

The ROC predictor is typically "smoothed" to avoid extreme rates of change (rates that are either too high or too close to zero, see Smoothed-ROC or S-ROC predictors in [47]). This interprets the traders' rationality that makes them diffident with extreme rates. We adopt in particular the confidence mechanism introduced in [41], where the ROC is combined with the last available price. Precisely, the price rate of change to be applied is a convex combination of the actual ROC (4.23a) and the unitary rate (corresponding to the last available price), with the ROC weight α_{ROC} that vanishes when the ROC attains extreme values (zero and infinity):

$$\begin{aligned} E_2[p_{t+1}] &= p_{t-1}\left(\alpha_{\mathrm{ROC}}\mathrm{ROC} + (1-\alpha_{\mathrm{ROC}})\right)^2, \\ f_2(\mathbf{x}_t) &= (\bar{p}+x_{t-1})\left(\alpha_{\mathrm{ROC}}\mathrm{ROC} + (1-\alpha_{\mathrm{ROC}})\right)^2 - \bar{p}. \end{aligned} \tag{4.23c}$$

The function

$$\alpha_{\mathrm{ROC}} = \frac{2}{\mathrm{ROC}^{\alpha} + \mathrm{ROC}^{-\alpha}} \tag{4.23d}$$

has been used in the analysis, where the parameter $1/\alpha$ measures how confident traders are with extreme rates.

The ROC predictor and the S-ROC predictor are not costly.

4.3 The effect of the uptick rule on shares price fluctuations: Analytical results

In this Section we report the stability analysis of fundamental and non-fundamental equilibria of models (4.12) and (4.17) for two pairs of traders' types. As traditionally done in the literature, type 1 is always the fundamental type (price predictor (4.21)), while type two is either the chartist in Sect. 4.3.1 (predictor (4.22)) or the nonlinear technical analyst (ROC trader) in Sect. 4.3.2 (predictor (4.23a,c,d)).

4.3.1 Fundamentalists vs chartists

Consider models (4.12) and (4.17) with predictors (4.21) and (4.22). Model (4.12, 4.21, 4.22) is the classical ARED model, proposed and fully analyzed in [30] for the case of zero supply of outside shares, i.e., $s = 0$, where short selling is intrinsically practiced at each trading period. The case with positive supply is analyzed in [16], where the effects of a negative bound on the traders' positions are also investigated.

Without any constraint on short selling, the existence and stability of the fixed points of model (4.12, 4.21, 4.22) are defined in the following lemma:

Lemma 4.3 *The following statements hold true for the dynamical system* (4.12) *with predictors* (4.21) *and* (4.22)*:*

1. *For* $1 < g < R$ *the fundamental equilibrium* $(0, \bar{m}^{(0)})$ *(see Lemma 4.1) is the only fixed point and is globally stable.*

2. *For $R < g < 2R - v$ there are the following possibilities:*

(a) *For* $0 \leq \beta < \beta_{\mathrm{LP}} = \dfrac{1}{C}\log\left(\dfrac{R-v}{g-R}\right)\left(1+\dfrac{a\sigma^2 s^2}{4C}\dfrac{g-v}{R-1}\right)^{-1} > 0$ *the fundamental equilibrium is the only fixed point and is stable.*

(LP) *At* $\beta = \beta_{\mathrm{LP}}$ *two equilibria appear (as* β *increases) at*

$$\bar{x}_{\mathrm{LP}} = \frac{a\sigma^2 s}{2(R-1)} > 0, \quad \bar{m} = 1 - 2\frac{R-v}{g-v},$$

through a saddle-node bifurcation (limit point, LP*).*

(b) *For* $\beta_{\mathrm{LP}} < \beta < \min\left\{\beta_{\mathrm{TR}} = \dfrac{1}{C}\log\left(\dfrac{R-v}{g-R}\right), \beta_{\mathrm{NS}}^{(+)}\right\} > \beta_{\mathrm{LP}}$ *the fundamental equilibrium is locally (asymptotically) stable and coexists with the two non-fundamental equilibria* $(\bar{x}^{(\pm)}, \bar{m})$*, with*

$$\bar{x}^{(\pm)} = \bar{x}_{\mathrm{LP}} \pm \sqrt{\left(\bar{x}_{\mathrm{LP}}^2 + \frac{a\sigma^2 C}{(R-1)(g-v)}\right)\left(1-\frac{\beta_{\mathrm{LP}}}{\beta}\right)}.$$

Equilibrium $(\bar{x}^{(+)}, \bar{m})$ *is locally (asymptotically) stable, whereas* $(\bar{x}^{(-)}, \bar{m})$ *is a saddle with 2-dimensional stable manifold separating the basins of attraction of the two stable equilibria.*

(TR) *At* $\beta = \beta_{\mathrm{TR}}$*,* $(\bar{x}^{(-)}, \bar{m})$ *collides and exchanges stability with the fundamental equilibrium (transcritical bifurcation,* TR*). The fundamental equilibrium is always at least locally (asymptotically) stable for* $\beta < \beta_{\mathrm{TR}}$ *and it is always unstable for* $\beta > \beta_{\mathrm{TR}}$*.*

(NS$^{(+)}$) *At* $\beta = \beta_{\mathrm{NS}}^{(+)}$ *the equilibrium* $(\bar{x}^{(+)}, \bar{m})$ *undergoes a Neimark-Sacker (NS) bifurcation. No explicit expression is available for* $\beta_{\mathrm{NS}}^{(+)}$*, but* $\beta_{\mathrm{TR}} \lesseqgtr \beta_{\mathrm{NS}}^{(+)}$ *if*

$$\beta_{\text{TR}} \lesseqgtr \frac{1}{a\sigma^2 s^2}\frac{(R-1)^2}{(g-R)(R-v)}.$$

(c) *For* $\beta_{\text{TR}} < \beta < \beta_{\text{NS}}^{(-)} > \beta_{\text{TR}}$ *the fundamental equilibrium is a saddle, with 2-dimensional stable manifold separating the positive from the negative dynamics, and the equilibrium* $(\bar{x}^{(-)}, \bar{m})$*, with* $\bar{x}^{(-)} < 0$*, is stable.*

(NS$^{(-)}$) *At* $\beta = \beta_{\text{NS}}^{(-)}$ *the equilibrium* $(\bar{x}^{(-)}, \bar{m})$ *undergoes a Neimark-Sacker bifurcation.*

3. For $g > 2R - v$ *there are the following possibilities:*

(a) *For* $0 \leq \beta < \beta_{\text{NS}}^{(\pm)}$ *the fundamental equilibrium is unstable and the equilibria* $(\bar{x}^{(\pm)}, \bar{m})$ *(*$\bar{x}^{(+)} > 0$ *and* $\bar{x}^{(-)} < 0$*) are stable.*

(NS) *At* $\beta = \beta_{\text{NS}}^{(\pm)}$ *the equilibria* $(\bar{x}^{(\pm)}, \bar{m})$ *undergo a Neimark-Sacker bifurcation.*

4. For $g > R^2$ *the dynamics can be unbounded for sufficiently large* β*.*

Lemma 4.3 generalizes Lemmas 2, 3, and 4 in [30] to the case $s > 0$, $0 < v < 1$, and Proposition 3.1 in [16]. In particular, for $s = 0$, note that the saddle-node and transcritical bifurcations concomitantly occur (case 2) at a so-called pitchfork bifurcation, whereas the mechanism making the fundamental equilibrium unstable is different for $s > 0$. First, the two non-fundamental equilibria $(\bar{x}^{(\pm)}, \bar{m})$ appear (as the traders' adaptability β increases) through the saddle-node bifurcation, and as β increases further a transcritical bifurcation occurs in which the saddle $(\bar{x}^{(-)}, \bar{m})$ exchanges stability with the fundamental equilibrium. Thus, for $\beta_{\text{LP}} < \beta < \beta_{\text{TR}}$, the fundamental equilibrium is stable, but coexists with an alternative stable fixed point of model (4.12, 4.21, 4.22).

