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On variational formulations and conservation laws for incompressible 2D Euler fluids

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Abstract. With the aim of presenting a unified viewpoint for the variational and Hamiltonian formalism of two-dimensional incompressible stratified Euler equations, we revisit some of the formulations currently discussed in the literature and examine their mutual relations. We concentrate on the example of two-layered systems and its one-dimensional reduction, and use it to illustrate general consequences of density stratification on conservation laws which have been partially overlooked until now. In particular, we focus on the conservation of horizontal momentum for stratified ideal fluid motion under gravity between rigid lids.

1. Introduction

We consider the Euler equations for an ideal incompressible and inhomogeneous fluid, subject to gravity, in two dimensions:

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{\nabla p}{\rho} - g\mathbf{k}\,, \qquad \nabla\cdot\mathbf{v} = 0\,, \qquad \rho_t + \nabla\cdot(\rho\,\mathbf{v}) = 0\,. \tag{1}$$

Here $\mathbf{v} = (u, w)$ is the velocity field, ρ and p are the density and pressure fields, respectively, g is the constant gravity acceleration, $\mathbf{k}$ is the unit vertical (upward) vector, and all physical variables depend on spatial coordinates (x, z) and time t. Besides their well known theoretical interest, this set of equations can be viewed as governing the motion of real fluids with sufficient accuracy whenever viscosity, compressibility and diffusivity effects can be neglected within the time scales of interest in the time evolution. These motion equations need to be supplemented with suitable boundary conditions. For instance, in the case of a fluid rigidly confined by horizontal "plates" of infinite length, located at $z = 0$ and $z = h$, the boundary conditions are those of no flux at infinity and across rigid boundaries

$$\mathbf{v}(x, z, t) \to 0 \quad \text{for } |x| \to \infty, \quad w(x, 0, t) = w(x, h, t) = 0, \tag{2}$$

and hydrostatic equilibrium at the far ends of the channel: $\dfrac{\partial p}{\partial z} = -g\rho$ for $|x| \to \infty$.

The variational formulation of the Euler equations for an ideal fluid has been the subject of several research efforts, as it presents some non-standard challenges with respect to classical field theory formalism such as that of the Maxwell equations, as first pointed out by Lin [9] (see also, e.g., the book [1], the review articles [10], [11], and references cited therein). We revisit this subject with the aim of pointing out some implications of the variational formalism, in the context of stratified fluids, that may have escaped some of the attention devoted to this subject. Our analysis stems from that of Benjamin [2], about the so-called Boussinesq model of 2D Euler fluids, which allows for an initial density distribution depending nontrivially on spatial coordinates, and assumes all subsequent changes to be incompressible. In [2] the conservation laws of such a system were determined, and, as a sort of side remark, it was pointed out that, for motion in a strip bounded by two rigid horizontal lids, total horizontal momentum is not conserved (despite the horizontal translation invariance of the system). Its time-variation is proportional to the pressure imbalance between the far ends of the channel, which however arises not as an imposed boundary condition but as a subtle consequence of the interplay between the fluid's incompressibility, stratification and inertia.

Recently, we have isolated and studied this phenomenon, substantiating Benjamin's observation with analytical and numerical results both for one-dimensional long-wave models and for the full two-dimensional Euler equations. In particular, in [3] we obtained some results on this pressure imbalance in the case of two-layer fluids. Along the same lines, a specific class of initial conditions — leading to momentum evolution for two-layer systems — was studied in [4] by means of an asymptotic expansions in the small-density-variation limit. The opposite case whereby the upper fluid has a very small density was the starting point of the analysis in [5]. This case suggested a topological selection mechanisms for conserved quantities, that turned out to be valid also in the case of continuous stratifications. We also briefly addressed the Hamiltonian aspects of these results, both in the sense of Benjamin and in that of Zakharov-Kuznetsov (see, for a review, [14]).

The focus of the present work is to give a more systematic account of the variational set-up of the Boussinesq model. In particular, after reviewing different Hamiltonian formulations of the problem, we point out their mutual relations, with a view towards our favorite test case of fluid motion in an infinite horizontal channel (which leads to lack of horizontal momentum invariance). The layout of the paper is as follows: Section 2 is devoted to Benjamin's approach, while Section 3 deals with Clebsch variables, both in the Zakharov-Kuznetsov representation and in that of Seliger and Whitham [12]. Lack of momentum conservation and its relation with the horizontal invariance of the system — and with pressure imbalance — are discussed in Section 4. In the final section we also quickly present a novel Dirac reduction whose outcome is the Hamiltonian structure of the dispersionless limit of two-layer systems, and discuss its relation with the 2D Hamiltonian picture.

