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Pension Fund Optimal Allocations

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Abstract

We address the problem of a private pension plan sponsor who has to decide the best pension funds that should be offered to the pension plan members. Starting from the analysis of the population of the plan in order to identify a set of representative subscribers, we focus on an individual optimal portfolio allocation in a pension perspective. Then, the optimal allocation for each representative will become a pension fund. For each representative, we propose a multistage stochastic program (MSP) which includes a multi-criteria objective function. The optimal choice is the portfolio allocation that minimizes the Average Value at Risk Deviation of the final wealth and satisfies a wealth target in the final stage. Stochasticity arises from investor's salary process and asset returns. The stochastic processes are assumed to be correlated. Numerical results show optimal dynamic portfolios with respect to investor's preferences and then the best pension funds the provider can offer.
Keywords: Pension fund · Portfolio management · Multistage stochastic programming · Cluster analysis

1 Introduction

The pension system has become more and more complex and structured all over Europe in the last decades. Because of the financial and social

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crisis, several countries implemented strong reforms in the state welfare in order to reduce the pension costs on the state budget balance. Furthermore, they encouraged the establishment of private pension facilities. In general, a private pension plan is composed of investment funds and receives periodical contributions from private investors and then provides an annuity during the retirement, see Consigli and Moriggia (2014). The main function of a pension plan is to face the risk that the subscriber might survive his/her savings. Then, a reasonable aim for this kind of investment would be to guarantee an integration of the public retirement pension so that the total income before and after retirement does not differ substantially. Typically, the plan is composed of pension funds which are sufficiently different each other in order to let the investors choose the optimal pension perspective investment among a well diversified offer. Often, such pension funds are issued following some standard investment allocations: a guaranteed capital, a low risk profile, a high yield investment, etc. The competition among the private pension plan providers is becoming stronger and stronger. They all would like to offer suitable and reliable pension funds for their contributors. This means that the offer could be somehow standard for huge providers, but for small and medium ones the investment strategies composition should be defined according to the needs of the future subscribers. The case of pension plans issued only for some workers categories or for the employees of a single company is a typical situation in which a policy decision following a rigorous analysis of the subscribers is required. For a given pension plan population, our goal is to identify the best pension funds that should be issued, i.e., their optimal allocations. We propose a two-step model. The former is a precise statistical analysis of a database containing the subscribers of the plan. The aim of this preliminary study is to cluster the population and identify a set of representative members. Thus, we assume that the optimal investment portfolio for each of these representatives will define a pension fund. Therefore, the second step consists in the formulation and implementation of a multistage stochastic program (MSP) in order to define the optimal investment allocation for each representative member.

The statistical analysis of the first step is briefly investigated in Section 2. The second step, analyzed in Section 3, explores the formulation of the MSP. In particular, we include a multi-criteria objective as suggested in Dupačová *et al.* (2002) and, considering the investor's risk aversion, the optimal portfolio allocation minimizes the Average Value at Risk Deviation of the final wealth. A wealth target is imposed in the final stage. Other constraints regard the pension funds rules, i.e., contribution constraints, portfolio balance, etc. Stochasticity arises from the investor's salary process and from the asset return processes. Finally, numerical results provide optimal dynamic portfolios with respect to investor's preferences.

2 First Step - Population analysis

The population analysis consists in a statistical description of a dataset composed of 5577 employees of a single company. Thus, they belong to an homogeneous population and represent the active population of the pension plan. The focus of the study is twofold: to give the pension plan sponsor a complete and rigorous view of the actual plan participants and investigate their main characteristics in order to have a reliable starting point for the latter clusterization. The considered members' features are:

- age and remaining working life
- gender
- accumulated wealth
- average annual contribution
- contribution choice as percentage of the salary
- diversification attitude
- withdraw and switch behavior.

The age analysis uses as input data the year when each member started to contribute to the plan. For the considered dataset the result shows a uniform distribution in the last decades. The male and female cardinality is almost equal. The accumulated wealth analysis highlights a huge variety starting from the younger employees with almost null wealth to the top manager positions which create a heavy right tail. The mean value is 70,000 euros, the standard deviation is 46,000 euros. To better analyze the accumulation process, we introduce a contribution ratio given by the accumulated wealth per year spent in the plan. The contribution ratio distribution is highly concentrated between 3,000 euros and 6,000 euros per year.

A particular focus is dedicated to the diversification choice. Up to now, the fund is composed of seven pension plans and we analyze the number of positions opened for each strategy and for each member. The first analysis investigates the preferred strategies, the second one shows the individual inclination to invest simultaneously in more than one strategy, i.e., to adopt for the pension perspective savings the same diversification strategies that are usually performed for investment portfolios.

The withdraw decision is studied both in terms of frequency and in terms of magnitude. In our dataset, according to the pension plan regulation, the minimum number of years between the entering date and the time of the first withdraw is eight. Considering the exact year in which each member required the withdraw, it is clear that he/she uses this option either because the regulatory eight years have expired or due to the 2008/2014 financial crisis. The average amount withdrawn depends on the age of the members: as far as the older ones are concerned they withdraw 45% of the amount, while the younger ones withdraw 62% of the amount. However, only the 22% of the pension plan population requires a withdraw.

The switch option between the strategies is not widely used. Only 4% of the pension plan population moved the accumulated wealth at least once. In those cases, the switch occurs typically from risky strategies to low risk ones.