With the uptick-rule, the existence and stability of the fixed points of model (4.17, 4.21, 4.22) are complemented by the following lemma:

Lemma 4.4 *The following statements hold true for the dynamical system* (4.17) *with predictors* (4.21) *and* (4.22)*:*

1. *The local and global stability of the fundamental equilibrium is as in Lemma 4.3. As long as equilibria $(\bar{x}^{(\pm)}, \bar{m})$ exist and are admissible, their local stability is as in Lemma 4.3. Equilibrium $(\bar{x}^{(+)}, \bar{m})$ is admissible iff $\bar{x}^{(+)} \leq \bar{x}^{(+)}_{\text{BC}} = a\sigma^2 s/(R-v)$. Equilibrium $(\bar{x}^{(-)}, \bar{m})$ is admissible iff $\bar{x}^{(-)}_{\text{BC}} = -a\sigma^2 s/(g-R) \leq \bar{x}^{(-)} \leq \bar{x}^{(+)}_{\text{BC}}$.*

2. *For $R < g < 2R - v$ there are the following possibilities:*

   (a) *If $R+v > 2$, equilibria $(\bar{x}^{(\pm)}, \bar{m})$ appear admissible at $\beta = \beta_{\text{LP}}$ and $(\bar{x}^{(+)}, \bar{m})$ becomes virtual (border-collision bifurcation) at $\beta = \beta^{(+)}_{\text{BC}}$, with*

   $$0 < \beta^{(+)}_{\text{BC}} = \frac{1}{C} \log\left(\frac{R-v}{g-R}\right)\left(1 + \frac{a\sigma^2 s^2}{C} \frac{(g-v)(1-v)}{(R-v)^2}\right)^{-1} < \beta_{\text{TR}}.$$

   (b) *If $R + v < 2$, equilibria $(\bar{x}^{(\pm)}, \bar{m})$ appear virtual at $\beta = \beta_{\text{LP}}$ and $(\bar{x}^{(-)}, \bar{m})$ becomes admissible at $\beta = \beta^{(+)}_{\text{BC}}$.*

   (c) *If $R+v = 2$, equilibria $(\bar{x}^{(\pm)}, \bar{m})$ appear on the border ∂Z_1 at $\beta = \beta_{\text{LP}} = \beta^{(+)}_{\text{BC}}$ and $(\bar{x}^{(+)}, \bar{m})$ and $(\bar{x}^{(-)}, \bar{m})$ are respectively virtual and admissible for larger β.*

(d) *If* $s < s_{\mathrm{BC}}^{(-)} = \left(\frac{C}{a\sigma^2} \frac{(g-R)^2}{(g-v)(g-1)} \right)^{1/2}$, *equilibrium* $(\bar{x}^{(-)}, \bar{m})$ *becomes virtual at* $\beta = \beta_{\mathrm{BC}}^{(-)}$, *with*

$$\beta_{\mathrm{BC}}^{(-)} = \frac{1}{C} \log\left(\frac{R-v}{g-R}\right)\left(1 - \frac{a\sigma^2 s^2}{C} \frac{(g-v)(g-1)}{(g-R)^2}\right)^{-1} > \beta_{\mathrm{TR}}.$$

3. *For* $g > 2R - v$ *there are the following possibilities:*

   a) *If* $s < s_{\mathrm{BC}}^{(-)}$ *equilibria* $(\bar{x}^{(\pm)}, \bar{m})$ *are virtual for any* $\beta \geq 0$.

   b) *If* $s > s_{\mathrm{BC}}^{(-)}$ *equilibrium* $(\bar{x}^{(+)}, \bar{m})$ *is virtual for any* $\beta \geq 0$, *whereas* $(\bar{x}^{(-)}, \bar{m})$ *is admissible for* $\beta > \beta_{\mathrm{BC}}^{(-)} > 0$.

4. *For* $g > R^2$ *the dynamics can be unbounded for sufficiently large* β.

Note that the uptick rule affects the price dynamics also when the supply of outside shares is large. Indeed, independently on s, there is always an equilibrium, $(\bar{x}^{(+)}, \bar{m})$ or $(\bar{x}^{(-)}, \bar{m})$, becoming virtual as β increases or decreases.

4.3.2 Fundamentalists vs ROC traders

Consider models (4.12) and (4.17) with the fundamental predictor (4.21) and with the ROC predictor (4.23a,b) or (4.23a,c,d).

Note that both predictors are such that $f_h(\bar{x}\mathbf{1})/\bar{x} < R$ for any possible equilibrium

$(\bar{x}, \bar{m})$ with $\bar{x} \neq 0$, so by means of Lemma 4.2 the fundamental equilibrium is the only fixed point. In the simplest case $L = 2$, its stability is characterized in the following lemma:

Lemma 4.5 *The following statements hold true for the dynamical systems* (4.12) *and* (4.17) *with predictors* (4.21) *and* (*4.23a,b*)*, as well as with predictors* (4.21) *and* (*4.23a,c,d*)*:*

1. *For $R \geq 2$ the fundamental equilibrium $(0, \bar{m}^{(0)})$ (see Lemma 4.1) is a stable fixed point for any $\beta > 0$.*

2. *For $R < 2$ the fundamental equilibrium is stable for*

$$0 < \beta < \beta_{\mathrm{NS}} = \frac{1}{C} \log\left(\frac{R}{2-R}\right)$$

and loses stability through a Neimark-Sacker bifurcation at $\beta = \beta_{\mathrm{NS}}$.

Similarly to the cases where chartists are paired with fundamentalists (Sect. 4.3.1), the stability of the fundamental equilibrium is guaranteed if the gross return R is sufficiently large.

The stability analysis for $L > 2$ is possible, following the lines indicated in [76], and the general conclusion is that rates of change calculated on larger windows of past prices stabilize the fundamental equilibrium, up to the point that the fixed point is stable for any value of β if L is sufficiently large.

4.4 The effect of the uptick rule on share price fluctuations: Numerical simulations.

In the first two Subsections of this Section we report several numerical analysis of models (4.12) and (4.17) for the two pairs of traders' types considered in Sect. 4.3, with the aim of characterizing the non-stationary (periodic, quasi-periodic, or chaotic) asymptotic regimes. This part contains technical considerations. In the last subsection, we discuss the effects of the uptick rule on the price dynamics.

As done in most of the related works in the literature, we use the traders' adaptability (or intensity of choice) β as a bifurcation parameter (two- or higher-dimensional bifurcation analyses are possible, see, e.g., [41], but will not be considered here). For each considered value of β, the transient dynamics is eliminated by computing the (largest) Lyapunov exponent associated to the orbit[11], i.e., we delete the number of initial iterations required to compute the largest Lyapunov exponent, whereas the asymptotic regime is discussed. To graphically study the bifurcations underwent by the different attractors, we vertically plot the deviations x_t in the attractor at the corresponding value of β, together with the associated largest Lyapunov exponent L (see, e.g., Figure (4.3)).

In each simulation, we set the initial condition as follows. The opening price deviation x_0 is randomly selected in a small, positive or negative neighborhood of zero to study the stability of the fundamental equilibrium; far from zero to study non-fundamental attractors. The initial traders' fractions are equally set ($m_1 = 0$, i.e., $n_{1,1} = n_{2,1} = 1/2$). For model (4.17), the initial values assigned to the traders' demands $z_{1,0}$ and $z_{2,0}$ are

[11] The largest Lyapunov exponent is a measure of the mean divergence of nearby trajectories; it is positive, zero, and negative in chaotic, quasi-periodic, and periodic (or stationary) regimes, respectively [2].

irrelevant, as the traders' fractions are not updated at $t = 1$.

4.4.1 Fundamentalists vs chartists

We first study the effects of a positive supply of outside shares ($s > 0$) on the dynamics of the original model (4.12, 4.20, 4.22) introduced by [30], then we study the effects of the uptick rule. Figure (4.3) reports the bifurcation diagrams and the corresponding largest Lyapunov exponent obtained for four different values of s (with s in the range of values commonly used in the literature, see [16, 69]). The first panel is the case with zero supply of outside shares ($s = 0$) and is included for comparison.

The bifurcation points β_{LP}, β_{TR}, and β_{NS} are indicated, together with the non-fundamental equilibria $(\bar{x}^{(\pm)}, \bar{m})$ (the gray parabola), and are in agreement with the analytical results of Lemma 4.3. In particular, $\beta_{\mathrm{LP}} = \beta_{\mathrm{TR}} = \beta_{\mathrm{PF}}$ indicates a pitchfork bifurcation when $s = 0$.

Note that the positive and negative deviation dynamics are separated in model (4.12, 4.20, 4.22). Indeed, due to the characteristics of the chartist predictor (4.22), if the opening price is above [below] the fundamental value ($x_0 > 0$ [$x_0 < 0$]), it will remain so forever (see Eq. (4.12a)). The positive and negative attractors therefore coexist (with basins of attraction separated by the linear manifold $x_{t-1} = 0$ in state space), so that two Lyapunov exponents (one for each of the two attractors) are plotted.