2. Physical variables: the Benjamin's picture

Benjamin [2] proposed and discussed a set-up for the Hamiltonian formulation of the Boussinesq model. His results can be summarized as follows. The basic variables are the density ρ together with kind of "momentum vorticity" σ defined by

$$\sigma = \nabla \times (\rho \boldsymbol{v}) = (\rho w)_x - (\rho u)_z. \tag{3}$$

The equations of motion for these two fields, ensuing from (1), are

$$\begin{aligned} &\rho_t + u\rho_x + w\rho_z = 0 \\ &\sigma_t + u\sigma_x + w\sigma_z + \rho_x\big(gz - \tfrac{1}{2}(u^2+w^2)\big)_z + \tfrac{1}{2}\rho_z\big(u^2+w^2\big)_x = 0 \ . \end{aligned} \tag{4}$$

They can be written in the form

$$\rho_t = -\left[\rho, \frac{\delta H}{\delta \sigma}\right] , \qquad \sigma_t = -\left[\rho, \frac{\delta H}{\delta \rho}\right] - \left[\sigma, \frac{\delta H}{\delta \sigma}\right] , \tag{5}$$

where, by definition, $[A,B] := A_x B_z - A_z B_x$, and $H = \int_{\mathcal{D}} \rho \left(\frac{1}{2}|\mathbf{v}|^2 + gz\right) \mathrm{d}x\,\mathrm{d}z$, $\mathcal{D}$ being the fluid domain. In this formalism, the Hamiltonian H is written in terms of the stream function ψ, which is related to Benjamin's variables (ρ, σ) via

$$\sigma = (\rho w)_x - (\rho u)_z = -(\rho\psi_x)_x - (\rho\psi_z)_z = -\rho\nabla^2\psi - \nabla\rho\cdot\nabla\psi. \tag{6}$$

More precisely, once ρ and σ are given, ψ is the unique solution of (6) vanishing on the plates, so that H turns out to be a functional of ρ and σ only. As shown by Benjamin, equations (5) are actually a Hamiltonian system with respect to a non-canonical Hamiltonian structure. This means that equations (4) can be written as

$$\rho_t = \{\rho, H\}, \qquad \sigma_t = \{\sigma, H\}$$

for the Poisson brackets defined by the Hamiltonian operator

$$J_B = -\begin{pmatrix} 0 & \rho_x\partial_z - \rho_z\partial_x \\ \rho_x\partial_z - \rho_z\partial_x & \sigma_x\partial_z - \sigma_z\partial_x \end{pmatrix}. \tag{7}$$

Indeed, the variational differential of the Hamiltonian H is $(\delta_\rho H\ ,\ \delta_\sigma H) = \big(g\,z - |\nabla\psi|^2/2\ ,\ \psi\big)$, so that equations (4) follow by applying the Hamiltonian operator (7).

We close this section by with two remarks. The first concerns the structure of the Poisson tensor (7), and provides a proof, somewhat alternative to the direct one given in the appendix of [2], of the Jacobi identities for (7). We notice that, being linear in the field variables, this is actually a Lie-Poisson structure. The related Lie algebra can be first identified with the vector space of pairs of functions $(\rho(x,z), \sigma(x,z))$ equipped with the Lie bracket

$$[(\rho_1,\sigma_1),(\rho_2,\sigma_2)] = (\nabla\rho_1\times\nabla\sigma_2 - \nabla\rho_2\times\nabla\sigma_1, \nabla\sigma_1\times\nabla\sigma_2). \tag{8}$$

This is the semidirect product between functions ρ and solenoidal vector fields $X^0_\sigma = (-\sigma_z, \sigma_x)$. Indeed, the RHS of (8) is expressed in terms of the natural action of vector fields on functions, $X^0_{\sigma_1}(\rho_2) - X^0_{\sigma_2}(\rho_1)$, with commutator $[X^0_{\sigma_1}, X^0_{\sigma_2}] = X^0_{\nabla\sigma_1\times\nabla\sigma_2}$. Second, as observed in [5], the previous construction heavily relies on the assumption that ρ (and σ) be constant along the plates (or, in general, on the boundary of the fluid domain). Otherwise, formula (7) does not even give rise to a skew-symmetric bracket.

3. Clebsch variables: the Zakharov-Kuznetsov picture

The Euler system (1) admits a variational formulation. As well known (see e.g., [12]), in the Euler representation the variation of the "physical" Lagrangian does not give rise to equations (1). The way out is to use the components of the Euler equations which are non-dynamical in the velocity as constraints in the Lagrangian. In the picture set forth by Zakharov and Kuznetsov (see, for a review, [14]) the action can be written as the integral of the difference between kinetic and potential energy, plus terms with Lagrange multipliers for constraints,

$$A = \int_{t_0}^{t_1} \left(\int_{\mathcal{D}} \mathcal{L}\,\mathrm{d}x\,\mathrm{d}z\right) \mathrm{d}t\,,$$

where

$$\mathcal{L} = \frac{\rho}{2}|\boldsymbol{v}|^2 - \rho g z + \Phi\nabla\cdot\boldsymbol{v} + \lambda\left(\rho_t + \nabla\cdot(\boldsymbol{v}\rho)\right). \tag{9}$$