From a cross analysis, we observe a strong correlation between the switch features and the diversification attitude. The participants who require a switch, are experiencing a diversified portfolio. Generally, we can distinguish between members with static and concentrated portfolios and members implementing dynamic and diversified strategies.

The strategies choice study shows also the risk attitude of the pension fund members. The lowest risk strategies represent the main investment. A few contributors switch to riskier positions for two reasons: the perspective of a long investment window if they are young members, or a natural attitude for risk which leads them to invest the savings seeking for an extra gain during the market high volatility periods.

Thanks to the previous results, we start the clustering analysis having as main characterizing features three elements: the accumulated wealth, the portfolio risk level, the remaining working age. The aim of the clusterization is to extract a set of representative participants among the whole population. For each of them, we propose an optimal portfolio allocation considering the investor's features and the stochastic environment. The pension plan sponsor wants to offer the best strategies to the active population of the plan. Then, the optimal obtained portfolios (one for each representative) will be suggested to the pension plan provider to become the pension funds that should be offered. Clearly, the cardinality of the representative member set and the number of pension funds will coincide. Therefore, the cluster cardinality must be decided by the provider, taking into account the suitability for the members and the manageability for the pension fund manager.

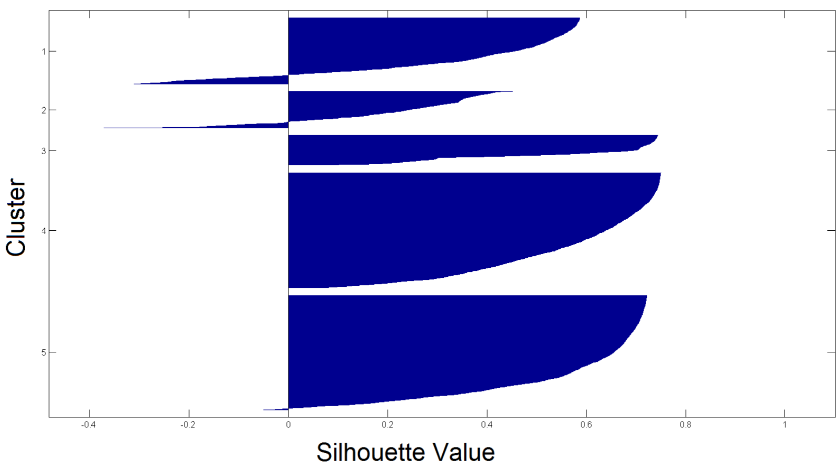
In order to create the cluster sets, we adopt the k-means Lloyd's algorithm using the cityblock distance which measures the distance between two elements x_i, x_j having p attributes as $d(x_i, x_j) = \sum_{k=1}^{p} |x_i^k - x_j^k|$, i.e., each centroids is the component-wise median of the points in that cluster, see

Lloyd (1982) and Kaufman and Rousseeuw (2009). As already mentioned, the number of clusters is a provider's decision. In the proposed case study, the pension plan sponsor chooses five clusters which produce the following centroids (wealth in euros - risk profile - remaining working years):

- 132,000 - very low - 9
- 78,500 - very low - 17
- 66,800 - medium - 28
- 35,000 - very low - 31
- 38,000 - high - 33

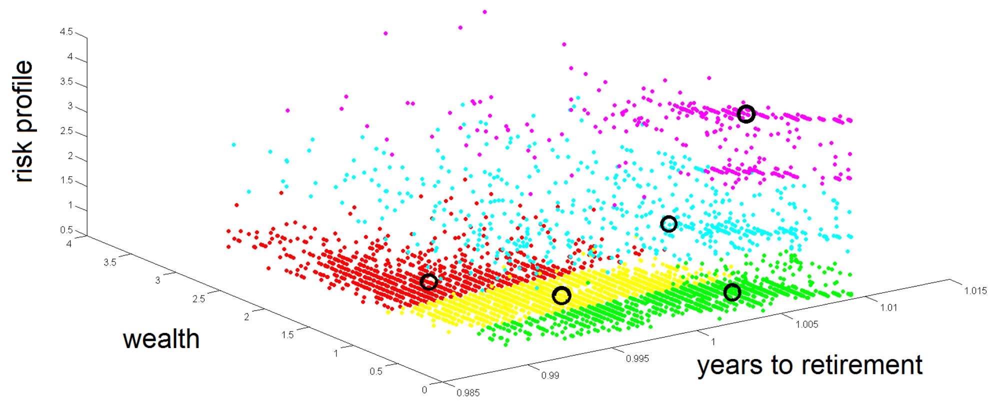
The silhouette value s_i describes how much each point i is similar to points in its own cluster and it is defined as $s_i = \frac{m_i - a_i}{\max(m_i, a_i)}$ where a_i is the average distance from the $i-$th point to points in the same cluster and m_i is the minimum average distance from the $i-$th point to points in different clusters, see Kaufman and Rousseeuw (2009). As distance measure we adopt the cityblock distance. Figure 1 shows the results of the silhouette analysis for the five-cluster case. Figure 2 represents the clusters: the five colors

Figure 1: Silhouette five clusters case



identify the different sets and the black circles are the clusters centroids. The values are normalized.