As expected, the fundamental equilibrium is destabilized for sufficiently high traders' adaptability and the amplitude of the price fluctuations increases as β increases. But the amplitude of fluctuations also increases with the supply of outside shares of the risky

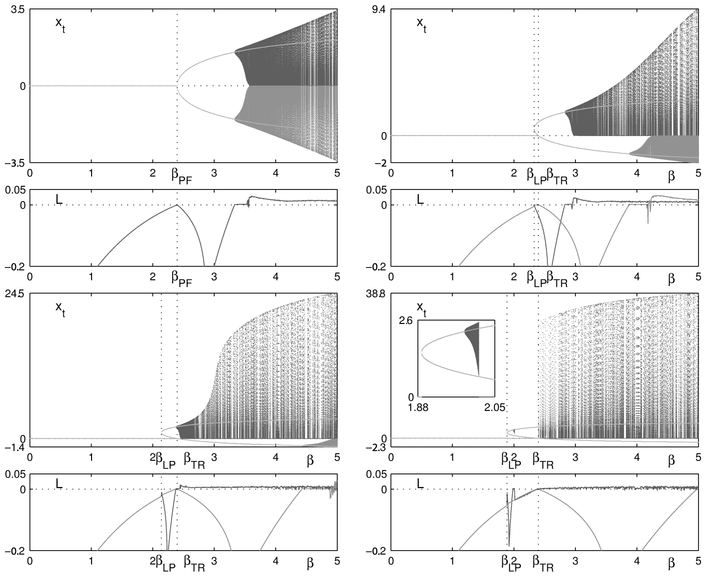

Figure 4.3: Bifurcation diagrams of model (4.12, 4.20, 4.22) for different values of s (the supply of outside shares of the risky asset): A (first row, left column) $s = 0$, in this case $\beta_{\text{TR}} = \beta_{\text{LP}} = \beta_{\text{PF}}$, related (largest) Lyapunov exponents in (second row, left column); B (first row, right column) $s = 0.1$, in this case $\beta_{\text{TR}} < \beta_{\text{NS}}^{(+)}$, related (largest) Lyapunov exponents in (second row, right column); C (third row, left column) $s = 0.2$, in this case $\beta_{\text{TR}} = \beta_{\text{NS}}^{(+)}$, related (largest) Lyapunov exponents (fourth row, left column); D (third row, right column) $s = 0.3$, in this case $\beta_{\text{TR}} > \beta_{\text{NS}}^{(+)}$, related (largest) Lyapunov exponents (fourth row, right column). $\beta_{\text{NS}}^{(+)}$ and $\beta_{\text{NS}}^{(-)}$ are not ticked because there is no analytical expression for them. However, the Neimark-Sacker bifurcation values can be clearly identified in the bifurcation diagrams. Other parameter values: $R = 1.1$, $a = 1$, $\sigma = 1$, $\bar{y} = 1$, $g = 1.2$, $C = 1$.

asset s (note the different vertical scales in Figure (4.3)). The latter effect is partially due to the risk premium, i.e. $(a\sigma^2 s)$, required by traders for holding extra shares that modifies the performance measures of the trading strategies.

Figure (4.3) shows another interesting dynamical phenomenon. For sufficiently high supply of outside shares of the risky asset ($s = 0.3$ in case D), the positive attractor, that appears at the Neimark-Sacker (NS) bifurcation ($\beta_{\text{NS}}^{(+)}$) of equilibrium $(\bar{x}^{(+)}, \bar{m})$, collapses through a "homoclinic" contact with the saddle equilibrium $(\bar{x}^{(-)}, \bar{m})$. The chaotic behavior is then reestablished when the fundamental equilibrium looses stability through the transcritical bifurcation (β_{TR}).

The analysis of the dynamics conducted up to now reveals that there are not substantial differences in the price dynamics for different values of s, except the amplitude of price fluctuations and some peculiar phenomena, such as the homoclinic contact just discussed. For this reason and to make the discussion more clear and concise, in the following we investigate the effects of the uptick rule on the price dynamics only for $s = 0.1$.

Figure (4.4) compares models (4.12) (left) and (4.17) (right) with predictors (4.20) and (4.22). The bifurcation diagram and the largest Lyapunov exponents associated with the different attractors are reported, together with two examples of state portraits (projections in the plane (x_t, x_{t-1}), see bottom panels). Blue and red dots identify the points in the attractor in which respectively fundamentalists and chartists have been prohibited from going short.

The first thing to remark is that multiple positive attractors are present in the constrained dynamics (right column in Figure (4.4)), i.e. when the uptick rule is in place. In particular a period-two cycle alternating unrestricted trading with restricted trading in which fundamentalists are forced by the uptick rule to take nonnegative positions (black and blue dots) is present for sufficiently large β. It appears through a nonsmooth saddle-node bifurcation and coexists with the chaotic attractor generated through the loss of fundamental stability. Before the bifurcation the fundamental equilibrium is globally stable (while it is globally stable in the unconstrained dynamics up to the saddle-node at $\beta = \beta_{\mathrm{LP}}$).

Also the bifurcation structure leading to the chaotic attractor is more involved. The

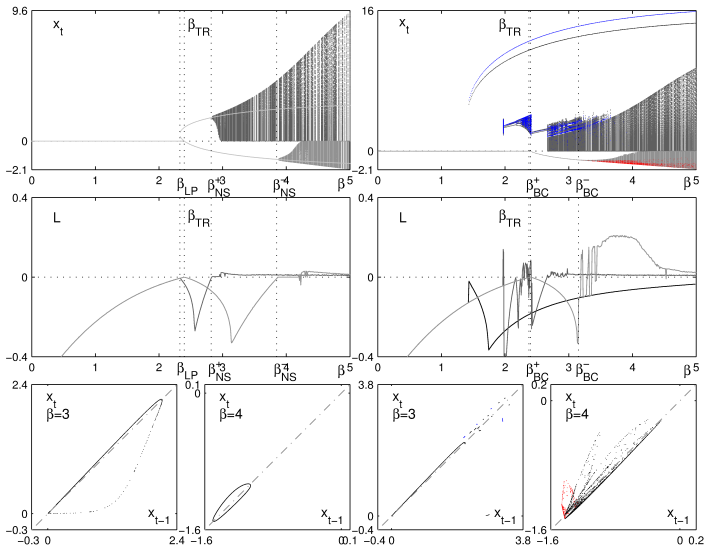

Figure 4.4: Bifurcation diagrams of models (4.12, 4.20, 4.22) (left) and (4.17, 4.20, 4.22) (right) and exemplary state portraits projections in the plane (x_t, x_{t-1}). Parameter values as in Figure (4.3)-case(B). The light gray represents the attractor of negative deviations and the corresponding largest Lyapunov exponent. The dark gray represents the attractor of positive deviations and the corresponding largest Lyapunov exponent. The black represents the attractor born through a nonsmooth saddle-node bifurcation and the corresponding largest Lyapunov exponent. The blue dots indicate that fundamentalists cannot have negative positions because of the uptick rule. The red dots indicate that chartists cannot have negative positions because of the uptick rule.

first branch of attractors suddenly appears as β increases. This is probably due to a homoclinic contact with the period-two saddle cycle, but this conjecture has not been verified. This first branch seems to collapse through a collision with the border ∂Z_1, in connection with the border collision of the non-fundamental equilibrium $(\bar{x}^{(+)}, \bar{m})$ at $\beta = \beta_{\mathrm{BC}}^{(+)}$. The remaining thinner attractor later explodes into a larger one, again due to a border collision with ∂Z_1. However, a deeper mathematical investigation would be required to confirm the above conjectures.

As for the negative equilibrium $(\bar{x}^{(-)}, \bar{m})$ (for $\beta > \beta_{\mathrm{TR}}$), it loses stability at $\beta = \beta_{\mathrm{BC}}^{(-)}$

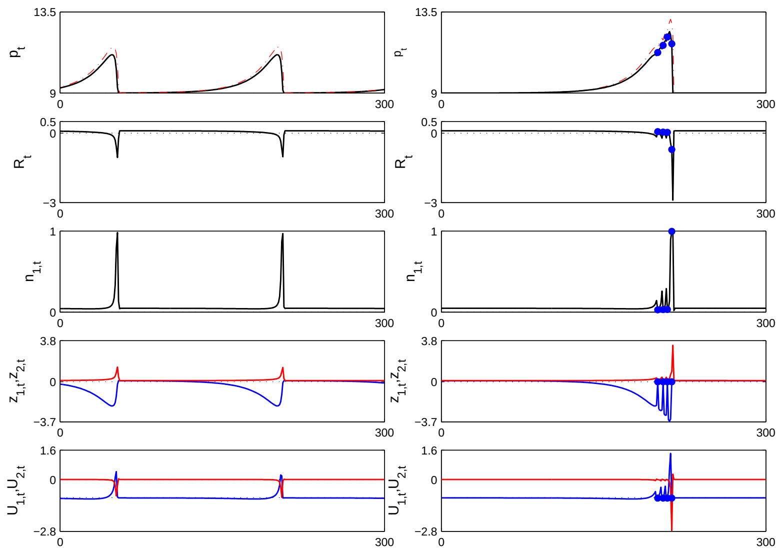

Figure 4.5: Positive (i.e. above fundamental) time series on the asymptotic regime of models (4.12, 4.20, 4.22) (left) and (4.17, 4.20, 4.22) (right) for $\beta = 3$. Other parameter values as in Figure (4.3)-case(B). The blue dots indicate that fundamentalists cannot have negative positions because of the uptick rule.

through a border collision with ∂Z_2, giving way to a chaotic attractor characterized by restrictions on chartists.

