Modified dispersion relations lead to a finite zero point gravitational energy

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We compute the zero point energy in a spherically symmetric background distorted at high energy as predicted by *Gravity's Rainbow*. In this context we setup a Sturm-Liouville problem with the cosmological constant considered as the associated eigenvalue. The eigenvalue equation is a reformulation of the Wheeler-DeWitt equation. With the help of a canonical decomposition, we find that the relevant contribution to one loop is given by the graviton quantum fluctuations around the given background. By means of a variational approach based on Gaussian trial functionals, we find that the ordinary divergences can here be handled by an appropriate choice of the rainbow's functions, in contrast to what happens in other conventional approaches. A final discussion on the connection of our result with the observed cosmological constant is also reported.

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I. INTRODUCTION

The idea of promoting general relativity to a quantum level, together with a solution of the cosmological constant problem, is one of the big challenges of our century. Indeed a satisfying *quantum gravity theory* does not exist yet and the enormous gap of 10^{120} orders of magnitude between the predicted theoretical value of the cosmological constant and the observed one has not yet found a compelling explanation. If the quantum description of nature is appropriate for every force, it should be applicable even to the gravitational force described by general relativity. But perhaps general relativity, as it stands, requires a change In this respect, various proposals on how the fundamental aspects of special relativity can be modified at very high energies have been done. Among these proposals, particularly promising appears to be the one known as *doubly special relativity* (DSR)[1]. One of the characterizing DSR effects is that the usual dispersion relation of a massive particle of mass m is modified into the following expression

$$E^2 g_1^2(E/E_P) - p^2 g_2^2(E/E_P) = m^2, \tag{1}$$

where $g_1(E/E_P)$ and $g_2(E/E_P)$ are two arbitrary functions which have the following property

$$\lim_{E/E_P\to 0} g_1(E/E_P) = 1 \quad \text{and} \quad \lim_{E/E_P\to 0} g_2(E/E_P) = 1. \tag{2}$$

Thus, the usual dispersion relation is recovered at low energies. Of course, the first ideas of DSR were minted for flat space. However, nothing forbids us to consider a curved background and therefore to enter into the realm of general relativity. From this point of view, Magueijo and Smolin [2] proposed that the energy-momentum tensor and the Einstein's field equations were modified with the introduction of a one-parameter family of equations

$$G_{\mu\nu}(E) = 8\pi G(E) T_{\mu\nu}(E) + g_{\mu\nu}\Lambda(E), \tag{3}$$

where $G(E)$ is an energy dependent Newton's constant, defined so that $G(0)$ is the low-energy Newton's constant. Similarly we have an energy dependent cosmological constant $\Lambda(E)$ leading to the *rainbow* version of the Schwarzschild line element

$$ds^2 = -\left(1 - \frac{2MG(0)}{r}\right)\frac{d\tilde{t}^2}{g_1^2(E/E_P)} + \frac{d\tilde{r}^2}{\left(1 - \frac{2MG(0)}{r}\right) g_2^2(E/E_P)} + \frac{\tilde{r}^2}{g_2^2(E/E_P)}(d\theta^2 + \sin^2\theta d\phi^2). \tag{4}$$

Since the functions $g_1(E/E_P)$ and $g_2(E/E_P)$ come into play when the energy E is comparable with E_P, it is likely that they modify the UV behavior in the same way as generalized uncertainty principle and noncommutative geometry (NCG) do, respectively. If the effect of generalized uncertainty principle and NCG is to modify the Liouville measure d^3xd^3k from one side, and to introduce a granularity from the other side, the *rainbow metric*) should be able to introduce a natural UV regulator hidden into the arbitrary functions $g_1(E/E_P)$ and $g_2(E/E_P)$. An encouraging partial answer has been obtained in Ref. [3], where an application of gravity's rainbow to black hole entropy computation has been considered. In that paper the UV

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regulator, namely, the *brick wall*, has been eliminated with the help of the following choice of $g_1(E/E_P)$ and $g_2(E/E_P)$

$$\frac{g_1(E/E_P)}{g_2(E/E_P)} = \exp\left(-\frac{E}{E_P}\right). \tag{5}$$

An interesting test to see gravity's rainbow at work again, should be the computation of zero point energy (ZPE). Nevertheless, we have to remark that any computation of ZPE leads to a regularization and subsequently to a renormalization process in order to have finite physical quantities. Therefore, the purpose of the paper is to show that, with the introduction of appropriate gravity's rainbow functions, it is possible to overpass the renormalization problem. Of course, this proposal does not represent a complete cure to have a finite theory of Quantum Gravity, but rather it suggests how the modification of some basic principles like the introduction of an energy dependent metric can lead to unexpected results such as the avoidance of a renormalization scheme. In ordinary gravity the computation of ZPE for quantum fluctuations of the *pure gravitational field* can be extracted by rewriting the Wheeler-DeWitt equation (WDW) [4] in a form which looks like an expectation value computation [5]. We remind the reader that the WDW equation is the quantum version of the classical constraint which guarantees the invariance under time reparametrization. Its original form with the cosmological term included is described by

$$\mathcal{H}\Psi = \left[(2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}({}^3R - 2\Lambda)\right]\Psi = 0. \tag{6}$$

Note that $\mathcal{H} = 0$ represents one of the classical constraints. The other one is the invariance by spatial diffeomorphism. If we multiply Eq. (6) by $\Psi^*[g_{ij}]$ and functionally integrate over the three spatial metric g_{ij}, we can write[1] [5]

$$\frac{1}{V}\frac{\int \mathcal{D}[g_{ij}]\Psi^*[g_{ij}]\int_\Sigma d^3x\hat{\Lambda}_\Sigma\Psi[g_{ij}]}{\int \mathcal{D}[g_{ij}]\Psi^*[g_{ij}]\Psi[g_{ij}]} = \frac{1}{V}\frac{\langle\Psi|\int_\Sigma d^3x\hat{\Lambda}_\Sigma|\Psi\rangle}{\langle\Psi|\Psi\rangle} = -\frac{\Lambda}{\kappa}, \tag{7}$$

where we have also integrated over the hypersurface Σ and we have defined

$$V = \int_\Sigma d^3x\sqrt{g} \tag{8}$$

as the volume of the hypersurface Σ with

$$\hat{\Lambda}_\Sigma = (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{g}{}^3R/(2\kappa). \tag{9}$$

[1] See also Ref. [6] for an application of the method to a $f(R)$ theory.

In this form, Eq. (7) can be used to compute ZPE, provided that Λ/κ be considered as an eigenvalue of $\hat{\Lambda}_\Sigma$; namely, the WDW equation is transformed into an expectation value computation. In Eq. (6), G_{ijkl} is the supermetric, π^{ij} is the supermomentum, ${}^3R$ is the scalar curvature in three dimensions, and Λ is the cosmological constant, while $\kappa = 8\pi G$ with G the Newton's constant. Nevertheless, solving Eq. (7) is a quite impossible task, therefore we are oriented to use a variational approach with trial wave functionals. The related boundary conditions are dictated by the choice of the trial wave functionals which, in our case, are of the Gaussian type. Different types of wave functionals correspond to different boundary conditions. The choice of a Gaussian wave functional is justified by the fact that ZPE should be described as a good candidate of the ''. To fix the ideas, a variant of the line element (4) will be considered

$$ds^2 = -N^2(r)\frac{dt^2}{g_1^2(E)} + \frac{dr^2}{(1 - \frac{b(r)}{r})g_2^2(E)} + \frac{r^2}{g_2^2(E)}(d\theta^2 + \sin^2\theta d\phi^2), \tag{10}$$

where N is the lapse function and $b(r)$ is subject to the only condition $b(r_t) = r_t$. Metric (10) will be the cornerstone of the whole paper, which is organized as follows. In Sec. II, we derive the Hamiltonian constraint in presence of the background (10), in Sec. III we compute the ZPE of quantum fluctuations around the background (10) and, with the help of an appropriate choice of the functions $g_1(E/E_P)$ and $g_2(E/E_P)$, we will show that the UV divergences of ZPE disappear. We summarize and conclude in Sec. IV. Units in which $\hbar = c = k = 1$ are used throughout the paper.