Figure 2: Cluster analysis five clusters case



3 Second Step - Pension Fund Allocations

The individual investment problem has been deeply studied during the last decades. The main feature of this class of models is to consider jointly all variables which characterize the investor's investment, i.e., the salary process, the consumption, the borrowings, etc. See for example Consiglio *et al.* (2004), Consiglio *et al.* (2007), Consigli (2007) and Medova *et al.* (2008). Moreover, many works focus on the individual investment in a pension perspective framework. Consigli *et al.* (2012) show an extension of a classical ALM individual investors model to include pension strategies. Given the representative employees defined in Section 2, each pension fund optimal strategy will coincide with the optimal portfolio allocation of each representative investor. Therefore, a model to describe the pension problem for a private investor is needed. The aim of the procedure is to define the optimal asset allocation for an employee in a retirement perspective. We deal with two main features: a long term horizon with a fixed and given sequence of portfolio rebalancing stages and an uncertainty environment regarding the asset returns and the salary evolution. These elements lead naturally to a Multistage Stochastic approach, see Dupačová *et al.* (2002). The considered framework is a defined contribution pension fund. The distinction between defined contribution and defined benefit, especially in terms of securities included, is described and analyzed in Consiglio *et al.* (2015). Several asset classes are involved in a pension fund portfolio. Nevertheless, the most suitable in terms of risk/reward profile are government and corporate bonds, see Lozza *et al.* (2013) and Abaffy *et al.* (2007). The robustness of the model can be measured analyzing its sensitivity as proposed in Bertocchi *et al.* (2000a) and Bertocchi *et al.* (2000b).

We suppose that the decision times correspond to all the stages but the last one in which we just compute the accumulated final wealth. The stochasticity arises from two sources: the asset returns and the salary process. The investment universe is composed by n assets which are the benchmarks that the fund manager is able to replicate. The asset returns and the salary stochastic processes are modeled as Geometric Brownian motions. The salary stochasticity is crucial in the definition of a consistent model for the private investor optimal allocation. We assume that the salary is correlated with the riskiest assets.The stochasticity is represented with a discrete scenario tree composed of S paths and characterized by a regular branching.

We define the nonnegative decision variables: $c_{i,t,s}$, $r^+_{i,t,s}$ and $r^-_{i,t,s}$, where $i = 1, ..., n$ represents the assets, $t = t_0, ..., T$ the stages and $s = 1, ..., S$ the scenarios. Thus, $c_{i,t,s}$ expresses the level of contribution we want to invest in the asset i, on the stage t, in the scenario s; the rebalancing variables $r^+_{i,t,s}$ and $r^-_{i,t,s}$ allow the redistribution of the accumulated wealth among the chosen assets quantifying how much we buy and how much we sell of each asset at the beginning of each stage, i.e., before adding the contribution. We denote with $\rho_{i,t,s}$ the asset returns process and with $\rho^{sal}_{t,s}$ the salary growth rate process. Then, having a mean μ_i and a standard deviation σ_i for each process and a correlation matrix $corr_{i,j}$, we assume the following structure for the asset price evolutions.

$$dP^i_t = \mu_i P^i_t dt + \sigma_i P^i_t dW^i_t, \ \ \forall i, \ \forall t$$

$$\mathbb{E}(dW^i_t, dW^j_t) = corr_{i,j} dt, \ \ \forall i, \ \forall j, \ \forall t$$

where W_t is the Wiener process.

Finally, we can list the set of constraints in order to express the regulatory bounds and the cash balance conditions.

Salary process
Fixing the initial level $sal_{t_0,s}$ equal to the actual salary of the employee, we can easily describe the salary process.

$$sal_{t,s} = sal_{t-1,s} \cdot (1 + \rho^{sal}_{t,s}), \ \ \forall t > t_0, \forall s \tag{1}$$

Maximum contribution level
In each stage the employee does not want to invest more than a certain

maximum percentage of his salary. Therefore, we introduce the parameter *propensity to save* denoted by ps and a coefficient e which represents a percentage supplementary contribution added by the employer. Moreover, the time structure of the problem defines the stages every Δt years, but in the real life the contribution is added yearly (sometimes also monthly) in the pension fund. Therefore, assuming the growth rate of the salary constant over each period and equal to the discount rate and assuming that the contribution is paid at the beginning of each year, we compute the actual value of a growing annuity paying one euro for Δt years simply multiplying by Δt. Thus, the constraint describing the maximum contribution level assumes the following form.

$$\sum_{i=1}^{n} c_{i,t,s} \leq sal_{t,s} \cdot ps \cdot (1+e) \cdot \Delta t, \quad \forall t, \forall s \tag{2}$$

Portfolio Balance

We define the set of constraints that describes the portfolio allocation, the rebalancing decisions and the wealth account. For this purpose, we introduce the holding variable $h_{i,t,s}$ which represents the amount we hold in each asset, and the total wealth variable $w_{t,s}$. Moreover, we define the *initial portfolio* vector ip_i in case the investor already has a position in the pension fund and the *initial cash* parameter iw if the investor wants to add an amount of money, i.e., a shift from another pension plan and/or an initial extra contribution.