Figure (4.5) shows an example of time series on the positive chaotic attractor obtained for $\beta = 3$ (left column: unconstrained dynamics; right column: dynamics constrained by the uptick rule). The top panels report the price dynamics (black) and the chartist prediction (red, dashed), with blue and red dots marking the periods in which fundamentalists and chartists are respectively prevented to go short by the uptick rule (right column). The remaining panels report, from top to bottom, the returns ($R_t = x_t - Rx_{t-1} + a\sigma^2 s$) and the traders' fractions ($n_{h,t}$), demands ($z_{h,t}$), and net profits ($U_{h,t} = R_t z_{h,t-1} - C_h$, $h = 1, 2$, blue for fundamentalists and red for chartists).

The price dynamics is characterized by recurrent peaks (financial bubbles), driven by chartists that expect a price rise and hold the shares of the risky asset and, at the same time, attenuated by fundamentalists which expect a devaluation and sell short (see the negative positions of fundamentalists in the unconstrained dynamics, left). In particular, when the price is closed to its fundamental value, the chartists' trading strategy is more profitable because their expectations are confirmed. Chartists dominate the market and the price is growing until the capital gain cannot compensate for the lower dividend yield. At this point, chartists start to suffer negative returns, while the short positions of the fundamentalists produce profits. Eventually most of traders adopt the fundamentalist's trading strategy and the price falls down close to the fundamental value. As soon the price starts to revert to the fundamental value, however, fundamentalists are prohibited from going short in the constrained dynamics because of the uptick rule (right column), and this triggers a further phase of rising prices. This happens several times, with the result of amplifying the price peak. At a certain point, when the price is very far away from its fundamental value, chartists suffer strong losses and the relative fraction of fundamentalists is almost one. The massive presence of fundamentalists pushes the price close to its fundamental value and the uptick rule cannot prevent this from happening. Indeed, if there are almost only fundamentalists, from the pricing equation we have that their demands must be equal to the supply of outside share, i.e. positive. When the price is close to its fundamental price chartists start to have a better performance and the story repeats. Despite this mispricing effect, the frequency of the price peaks is slowed down by the short selling restriction.

Similarly, Figure (4.6) shows an example of time series on the negative attractor obtained for $\beta = 4$ (here the unconstrained dynamics is quasi-periodic, see Figure (4.4)). In this case of negative price deviations, the chartists go short and have, on average, higher profits than fundamentalists. In the unconstrained dynamics chartists are predominant and drive prices down. This phenomenon is attenuated by the uptick rule, which limit the downward price movements and increases the performance of fundamentalists. The

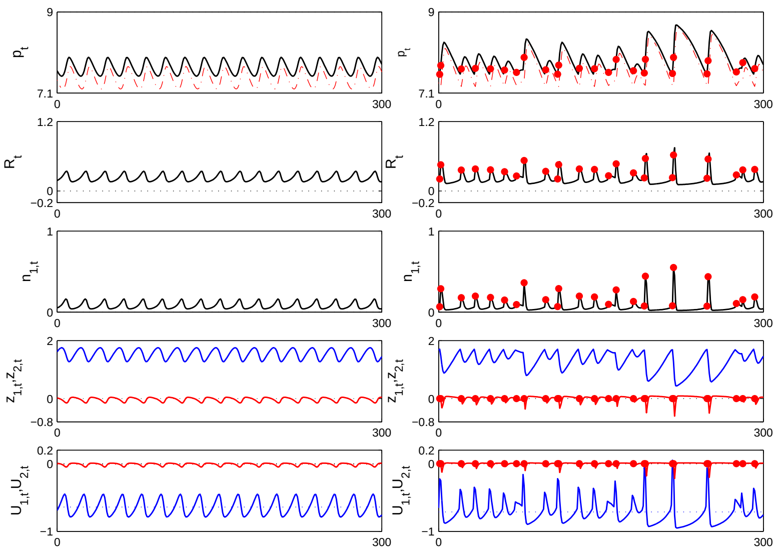

Figure 4.6: Negative (i.e., below fundamental) time series on the asymptotic regime of models (4.12, 4.20, 4.22) (left) and (4.17, 4.20, 4.22) (right) for $\beta = 4$. Other parameter values as in Figure (4.3)-case(B). The red dots indicate that chartists cannot have negative positions because of the uptick rule.

frequencies of the negative peaks is slightly lowered by the short selling restriction, but they are more irregular due to the presence of chaotic dynamics as indicated by the positivity of the largest Lyapunov exponent, see Figure (4.4).

The different types of price dynamics, quasi-periodic for the unconstrained model and chaotic for the constrained one, are due to the different types of bifurcations through which the non-fundamental equilibrium $(\bar{x}^{(-)}, \bar{m})$ losses stability, a Neimark-Sacker bifurcation in the first case and a border-collision bifurcation in the second. Indeed, as typical in piecewise linear model, at the border-collision bifurcation we have sudden transition from a stable fixed point to a fully developed *robust* (i.e., without periodic windows) chaotic attractor, see, e.g. [42].

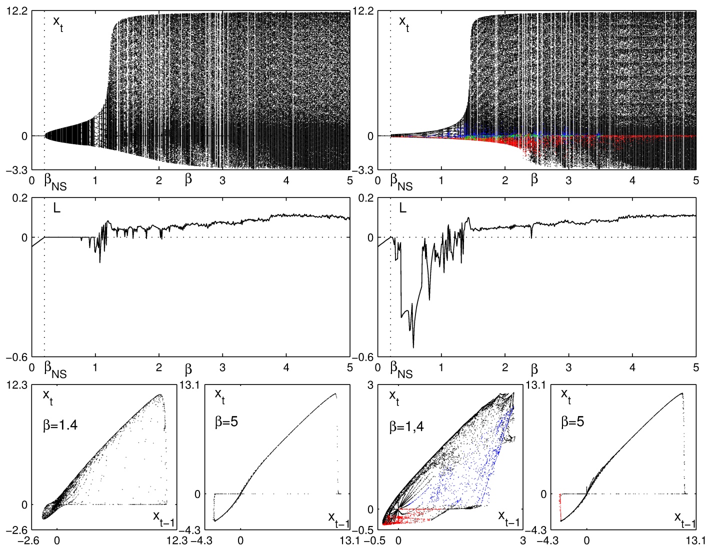

Figure 4.7: Bifurcation diagrams of models (4.12, 4.20, 4.23a,c,d) (left) and (4.17, 4.20, 4.23a,c,d) (right) and exemplary state portraits projections in the plane (x_t, x_{t-1}). Parameter values: $\alpha = 10$, $L = 2$, other values as in Figure (4.3)-case(B). The blue dots indicate that fundamentalists cannot have negative positions because of the uptick rule. The red dots indicate that chartists cannot have negative positions because of the uptick rule.

4.4.2 Fundamentalists vs ROC traders

Figure (4.7) reports the bifurcation analysis of models (4.12) (left) and (4.17) (right) with predictors (4.20) and (4.23a,c,d), while Figure (4.8) shows the time series on the chaotic attractor obtained for $\beta = 1.4$.

Here the fundamental equilibrium is globally stable up to the Neimark-Sacker bifurcation at $\beta \leq \beta_{\text{NS}}$. Interestingly, the price fluctuations showed by the non-stationary attractors originated for larger β (quasi-periodic with periodic windows and later chaotic) have a remarkably smaller amplitude in the constrained dynamics, than in the unrestricted case. In this sense, the uptick-rule shows a rather robust stabilizing effect, at least as long β is not too large, i.e., traders are not fast enough in changing their beliefs to react

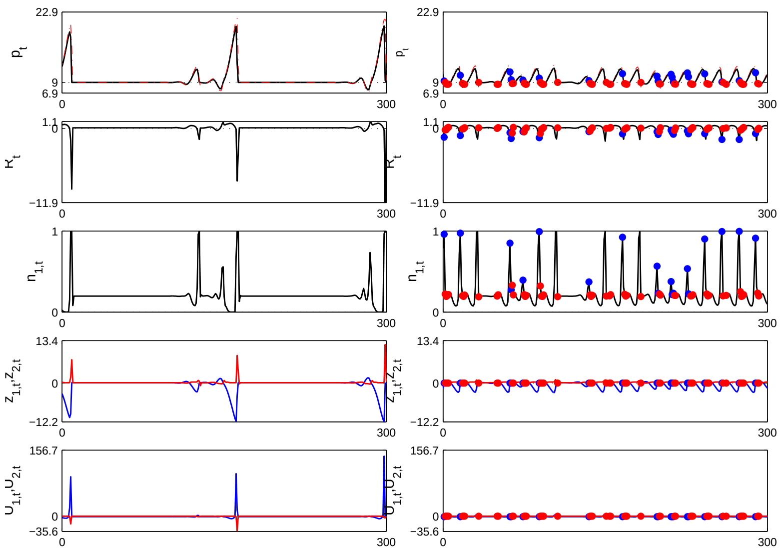

Figure 4.8: Time series on the asymptotic regime of models (4.12, 4.20, 4.23a,c,d) (left) and (4.17, 4.20, 4.23a,c,d) (right) for $\beta = 1.4$. Other parameter values as in Figure (4.7). The blue dots indicate that fundamentalists cannot have negative positions because of the uptick rule. The red dots indicate that chartists cannot have negative positions because of the uptick rule.

to past performances.