II. THE HAMILTONIAN CONSTRAINT IN GRAVITY'S RAINBOW

In order to use Eq. (7) for the metric (10), we need to understand how the WDW modifies when the functions $g_1(E/E_P)$ and $g_2(E/E_P)$ distort the background. It is therefore necessary to understand how some basic ingredients change under the transformation of the line element (10). The form of the background is such that the *shift function*

$$N^i = -Nu^i = g_0^{4i} = 0 \tag{11}$$

vanishes, while N is the previously defined *lapse function*. Thus the definition of K_{ij} implies

$$K_{ij} = -\frac{\dot{g}_{ij}}{2N} = \frac{g_1(E)}{g_2^2(E)}\tilde{K}_{ij}, \tag{12}$$

where the dot denotes differentiation with respect to the time t and the tilde indicates the quantity computed in absence of rainbow's functions $g_1(E)$ and $g_2(E)$. For simplicity, we have set $E_P = 1$ in $g_1(E/E_P)$ and $g_2(E/E_P)$

throughout the paragraph. The trace of the extrinsic curvature, therefore, becomes

$$K = g^{ij}K_{ij} = g_1(E)\tilde{K} \tag{13}$$

and the momentum π^{ij} conjugates to the three-metric g_{ij} of Σ is

$$\pi^{ij} = \frac{\sqrt{g}}{2\kappa}(Kg^{ij} - K^{ij}) = \frac{g_1(E)}{g_2(E)}\tilde{\pi}^{ij}. \tag{14}$$

Since the rainbow's functions distort the classical constraint, it could be a good test to verify if it is satisfied for the background (10). The distorted classical constraint becomes

$$\mathcal{H} = (2\kappa)\frac{g_1^2(E)}{g_2^3(E)}\tilde{G}_{ijkl}\tilde{\pi}^{ij}\tilde{\pi}^{kl} - \frac{\sqrt{\tilde{g}}}{2\kappa g_2(E)}\left(\tilde{R} - \frac{2\Lambda_c}{g_2^2(E)}\right) = 0, \tag{15}$$

where we have used the following property on R

$$R = g^{ij}R_{ij} = g_2^2(E)\tilde{R} \tag{16}$$

and where

$$G_{ijkl} = \frac{1}{2\sqrt{g}}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}) = \frac{\tilde{G}_{ijkl}}{g_2(E)}. \tag{17}$$

Note that the classical constraint (15) can be extracted directly from the distorted Einstein's field Eqs. (3) with the help of the following procedure

$$G_{\mu\nu}(E)u^\mu u^\nu = 8\pi G(E)T_{\mu\nu}(E)u^\mu u^\nu + g_{\mu\nu}\Lambda(E)u^\mu u^\nu, \tag{18}$$

where $g_{\mu\nu}u^\mu u^\nu = -1$ and u^μ is a timelike vector normal to the spatial hypersurface. Since the metric is static, the term containing $\tilde{\pi}^{ij}$ vanishes. This can be easily verified from the definition of the extrinsic curvature K_{ij} of Eq. (12). From Eq. (10) and from the property (16), we find

$$R = g^{ij}R_{ij} = g_2^2(E)\tilde{R} = 2g_2^2(E)\frac{b'(r)}{r^2}, \tag{19}$$

where we have used the mixed Ricci tensor $\tilde{R}_j^a$ whose components are:

$$\tilde{R}_j^a = \left\{\frac{b'(r)}{r^2} - \frac{b(r)}{r^3}, \frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3}, \frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3}\right\}. \tag{20}$$

Therefore the Hamiltonian constraint (15) is reduced to

$$\mathcal{H} = 2\frac{b'(r)}{r^2} - \frac{2\Lambda_c}{g_2^2(E)} = 0. \tag{21}$$

For the Schwarzschild case, we find $b(r) = 2MG$ and $\mathcal{H} = 0$ if and only if $\Lambda_c = 0$, for every choice of $g_2(E)$ as it should be. With regard to the de Sitter (dS) and anti-de Sitter (AdS) cases, we find that the classical constraint (15) is satisfied if

$$\begin{aligned} \mathcal{H} &= \Lambda_{\mathrm{dS}} - \frac{\Lambda_c}{g_2^2(E)} = 0 \Rightarrow \Lambda_{\mathrm{dS}}g_2^2(E) = \Lambda_c \\ b(r) &= \frac{\Lambda_{\mathrm{dS}}}{3}r^3, \end{aligned} \tag{22}$$

for the dS background and

$$\begin{aligned} \mathcal{H} &= \Lambda_{\mathrm{AdS}} + \frac{\Lambda_c}{g_2^2(E)} = 0 \Rightarrow -\Lambda_{\mathrm{AdS}}g_2^2(E) = \Lambda_c \\ b(r) &= -\frac{\Lambda_{\mathrm{AdS}}}{3}r^3 \end{aligned} \tag{23}$$

for the AdS background. Now that we have investigated the classical part, we can define the WDW equation for the background (10). From Eq. (6) we find that $\mathcal{H}\Psi = 0$ becomes

$$\mathcal{H}\Psi = \left[(2\kappa)\frac{g_1^2(E)}{g_2^3(E)}\tilde{G}_{ijkl}\tilde{\pi}^{ij}\tilde{\pi}^{kl} - \frac{\sqrt{\tilde{g}}}{2\kappa g_2(E)}\left(\tilde{R} - \frac{2\Lambda_c}{g_2^2(E)}\right)\right]\Psi = 0 \tag{24}$$

and the Eq. (7) transforms into

$$\frac{g_2^3(E)}{\tilde{V}}\frac{\langle\Psi|\int_\Sigma d^3x\tilde{\Lambda}_\Sigma|\Psi\rangle}{\langle\Psi|\Psi\rangle} = -\frac{\Lambda_c}{\kappa}, \tag{25}$$

where

$$\tilde{\Lambda}_\Sigma = (2\kappa)\frac{g_1^2(E)}{g_2^3(E)}\tilde{G}_{ijkl}\tilde{\pi}^{ij}\tilde{\pi}^{kl} - \frac{\sqrt{\tilde{g}}\tilde{R}}{(2\kappa)g_2(E)}. \tag{26}$$

We can gain more information if we consider $g_{ij} = \bar{g}_{ij} + h_{ij}$, where $\bar{g}_{ij}$ is the background metric and h_{ij} is a quantum fluctuation around the background. Thus, Eq. (25) can be expanded in terms of h_{ij}. Since the kinetic part of $\hat{\Lambda}_\Sigma$ is quadratic in the momenta, we only need to expand the three-scalar curvature $\int d^3x\sqrt{g}^3R$ up to the quadratic order. However, to proceed with the computation, we also need an orthogonal decomposition on the tangent space of 3-metric deformations [7,8]:

$$h_{ij} = \frac{1}{3}(\sigma + 2\nabla\cdot\xi)g_{ij} + (L\xi)_{ij} + h_{ij}^\perp. \tag{27}$$

The operator L maps ξ_i into symmetric tracefree tensors

$$(L\xi)_{ij} = \nabla_i\xi_j + \nabla_j\xi_i - \frac{2}{3}g_{ij}(\nabla\cdot\xi), \tag{28}$$

$h_{ij}^\perp$ is the traceless-transverse component of the perturbation (TT), namely

$$g^{ij}h_{ij}^\perp = 0, \qquad \nabla^i h_{ij}^\perp = 0 \tag{29}$$

and h is the trace of h_{ij}. It can immediately be recognized that the trace element $\sigma = h - 2(\nabla\cdot\xi)$ is gauge-invariant. It is straightforward to see that the gauge-invariant decomposition (27) does not change, when we consider the rainbow's metric (10). Therefore, following the results of Ref. [9], we can use the final expression

$$\frac{1}{V}\frac{\langle\Psi^{\perp}|\int_{\Sigma}d^3x[\hat{\Lambda}_{\Sigma}^{\perp}]^{(2)}|\Psi^{\perp}\rangle}{\langle\Psi^{\perp}|\Psi^{\perp}\rangle}+\frac{1}{V}\frac{\langle\Psi^{\sigma}|\int_{\Sigma}d^3x[\hat{\Lambda}_{\Sigma}^{\sigma}]^{(2)}|\Psi^{\sigma}\rangle}{\langle\Psi^{\sigma}|\Psi^{\sigma}\rangle}=-\frac{\Lambda_c}{\kappa}. \tag{30}$$

Note that in the expansion of $\int_{\Sigma}d^3x\sqrt{g}R$ to second order in terms of h_{ij}, a coupling term between the TT component and the scalar one remains. However, the Gaussian integration does not allow such a mixing, which has to be introduced with an appropriate wave functional. Extracting the TT tensor contribution from Eq. (25), we find

$$\hat{\Lambda}_{\Sigma}^{\perp}=\frac{g_2^3(E)}{4\tilde{V}}\int_{\Sigma}d^3x\sqrt{\sim\bar{g}}\tilde{G}^{ijkl}\Big[(2\kappa)\frac{g_1^2(E)}{g_2^3(E)}\tilde{K}^{-1\perp}(x,x)_{ijkl}+\frac{1}{(2\kappa)g_2(E)}(\tilde{\triangle}_L^m\tilde{K}^{\perp}(x,x))_{ijkl}\Big]. \tag{31}$$