$$h_{i,t_0,s} = ip_i + r^+_{i,t_0,s} - r^-_{i,t_0,s} + c_{i,t_0,s}, \quad \forall i, \forall s \tag{3}$$

$$\sum_{i=1}^{n} r^+_{i,t_0,s} = \sum_{i=1}^{n} r^-_{i,t_0,s} + iw, \quad \forall s \tag{4}$$

$$r^-_{i,t_0,s} \leq ip_i, \quad \forall i, \forall s \tag{5}$$

$$\sum_{i=1}^{n} r^-_{i,t_0,s} \leq \theta \sum_{i=1}^{n} ip_i, \quad \forall s \tag{6}$$

$$h_{i,t,s} = h_{i,t-1,s} \cdot (1+\rho_{i,t,s}) + r^+_{i,t,s} - r^-_{i,t,s} + c_{i,t,s}, \quad \forall i, t_0 < t < T, \forall s \tag{7}$$

$$\sum_{i=1}^{n} r^+_{i,t,s} = \sum_{i=1}^{n} r^-_{i,t,s}, \quad t_0 < t < T, \forall s \tag{8}$$

$$r^-_{i,t,s} \leq h_{i,t-1,s} \cdot (1+\rho_{i,t,s}), \quad \forall i, t_0 < t < T, \forall s \tag{9}$$

$$\sum_{i=1}^{n} r^-_{i,t,s} \leq \theta \cdot w_{t,s}, \quad t_0 < t < T, \forall s \tag{10}$$

$$w_{t,s} = \sum_{i=1}^{n} \left(h_{i,t-1,s} \cdot (1 + \rho_{i,t,s})\right), \quad t > t_0, \forall s \tag{11}$$

Equation (3) defines the holding in the first stage for each asset equal to the initial portfolio allocation ip_i plus $r^+_{i,t_0,s}$ and $r^-_{i,t_0,s}$, which are the buying and selling of the initial portfolio and the buying and selling of the initial wealth, plus the first period contribution $c_{i,t_0,s}$. The initial portfolio reallocation is defined using equations (4)-(6). In particular, equation (4) defines the buying as reallocation of the initial portfolio plus the allocation of the initial wealth. For next stages, equation (7) defines the holding as capitalization of the previous holding for each asset plus the reallocation of the accumulated wealth and plus the contribution. The portfolio reallocation follows equations (8), (9) and (10). Equations (6) and (10) expresses the turnover constraints through the parameter θ which states that it is not possible to sell more than a fixed percentage θ of the portfolio. Finally, equation (11) computes the accumulated wealth in each stage and for each scenario. According to this wealth variable we build the target constraint and the objective function. Moreover, we include a risk exposure constrain. In order to have a linear problem, we assume that each asset has an associated risk coefficient rc_i, and we set a risk level R the portfolio can not exceed in average.

$$\sum_{i=1}^{n} h_{i,t,s} \cdot rc_i \leq R \cdot \sum_{i=1}^{n} h_{i,t,s}, \quad \forall t, \forall s \tag{12}$$

Since we use a stochastic tree structure, see Dupačová *et al.* (2009), we include the set of all the nonanticipativity constraints on the decision variables. As suggested in Kilianová and Pflug (2009), we define the multicriteria objective function including two wealth targets and the Average Value at Risk Deviation ($AV@RD$) as risk measure, where $AV@RD(x) = \mathbb{E}(x) - AV@R(x)$. We adopt the ϵ-Constrained Approach:

$$\begin{aligned}
\min \quad & \sum_{s=1}^{S} (w_{T,s} \cdot p_s) - a + \frac{1}{\alpha} \sum_{s=1}^{S} (z_s \cdot p_s) && (13)\\
\text{s.t.} \quad & -a + w_{T,s} + z_s \geq 0, \quad z_s \geq 0, \quad \forall s && (14)\\
& \sum_{s=1}^{S} w_{T,s} \cdot p_s \geq \Pi_T && (15)\\
& (2) - (12) && (16)
\end{aligned}$$

In (13) we minimize the $AV@RD$ on the last stage, i.e., on the final wealth, for the given confidence level α. According to Rockafellar and Uryasev (2000) and Rockafellar and Uryasev (2002), the discrete definition of the $AV@RD$ needs the inequality (14) in order to define jointly the variables a and z_s. The final wealth target (15) forces the average of the accumulated wealth on the final stage to be greater or equal than a fixed amount Π_T. In order to compute this value, we suppose to have another portfolio which grows with the same total contribution described in (2) and meanwhile invests uniformly only in the assets which singly satisfy the risk exposure. Then, the target value is the expected wealth accumulated by this portfolio in the final stage. The described formulation produces a linear programming problem.

4 Settings and results

The proposed model is applied to the five representative members defined with the cluster analysis in order to identify the five optimal investment strategies that should be issued by the pension plan. Let assume that the pension fund manager is able to replicate artificially six different securities which compose the investment universe we deal with. The pension funds are a combination of these assets, which are: a guaranteed capital security, two low risk, a medium risk and two high risk assets. Their risky level is described by the associated risk coefficient

$$rc_i = [0 \;\; 1 \;\; 2 \;\; 3 \;\; 7 \;\; 8]$$

We assume that the return processes for the assets and the salary (*) follow a multivariate normal distribution characterized by the following statistics

$$\mu = \begin{bmatrix} 0 \\ 1.5\% \\ 2.0\% \\ 4.5\% \\ 5.0\% \\ 5.5\% \\ 1.0\%^* \end{bmatrix} \qquad \sigma = \begin{bmatrix} 0 \\ 1.5\% \\ 2.0\% \\ 9.5\% \\ 10.0\% \\ 10.5\% \\ 1.0\%^* \end{bmatrix}$$

$$corr = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0^* \\ 0 & 1 & 0.9 & -0.1 & -0.1 & -0.1 & 0^* \\ 0 & 0.9 & 1 & 0 & 0 & 0 & 0^* \\ 0 & -0.1 & 0 & 1 & 0.9 & 0.8 & 0.9^* \\ 0 & -0.1 & 0 & 0.9 & 1 & 0.9 & 0.9^* \\ 0 & -0.1 & 0 & 0.8 & 0.9 & 1 & 0.9^* \\ 0^* & 0^* & 0^* & 0.9^* & 0.9^* & 0.9^* & 1^* \end{bmatrix}$$

In (2) the propensity to save parameter ps is 7%, while the employer contribution e is 50%. Moreover, we let the solver free to find the best here-and-now solution by setting null the initial portfolio, i.e., $ip_i = 0 \;\; \forall i$, and accumulating the whole wealth as extra initial contribution iw. The initial salary is settled for each representative participant to 15,000 euros, i.e., $sal_{t_0,s} = 15000$. This choice is driven by the evidence of a highly dishomogeneous salary level among the cluster elements, thus, we adopt as fixed initial salary the average net salary of the whole population.