This is also evident in the time series of Figure (4.8), where the short selling restriction also intensifies the frequency of the price peaks.

4.4.3 Economic discussion of the numerical results

On the basis of the three declared objectives of the uptick rule, this subsection provides a discussion of the effects of this short selling regulation in the light of the analytical and by numerical analysis reported in the previous Sections.

Let us start to discuss the scenario described by models (4.12) and (4.17) with pre-

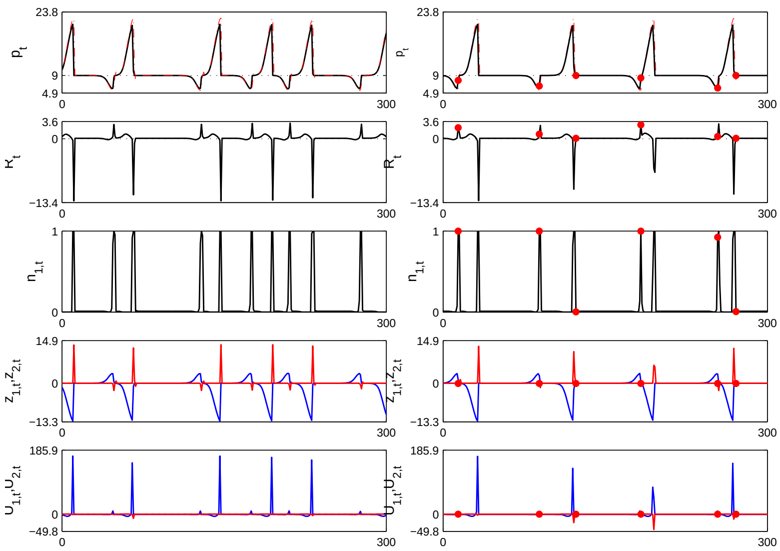

Figure 4.9: Time series on the asymptotic regime of models (4.12, 4.20, 4.23a,c,d) (left) and (4.17, 4.20, 4.23a,c,d) (right) for $\beta = 4.5$. Other parameter values as in Figure (4.7). The red dots indicate that chartists cannot have negative positions because of the uptick rule.

dictors (4.21) and (4.22) (fundamentalists vs chartists). We first consider the case of negative price deviations (the negative attractor in Figure (4.4) and the time series in Figure (4.6)), where we see that chartists take short positions when prices fall, whereas fundamentalists take long positions believing that the stock price will rise to reach the fundamental value (see the unconstrained dynamics in the left column of Figure (4.6)). It is important to point out that whenever an asset is undervalued (price below its fundamental value) it should be better to hold it rather than to sell it because the return is always positive: dividend yield outweighs the capital gain effect. Indeed, by going short chartists obtain negative excess returns, see the dynamic of the net profits $U_{2,t}$ in the last row of Figure (4.6). However chartists do better than fundamentalists here because the latter are charged a cost, (C), which is larger than the profit they obtain by holding the assets, compare the net profits of the two trading strategies ($U_{1,t}$ and $U_{2,t}$) again in the last row of Figure (4.6). It follows that chartists are predominant in the market, as indicated by the low value of the fraction n_1, and their trading strategy causes downward movements of the stocks' price. In the constrained dynamics (right column), the uptick rule limits the possibility to go short for chartists. This helps to revert the stock prices toward the fundamental value. It follows a better performance for the fundamentalists and their presence in the market increases. As a result, the short selling restriction reduces the negative peaks reached by stock prices. From this example it is clear the effectiveness of the regulation to meet the last two goals established by the SEC. Moreover, from the analysis of the bifurcation diagram, it is interesting to note that the negative non-fundamental equilibrium $(\bar{x}^{(-)}, \bar{m})$ looses stability in the constrained dynamics at a lower traders' adaptability, i.e., at a lower value of the parameter β, respect to the unconstrained dynamics. This is due to the border collision bifurcation at $\beta = \beta_{\mathrm{BC}}^{(-)}$, that occurs before the Neimark-Sacker bifurcation at $\beta_{\mathrm{NS}}^{(-)}$ responsible of the instability in the unconstrained dynamics. This might be mathematically interpreted as a destabilizing effect of the uptick rule. However, the chaotic fluctuations established after the bifurcation move the prices, on average, closer to the fundamental value with respect to the equilibrium deviation $\bar{x}^{(-)}$.

Considering the same model and predictors, by the time series of Figure (4.5) it is possible to notice that the dynamics of positive price deviations are characterized (for $\beta > \beta_{\mathrm{NS}}^{(+)}$) by frequently financial bubbles. These bubbles are enforced by chartists which overvalued the stock prices in the upward trend and are attenuated by fundamentalists. In fact, in these phases of rising prices fundamentalists go short driven by the belief that the price will revert to its fundamental value in the next period. This increases the supply of shares helping to curb rising prices. At a certain point of the upward trend, the fundamentalists' strategy take over and prices are driven to the fundamental value of the stock. During these upward trends of the market the uptick rule prevents the fundamentalists to take short positions, see blue dots in the fourth row, right column of Figure (4.5). This phenomenon is also emphasized by empirical analysis (see, e.g.,

[1] and [26]), and represents a flaw of the regulation. The result is an increase of the amplitude of the financial bubbles. As a positive effect of the regulation, it is possible to notice a reduction in the frequency of occurrences of these bullish divergences.

The main point is that every time fundamentalists go short they force prices to converge to the fundamental value. If it were possible to discriminate between the beliefs of the traders, fundamentalists should not be forbidden to go short in this situation. However, this is a very difficult task and it is not even so easy to correctly determine the fundamental values of a risky asset in the real market. A possible solution could be to allow short sales after a long period of rising prices. This should avoid pushing up prices because fundamentalists (and contrarians, but these traders are not taken into consideration in this work) are driven out of the market. Another possible solution could be to restrict short selling only in cases of sharp and sudden falls in stock prices. This should produce two effects, to let agents believing in the fundamental price go short when the price increases, reducing positive oscillations of price deviations from the fundamental value, and to reduce sharp drops in prices observed when the stock market bubble breaks. An important step in this direction has already been done, the SEC adopted Rule 201 which was implemented on February 28, 2011[12]. This new short selling regulation prohibits short selling operations if the value of the stock decreases by more than 10% in two consecutive trading sections.

Summarizing the analysis of the positive and negative price deviations from the fundamental value for the model with predictors (4.20) and (4.22) for the cases of unrestricted and restricted short sales, we can conclude that the second goal of the regulation is ensured, but some distorting effects produced by this rule are observed, such as overvaluation of shares. Moreover, the short selling restriction can trigger distorting mechanisms

[12]See, Securities Exchange Act Release No. 61595 (Feb. 26, 2010).

that support dynamics of overpricing, otherwise not feasible in the long run. This is indicted by the presence of multiple attractors in the region of positive price deviations in the bifurcation diagram in the right column of Figure (4.4).

With the intent to provide a more detailed and comprehensive description of the effects of the uptick rule, the same asset pricing model has been analyzed with a different non-fundamental predictor (the Smoothed-ROC predictor, see Subsection 4.2.4). Differently from other trend-following indicators, the Smoothed-ROC predictor is particularly useful to detect fast and short-term upward and downward movements of the stock price and gives different trading signals to agents than predictor (4.22), such as gain and lose of speed in the trend (see [47]). In this case, we obtain an interesting and surprising result, i.e., the uptick rule helps to reduce the amplitude of price fluctuations, and causes an increase of the frequency of oscillations above and below the fundamental value, this is made clear by comparing the time series of prices in the first row of Figure (4.8). The explanation of this lies on the higher degree of rationality (compared to the one assumed for chartists) of the non-fundamentalist agents that the use of the non-linear predictor (S-ROC predictor) implies. These agents are uncomfortable with extreme assessments of the value of the shares and when the shares are forced to be overvalued due to the short selling constraints, non-fundamental agents react and become more confidential in the fundamental price. This changes their trading strategies and, as a results, the amplitude of the price fluctuations decreases instead of increasing as might be expected. The choice of the value of α plays an important rule in this. We can conclude that by using this couple of predictors it is possible to observe, at least when evolutionary pressure, β, is not excessively large, all the main goals of the uptick rule: short selling restriction does not produce mispricing, prevents chartists from going short during downward price movements to avoid reaching negative price variation peaks and prevents fundamentalists from going short only in a downward price movement for positive price deviations, reducing the speed of convergence to the fundamental value and preventing sharps drops in prices. As a further observation, it is worth noticing that the uptick rule prevents extremely negative excess returns and at the same time makes the strategy of fundamentalists on the average more profitable, changing the beliefs of traders, compare $n_{1,t}$ and $n_{2,t}$ in Figure (4.8). Moreover, looking at the bifurcation diagram of Figure (4.7) right-column, it is clear that for relatively high values of the intensity of choice, β, the uptick rule does not have any effect on the amplitude of price fluctuations. As revealed by the analysis of the time series in Figure (4.9), this is due to the fact that for high levels of β the market is always dominated by one trading strategy. It follows that the short selling constraint can apply either to a trading strategy adopted almost by any trader or to a trading strategy adopted by almost all the traders. In the first case the regulation does not have any effect on the price, in the second case it reverts the price toward the fundamental value with the results of reducing the frequency of market-bubbles but without reducing the amplitude of them. Compare the two panels in the first row of Figure (4.9). For β large enough, at each trading section there is only one type of trader that dominates the market and its demand of shares, being equal to the supply of outside share per trader, must be positive, then the uptick rule does not affect the dynamics of prices.