The origin of the operator $\tilde{\triangle}_L^m$ comes from

$$(\hat{\triangle}_L^m h^{\perp})_{ij}=(\triangle_L h^{\perp})_{ij}-4R_i^k h_{kj}^{\perp}+{}^3Rh_{ij}^{\perp}, \tag{32}$$

which is the modified Lichnerowicz operator where $\triangle_L$ is the Lichnerowicz operator defined by

$$(\triangle_L h)_{ij}=\triangle h_{ij}-2R_{ikjl}h^{kl}+R_{ik}h_j^k+R_{jk}h_i^k$$
$$\triangle=-\nabla^a\nabla_a. \tag{33}$$

G^{ijkl} represents the inverse DeWitt metric without the $\sqrt{g}$ factor and all indices run from one to three. Note that the term

$$-4R_i^k h_{kj}^{\perp}+{}^3Rh_{ij}^{\perp} \tag{34}$$

disappears in four dimensions when we use a background which is a solution of the Einstein's field equations without matter contribution. The "$\sim$" symbol in Eq. (31) means that we have rescaled every piece in Eq. (25), evaluated at second order. Moreover, although the expression of $\tilde{\Lambda}_{\Sigma}$ explicitly shows how the operator $\hat{\Lambda}_{\Sigma}$ globally changes, when we consider the following eigenvalue equation

$$(\hat{\triangle}_L^m h^{\perp})_{ij}=E^2h_{ij}^{\perp}, \tag{35}$$

we find that

$$(\tilde{\triangle}_L^m\tilde{h}^{\perp})_{ij}=\frac{E^2}{g_2^2(E)}\tilde{h}_{ij}^{\perp}, \tag{36}$$

in order to reestablish the correct way of transformation of the perturbation. Then, the propagator $K^{\perp}(x,x)_{iakl}$ can be represented as

$$K^{\perp}(\vec{x},\vec{y})_{iakl}=\tilde{K}^{\perp}(\vec{x},\vec{y})_{iakl}=\sum_{\tau}\frac{\tilde{h}_{ia}^{(\tau)\perp}(\vec{x})\tilde{h}_{kl}^{(\tau)\perp}(\vec{y})}{2\lambda(\tau)g_2^4(E)}, \tag{37}$$

where $\tilde{h}_{ia}^{(\tau)\perp}(\vec{x})$ are the eigenfunctions of $\tilde{\triangle}_L^m$. τ denotes a complete set of indices and $\lambda(\tau)$ are a set of variational parameters to be determined by the minimization of Eq. (31). The expectation value of $\hat{\Lambda}_{\Sigma}^{\perp}$ is easily obtained by inserting the form of the propagator into Eq. (31) and minimizing with respect to the variational function $\lambda(\tau)$. Thus the total one-loop energy density for TT tensors becomes[2]

$$\frac{\Lambda}{8\pi G}=-\frac{1}{2}\sum_{\tau}g_1(E)g_2(E)[\sqrt{E_1^2(\tau)}+\sqrt{E_2^2(\tau)}]. \tag{39}$$

The above expression makes sense only for $E_i^2(\tau)>0$, where E_i are the eigenvalues of $\tilde{\triangle}_L^m$. With the help of Regge and Wheeler representation [10], the eigenvalue Eq. (35) can be reduced to

$$\Big[-\frac{d^2}{dx^2}+\frac{l(l+1)}{r^2}+m_i^2(r)\Big]f_i(x)=\frac{E_{i,l}^2}{g_2^2(E)}f_i(x)\qquad i=1,2, \tag{40}$$

where we have used reduced fields of the form $f_i(x)=F_i(x)/r$ and where we have defined two r-dependent effective masses $m_1^2(r)$ and $m_2^2(r)$

$$m_1^2(r)=\frac{6}{r^2}\Big(1-\frac{b(r)}{r}\Big)+\frac{3}{2r^2}b'(r)-\frac{3}{2r^3}b(r)$$
$$m_2^2(r)=\frac{6}{r^2}\Big(1-\frac{b(r)}{r}\Big)+\frac{1}{2r^2}b'(r)+\frac{3}{2r^3}b(r)\qquad(r\equiv r(x)). \tag{41}$$

In order to use the WKB approximation, from Eq. (40) we can extract two r-dependent radial wave numbers

$$k_i^2(r,l,\omega_{i,nl})=\frac{E_{i,nl}^2}{g_2^2(E)}-\frac{l(l+1)}{r^2}-m_i^2(r)\qquad i=1,2. \tag{42}$$

III. ONE-LOOP ENERGY IN AN ORDINARY SPHERICALLY SYMMETRIC BACKGROUND

It is now possible to explicitly evaluate Eq. (39) in terms of the effective mass. To further proceed, we use the WKB

[2]Note that one could discuss the following eigenvalue equation $(\hat{\triangle}_L^m h^{\perp})_{ij}=\omega h_{ij}^{\perp}$ rather than Eq. (35), with $\omega\neq E$ having nonetheless energy square dimensions. This choice leads to an induced $\Lambda/8\pi G$, which cannot be regularized by any choice of the rainbow's functions. This can be easily understood by looking at Eq. (39). Indeed, if we set $\omega\neq E$, the total one-loop energy density becomes

$$\frac{\Lambda}{8\pi G}=-\frac{1}{2}\sum_{\tau}g_1(E)g_2(E)\Big[\sqrt{\omega_1(\tau)}+\sqrt{\omega_2(\tau)}\Big] \tag{38}$$

and the expression is divergent. Therefore this option will be discarded.

method used by 't Hooft in the brick wall problem [11] and we count the number of modes with frequency less than ω_i, $i = 1, 2$. This is given approximately by

$$\tilde{g}(E_i) = \int_0^{l_{\max}} \nu_i(l, E_i)(2l+1)dl, \quad (43)$$

where $\nu_i(l, E_i)$, $i = 1, 2$ is the number of nodes in the mode with (l, E_i), such that $(r \equiv r(x))$

$$\nu_i(l, E_i) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dx\sqrt{k_i^2(r, l, E_i)}. \quad (44)$$

Here it is understood that the integration with respect to x and $l_{\max}$ is taken over those values which satisfy $k_i^2(r, l, E_i) \geq 0$, $i = 1, 2$. With the help of Eqs. (43), (44), and (39), this leads to

$$\frac{\Lambda}{8\pi G} = -\frac{1}{\pi}\sum_{i=1}^{2}\int_0^{+\infty} E_i g_1(E) g_2(E)\frac{d\tilde{g}(E_i)}{dE_i}dE_i. \quad (45)$$

This is the graviton contribution to the induced cosmological constant to one loop. The explicit evaluation of the density of states yields

$$\begin{aligned}\frac{d\tilde{g}(E_i)}{dE_i} &= \int\frac{\partial\nu(l, E_i)}{\partial E_i}(2l+1)dl \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty}dx\int_0^{l_{\max}}\frac{(2l+1)}{\sqrt{k^2(r,l,E)}}\frac{d}{dE_i}\left(\frac{E_i^2}{g_2^2(E)} - m_i^2(r)\right)dl \\ &= \frac{2}{\pi}\int_{-\infty}^{+\infty}dx r^2\frac{d}{dE_i}\left(\frac{E_i^2}{g_2^2(E)} - m_i^2(r)\right)\sqrt{\frac{E_i^2}{g_2^2(E)} - m_i^2(r)} \\ &= \frac{4}{3\pi}\int_{-\infty}^{+\infty}dx r^2\frac{d}{dE_i}\left(\frac{E_i^2}{g_2^2(E)} - m_i^2(r)\right)^{3/2}. \end{aligned} \quad (46)$$