By varying the action with respect to all the fields in $\mathcal{L}$, we get

$$\begin{aligned} &\frac{\delta A}{\delta \Phi} = 0 \Rightarrow \nabla \cdot \boldsymbol{v} = 0\,, \qquad \frac{\delta A}{\delta \lambda} = 0 \Rightarrow \rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0\,, \\ &\frac{\delta A}{\delta \rho} = 0 \Rightarrow \lambda_t + \boldsymbol{v} \cdot \nabla \lambda - \frac{1}{2}|\boldsymbol{v}|^2 + g\,z = 0, \qquad \frac{\delta A}{\delta \boldsymbol{v}} = 0 \Rightarrow \rho \boldsymbol{v} - \nabla \Phi - \rho \nabla \lambda = 0. \end{aligned} \tag{10}$$

These equations need to be augmented by the appropriate boundary conditions. Next, it is interesting to show how solutions of equations (10) are related to solutions of the original Euler equations (1). Taking the gradient of the third equation in the system (10), and the convective derivative $D/Dt \equiv \partial/\partial t + \boldsymbol{v} \cdot \nabla$ of the fourth one, yields $\frac{D(\rho \boldsymbol{v})}{Dt} = \nabla\left(\frac{D\Phi}{Dt}\right) - \rho g\, \boldsymbol{k}$, whence Euler equations follow, once we enforce the relation

$$\nabla\left(\frac{D\Phi}{Dt}\right) = -\nabla p\ . \tag{11}$$

It is important to notice that only a specific class of solutions of the Euler equations can be obtained by using this Clebsch representation. Indeed, solving (10) for the velocity gives

$$\boldsymbol{v} = \frac{\nabla \Phi}{\rho} + \nabla \lambda\ \Rightarrow \boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \frac{1}{\rho^2} \nabla \Phi \times \nabla \rho\,. \tag{12}$$

Thus, the Clebsch representation (10) is compatible only with those fluid motions whose vorticity vanishes in the regions where the density ρ is constant. In particular, for a two-layer fluid, the vorticity may be non-zero only along the interface between the two homogeneous layers.

3.1. The Hamiltonian picture

With this proviso in mind, let us now discuss how, by using these Clebsch variables, one can obtain a canonical Hamiltonian formulation in which ρ and λ are conjugate variables. The Lagrangian density (9), thought of as a function of the three field-variables (ρ, Φ, λ) and their "generalized" velocities $(\rho_t, \Phi_t, \lambda_t)$, can be also written (up to an integration by parts) as

$$\mathcal{L}' = \lambda \rho_t + \rho\left(\frac{|\boldsymbol{v}|^2}{2} - g\,z\right) - \boldsymbol{v} \cdot (\nabla \Phi + \rho \nabla \lambda)\,. \tag{13}$$

Since here $\boldsymbol{v}$ is to be considered a shorthand notation for its expression (12) in terms of (ρ, Φ, λ), we obtain, by taking the (partial) Legendre transformation w.r.t. the variable ρ, the Routhian density

$$\mathcal{R} = \frac{\partial \mathcal{L}'}{\partial \rho_t} \rho_t - \mathcal{L}' = \frac{1}{2\rho} |\nabla \Phi + \rho \nabla \lambda|^2 + \rho\, g\, z. \tag{14}$$

According to the Routhian formalism, the variational equations (10) are — as it can be easily verified — written as

$$\begin{aligned} \rho_t &= \frac{\delta R}{\delta \lambda} = -\nabla \cdot (\nabla \Phi + \rho \nabla \lambda) \\ \lambda_t &= -\frac{\delta R}{\delta \rho} = -\left(\frac{1}{\rho} \nabla \Phi + \nabla \lambda\right) \cdot \nabla \lambda - gz + \frac{1}{2} \left|\frac{1}{\rho} \nabla \Phi + \nabla \lambda\right|^2 \\ 0 &= \frac{\delta R}{\delta \Phi} = \nabla^2 \lambda + \nabla \cdot \frac{\nabla \Phi}{\rho} \end{aligned} \tag{15}$$

the last equation being just the zero-divergence equation $\nabla \cdot \boldsymbol{v} = 0$. (We will always use calligraphic letters for the densities and roman letters for the corresponding functionals.)

To obtain a fully fledged Hamiltonian formalism, we can proceed as follows. We regard the third of (15) as a constraint allowing the variable Φ to be determined by the canonical variables λ and ρ. Indeed, the incompressibility equation $\nabla \cdot \boldsymbol{v} = 0$ reads as $\nabla \cdot ((\nabla\Phi)/\rho + \nabla\lambda) = 0$ that is, an elliptic problem for Φ the solution of which — supplemented by suitable boundary conditions — yields $\Phi = \Phi(\lambda, \rho)$.