The conducted analysis points out that the uptick rule meets part of its goals, but which of them often depends on the condition of the markets. Moreover, the regulation can produce several side-effects which may be different according once again to the market's conditions and investor's beliefs which may strongly influence the effectiveness of the regulation itself. Due to these findings, studying the impact of the uptick rule on financial markets does not seem to be an easy task. Nonetheless, it is possible to isolate some remarkable effects regardless of market conditions and traders' beliefs. First, the uptick rule ensures a reduction of the downward market movements when the shares are undervalued, i.e. when the shares are priced below their fundamental value. Second, the intensity of choice β to switch predictors affects the effectiveness of the short selling regulation. By using the bifurcation analysis, it possible to observe that the effectiveness of the regulation fades away increasing the value of β (increasing β, agents tend to overreact to the market's information, changing their beliefs quickly to react to past performances). In other words, the switching destabilizing effect prevails over the regulation's effects, i.e. when agents overreact to the differences in performance related to different beliefs, the regulation does not affect the dynamics of stock prices. This is consistent with many interesting empirical results testifying that there is no statistical effect of the uptick rule on price fluctuations in turbulent financial markets (see, e.g., [45]).

As a final remark, it is important to clarify that the simple asset pricing model used in this Chapter can reproduce only some possible "stylized effects" which are a direct consequence of the regulation. However, in the real financial markets many more different trading strategies and emotional actions are present, which can modify the effectiveness of the uptick rule.

4.5 Conclusions and future directions

In this Chapter, we have investigated the effect of the "uptick rule" on an asset price dynamics by means of an asset pricing model with heterogeneous, adaptive beliefs. The analysis has suggested the effectiveness of the regulation in reducing the downward price movements of undervalued shares avoiding speculative behavior whenever the market is characterized by not too many aggressive traders, i.e. when the agents' propensity of changing trading strategy is relatively low. On the contrary, when the agents have a high propensity of changing trading strategy according to the past trading performances, which is a sign of turbulent markets according to the model, the effects of the regulation tend to fade. As a side effect of the regulation, an amplification of the market bubbles in the case of overvalued shares is possible.

This work represents only a starting point. There are still several aspects that can and deserve to be analyzed. First of all, it is interesting to analyze the effect of the uptick regulation using the same asset pricing model with a greater number of investor types, for example contrarians, chartists and fundamentalists. In fact, there is empirical evidence in the literature about the switch in trading style by short-sellers. [45] found that under the uptick rule most of the short positions are opened by contrarians, on the contrary, when the uptick rule is not imposed are chartists, the ones who prefer to go short (see, e.g., [26]). There is a hypothetical explanation for this. Chartists take short positions in declining price trends and the uptick rule makes this operation more difficult, on the contrary the restriction does not effect the contrarians' short strategy. They usually go short in upward price trend betting on a change in price movement with the effect of stabilizing the market. Investigating the validity of this hypothesis provides a better understanding of the issue.

The regulation should also be evaluated in the contest of the multi-assets market to discover how the short selling restriction for one stock influences the price of the others. It is reasonable to expect that traders will switch to trade stocks that are not effected by the restriction in that specific moment and this can produce effects on prices which are not easy to predict without a deep analysis.

Another aspect that deserves to be investigated is the effect of the regulation when there is a fraction of investors which does not change the trading strategy (or belief) as in [44]. As highlighted in this Chapter, the uptick rule loses its effectiveness due to a high propensity to switch trading strategy by agents. It follows that, if there are constraints on the possibility to change trading strategy, we expect an increase of the effectiveness of the regulation and a reduction of unwelcome effects on price dynamics. Last but not least, the piecewise continuous model here proposed can be easily adapted according to the new short selling regulation imposed by the SEC, i.e. Rule 201. Comparing the two cases can help to understand the pros and cons of the new regulation.

4.6 Appendix

Proof of Lemma 4.1. The existence of the fundamental equilibrium immediately follows by substituting $f_h(\mathbf{0}) = 0$ into Eqs. (4.10) and (4.12). For the definition of the associated eigenvalues, let us substitute Eq. (4.10b) written for $n_{h,t}$ into Eq. (4.10a) (and Eq. (4.12b) written for m_t into Eq. (4.12a)). Then, we can consider $(x_{t-1}, \ldots, x_{t-(L+2)})$ as the state variables for both models, and the Jacobian of the system at the fundamental equilibrium is equal to

$$\left[\begin{array}{ccccc|cc}
\gamma_1 & \gamma_2 & \cdots & & \gamma_L & 0 & 0 \\
1 & 0 & \cdots & & 0 & 0 & 0 \\
0 & 1 & \cdots & & 0 & 0 & 0 \\
\vdots & & \ddots & & 0 & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & \cdots & 0 & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 0
\end{array}\right],$$

which proves the result. ■

Proof of Lemma 4.2. At the equilibrium price deviation $\bar{x}$, we obviously have $\mathbf{x}_t = \bar{x}\mathbf{1}$ and, from Eq. (4.10a), $\bar{x}$ must satisfy

$$R = \sum_{h=1}^{H} n_{h,t} f_h(\bar{x}\mathbf{1})/\bar{x}.$$

Being $n_{h,t} \in (0,1)$ for all $h = 1, \ldots, H$, this is possible only if $f_h(\bar{x}\mathbf{1})/\bar{x} < R$ for some h and $f_k(\bar{x}\mathbf{1})/\bar{x} > R$ for some $k \neq h$. ■

Proof of Lemma 4.3.

1. The uniqueness follows from Lemma 4.2, while the global stability from (4.12a), which can be rewritten as

$$x_t = \frac{1}{R}\left(n_{1,t}v + n_{2,t}g\right)x_{t-1},$$

and contracts the deviation x_t as t goes to infinity.

2. From Lemma 4.1, the non-zero eigenvalue associated to the fundamental equilibrium is

$$\lambda^{(0)} = \gamma_1 = \frac{1}{2R}\left((1+\bar{m}^{(0)})v + (1-\bar{m}^{(0)})g\right) > 0, \quad \bar{m}^{(0)} = \tanh\left(-\beta C/2\right).$$

Thus the fundamental equilibrium can loose stability only when $\lambda^{(0)} = 1$ at a transcritical (or pitchfork) bifurcation (being fixed point of model (4.12, 4.21, 4.22) for any admissible parameter setting, it cannot disappear through a saddle-node bifurcation). Solving $\lambda^{(0)} = 1$ for β gives β_{TR}.

Evaluating Eqs. (4.10) at the generic equilibrium $(\bar{x}, \bar{m})$, solving Eq. (4.12a) for $\bar{m}$, and equating the result to Eq. (4.12b), we get

$$\bar{m} = 1 - 2\frac{R-v}{g-v} = \tanh\left(-\frac{\beta}{2}\left((-(R-1)\bar{x} + a\sigma^2 s)\frac{g-v}{a\sigma^2}\bar{x} + C\right)\right),$$

which solved for $\bar{x}$ gives $\bar{x}^{(\pm)}$.

Solving $\bar{x}^{(+)} = \bar{x}^{(-)}$ for β gives β_{LP} and the equilibrium deviations $\bar{x}^{(\pm)}$ are defined only for $\beta > \beta_{\text{LP}}$. There are therefore no other equilibria and this concludes the proof of points (a), (LP), and (TR). Note that $\beta_{\text{LP}} = \beta_{\text{TR}}$ when $s = 0$ (the transcritical and saddle-node bifurcations coincide at a pitchfork bifurcation).