Plugging expression (46) into Eq. (45) and dividing for a volume factor, we obtain

$$\frac{\Lambda}{8\pi G} = -\frac{1}{3\pi^2}\sum_{i=1}^{2}\int_{E^*}^{+\infty} E_i g_1(E) g_2(E)\frac{d}{dE_i} \times\sqrt{\left(\frac{E_i^2}{g_2^2(E)} - m_i^2(r)\right)^3}dE_i, \quad (47)$$

where E^* is the value which annihilates the argument of the root. In the previous equation, we have included an additional 4π factor coming from the angular integration and we have assumed that the effective mass does not depend on the energy E. To further proceed, we can see what happens to the expression (47) for some specific forms of $g_1(E/E_P)$ and $g_2(E/E_P)$. One popular choice is given by

$$g_1(E/E_P) = 1 - \eta(E/E_P)^n \quad \text{and} \quad g_2(E/E_P) = 1, \quad (48)$$

where η is a dimensionless parameter and n is an integer [12]. Nevertheless, the above choice does not allow a finite result in Eq. (47) and therefore will be discarded. Thus the choice of the possible forms of $g_1(E/E_P)$ and $g_2(E/E_P)$ is strongly restricted by convergence criteria. We have hitherto used a generic form of the background. We now fix the attention on some backgrounds which have the following property

$$m_0^2(r) = m_2^2(r) = -m_1^2(r), \quad \forall\ r \in (r_t, r_1). \quad (49)$$

For example, the Schwarzschild background represented by the choice $b(r) = r_t = 2MG$ satisfies the property (49) in the range $r \in [r_t, 5r_t/2]$. Similar backgrounds are the Schwarzschild-de Sitter and Schwarzschild–anti-de Sitter. On the other hand, other backgrounds, like dS, AdS and Minkowski have the property

$$m_0^2(r) = m_2^2(r) = m_1^2(r), \quad \forall\ r \in (r_t, \infty). \quad (50)$$

If case (49) holds, Eq. (47) becomes

$$\frac{\Lambda}{8\pi G} = -\frac{1}{3\pi^2}(I_+ + I_-), \quad (51)$$

where

$$I_+ = \int_0^{\infty}(Eg_1(E/E_P)g_2(E/E_P)) \times\frac{d}{dE}\left(\frac{E^2}{g_2^2(E/E_P)} + m_0^2(r)\right)^{3/2}dE \quad (52)$$

and

$$I_- = \int_{E^*}^{\infty}(Eg_1(E/E_P)g_2(E/E_P)) \times\frac{d}{dE}\left(\frac{E^2}{g_2^2(E/E_P)} - m_0^2(r)\right)^{3/2}dE. \quad (53)$$

Instead, in case condition (50) holds, Eq. (47) becomes

$$\frac{\Lambda}{8\pi G} = -\frac{2}{3\pi^2}I_-. \quad (54)$$

We begin to look at Eq. (51). It can immediately be seen that integrals I_+ and I_- can be easily solved for a very particular choice. Indeed, if we set

$$g_2^{-2}(E/E_P) = g_1(E/E_P) \quad (55)$$

we find that I_+ and I_- take the form:

$$I_+ = 3\int_0^{\infty}\left(\frac{E}{g_2(E/E_P)}\right)^2\frac{d}{dE}\left(\frac{E}{g_2(E/E_P)}\right) \times\sqrt{\left(\frac{E}{g_2(E/E_P)}\right)^2 + m_0^2(r)}dE \quad (56)$$

and

$$I_- = 3\int_{E^*}^{\infty}\left(\frac{E}{g_2(E/E_P)}\right)^2 \frac{d}{dE}\left(\frac{E}{g_2(E/E_P)}\right) \times \sqrt{\left(\frac{E}{g_2(E/E_P)}\right)^2 - m_0^2(r)}dE. \tag{57}$$

The above integrals can be easily evaluated using the auxiliary variable

$$z(E/E_P) = \frac{E/E_P}{g_2(E/E_P)} \tag{58}$$

so that Eq. (47) becomes:

$$\frac{\Lambda}{8\pi G} = -\frac{E_P^4}{\pi^2}\left\{\int_x^{z_\infty} z^2\sqrt{z^2 - x^2}dz + \int_0^{z_\infty} z^2\sqrt{z^2 + x^2}dz\right\}, \tag{59}$$

where $z_\infty = \lim_{E\to\infty} z(E/E_P)$ and $x = \sqrt{m_0^2(r)/E_P^2}$. The integrals involved in Eq. (59) can be calculated straightforwardly being

$$I_{1,x}(z) = \int z^2\sqrt{z^2 - x^2}dz = \frac{1}{8}\{z(2z^2 - x^2)\sqrt{z^2 - x^2} - x^4 \log[2(z + \sqrt{z^2 - x^2})]\} \tag{60}$$

and

$$I_{2,x}(z) = \int z^2\sqrt{z^2 + x^2}dz = \frac{1}{8}\{z(2z^2 + x^2)\sqrt{z^2 + x^2} - x^4 \log[2(z + \sqrt{z^2 + x^2})]\}. \tag{61}$$

Thus we get the final expression:

$$\frac{\Lambda}{8\pi G} = -\frac{E_P^4}{\pi^2}\{I_{1,x}(z_\infty) - I_{1,x}(x) + I_{2,x}(z_\infty) - I_{2,x}(0)\}, \tag{62}$$

for the case with $x < z_\infty$, and the expression

$$\frac{\Lambda}{8\pi G} = -\frac{E_P^4}{\pi^2}\{I_{2,x}(z_\infty) - I_{2,x}(0)\}, \tag{63}$$

for the case with $z_\infty < x$. In particular, for the class of rainbow functions that satisfy the condition $z_\infty = 0$, we get a vanishing cosmological constant:

$$\frac{\Lambda}{8\pi G} = 0. \tag{64}$$

Although very appealing, the result (64) presents the unpleasant feature of being always negative, even in the region of space where we would expect a positive cosmological constant. Moreover, it is independent on the choice of $g(E/E_P)$, provided that this last one can guarantee the convergence of the integral and the absence of imaginary factors. For these reasons, we are led to investigate other forms of $g_1(E/E_P)$ and $g_2(E/E_P)$ even if they have less symmetry with respect to proposal (55). The only restrictions we have are the low-energy limit (2) and the convergence requirement for the integrals (52) and (53). To do calculations in practice, a useful choice is the following

$$g_1(E/E_P) = \sum_{i=0}^{n} \beta_i \frac{E^i}{E_P^i} \exp\left(-\alpha \frac{E^2}{E_P^2}\right), \quad g_2(E/E_P) = 1; \qquad \alpha > 0, \qquad \beta_i \in \mathbb{R}. \tag{65}$$

The use of a "Gaussian" form is dictated by the possibility of doing a comparison with NCG models. Indeed, in Ref. [13] the authors have considered a distortion induced by an underlying NCG on the counting of states. Basically, one finds that the number of states is modified in the following way

$$dn = \frac{d^3xd^3k}{(2\pi)^3} \Rightarrow dn_i = \frac{d^3xd^3k}{(2\pi)^3}\exp\left(-\frac{\theta}{4}(\omega_{i,nl}^2 - m_i^2(r))\right), \qquad i = 1, 2, \tag{66}$$

where the UV cutoff is triggered only by higher momenta modes $\gtrsim 1/\sqrt{\theta}$ which propagate over the background geometry. Then the induced cosmological constant becomes

$$\frac{\Lambda}{8\pi G} = \frac{1}{6\pi^2}\left[\int_{\sqrt{m_0^2(r)}}^{+\infty} \sqrt{(\omega^2 - m_0^2(r))^3} e^{-(\theta/4)(\omega^2 - m_0^2(r))} d\omega + \int_0^{+\infty} \sqrt{(\omega^2 + m_0^2(r))^3} e^{-(\theta/4)(\omega^2 + m_0^2(r))} d\omega\right]. \tag{67}$$

The analogy with the choice (65) can be seen immediately. However, Eq. (67) leads directly to a positive-induced cosmological constant, while Eq. (51) needs an appropriate choice of $g_1(E/E_P)$ and $g_2(E/E_P)$ to induce a positive part. After choice (65), the graviton contribution terms (52) and (53) become

$$I_+ = 3\int_0^{\infty}\left(\sum_{i=0}^{n} \beta_i \frac{E^i}{E_P^i} \exp\left(-\alpha \frac{E^2}{E_P^2}\right)\right) E^2\sqrt{E^2 + m_0^2(r)}dE \tag{68}$$

and

$$I_- = 3\int_{\sqrt{m_0^2(r)}}^{\infty}\left(\sum_{i=0}^{n} \beta_i \frac{E^i}{E_P^i} \exp\left(-\alpha \frac{E^2}{E_P^2}\right)\right) E^2\sqrt{E^2 - m_0^2(r)}dE. \tag{69}$$

In the appendix, we explicitly compute the integrals (68) and (69) for every n. In order to motivate choice (65), we have to observe that the case with $n = 0$ leads to a negative value of $\Lambda/8\pi G$ for every kind of background as one can see from Eq. (51). Thus, it is necessary to make a correction on the pure Gaussian choice in such a way that we have a possible change of sign in $\Lambda/8\pi G$. For our purposes, it is sufficient to discuss the case with $n = 1$ and $n = 3$. We begin with $n = 1$.