Substituting this in $\mathcal{R}$ yields the Hamiltonian density $\mathcal{H} = \mathcal{H}(\lambda, \rho)$. The two conjugated variables λ, ρ still evolve according to the system

$$
\begin{aligned}
\rho_t &= \frac{\delta H}{\delta \lambda} = -\nabla \cdot \Big(\nabla\Phi(\lambda,\rho) + \rho\nabla\lambda\Big), \\
\lambda_t &= -\frac{\delta H}{\delta \rho} = -\left(\frac{1}{\rho}\nabla\Phi(\rho,\lambda) + \nabla\lambda\right)\cdot\nabla\lambda - gz + \frac{1}{2}\left|\frac{1}{\rho}\nabla\Phi(\rho,\lambda) + \nabla\lambda\right|^2
\end{aligned}
\tag{16}
$$

with, now,

$$
H = \int_{\mathcal{D}} \left(\frac{1}{2\rho}|\nabla\Phi(\rho,\lambda) + \rho\nabla\lambda|^2 + g\rho\, z\right) \mathrm{d}x\,\mathrm{d}z\,. \tag{17}
$$

We can summarize this three-step procedure in the following diagram:

$$
\begin{tikzcd}
\text{Lagrangian } L \text{ (Eq.(9))} \arrow[rr, "\substack{\text{Hamilton}\\ \text{principle}}"] \arrow[d, "\substack{1^{st}\text{ reduction}\\ \mathbf{v} = \frac{1}{\rho}\nabla\Phi + \nabla\lambda}"'] & & \text{Eqns.(10)} \\
\text{Lagrangian } L' \text{ (Eq.(13))} \arrow[r, "\substack{\text{Partial}\\ \text{Legendre}}"] \arrow[d, "\substack{2^{nd}\text{ reduction}\\ \nabla\cdot\frac{\nabla\Phi(\rho,\lambda)}{\rho} = -\nabla^2\lambda}"'] & \text{Routhian (Eq.(14))} \arrow[r, "\substack{\text{Hamilton}\\ \text{principle}}"] & \text{Eqns.(15)} \\
\text{Lagrangian } L'|_{\Phi=\Phi(\lambda,\rho)} \arrow[r, "\substack{\text{Full}\\ \text{Legendre}}"] & \text{Hamiltonian (Eq.(17))} \arrow[r, "\substack{\text{Hamilton}\\ \text{principle}}"] & \text{Eqns.(16)}
\end{tikzcd}
$$

A straightforward computation, based on the fact that the Jacobian matrix of the "coordinate transformation"

$$
(\rho, \lambda) \mapsto (\rho, \sigma) = (\rho, \rho_x\lambda_z - \rho_z\lambda_x) \tag{18}
$$

can be represented by the matrix of differential operators

$$
\begin{pmatrix} 1 & 0 \\ \lambda_z\partial_x - \lambda_x\partial_z & \rho_x\partial_x - \rho_z\partial_x \end{pmatrix},
$$

shows how the Lie–Poisson structure (7) can be formally obtained from the canonical bracket.

3.2. Clebsch variables: the Seliger-Whitham picture

A variational Lagrangian formalism for the Euler fluids under consideration, allowing for arbitrary values of the vorticity, was set forth by Seliger and Whitham [12]. Its main features are the following. Rather than (as we did in the previous section) fixing one of the Clebsch variable to be the density ρ, one can consider (in the case of a general 3D Euler fluid) a full set of Clebsch variables $(\vartheta, \eta, \alpha, \beta)$ and write the constrained Lagrangian density as

$$
\mathcal{L}_{SW} = \mathcal{T} + \rho g z - \rho\epsilon(\rho, S) + \vartheta\frac{D}{Dt}(\rho\boldsymbol{v}) + \eta\frac{D}{Dt}(\rho S) + \beta\frac{D}{Dt}(\rho\alpha).
$$

Here, $\rho\epsilon$ is the internal energy (which can be set to a constant in the incompressible case), S is the entropy density, and α, according to Lin's recipe [9], is one of the fluid's initial Lagrangian coordinate, while ϑ, η, β are Lagrange multipliers.