In the multicriteria objective function (13) the Average Value at Risk Deviation ($AV@RD$) is computed considering a confidence level $\alpha = 5\%$.

The stochastic tree grows on six stages and the tree branching is 8-5-5-5-5, then the tree is composed of 5,000 scenarios. We propose different time lengths between stages according to the representative member we are going to consider.

The first one is characterized by an initial wealth of 132,000 euros, $iw = 132000$, by a very low risk profile, $R = 1$, and by nine remaining working years. We assume the time length between the six stages as follows: 1, 2, 2, 2 and 2 years respectively. The dynamic optimal allocation evolution is depicted in Figure 3, where the white asset is the guaranteed capital security, and then the asset risk level is identified by the color: from dark green the less risky, to dark red the most risky. Figure 4 describes the distribution of the final wealth and its basic statistics for the first representative member. The here-and-now allocation highlights a prudential strategy. The riskier assets represents less than 5% of the portfolio while the risk free asset is 25%. Then, getting close to the final horizon, the portfolio moves to a safer allocation increasing the portion invested in the risk free asset.

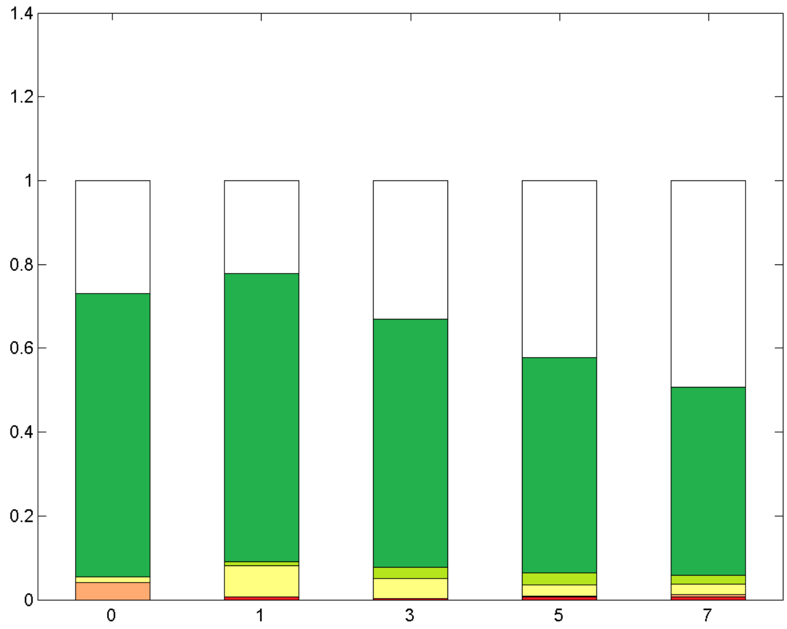
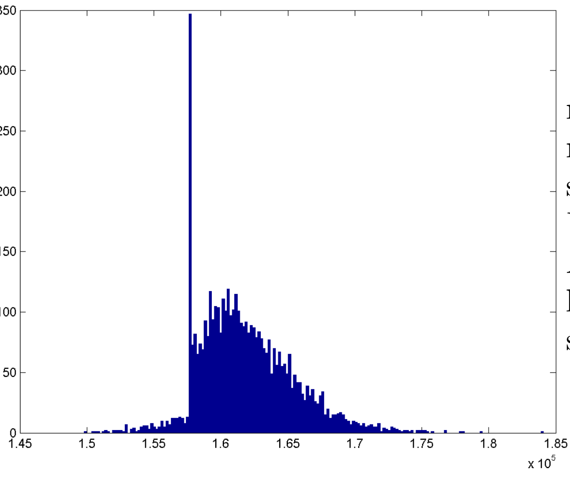


Figure 3: First representative member - Allocation evolution



mean	161,180
median	161,280
st. dev.	3,602
$V@R_{0.05}$	157,690
$AV@R_{0.05}$	156,850
kurtosis	4.19
skewness	0.72

Figure 4: Final wealth distribution and related statistics for the first representative member

The final wealth statistics show a low risk distribution and a high quality of the whole solution due to a low distance between the $AV@R$ and the mean

of the distribution.
The second representative member is characterized by an initial wealth of 78,500 euros, $iw = 78500$, by a very low risk profile, $R = 1$, and by seventeen remaining working years. We assume the time length between the six stages as follows: 2, 2, 4, 4 and 5 years respectively. The dynamic optimal allocation evolution is depicted in Figure 5. As for the first representative, the here-