A. Example a) $n = 1$

After integration, for $n = 1$, Eq. (51) can be rearranged in the following way

$$\frac{\Lambda}{8\pi G E_P^4} \equiv \frac{\Lambda}{8\pi G E_P^4}(\alpha;\beta;x) = -\frac{1}{2\pi^2}\left[\frac{x^2}{\alpha}\cosh\left(\frac{\alpha x^2}{2}\right)K_1\left(\frac{\alpha x^2}{2}\right) - \beta\left(\frac{3x}{2\alpha^2} - \frac{x^2\sqrt{\pi}}{\alpha^{3/2}}\sinh(\alpha x^2) + \frac{3\sqrt{\pi}}{2\alpha^{5/2}}\cosh(\alpha x^2) + \frac{\sqrt{\pi}}{2\alpha^{3/2}}\left(x^2 - \frac{3}{2\alpha}\right)e^{\alpha x^2}\operatorname{erf}(\sqrt{\alpha}x)\right)\right], \tag{70}$$

where, again, $x = \sqrt{m_0^2(r)/E_P^2}$, $\beta_1 \equiv \beta$ and where $K_0(x)$ is the Bessel function and erf (x) is the error function. It is clear that for every choice of the couple (α, β) there exists a curve with a different behavior. Therefore, to fix ideas, we will fix the Gaussian factor α to the same one proposed by the NCG setting of Eq. (67). Before doing this, it is useful to compute the series expansion for small and large x. For large x one gets

$$\frac{\Lambda}{8\pi G E_P^4} \simeq -\frac{(2\beta\alpha^{3/2} + \sqrt{\pi}\alpha^2)x}{4\pi^2\alpha^{7/2}} - \frac{8\beta\alpha^{5/2} + 3\sqrt{\pi}\alpha^3}{16\pi^2\alpha^{11/2}x} + \frac{3}{128\pi^2}\frac{16\beta\alpha^{7/2} + 5\sqrt{\pi}\alpha^4}{\alpha^{15/2}x^3} + O(x^{-4}), \tag{71}$$

while for small x we obtain

$$\frac{\Lambda}{8\pi G E_P^4} \simeq -\frac{4\alpha^{5/2} + 3\sqrt{\pi}\beta\alpha^2}{4\pi^2\alpha^{9/2}} + O(x^3). \tag{72}$$

It is straightforward to see that if we set

$$\beta = -\frac{\sqrt{\alpha\pi}}{2}, \tag{73}$$

then the linear divergent term of the asymptotic expansion (71) disappears and Eq. (70) vanishes for large x. Plugging Eq. (73) into expansion (72), we obtain

$$\frac{\Lambda}{8\pi G E_P^4} \simeq \frac{3\pi - 8}{8\pi^2\alpha^2} + O(x^3), \tag{74}$$

which means that for $x = 0$, the induced cosmological constant never vanishes and therefore cannot be a good candidate to reproduce the Minkowskian limit. Indeed, we have to recall that the variable x expresses the curvature of the background through the shape function $b(r)$, which for Minkowski vanishes. On the other hand, the vanishing of expression (71) and consequently Eq. (70) for $x \to \infty$ offers a good candidate for large distance estimates. Alternatively, by imposing that

$$\beta = -\frac{4}{3}\sqrt{\frac{\alpha}{\pi}}, \tag{75}$$

the expression (72) vanishes for small x, while for large x, the leading term becomes

$$\frac{\Lambda}{8\pi G E_P^4} \simeq -\frac{(3\pi - 8)x}{12\sqrt{(\pi\alpha)^3}}. \tag{76}$$

This means that Eq. (70) diverges towards negative values. It is straightforward to see that we cannot simultaneously fix both the conditions (73) and (75) for the same α in order to have a vanishing expectation value of $\Lambda/8\pi G$ for small and large x, unless we consider different values for α for the different behaviors. The idea is to find a point where a transition from one parametrization to the other one exists. To begin, we have to observe that if we fix one couple of parameters to

$$\alpha_1 = \frac{1}{4}, \qquad \beta = -\frac{4}{3}\sqrt{\frac{\alpha_1}{\pi}}, \tag{77}$$

where α_1 has the same value of the numerical factor appearing in Eq. (67) and the second couple with generic values, one discovers multiple roots where a smooth transition from one parametrization to the other one can happen. This is illustrated in Fig. 1, where the couple (77) together with some generic values of the couple satisfying condition (73) are shown. It is visible the presence of multiple roots. It can also immediately be seen that there

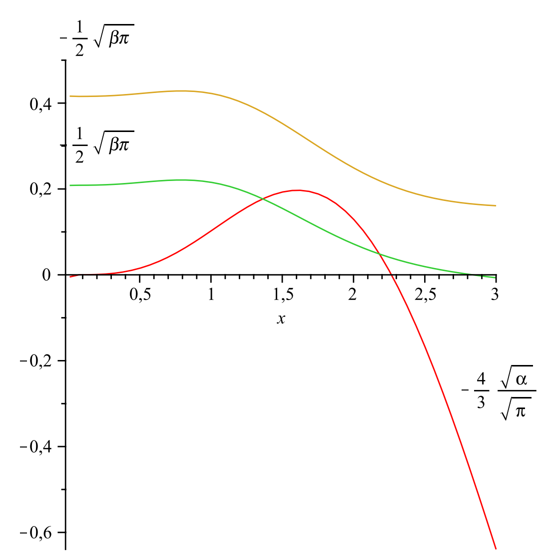

FIG. 1 (color online). Plot of $\Lambda/8\pi G$ as a function of the scale-invariant x. Choosing parametrization (77), we obtain the vanishing of $\Lambda/8\pi G$ when $x \to 0$. The other curves satisfy condition (73) for different values of α, with $\alpha \neq \alpha_1$. It is visible the presence of multiple roots.

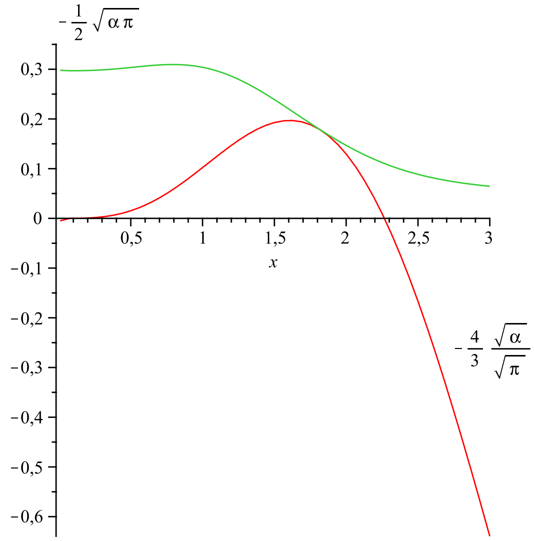

FIG. 2 (color online). Plot of $\Lambda/8\pi G$ as a function of the scale-invariant x. For $\alpha_1 = 1/4$, there exists $\alpha_2 = .7744164292$ where a smooth transition between the two asymptotic behaviors is possible. The transition appears for $x = 1.818231873$.

exists one and only one transition point, which can be found by imposing the existence of a tangent point between the curves parametrized by the values in (77) and the curves parametrized by

$$\alpha_2, \qquad \beta = -\frac{\sqrt{\alpha_2\pi}}{2}, \tag{78}$$

where α_2 is to be determined. We end up with the following choice

$$\alpha_3 = .7744164292, \qquad \beta = -\frac{\sqrt{\alpha_3\pi}}{2}, \tag{79}$$

where the common point is located in $x = 1.818231873$ as shown in Fig. 2.