Among other properties of such general Clebsch formulation, Seliger and Whitham show that, by using the equations of motion (that is, in field-theoretical parlance, *on-shell*), the Lagrangian density is nothing but the pressure field p. Also (see §6 of [12]), for the case of (2D) incompressible evolutions, the number of Clebsch variables can be reduced to three, with the variational equations taking the form:

$$\boldsymbol{v} = \nabla\chi + \alpha\nabla\mu, \qquad \frac{D\alpha}{Dt} = 0, \qquad \frac{D\mu}{Dt} = p\frac{\rho'(\alpha)}{\rho^2(\alpha)}. \tag{19}$$

Here, the density ρ is a function of the sole variable α, while μ and χ are suitable functions of the original Clebsch variables $(\vartheta, \eta, \alpha, \beta)$ and of the entropy density S. More importantly, the Lagrangian density — i.e., the pressure — is expressed through χ, α, μ as

$$\mathcal{L}_{SW} = p = -\rho(\alpha)\left(\chi_t + \alpha\mu_t + \frac{1}{2}|\boldsymbol{v}|^2 + g\,z\right). \tag{20}$$

Let us compare the Seliger & Whitham with the Zakharov & Kuznetsov approaches (hereafter referred to as SW and ZK, respectively) in the Lagrangian set-up (in the case $\rho'(\alpha) \neq 0$). We re-define the Clebsch variable of Section 3 as

$$\varphi = \Phi + \lambda\rho\,. \tag{21}$$

Taking the convective derivative of both sides of the first relation we get

$$\frac{D\varphi}{Dt} = \frac{D\Phi}{Dt} + \lambda\frac{D\rho}{Dt} + \rho\frac{D\lambda}{Dt} = \frac{D\Phi}{Dt} + \lambda(\rho_t + \nabla\cdot(\rho\boldsymbol{v})) - \lambda\rho\nabla\boldsymbol{v} + \rho\left(\frac{|\boldsymbol{v}|^2}{2} - g\ z\right), \tag{22}$$

where we have used the third of equations (10). Hence, by taking into account that thanks to (11) we can identify (up to constants) the pressure p with $-\frac{D\Phi}{Dt}$, and we set $\varphi - \lambda\rho = \Phi$, we get

$$\mathcal{L} - \varphi_t - \nabla\cdot(\varphi\boldsymbol{v}) = p. \tag{23}$$

In analogy with the SW case, we obtain that the (on shell) ZK Lagrangian density differs from the pressure by terms that do not affect the action.

4. Pressure imbalances and momentum non conservation: the case of the channel

When the two-dimensional domain, as in our example case, is the infinite strip $\mathcal{S} = \mathbb{R}\times[0, h]$, translation along the x-axis is a symmetry of the system, hence by Noether's (first) theorem we expect to obtain a conservation law for the Euler equations.

Let us first examine the problem from the Hamiltonian viewpoint of ZK, in which the density ρ and the Clebsch variable λ are canonically conjugate variables. With respect to the canonical brackets, the functional generating translations along x is

$$I_{ZK} = \int_{\mathcal{S}} \rho\lambda_x \,\mathrm{d}x\,\mathrm{d}z\,. \tag{24}$$

As can be easily verified, this quantity is a constant of the motion.

Benjamin's formalism [2] is explicity taylored to symmetries. The generator of translations along the horizontal directions (the "impulse") is

$$I_B = \int_{\mathcal{D}} z\sigma(x, z)\,\mathrm{d}x\,\mathrm{d}z\,. \tag{25}$$

Under the transformation (18) it agrees, up to divergences, with the generator (24). In particular, for motion between two rigid horizontal lids, I_B can be written as the sum of a bulk and a boundary term (called B_6 by Benjamin),

$$I_B = \int_{\mathcal{S}} \rho\, u \,\mathrm{d}x\,\mathrm{d}z - B_6,$$

where the first term is the ordinary total horizontal momentum, while the second term B_6 equals $\int_{\mathbb{R}} [z\rho u]_{z=0}^{z=h}\,\mathrm{d}x$. The Euler equation for the horizontal momentum

$$(\rho u)_t = -\frac{1}{2}\rho_x(u^2+w^2) - (\frac{1}{2}\rho(u^2+w^2)+p)_x + w\,\sigma\ , \tag{26}$$

in the case of a horizontally constant density at the top lid (where $w=0$) yields, as noticed by Benjamin,

$$\frac{dB_6}{dt}\left(= \frac{d}{dt}\int_{\mathcal{S}} \rho u \,\mathrm{d}x\,\mathrm{d}z\right) = -h\left(p(+\infty,h) - p(-\infty,h)\right). \tag{27}$$

Hence, total horizontal momentum does not necessarily coincide with the constant of the motion I_B; it can evolve in time due to asymptotic pressure imbalances (see [4] for further details).