Substituting Eq. (4.12b) written for m_t into Eq. (4.12a) and using $(x_{t-1}, x_{t-2}, x_{t-3})$ as state variables, the Jacobian of the systems at equilibria $(\bar{x}^{(\pm)}, \bar{m})$ is given by

$$\begin{bmatrix} \gamma_1^{(\pm)} & \gamma_2^{(\pm)} & \gamma_3^{(\pm)} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

with

$$\gamma_1^{(\pm)} = 1 + \frac{\gamma}{R}\bar{x}^{(\pm)}, \quad \gamma_2^{(\pm)} = -\gamma\bar{x}^{(\pm)}, \quad \gamma_3^{(\pm)} = \frac{\gamma}{R}\left(-(R-1)\bar{x}^{(\pm)} + a\sigma^2 s\right),$$

and

$$\gamma = \beta\bar{x}^{(\pm)}\frac{(g-v)^2}{4a\sigma^2}\,\text{sech}^2\!\left(-\frac{\beta}{2}\left((-(R-1)\bar{x}^{(\pm)} + a\sigma^2 s)\frac{g-v}{a\sigma^2}\bar{x}^{(\pm)} + C\right)\right),$$

so the three associated eigenvalues λ_1, λ_2, λ_3 are the roots of the characteristic equation

$$\lambda^3 - \gamma_1^{(\pm)}\lambda^2 - \gamma_2^{(\pm)}\lambda - \gamma_3^{(\pm)} = 0.$$

In particular, imposing $\lambda = 1$ and solving the characteristic equation for $\bar{x}^{(\pm)}$, gives only zero and x_{LP} as solutions, so no other transcritical, saddle-node (or pitchfork) bifurcation is possible. In contrast, imposing $\lambda_1 = -1$ and taking into account that $\lambda_1 + \lambda_2 + \lambda_3 = \gamma_1$, we get $\lambda_2 + \lambda_3 = 2 + \gamma\bar{x}^{(\pm)}/R > 2$ (note that $\gamma\bar{x}^{(+)} > 0$ and that $\gamma\bar{x}^{(-)} \geq 0$ only vanishes at the transcritical bifurcation), so that equilibrium $(\bar{x}^{(\pm)}, \bar{m})$ would be unstable at a period-doubling (flip) bifurcation. However, both equilibria $(\bar{x}^{(\pm)}, \bar{m})$ loose stability by increasing β, because the coefficient γ_1 linearly diverges with β (the limit as $\beta \to \infty$ of the sech argument is finite and equal to $-\log((R-v)/(g-R))/2 < 0$), so the same does the sum of the eigenvalues. Stability is therefore lost through a Neimark-Sacker bifurcation and this concludes the proof of points (b), (c), and (NS$^{(\pm)}$).

3. For $g > 2R - v$, the non-zero eigenvalue $\lambda^{(0)}$ associated to the fundamental equilibrium is larger than one for any $\beta > 0$. We also have $\beta_{\mathrm{LP}} < 0$, $\beta_{\mathrm{TR}} < 0$, and equilibria $(\bar{x}^{(\pm)}, \bar{m})$ are defined for any $\beta > 0$ with $\bar{x}^{(+)} > 0$ and $\bar{x}^{(-)} < 0$. Similarly to point 2, they loose stability through a Neimark-Sacker bifurcation.

4. In the limiting case $\beta \to \infty$, from Eq. (4.12b) we have

$$m_{t+1} = \begin{cases} 1 & \text{if} \quad (Rx_{t-1} - x_t - a\sigma^2 s)\dfrac{g-v}{a\sigma^2}x_{t-1} > C, \\ -1 & \text{if} \quad (Rx_{t-1} - x_t - a\sigma^2 s)\dfrac{g-v}{a\sigma^2}x_{t-1} < C. \end{cases}$$

Starting at $x_0 = \pm\epsilon$ with sufficiently small $\epsilon > 0$ and $m_1 = -1$, we therefore have

$x_t = \pm\epsilon(g/R)^t$ (see Eq. (4.12a)) as long as m_{t+1} stays at -1. Thus, x_t diverges if

$$\epsilon^2(g/R)^{2t-1}\left(\frac{R^2}{g} - 1 \mp a\sigma^2 s\epsilon^{-1}(g/R)^{-t}\right)\frac{g-v}{a\sigma^2} > C,$$

is never satisfied for increasing t, i.e., when $g > R^2$.

■

Proof of Lemma 4.4.

1. At the fundamental equilibrium, the traders' demands are positive ($\bar{z}_1^{(0)} = \bar{z}_2^{(0)} = s$), so $(0, \bar{m}^{(0)})$ is an admissible fixed point of the unconstrained dynamics for any admissible parameter setting. Its local stability is therefore ruled by Lemma 4.3.

   The global stability for $1 < g < R$ is a consequence of the following arguments. First, the price deviation is contracted from period $(t-1)$ to period t (i.e., $|x_t| < |x_{t-1}|$) whenever the unconstrained dynamics is applied (see Lemma 4.3, point 1). Second, the unconstrained deviation $x_t^{(0)}$ following x_{t-1} is smaller than any of the constrained deviations $x_t^{(1)}$ and $x_t^{(2)}$ given by Eq. (4.17a) in region Z_1 and Z_2, respectively. This is graphically clear from Figure (4.1), and is analytically shown by noting that

$$x_t^{(h)} - x_t^{(0)} = -\frac{a\sigma^2}{R}\frac{n_h}{n_k} z_{h,t}^{(0)} > 0, \quad h = 1, 2,\ k \neq h$$

   (see (4.16b), and recall that $z_{h,t}^{(0)} < 0$ in region Z_h). Third, from Eq. (4.17a) we get that $0 < x_t^{(0)} < x_t^{(h)} < (g/R)x_{t-1}$ when $x_{t-1} > 0$ and $x_t^{(0)} < x_t^{(h)} < (v/R)x_{t-1} < 0$ when $x_{t-1} < 0$. Thus, the constrained dynamics in regions Z_1 and Z_2 also contracts the price deviation from period $(t-1)$ to period t.

Being $\bar{x}^{(+)} > 0$, equilibrium $(\bar{x}^{(+)}, \bar{m})$ can only collide with border ∂Z_1 (see (4.19)), at which $\bar{x}^{(+)} = \bar{x}_{\mathrm{BC}}^{(+)}$ (the expression for $\bar{x}_{\mathrm{BC}}^{(+)}$ can be easily obtained by solving the first equation in (4.16b) with $z_{1,t}^{(0)} = 0$ for x_{t-1}). It is obviously admissible iff $\bar{x}^{(+)} \leq \bar{x}_{\mathrm{BC}}^{(+)}$.

Depending on the parameter setting, equilibrium $(\bar{x}^{(-)}, \bar{m})$ can be either positive or negative, and can therefore collide with both borders ∂Z_1 and ∂Z_2. At the border ∂Z_2, $\bar{x}^{(-)} = \bar{x}_{\mathrm{BC}}^{(-)}$ (the expression for $\bar{x}_{\mathrm{BC}}^{(-)}$ is obtained by solving the second equation in (4.16b) with $z_{2,t}^{(0)} = 0$ for x_{t-1}), so that $(\bar{x}^{(-)}, \bar{m})$ is admissible iff $\bar{x}_{\mathrm{BC}}^{(-)} \leq \bar{x}^{(-)} \leq \bar{x}_{\mathrm{BC}}^{(+)}$.

2. If $R+v > 2$ (case (a)), $\bar{x}_{\mathrm{LP}}$ from Lemma 4.3 is smaller than $\bar{x}_{\mathrm{BC}}^{(+)}$, so that equilibria $(\bar{x}^{(\pm)}, \bar{m})$ are admissible at the saddle-node bifurcation. The price deviation $\bar{x}^{(+)}$ increases as β increases and reaches $\bar{x}_{\mathrm{BC}}^{(+)}$ at $\beta = \beta_{\mathrm{BC}}^{(+)}$ (collision with border ∂Z_1). If $R+v < 2$ (case (b)), equilibria $(\bar{x}^{(\pm)}, \bar{m})$ are virtual at the saddle-node bifurcation. The price deviation $\bar{x}^{(-)}$ decreases as β increases and reaches $\bar{x}_{\mathrm{BC}}^{(+)}$ at $\beta = \beta_{\mathrm{BC}}^{(+)}$. If $R+v = 2$ (case (c)), then $\beta_{\mathrm{BC}}^{(+)} = \beta_{\mathrm{LP}}$.

   Equilibrium $(\bar{x}^{(-)}, \bar{m})$ collides with the border ∂Z_2 only if the limit of $\bar{x}^{(-)}$ as $\beta \to \infty$ is below $\bar{x}_{\mathrm{BC}}^{(-)}$. This yields the condition on s and the border collision at $\beta = \beta_{\mathrm{BC}}^{(-)}$ in point (d).

3. For $g > 2R - v$, $\bar{x}^{(+)}$ increases as β increases, whereas $\bar{x}^{(-)}$ decreases, and their limiting value for $\beta \to \infty$ are as in point 2. Equilibrium $(\bar{x}^{(+)}, \bar{m})$ is always virtual, because its limiting value is above $\bar{x}_{\mathrm{BC}}^{(+)}$ for any admissible parameter setting.