This choice corresponds to the following setting

$$\begin{aligned} g_1(E/E_P) &= \exp\left(-\alpha_1\frac{E^2}{E_P^2}\right)\left(1-\sqrt{\frac{\alpha_1}{\pi}}\frac{4E}{3E_P}\right), \\ g_2(E/E_P) &= 1 \quad 0 \le x \le 1.818231873 \\ g_1(E/E_P) &= \exp\left(-\alpha_2\frac{E^2}{E_P^2}\right)\left(1-\frac{\sqrt{\alpha_2\pi}E}{2E_P}\right), \\ g_2(E/E_P) &= 1 \quad x \ge 1.818231873. \end{aligned} \tag{80}$$

The setting (80) allows the expression (51) to have finite values for every kind of background of the spherically symmetric type. Let us apply our result to the Schwarzschild background. In terms of the variable x, we find that

$$x = \sqrt{\frac{m_0^2(r)}{E_P^2}} = \sqrt{\frac{3MG}{r^3E_P^2}} = \begin{cases} \frac{3MG}{r^3E_P^2} & r > 2MG \\ \frac{3}{8(MG)^2E_P^2} & r = 2MG \end{cases}. \tag{81}$$

Its behavior is

$$x \to \begin{cases} \infty & \text{when} \quad M \to 0 \quad \text{for } r = 2MG \\ \infty & \text{when} \quad M \to 0 \quad \text{for } r > 2MG \end{cases}, \tag{82}$$

while

$$x \to \begin{cases} 0 & \text{when} \quad M \to \infty \quad \text{for } r = 2MG \\ \infty & \text{when} \quad M \to \infty \quad \text{for } r > 2MG \end{cases}. \tag{83}$$

The situation with $M \to \infty$ describes a wormhole incorporating the whole universe which is not a physical situation, while for $M \to 0$ we approach the Minkowski limit, which should predict a vanishing induced cosmological constant. Note that for both settings (73) and (75), we find that the whole behavior can be summarized by the following double limit

$$\lim_{M\to 0}\lim_{r\to 2MG}\frac{\Lambda(r)}{8\pi G} \neq \lim_{r\to 2MG}\lim_{M\to 0}\frac{\Lambda(r)}{8\pi G}, \tag{84}$$

suggesting that a sort of noncommutativity emerges in proximity of the throat. Therefore, when we adopt the parametrization (80), the Minkowskian limit is recovered for *every value* of M. Turning now to the case of Eq. (54), we find that it is possible to have only one parametrization to obtain the desired behavior as shown in Fig. 3.

B. Example b) $\quad n = 3$

The example we want to analyze corresponds to the case $n = 3$. Of course, we are not going to discuss all the possible cases. However, $n = 3$ represents a fair compromise of generalization. In the region where relation (49) is valid, the integration of Eq. (51) gives

$$\begin{aligned} \frac{\Lambda}{8\pi G E_P^4} = \frac{e^{-x^2\alpha}}{16\pi^2\alpha^{7/2}}\Big\{ & -\sqrt{\pi}(15\gamma + 4x^2\alpha^2(\beta + x^2\gamma) + 6\alpha(\beta + 2x^2\gamma)) - 2e^{x^2\alpha/2}(1+e^{x^2\alpha})x^4\alpha^{5/2}\delta K_0\left(\frac{x^2\alpha}{2}\right) \\ & + e^{2x^2\alpha}\sqrt{\pi}(2\alpha(-3+2x^2\alpha)\beta + (-15 - 4x^2\alpha(-3+x^2\alpha))\gamma)\mathrm{erf}(x\sqrt{\alpha}) \\ & + 2e^{x^2\alpha}x\sqrt{\alpha}\left(-6\alpha\beta - 15\gamma + 2x^2\alpha\gamma + 2x\alpha K_1\left(\frac{x^2\alpha}{2}\right)\left(-2(\alpha+2\delta)\cosh\left(\frac{x^2\alpha}{2}\right) + x^2\alpha\delta\sinh\left(\frac{x^2\alpha}{2}\right)\right)\right)\Big\}, \end{aligned} \tag{85}$$

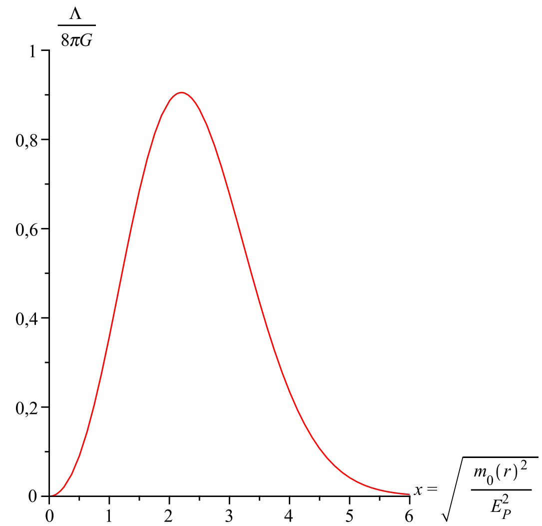

FIG. 3 (color online). Plot of $\Lambda/8\pi G$ as a function of the scale-invariant x for $\alpha = 1/4$. The plot works well for backgrounds of the dS, AdS and Minkowski type. Note that to obtain a positive induced cosmological constant vanishing at small and large x, we need only one parametrization.

where $K_\nu(x)$ ($\nu = 0, 1$) are the Bessel functions, $\delta \equiv \beta_2$ and $\gamma \equiv \beta_3$. Instead, in the region where

$$m_0^2(r) = m_2^2(r) = m_1^2(r), \quad \forall\ r \in (r_1, \infty), \tag{86}$$

we get

$$\frac{\Lambda}{8\pi G E_P^4} = \frac{e^{-x^2\alpha}}{8\pi^2\alpha^{7/2}}\Big\{-\sqrt{\pi}(15\gamma + 4x^2\alpha^2(\beta + x^2\gamma) + 6\alpha(\beta + 2x^2\gamma)) + 2e^{x^2\alpha/2}x^2\alpha^{3/2} \times\Big[-x^2\alpha\delta K_0\Big(\frac{x^2\alpha}{2}\Big) - (4\delta + \alpha(2 + x^2\delta))K_1\Big(\frac{x^2\alpha}{2}\Big)\Big]\Big\}. \tag{87}$$

The asymptotic expansion of Eq. (85) in the small x regime is:

$$\frac{\Lambda}{8\pi G E_P^4} = -\frac{8\alpha^{3/2} + 6\sqrt{\pi}\alpha\beta + 15\sqrt{\pi}\gamma + 16\sqrt{\alpha}\delta}{8\pi^2\alpha^{7/2}} + O(x^3), \tag{88}$$

whereas the leading contributions to Eq. (85) for large x are:

$$\frac{\Lambda}{8\pi G E_P^4} = -\frac{x(2\sqrt{\pi}\alpha^{3/2} + 4\alpha\beta + 8\gamma + 3\sqrt{\pi}\sqrt{\alpha}\delta)}{8\pi^2\alpha^3} - \frac{6\sqrt{\pi}\alpha^{3/2} + 16\alpha\beta + 48\gamma + 15\sqrt{\pi}\sqrt{\alpha}\delta}{32\pi^2 x\alpha^4} + \frac{3(40\sqrt{\pi}\alpha^{3/2} + 128\alpha\beta + 512\gamma + 105\sqrt{\pi}\sqrt{\alpha}\delta)}{1024\pi^2x^3\alpha^5} + O(x^{-4}). \tag{89}$$

Again, as in the case of $n = 1$, we find that there is in principle a leading linear divergency in the large x regime. However, we can choose the parameters satisfying the Minkowski limit (i.e. the limit of vanishing cosmological constant density). This time, as opposed to the $n = 1$ case, we can ask that the Minkowski limit is satisfied both in the $x \to 0$ and in the $x \to \infty$ region, with a unique choice of the parameters. From Eqs. (88) and (89), it follows that the parameters that satisfy these requests have to solve the system

$$\begin{aligned} &2\sqrt{\pi}\alpha^{3/2} + 4\alpha\beta + 8\gamma + 3\sqrt{\pi}\sqrt{\alpha}\delta = 0 \\ &8\alpha^{3/2} + 6\sqrt{\pi}\alpha\beta + 15\sqrt{\pi}\gamma + 16\sqrt{\alpha}\delta = 0. \end{aligned} \tag{90}$$

Notice that already, in the case of three parameters (i.e. $\gamma = 0$), the system (90) can be solved but one gets negative values of the cosmological constant density in a large $x > 1$ zone (see e.g. Fig. 4).

Finally, in the full four-parameter case (i.e. $\gamma \neq 0$), the system (90) can be solved with the further request of approaching the Minkowski limit maintaining positive values of Λ. A solution of Eq. (90) satisfying this further request is:

$$\begin{aligned} \alpha &= 1/4, \\ \beta &= \frac{13635\pi + 2048\sqrt{\pi} - 38784}{1024(9\pi - 32)}, \\ \delta &= -\frac{768\pi + 909\sqrt{\pi} - 2048}{512(9\pi - 32)}, \\ \gamma &= -\frac{303}{2048}. \end{aligned} \tag{91}$$

The resulting Λ as a function of x is plotted in Fig. 5. A small zone in which the cosmological constant density maintains a negative value is, however, still present.