To discuss this issue in the Lagrangian formalisms of Section 2, let us recall that for a Lagrangian system in two spatial dimensions with N fields $(\varphi_1, \varphi_2, \dots, \varphi_N)$, the expression for the conservation law associated with horizontal translations is

$$\sum_{\alpha=1}^{N}\left(\frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial \varphi_{\alpha,t}}\varphi_{\alpha,x}\Big) + \frac{\partial}{\partial x}\Big(\frac{\partial L}{\partial \varphi_{\alpha,x}}\varphi_{\alpha,x} - L\Big) + \frac{\partial}{\partial z}\Big(\frac{\partial L}{\partial \varphi_{\alpha,z}}\varphi_{\alpha,x}\Big)\right) = 0, \tag{28}$$

so that for the Lagrangian (9) we get

$$\frac{\partial}{\partial t}(\lambda\rho_x) + \nabla\cdot\boldsymbol{J} = 0, \quad \boldsymbol{J} = (\Phi u_x - u(\lambda\rho)_x - \frac{1}{2}\rho|\mathbf{v}|^2 + \rho g z, \Phi w_x - w(\lambda\rho)_x). \tag{29}$$

This identifies the conserved density of the (Lagrangian) ZK formalism with $\lambda\rho_x$, which is consistent with the outcome of the Hamiltonian formalism (24).

A corresponding result can be obtained in the Seliger-Whitham formalism. Consider the Lagrangian density $\mathcal{L}_{SW}$ of equation (20) and apply the general formula (29); we get

$$\frac{\partial J_0}{\partial t} + \frac{\partial J_{(x)}}{\partial x} + \frac{\partial J_{(z)}}{\partial z} = 0, \qquad \text{where } J_0 = -\rho\chi_x - \rho\alpha\mu_x = -\rho u \text{ by Eq.(19)}.$$

Since $\mathcal{L}_{SW}$ is independent of spatial derivatives, $(J_{(x)}, J_{(z)}) = (-p, 0)$. Thus,

$$\frac{d}{dt}\int_{\mathcal{S}} \rho u \,\mathrm{d}x\,\mathrm{d}z = -\int_0^h \left(p(+\infty,z) - p(-\infty,z)\right)\mathrm{d}z, \tag{30}$$

which reduces to (27) under the assumption that the pressure be hydrostatic at the far ends of the channel.

5. Two-layer fluids in a channel: the dispersionless case

In this final section we shall briefly discuss the Hamiltonian structures that can be used to describe a sharply stratified fluid constrained within an infinite channel. The set-up we consider (see figure 1) is as follows. The upper (resp. lower) fluid has thickness $\eta_1(x,t)$ (resp. $\eta_2(x,t)$), constant density ρ_1 (resp. ρ_2), and velocity field (u_1, w_1) (resp. (u_2, w_2)). The total height of the channel is $h = \eta_1(x,t) + \eta_2(x,t)$, while $P(x,t) = p(x, \eta_2(x,t), t)$ is the interfacial pressure.

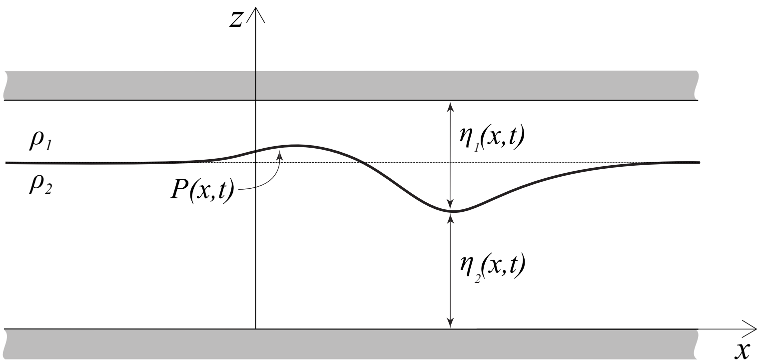

Figure 1: A two-layer system. Asymptotically the interface has the same height at $+\infty$ and at $-\infty$. The asymptotic velocities are zero.

By putting

$$\overline{f}_1(x,t) = \frac{1}{\eta_1(x,t)} \int_{\eta_2(x,t)}^{h} f_1(x,t,z)\,\mathrm{d}z, \qquad \overline{f}_2(x,t) = \frac{1}{\eta_2(x,t)} \int_0^{\eta_2(x,t)} f_2(x,t,z)\,\mathrm{d}z,$$

and vertically averaging the Euler equations in the channel (see [13, 6, 7]), we obtain

$$\begin{aligned} &\eta_{j\,t} + \left(\eta_j \overline{u}_j\right)_x = 0, \\ &\rho_j(\eta_j \overline{u}_j)_t + \rho_j\left(\eta_j \overline{u_j u_j}\right)_x = -(\eta_j \overline{p}_j)_x + (-1)^j (\eta_2)_x P\,, \qquad j = 1,2\,. \end{aligned}$$