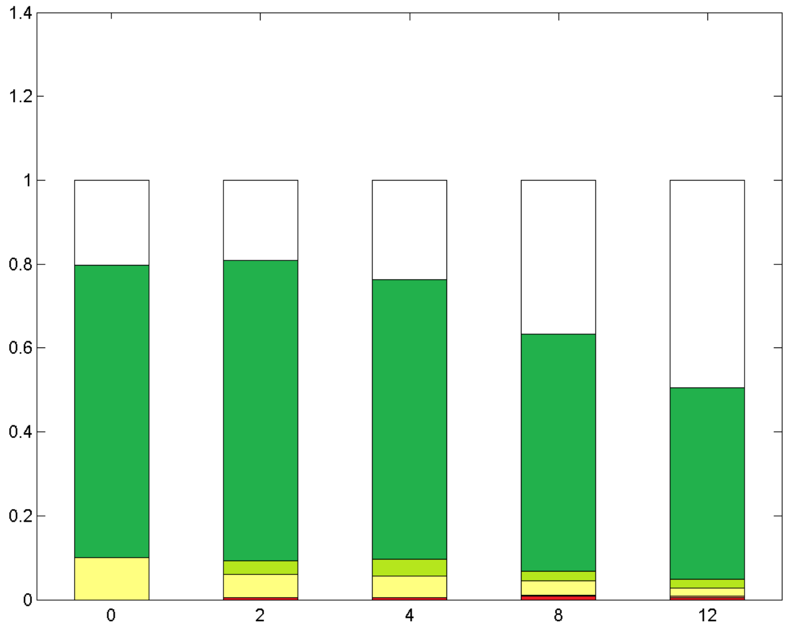
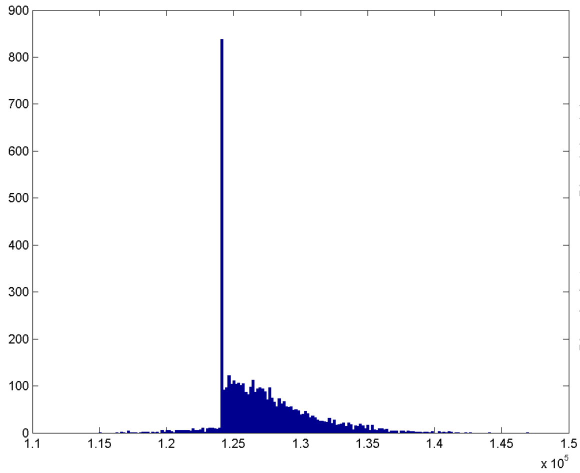


Figure 5: Second representative member - Allocation evolution

and-now allocation is concentrated in the low risk assets. It is clear that the strategy is strongly influenced by the risk attitude of the investor. The time horizon produces a visible effect only in terms of final return as shown in Figure 6 where we show the distribution of the final wealth and its basic statistics.



mean	127,140
median	126,400
st. dev.	3,447
$V@R_{0.05}$	124,070
$AV@R_{0.05}$	123,310
kurtosis	4.73
skewness	1.04

Figure 6: Final wealth distribution and related statistics for the second representative member

The final wealth is again very conservative with a standard deviation of 3,447. The distribution has mean, median and $AV@R$ values very close from each other.

The third representative member is characterized by an initial wealth of 66,800 euros, $iw = 66000$, by a medium risk profile, $R = 5$, and by twenty eight remaining working years. We define the time length between the six stages as follows: 5, 5, 5, 5 and 8 years respectively. The dynamic optimal allocation evolution is depicted in Figure 7. Figure 8 describes the distribution of the final wealth and its basic statistics for the third representative member.

Since the risk profile is slightly higher that the previous two representatives, the here-and-now solution is more aggressive and the two riskiest assets represent more than the 30% of the portfolio. The risk free asset is included from the second stage and then its portion is increased till the last decisional stage.

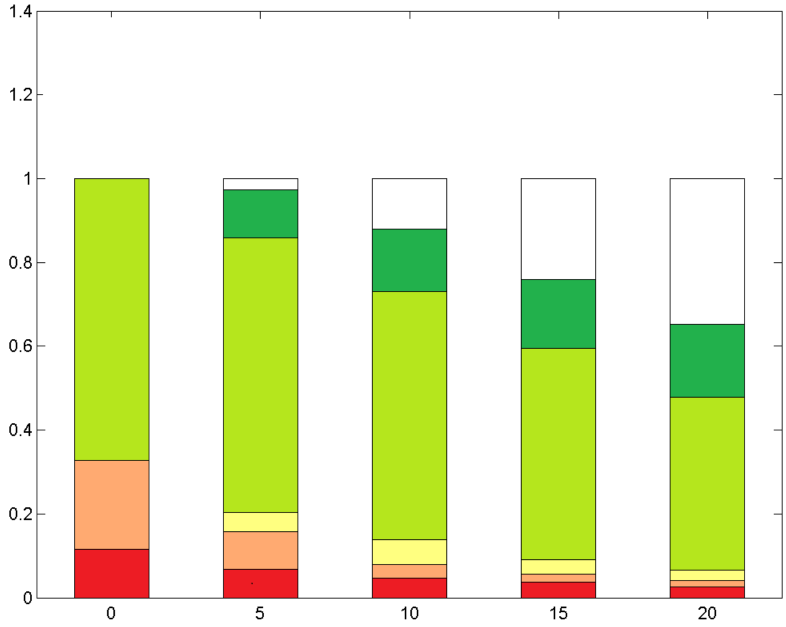
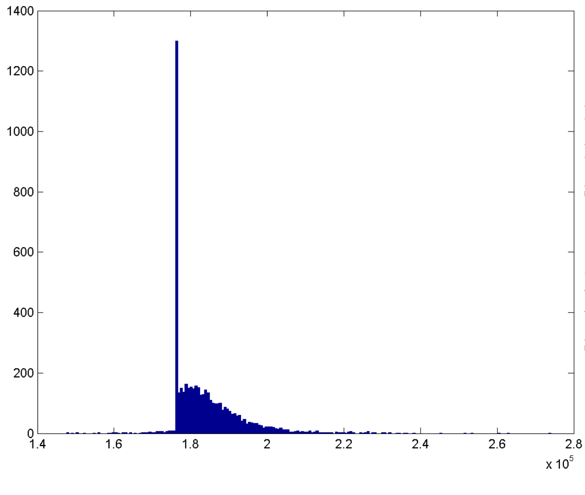