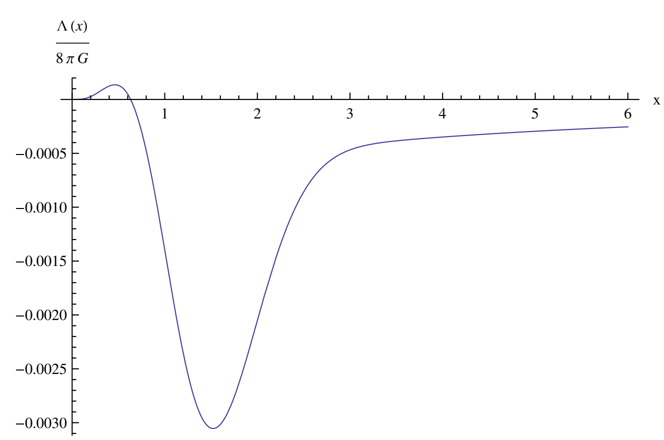

FIG. 4 (color online). Plot of $\Lambda/8\pi G$ as a function of the scale-invariant x for $\alpha = 1/4$, $\beta = 2\sqrt{\pi}/(9\pi - 32)$, $\delta = (8 - 3\pi)/(18\pi - 64)$, $\gamma = 0$. The plot shows that the Minkowski limit is satisfied both in the $x \to 0$ and in the $x \to \infty$ limit, but a large $\Lambda < 0$ zone is present.

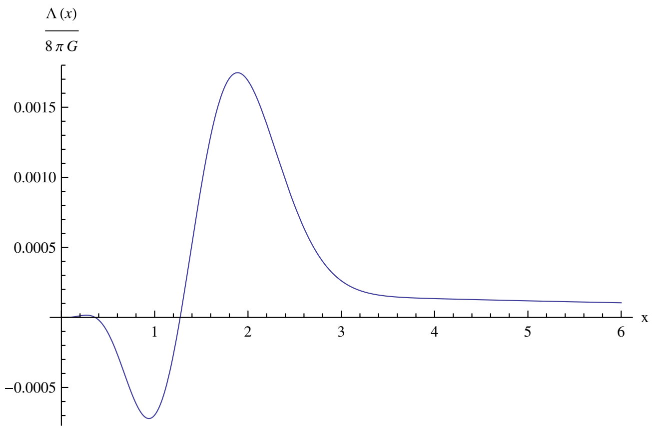

FIG. 5 (color online). Plot of $\Lambda/8\pi G$ as a function of the scale-invariant x for values of the parameters given by Eq. (91). The plot shows that the Minkowski limit is satisfied both in the $x \rightarrow 0$ and in the $x \rightarrow \infty$ limit. Both limits are approached maintaining a positive Λ. A small $\Lambda < 0$ zone is, however, still present.

IV. SUMMARY AND DISCUSSION

Motivated by the promising results obtained in the application of gravity's rainbow to black hole entropy computation [3] and, on the other side in NCG application of ZPE evaluation [13], in this paper we have considered how gravity's rainbow influences the UV behavior of ZPE. We have found that, due to the arbitrariness of $g_1(E/E_P)$ and $g_2(E/E_P)$, it is always possible to find a form of the rainbow functions in such a way that the expression in (7) is UV finite. As introduced in Ref. [5], the finite result $\Lambda/8\pi G$ is interpreted as an induced cosmological constant but without a regularization and a renormalization to keep the UV divergences under control. To fix ideas, we have used Gaussian regulators. In this way our approach is directly comparable to NCG. The first evident difference is that, in NCG, the regulator comes into play in the counting of nodes, while in gravity's rainbow it appears in both the sum over eigenvalues and in the counting of nodes. If one fixes the attention on the pure Gaussian regulator one discovers that the ZPE is always negative for gravity's rainbow. This unpleasant feature can be corrected with the introduction of a polynomial with real arbitrary coefficients, as in Eq. (65). By imposing the positivity of the result for every $x = \sqrt{m_0^2(r)/E_P^2}$ we find that, if condition (50) is satisfied, the parametrization (75) is sufficient to guarantee that $\Lambda/8\pi G > 0$, as shown in Fig. 4. On the other hand, when condition (49) is satisfied we need two different parametrizations to guarantee a correct behavior of $\Lambda/8\pi G$ for $x \in (0, +\infty)$ and most importantly a point of connection where a smooth transition can happen as shown in parametrization (80) and in Fig. 3. In summary, the final plot becomes the one depicted in Fig. 6.

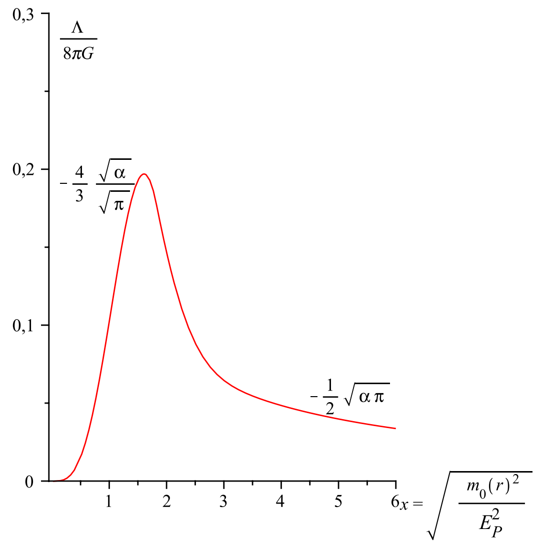

FIG. 6 (color online). Plot of $\Lambda/8\pi G$ as a function of the scale-invariant x for values of the parameters given by Eq. (80). The plot shows that the Minkowski limit is satisfied at the boundaries of the range $(0, +\infty)$. Both limits are approached maintaining a positive Λ. The undesired part of the plot has been eliminated to visualize the global behavior.

Note that the transition clearly highlights the factthat we need two metrics with the same background $b(r)$ but with two different choices of $g_1(E/E_P)$ and $g_2(E/E_P)$. Of course, this transition happens because we insist on having positivity and a vanishing behavior at the boundary of the range $(0, +\infty)$. The vanishing behavior for small x is a guarantee that the Minkowski limit is reproduced. Note that the reproduction of a Minkowski limit in NCG (i.e. the $\sqrt{\theta} \rightarrow 0$ limit) is less trivial because of the existence of the IR/UV mixing [14]. It also appears that the case $n = 1$ seems to be special because it is the only one that has a positive $\Lambda/8\pi G$ for $x \in (0, +\infty)$ when condition (49) is satisfied. We can focus our attention on this case andconsider the dS background, which is the static representation of the Friedmann-Robertson-Walker model. For this choice, the shape function $b(r)$ is

$$b(r) = \frac{\Lambda_{\rm dS}}{3} r^3 \tag{92}$$

and the effective masses (41) become on the cosmological throat $r_c = \sqrt{3/\Lambda_{\rm dS}}$

$$m_0^2(r) = m_2^2(r) = m_1^2(r) = \Lambda_{\rm dS}. \tag{93}$$

Since the behavior of $\Lambda/8\pi G$ for the dS universe is described by Fig. 3, we can compare our results with observation. Since $\Lambda/8\pi G$ represents the observed cosmological constant induced by quantum fluctuations of the pure gravitational field, we can fix its value at the present day as

$$\frac{\Lambda}{8\pi G} \simeq 10^{-11}\ \mathrm{eV}^4, \tag{94}$$

which can be described either for $x \ll 1$ or $x \gg 1$. However, for the dS case $x = \sqrt{\Lambda_{\mathrm{dS}}}/E_P$ and $x \ll 1$ means $r_c \gg 1$, which is in agreement with our present universe. On the other hand, when $x \gg 1$ means that $r_c \ll 1$, which could be in agreement with the very early universe except for the disagreement with the expected theoretical prediction, which for our plot in units of E_P^4 should be $O(1)$. Therefore it appears that only the left branch of Fig. 3 from the bottom to the hilltop can be interpreted as a sort of a *backward evolution* in the radial coordinate r. However, to follow the curve from $x \simeq 2.18$ to $x \simeq 0$, one should have a variable Λ_{dS}, namely $\Lambda_{\mathrm{dS}} \equiv \Lambda_{\mathrm{dS}}(r)$. The same situation appears to exist for Minkowski space in radial coordinates and for the AdS space. Fortunately, Minkowski space does not have a preferred scale and Fig. 3 has the correct asymptotic behavior, except for an unpleasant peak in correspondence of the peak location of the dS space. It is likely that this spurious prediction is due to the coordinate choice. On the other side one, can verify that $\Lambda/8\pi G \to 0$ when $r \to 0$ for Minkowski, dS, and AdS spaces. Coming back on the AdS space, we have to note that this background is not endowed with a horizon, which makes it difficult to find a significant point. Nevertheless, looking once again Fig. 3, we can claim that for $r \to \infty$ and very small Λ_{AdS} one gets

$$b(r) = -\frac{\Lambda_{\mathrm{AdS}}}{3} r^3 \quad \text{and} \quad x = \sqrt{\frac{6/r^2 + \Lambda_{\mathrm{AdS}}}{E_P^2}} \to 0, \tag{95}$$

namely a vanishing $\Lambda/8\pi G$, while in the other regime, i.e. $r \to 0$, $x \to \infty$ and once again one obtains a vanishing $\Lambda/8\pi G$, which means that, against all odds, we have regularity on the singularity $r = 0$.