Disregarding terms of order greater than zero in the parameter ϵ (the ratio between vertical and horizontal length scales), the previous set of equations becomes, in terms of the momenta $m_j = \rho_j \eta_j \overline{u}_j$,

$$\begin{aligned} \eta_{j_t} &= -(m_j/\rho_j)_x \\ m_{j_t} &= -\left(m_j^2/(\rho_j\,\eta_j)\right)_x - (-1)^j g\rho_j\eta_j\eta_{j_x} - \eta_j P_x. \end{aligned} \tag{31}$$

These equations are actually a system of Hamiltonian equations in one spatial dimension with Hamiltonian density

$$H_{dl} = \int_{-\infty}^{+\infty} \left(\frac{1}{2}\left(\frac{{m_1}^2}{\rho_1\eta_1} + \frac{{m_2}^2}{\rho_2\,\eta_2}\right) + (\eta_1+\eta_2-h)P + \frac{g}{2}\left(\rho_2\eta_2^2 - \rho_1\eta_1^2\right)\right)\mathrm{d}x \tag{32}$$

and the product Lie-Poisson Hamiltonian operator in the variables $(m_1, \eta_1, m_2, \eta_2)$

$$J := -\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}, \quad J_k := \begin{pmatrix} m_k\partial_x + \partial_x m_k & \eta_k\partial_x \\ \partial_x\eta_k & 0 \end{pmatrix}, \quad k = 1,2. \tag{33}$$

Equations (31) are the dispersionless limit of the set of equations obtained in [7]. We will refer to them as *dispersionless* strongly nonlinear system.

Note that, in the Hamiltonian (32), the interfacial pressure P (the only quantity that couples the two set of equations (31)) plays the role of Lagrangian multiplier enforcing the constraint (a primary constraint, in Dirac's terminology [8])

$$\phi_1 \equiv \eta_1 + \eta_2 - h = 0. \tag{34}$$

The time evolution of this primary constraint yields $m_{1x}/\rho_1 + m_{2x}/\rho_2 = 0$, which, thanks to the null-velocity (and hence momentum) conditions at the far ends of the channel, yields

$$\phi_2 \equiv \frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} = 0. \tag{35}$$

It turns out that these two constraints actually form a complete consistent set. Indeed, on the constrained "surface" defined by $\phi_i = 0$, for $i = 1, 2$, we have $\phi_{1t} = -\phi_{2x} = 0$; by evaluating $(\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2})_t$ for $x \to \infty$, we see that no further dynamical constraints arise.

These two constraints are actually of "second class," so that they allow the definition of Dirac reduced brackets. To check this, we need only to compute the matrix of Poisson brackets $C_{ij} := \{\phi_i, \phi_j\}$. Since

$$\{\phi_1, \phi_1\} = 0, \quad \{\phi_1, \phi_2\} = -\partial_x\left(\frac{\eta_1}{\rho_1} + \frac{\eta_2}{\rho_2}\right), \quad \{\phi_2, \phi_2\} = -\left(\frac{m_1}{\rho_1^2} + \frac{m_2}{\rho_2^2}\right)\partial_x - \partial_x\left(\frac{m_1}{\rho_1^2} + \frac{m_2}{\rho_2^2}\right),$$

the matrix C can proven to be invertible (as a matrix with elements in the ring of pseudo-differental operators). A direct computation shows that, by choosing on the constrained "surface" the coordinates

$$\mu = \left(1 + \frac{\eta_1/\rho_1}{(h - \eta_1)/\rho_2}\right) m_1, \qquad \zeta = \eta_1 - h = -\eta_2, \tag{36}$$

the reduced Dirac Poisson tensor acquires the local Lie-Poisson form

$$J^D = -\begin{pmatrix} \mu\partial_x + \partial_x\mu & \zeta\partial_x \\ \partial_x\zeta & 0 \end{pmatrix}, \tag{37}$$

while the (reduced) Hamiltonian density is

$$\mathcal{H}^D(\mu, \zeta) = -\frac{\zeta\mu^2}{2(h + \zeta)(\rho_2(h + \zeta) - \rho_1\zeta)} + \frac{g}{2}\left(\rho_2\zeta^2 - \rho_2(h + \zeta)^2\right). \tag{38}$$

Let us also remark that the generator of the x-translation for the tensor (37) is the coordinate μ, which does not coincide with the total "horizontal" momentum $m_1 + m_2$. Hence, at the dispersionless level in the layer averaged formalism, we again obtain that the total horizontal momentum is not the conserved quantity related by Noether's theorem with translational symmetry.