Figure 7: Third representative member - Allocation evolution



mean	183,790
median	181,360
st. dev.	9,529
$V@R_{0.05}$	176,140
$AV@R_{0.05}$	175,600
kurtosis	11.25
skewness	2.01

Figure 8: Final wealth distribution and related statistics for the third representative member

The final distribution is riskier with a standard deviation which is almost three times the previous ones. Nevertheless, the quality of the solution is very good observing the distance between the distribution mean and the $AV@R$.

The fourth representative member is characterized by an initial wealth of 35,000 euros, $iw = 35000$, by a low risk profile, $R = 1$, and by thirty one remaining working years. We assume that the time length between the six stages is 5, 6, 6, 6 and 8 years respectively. The dynamic optimal allocation evolution is depicted in Figure 9. Figure 10 describes the distribution of the final wealth and its basic statistics for the fourth representative member. As

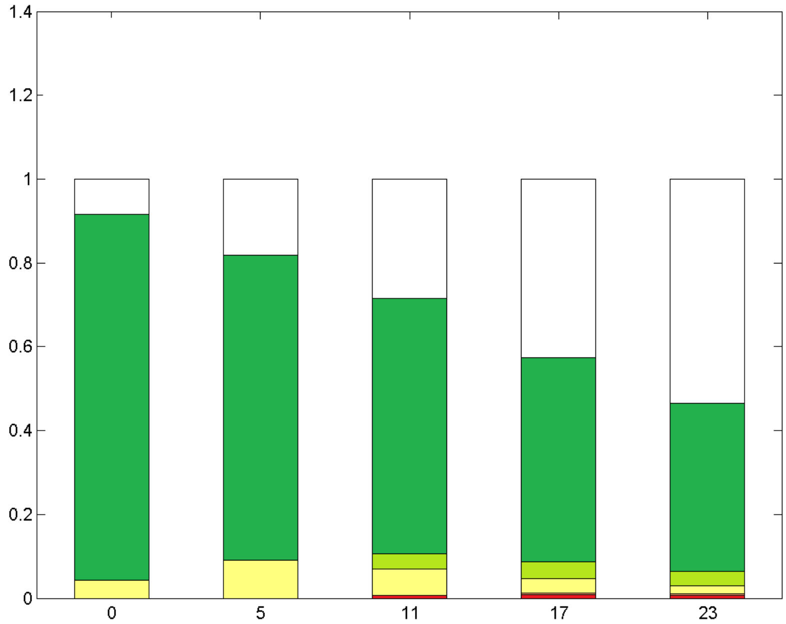
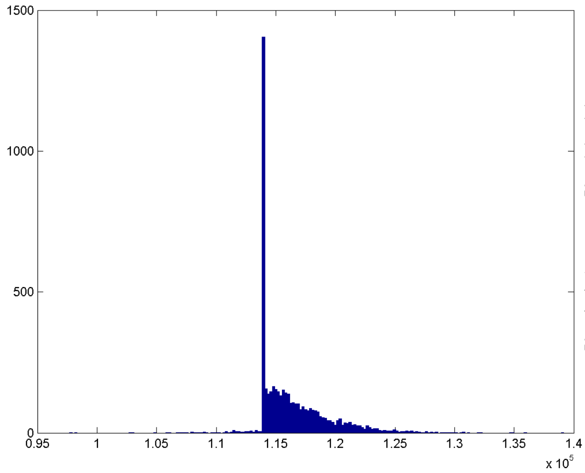


Figure 9: Fourth representative member - Allocation evolution

for the first two representatives, the low risk profile of the fourth representative induces a very prudential allocation, both in the here-and-now solution and in whole the dynamic strategy.



mean	116,370
median	115,430
st. dev.	3,128
$V@R_{0.05}$	113,990
$AV@R_{0.05}$	113,740
kurtosis	7.56
skewness	1.39

Figure 10: Final wealth distribution and related statistics for the fourth representative member

The final distribution highlights results similar to the first two representative members.
The fifth representative member is characterized by an initial wealth of 38,000 euros, $iw = 38000$, by a high risk profile, $R = 10$, and by thirty three remaining working years. We define the time length between the six stages as follows: 6, 6, 6, 7 and 8 years respectively. The dynamic optimal allocation evolution is depicted in Figure 11. The final wealth distribution and its statistics are in Figure 12. The last representative has the highest risk profile and the related allocation is consequently the most aggressive. The here-and-now solution invests more than 50% in the two riskiest assets and the risk free asset is included from the third stage.