APPENDIX A: INTEGRALS

In this appendix, we explicitly compute the integrals coming from Eq. (51). We begin with

$$I_+ = 3\int_0^\infty \left[\sum_{i=0}^n c_i \frac{E^i}{E_P^i} \exp\left(-\alpha \frac{E^2}{E_P^2}\right)\right] E^2 \sqrt{E^2 + m_0^2(r)}\, dE. \tag{A1}$$

It is useful to divide I_+ into two pieces with i odd and i even, thus we can write

$$I_+ = I_+^e + I_+^o, \tag{A2}$$

where

$$I_+^e = 3/2 E_P^4 \sum_{i=0}^n c_i (-)^i \lim_{\beta\to 0} \frac{d^i}{d\beta^i} \int_0^\infty x^{1/2} \exp[-(\alpha+\beta)x] \times \sqrt{x + m_0^2(r)/E_P^2}\, dx, \tag{A3}$$

$$I_+^o = 3/2 E_P^4 \sum_{i=0}^n c_i (-)^i \lim_{\beta\to 0} \frac{d^i}{d\beta^i} \int_0^\infty x \exp[-(\alpha+\beta)x] \times \sqrt{x + m_0^2(r)/E_P^2}\, dx \tag{A4}$$

are expressed in terms of the variable $x = E^2/E_P^2$.

The integrals involved in the expressions of I_+^e and I_+^o can be evaluated using the formulas

$$\int_0^\infty dx (x+t)^{1/2} x^{1/2} \exp(-\mu x) = \frac{t}{2\mu} \exp\left(\frac{t\mu}{2}\right) K_1\left(\frac{t\mu}{2}\right) \quad t>0, \mu>0 \tag{A5}$$

$$\int_0^\infty dx (x+t)^{1/2} x \exp(-\mu x) = \frac{3}{2}\frac{\sqrt{t}}{\mu^2} + \frac{\sqrt{\pi}}{4} \mu^{-5/2} \exp(t\mu)(3 - 2t\mu) \mathrm{Erfc}[\sqrt{t\mu}] \quad t>0,\ \mu>0 \tag{A6}$$

obtaining

$$I_+^e = 3/2 E_P^4 \sum_{i=0}^n c_i (-)^i \lim_{\beta\to 0} \frac{d^i}{d\beta^i} \left\{ \frac{m_0^2(r)}{2E_P^2(\alpha+\beta)} \times \exp\left[\frac{m_0^2(r)(\alpha+\beta)}{2E_P^2}\right] K_1\left[\frac{m_0^2(r)}{2E_P^2}(\alpha+\beta)\right]\right\}, \tag{A7}$$

$$I_+^o = +3/2 E_P^4 \sum_{i=0}^n c_i (-)^i \lim_{\beta\to 0} \frac{d^i}{d\beta^i} \left\{ \frac{3m_0(r)}{2E_P(\alpha+\beta)^2} + -\frac{\sqrt{\pi}}{4} \exp\left[\frac{m_0^2(r)(\alpha+\beta)}{E_P^2}\right]\left[\frac{2m_0^2(r)}{E_P^2(\alpha+\beta)^{3/2}} - \frac{3}{(\alpha+\beta)^{5/2}}\right] \mathrm{Erfc}\left[\frac{m_0(r)}{E_P}\sqrt{(\alpha+\beta)}\right]\right\}. \tag{A8}$$

The same procedure can be followed to evaluate

$$I_- = 3\int_{\sqrt{m_0^2(r)}}^\infty \left[\sum_{i=0}^n c_i \frac{E^i}{E_P^i} \exp\left(-\alpha \frac{E^2}{E_P^2}\right)\right] E^2 \sqrt{E^2 - m_0^2(r)}\, dE. \tag{A9}$$

Even in this case, it is useful to divide I_- into two pieces with i odd and i even. Thus, we can write

$$I_- = I_-^o + I_-^e, \tag{A10}$$

where I_-^e and I_-^o are given by

$$I_-^e = 3/2 E_P^4 \sum_{i=0}^n c_i (-)^i \lim_{\beta\to 0} \frac{d^i}{d\beta^i} \int_{m_0^2/E_P^2}^\infty x^{1/2} \exp[-(\alpha+\beta)x] \times \sqrt{x - m_0^2(r)/E_P^2}\, dx, \tag{A11}$$

$$I_{-}^{o} = 3/2E_P^4 \sum_{i=0}^{n} c_i(-)^i \lim_{\beta\to 0} \frac{d^i}{d\beta^i} \int_{m_0^2/E_P^2}^{\infty} x\exp[-(\alpha+\beta)x] \times \sqrt{x - m_0^2(r)/E_P^2}dx. \quad \text{(A12)}$$

Using now the formulas

$$\int_t^\infty dx(x-t)^{1/2}x^{1/2}\exp(-\mu x) = \frac{t}{2\mu}\exp\left(-\frac{t\mu}{2}\right)K_1\left(\frac{t\mu}{2}\right) \qquad t>0,\ \mu>0 \quad \text{(A13)}$$

$$\int_t^\infty dx(x-t)^{1/2}x\exp(-\mu x) = \frac{\sqrt{\pi}}{4}\mu^{-5/2}(3+2\mu t)\exp(-\mu t) \quad t>0,\ \mu>0 \quad \text{(A14)}$$

I_{-}^{e} and I_{-}^{o} can be rewritten in the form

$$I_{-}^{e} = 3/2E_P^4 \sum_{i=0}^{n} c_i(-)^i \lim_{\beta\to 0} \frac{d^i}{d\beta^i}\left\{\frac{m_0^2(r)}{2(\alpha+\beta)E_P^2} \times \exp\left[-\frac{m_0^2(r)(\alpha+\beta)}{2E_P^2}\right]K_1\left[\frac{m_0^2(r)(\alpha+\beta)}{2E_P^2}\right]\right\}, \quad \text{(A15)}$$

$$I_{-}^{o} = 3/2E_P^4 \sum_{i=0}^{n} c_i(-)^i \lim_{\beta\to 0} \frac{d^i}{d\beta^i}\left\{\frac{\sqrt{\pi}}{4}(\alpha+\beta)^{-5/2} \times \left[3 + 2(\alpha+\beta)\frac{m_0^2(r)}{E_P^2}\right]\exp\left[-\frac{m_0^2(r)(\alpha+\beta)}{E_P^2}\right]\right\}. \quad \text{(A16)}$$

[1] G. Amelino-Camelia, Int. J. Mod. Phys. D **11**, 35 (2002); G. Amelino-Camelia, Phys. Lett. B **510**, 255 (2001).

[2] J. Magueijo and L. Smolin, Classical Quantum Gravity **21**, 1725 (2004).

[3] R. Garattini, Phys. Lett. B **685**, 329 (2010).

[4] B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).

[5] R. Garattini, Vestn. Tomsk State Pedagog. Univ. **44**, 72 (2004); R. Garattini, J. Phys. A **39**, 6393 (2006); J. Phys. Conf. Ser. **33**, 215 (2006).

[6] S. Capozziello and R. Garattini, Classical Quantum Gravity **24**, 1627 (2007).

[7] D. V. Vassilevich, Int. J. Mod. Phys. A **8**, 1637 (1993).

[8] M. Berger and D. Ebin, J. Diff. Geom. **3**, 379 (1969); J. W. York, Jr., J. Math. Phys. (N.Y.) **14**, 4 (1973); Ann. Inst. Henri Poincaré A **21**, 319 (1974).

[9] R. Garattini, arXiv:0910.1735.

[10] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).

[11] G. 't Hooft, Nucl. Phys. **B256**, 727 (1985).

[12] Y. Ling, J. Cosmol. Astropart. Phys. 17 (2007) 708; Y. Ling, X. Li, and H. Zhang, Mod. Phys. Lett. A **22**, 2749 (2007); Y. Ling, B. Hu, and X. Li, Phys. Rev. D **73**, 087702 (2006); U. D. Machado and R. Opher, arXiv:1102.4828.

[13] R. Garattini and P. Nicolini, Phys. Rev. D **83** (2011) 064021.

[14] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, J. High Energy Phys. 02 (2000) 020; G. Amelino-Camelia, G. Mandanici, and K. Yoshida, J. High Energy Phys. 01 (2004) 037.