Let us finally show that, in the long-wave regime, the Poisson bracket from the Dirac reduction is actually related with that of Benjamin (7), discussed in Section 2. To this end, let us write the physical variables of the two-layer fluid with the aid of the Heaviside θ-function as

$$\begin{aligned} &\rho = \rho_2 + (\rho_1 - \rho_2)\theta(z - \eta_2(x, t)) \\ &u(x, z, t) = u_2(x, z, t) + (u_1(x, z, t) - u_2(x, z, t))\theta(z - \eta_2(x, t)) \\ &w(x, z, t) = w_2(x, z, t) + (w_1(x, z, t) - w_2(x, z, t))\theta(z - \eta_2(x, t)). \end{aligned} \tag{39}$$

Assuming that the motion is irrotational in the bulk of each of the layers (which is consistent with the long wave approximation of the strongly nonlinear model), we get at first

$$\sigma = -\big(\rho_1 u_1(x, z, t) - \rho_2 u_2(x, z, t) + \eta_{2x}(\rho_1 w_1(x, z, t) - \rho_2 w_2(x, z, t))\big)\delta(z - \eta_2(x, t)). \tag{40}$$

The first order approximation in the expansion in $\epsilon = h/L \simeq w_i/u_i$ is, by replacing u_i with the layer mean velocities $\bar{u}_i(x, t)$,

$$\sigma = -\big(\rho_1\bar{u}_1(x, t) - \rho_2\bar{u}_2(x, t)\big)\delta(z - \eta(x, t)). \tag{41}$$

Integrating over the fluid layer gives the layer-mean

$$\langle\sigma\rangle = \frac{1}{h}\int_0^h \sigma(x, z)\,\mathrm{d}z = -\frac{1}{h}\big(\rho_1\bar{u}_1(x, t) - \rho_2\bar{u}_2(x, t)\big) = \frac{1}{h}\left(\frac{m_2}{\eta_2} - \frac{m_1}{\eta_1}\right). \tag{42}$$

By using the constraints (34, 35) it turns out that $\langle\sigma\rangle = -\mu/(h\,\zeta)$, so that the pair $(\zeta, \langle\sigma\rangle)$ are a system of coordinates for the spatial 1-D system of the averaged dispersionless equations obtained by the reduced Hamiltonian (38). In these coordinates the Poisson tensor is turned into the constant tensor

$$J^D = \frac{1}{h}\begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \tag{43}$$

and the equations of motion are written as the conservation laws

$$\zeta_t = -\left(\frac{h\zeta(h+\zeta)\langle\sigma\rangle}{\rho_2(h+\zeta)-\rho_1\zeta}\right)_x, \quad \langle\sigma\rangle_t = -\left(\frac{h\langle\sigma\rangle^2(\rho_2(h+\zeta)^2-\rho_1\zeta^2)}{2(\rho_2(h+\zeta)-\rho_1\zeta)^2} - \frac{g}{h}(\rho_2-\rho_1)\zeta\right)_x.$$

Let us look at the relations (the first one is obtained by integrating the first of (39) along z)

$$\zeta(x) = \frac{1}{\rho_1-\rho_2}\int_0^h (\rho(x,z)-\rho_1)\,\mathrm{d}z\;, \qquad \langle\sigma\rangle(x) = \frac{1}{h}\int_0^h \sigma(x,z)\,\mathrm{d}z \tag{44}$$

as defining a map from the manifold M^2 spanned by Benjamin's variables $(\rho(x,z), \sigma(x,z))$ to the "reduced" manifold M^1 of the strongly nonlinear dispersionless equations, spanned, as we have just seen, by the coordinates $(\zeta(x), \langle\sigma\rangle(x))$.

By using standard techniques of Hamiltonian reduction we can now be shown that the Poisson tensor J^D is the Hamiltonian reduction (under the map (44)) of Benjamin's Poisson tensor (7).

Indeed, let $(\alpha_\zeta(x), \alpha_{\langle\sigma\rangle}(x))$ be a covector on M^1; the pair $\left(\frac{1}{\rho_1-\rho_2}\alpha_\zeta(x),\ \frac{1}{h}\alpha_{\langle\sigma\rangle}(x)\right)$ is its pull-back along the map (44) to M^2; applying the tensor (7) to this pull-back we get the vector

$$(\dot\rho, \dot\sigma) = \left(\frac{\rho_1-\rho_2}{h}\delta(z-\eta_2)\partial_x(\alpha_{\langle\sigma\rangle}),\ \delta(z-\eta_2)\partial_x(\alpha_\zeta) + \frac{1}{h}(\rho_2\overline{u}_2 - \rho_1\overline{u}_1)\delta'(z-\eta_2)\partial_x(\alpha_{\langle\sigma\rangle})\right).$$

Pushing this vector to M^1 via (44), since $\int_0^h \delta'(z-\eta_2)\,\mathrm{d}z = 0$, yields the vector

$$(\dot\zeta, \dot{\langle\sigma\rangle}) = \left(\frac{1}{h}\partial_x(\alpha_{\langle\sigma\rangle})\ \ \frac{1}{h}\partial_x(\alpha_\zeta)\right),$$

which is nothing but the image under the Hamiltonian operator (43) of the covector $(\alpha_\zeta(x), \alpha_{\langle\sigma\rangle}(x))$.

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