Equilibrium $(\bar{x}^{(-)}, \bar{m})$ becomes admissible at $\beta = \beta_{\mathrm{BC}}^{(-)}$ only if its limiting value is above $\bar{x}_{\mathrm{BC}}^{(-)}$, which gives the conditions at points (a) and (b).

4. In the limiting case $\beta \to \infty$, m_t switches between ± 1, so that only one type of trader is present and the three options in Eq. (4.17a) give the same price deviation x_t. The result therefore follows from Lemma 4.3.

■

Proof of Lemma 4.5. From Lemma 4.1, the characteristic equation associated with the nontrivial eigenvalues of the fundamental equilibrium is $\lambda^2 - \gamma_1 \lambda - \gamma_2 = 0$, with

$$\gamma_1 = \frac{1}{2R}\left((1+\bar{m}^{(0)})v + 3(1-\bar{m}^{(0)})\right), \quad \gamma_2 = -\frac{1}{R}(1-\bar{m}^{(0)}).$$

Note that the same characteristic equation is obtained for both predictors (4.23a,b) and (4.23a,c,d), and also for different choices of the confidence function (4.23d), as long as $\partial \alpha_{\mathrm{ROC}} / \partial \mathrm{ROC}|_{\mathrm{ROC}=1} = 0$.

The fundamental equilibrium is stable at $\beta = 0$ (the Routh-Hurwitz-Jury test for 2nd-order polynomials requires $-\gamma_2|_{\beta=0} = 1/R < 1$ and $\gamma_1|_{\beta=0} = (v+3)/(2R) < 1 - \gamma_2|_{\beta=0} = (R+3+R-1)/(2R)$, which are readily satisfied).

As β increases, transcritical and saddle-node (or pitchfork) bifurcations are not possible. In fact, substituting $\lambda_1 = 1$ into the constraints:

$$\lambda_1 + \lambda_2 = \gamma_1, \quad \lambda_1 \lambda_2 = -\gamma_2,$$

and eliminating λ_2, we get the contradiction

$$R = \frac{v}{2}(1 + \bar{m}^{(0)}) + \frac{1}{2}(1 - \bar{m}^{(0)})$$

(with left-hand side larger than one and right-hand side smaller than 1).

Similarly, we exclude flip bifurcations: imposing $\lambda_1 = -1$ in the above constraints and eliminating λ_2, we get the contradiction

$$-R = \frac{v}{2}(1 + \bar{m}^{(0)}) + \frac{5}{2}(1 - \bar{m}^{(0)}).$$

(with left-hand side positive and right-hand side negative).

To look for a Neimark-Sacker bifurcation, we impose $\lambda_1\lambda_2 = 1$ and $|\lambda_1\lambda_2| < 2$. Under the first condition, the second turns into $v(1 + \bar{m}^{(0)}) < R$ that is always satisfied (recall that $\bar{m}^{(0)} < 0$, see Lemma 4.1). Solving the first condition for β gives β_{NS}. ■

5 Conclusions

We conclude this Thesis by making some final remarks about the type of models introduced and studied, why they are useful in economics and finance, their peculiarities and what might be the related critical points.

Adaptive evolutionary models of bounded rationality and heterogeneity of agents have been applied in different contexts to a wide range of fields in economics and finance in recent years. These models are mainly nonlinear dynamical systems and exhibit complex dynamical behaviors. According to the specific economic variables to analyze and the time-changing pattern, they are formulated either in terms of differential or difference equations. When simple, these models have some nice smoothness properties. However, due to regulatory policies or to some constraints in the state variables the dynamics of these models can appear more complicated and characterized by discontinuities. Their peculiarity is given to their adaptive evolutionary processes, but the way in which we model these processes is not unique. There are a variety of dynamical equations that are commonly used. The choice depends on the specific selection or adaptive process that we have to model and on the type of agents involved and on the mathematical properties of these dynamical equations. The choice of a specific evolutionary equation is not without consequences. Different evolutionary equations have different dynamics and sometimes lead to different equilibria. A reasonable choice would be to use the equation that better fits the specific problem and makes the analysis of the model as simple as possible, i.e. to ensure higher mathematical tractability, without losing the main features of the real process.

Models of heterogeneous agents and bounded rationality are increasingly used to evaluate different measures of economic policy and regulations, such as taxes and prohibitions, in different fields of economics and finance, for examples see [109], [17] and [87]. Compared to the traditional general-equilibrium approach, models based on an evolutionary approach are formulated in order to study the effectiveness of regulatory policies when economic behavior evolves through social interactions and learning. Moreover, the evolutionary models are appropriate for studying new forms of regulatory policies or "weak" forms of regulations where the aim of the policy-maker is to stimulate the agents to choose the best strategy among the set of those available without imposing a specific behavior.

All the models proposed in this Thesis can be seen as an attempt to evaluate different forms of regulatory policies by means of models of bounded rationality and heterogeneity of agents. Indeed, each model is related to a specific regulatory policy adopted or possibly adoptable in order to tackle some of the actual economic problems. In particular, in the Thesis we have implemented and analyzed new forms of regulatory policies based on self-selection or self-adjustment mechanisms as in Chapters 2 and 3 as well as strong forms of regulations as in Chapter 4. In Chapter 2 we have demonstrated that a new mechanism of fund raising introduced by a policy-maker and based on financial options can be a suitable solution to increase the quality of life in a city. In Chapter 3 we have seen that weak forms of regulation based on a self-adjustment mechanism, where agents are free to decide which strategy to adopt among a set of those available, can be more effective in reaching the intended objective and are less sensitive to changes in the values of the parameters of the model, in respect to strong forms of regulation, where an authority exogenously imposes the strategy that the agents should adopt. In Chapter 4 we illustrate how a strong form of regulation which bans short selling according to price testing, as it occurs in the New York Stock Exchange, can stabilize a stock market. In each Chapter, the emphasis is on the complex dynamics generated by the models and by the regulatory policies and how these complex dynamics can affect the achievement of the intended objectives. This is justified by the fact that the effectiveness of a regulatory policy changes according to the values for the parameters of the model. There are regions in the parameter space that allow the best performance for the regulation at stake, yet others do not. It follows that understanding the dynamical properties of the system is essential when we evaluate whether to introduce or not a specific regulatory policy. This makes analyzing the dynamics of the model extremely important.

With this Thesis we attempt to show that the analysis of a specific regulatory policy by means of evolutionary models with heterogeneous agents is quite complicated. Although most of the times the execution of a specific regulatory policy allows for the existence of a virtuous equilibrium, where the wellbeing of all the agents is improved, such as the "welfare-improving Nash equilibrium" for the model from Chapter 2 or the equilibrium where fishermen improve their profits without excessively depleting the fish stock for the model from Chapter 3, the way of reaching this equilibrium is not so obvious and it often depends on the initial distribution of the strategies in the populations. Moreover, the virtuous equilibrium can be unstable at times or even if it is stable the probability of reaching it can be quite low due to multiple attractors, path dependence and intermingled basins of attraction. In such cases, the risk of having undesired situations is high with the consequence of losing the financial resources deployed to rich the target, as observed and discussed in Chapter 2. Another important aspect to emphasize is that we are usually forced to introduce discontinuities in the dynamics of the models as we try to model some specific regulations. As a result, we obtain models that are piecewise-smooth. The peculiarity of these models is that *border-collision* bifurcations can generate scenarios of remarkable complexity, including sudden transition from a stable fixed point to a fully developed *robust* (i.e. without period windows) chaotic attractor. This aspect is emerged and discussed in Chapter 4. Other forms of regulatory policies can generate hybrid dynamics, as is the case in Chapter 3. In this context problems of overshooting could arise and a thorough analysis of the model is necessary in order to fully understand under what conditions do these specific regulations bring the desired benefits to the system. All these examples suggest that in a model of adaptive and bounded rationality both forms of weak or strong regulations can be a proxy for complex dynamics. In such a context, the risk of setting down a wrong scheme of incentives is quite high and if we want to reduce and control this risk we need to fully understand the dynamics of the models. In particular, we need to know how the parameters influence the stability of the equilibria and how to modify their values to increase the stability regions. It is at this point that the "Qualitative theory of smooth and non-smooth dynamical systems", the "Bifurcation theory" and some numerical tools for the analysis of dynamical systems, such as "Bifurcation diagrams", become very important for economic analysis.

As a general conclusion of this Thesis, we argue that the difficulty of studying and drawing general conclusions about the validity of a specific regulatory policy in the context of evolutionary selection of boundedly rational strategies is specifically related to the complexity of the dynamics that is partially intrinsic in these kinds of models, but that most of the times it is induced by the specific regulatory policy itself. It follows that studying complex dynamics generated by a specific regulatory policy is the first step for making it effective.

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