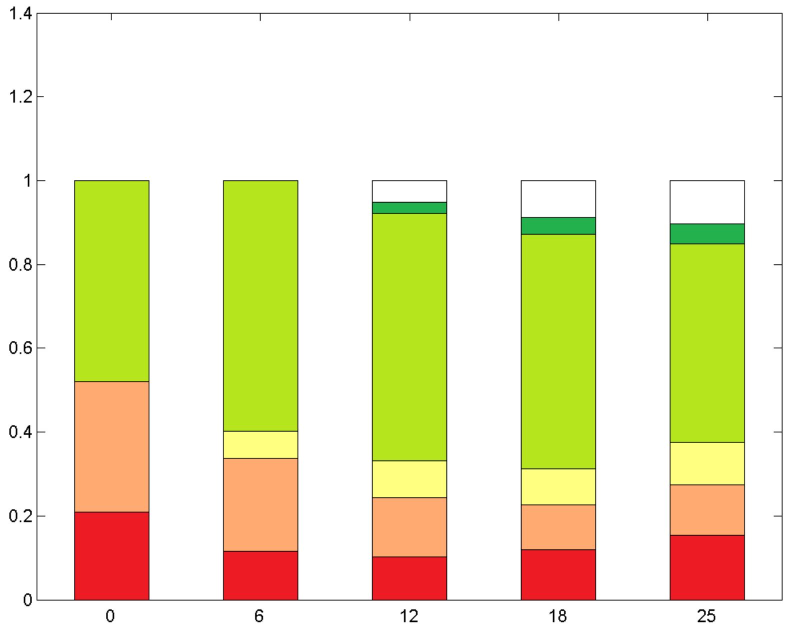
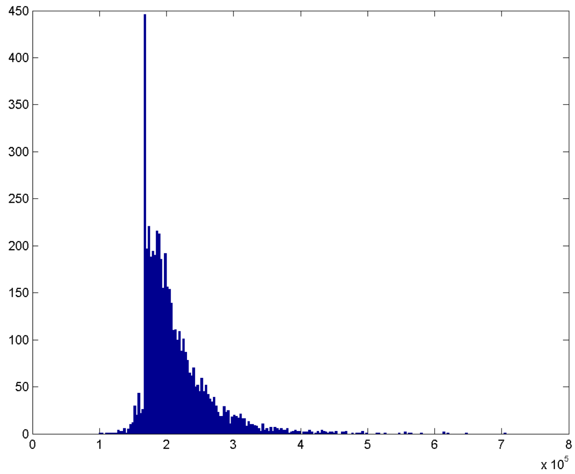


Figure 11: Fifth representative member - Allocation evolution



mean	213,020
median	198,340
st. dev.	53,185
$V@R_{0.05}$	167,400
$AV@R_{0.05}$	162,010
kurtosis	13.21
skewness	2.48

Figure 12: Final wealth distribution and related statistics for the fifth representative member

The statistics of the distribution of the fifth representative final wealth re-

mark a very aggressive strategy. the standard deviation is extremely high and the distance between the mean and the $AV@R$ is remarkable. However, the $V@R$ and $AV@R$ are sufficiently high and suitable for a pension perspective investment.
To summarize, the five optimizations produce five dynamic investment strategies, one for each representative. According to the aim of the proposed analysis, the optimal pension funds are the here-and-now solutions (the first stage decisions) of each strategy and are called fund A, B, C, D and E. The investment strategies compositions are reported in Table 1. The percentage allocation is similar for strategies A, B and D which invest a high percentage in the two lower risk assets and only a residual percentage in a high risk asset. Fund C moves to a more balanced allocation by investing the 67% in a conservative asset and the 33% in high return securities. The most aggressive fund is E which allocates more than 50% in the two riskiest assets. Analyzing the wealth evolutions, it is clear that young representative members can afford riskier positions and then achieve higher returns than older investors. Consequently, the final wealth distribution reflects the portfolio risk attitude. For the fifth investment strategy the distance between the mean and the $AV@R$ is large, while for other allocations is quite small. The kurtosis values highlight fatter tails for the third and the fifth investor. The features of the dynamic allocations and the statistics of the final wealth distributions represent the risk/reward level of each strategy.

Table 1: Pension funds allocation summary

	fund A	fund B	fund C	fund D	fund E
guaranteed capital	25%	20%	0%	9%	0%
low risk 1	69%	69%	0%	87%	0%
low risk 2	0%	0%	67%	0%	48%
medium risk	2%	11%	0%	4%	0%
high risk 1	4%	0%	22%	0%	31%
high risk 2	0%	0%	11%	0%	21%

5 Conclusion

In conclusion, nowadays a quantitative approach is strongly recommended not only to manage a pension fund in an ALM perspective, but also to define the optimal pension provider's offer in terms of pension funds. Indeed, the constitution of a suitable set of pension funds is a crucial goal for a pension plan provider. We propose a two-step approach to address and solve this problem considering the case in which the fund is offered to a homogeneous group of people. The provider should analyze the population in order to offer a set of investment strategies accordingly to its features and needs. Using a clusterization, the statistical analysis produces a set of representative members. Then, for each representative, the optimal here-and-now allocation in a dynamic pension perspective strategy corresponds to each pension fund that the pension plan sponsor should issue. The number of investment strategies follows the cluster cardinality. Clearly, the pension plan provider has to decide if the proposed pension funds are sufficiently different each other, in order to justify the implementation of all of them. The correct balance between the pension plan effort and the members satisfaction is hard to reach, but a quantitative and stochastic formulation of the problem ensures a more reliable decision.

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