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Asymptotic stochastic dominance rules for sums of i.i.d. random variables

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Abstract

In this paper, we deal with stochastic dominance rules under the assumption that the random variables are stable distributed. The stable Paretian distribution is generally used to model a wide range of phenomena. In particular, its use in several applicative areas is mainly justified by the generalized central limit theorem, which states that the sum of a number of i.i.d. random variables with heavy tailed distributions tends to a stable Paretian distribution. We show that the asymptotic behaviour of the tails is fundamental for establishing a dominance in the stable Paretian case. Moreover, we introduce a new weak stochastic order of dispersion, aimed at evaluating whether a random variable is more "risky" than another under condition of maximum uncertainty, and a stochastic order of asymmetry, aimed at evaluating whether a random variable is more or less asymmetric than another. The theoretical results are confirmed by a financial application of the obtained dominance rules. The empirical analysis shows that the weak order of risk introduced in this paper is generally a good indicator for the second order stochastic dominance.

Keywords: Asymmetry, heavy tails, stable Paretian distribution, stochastic dominance.
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1. Introduction

Historically, the Gaussian distribution has always been considered the most relevant probability distribution, due to its important role in statistical inference and to its use in approximating real-valued random variables (r.v.) in many fields of study.

The key reason why the normal distribution is so important is because of the Central Limit Theorem, which states that the sum of a sufficiently large number of independent and identically distributed (i.i.d.) random variables is approximately normally distributed (regardless of the underlying distribution) provided that each random variable has finite variance. When we deal with random variables which do not have finite variance, and thereby when the Gaussian distribution in unsuitable as a limit distribution, we rely on the generalized central limit theorem [1], which states that the sum of a number of i.i.d. random variables with a Paretian tail distribution (decreasing as $|x|^{-\alpha-1}$, where $0 < \alpha < 2$, and therefore having infinite variance) tends to a stable Paretian distribution as the number of summands grows.

Nevertheless, in real world problems the assumption of finite variance is not always appropriate, because several phenomena are well described by random variables which are generally not square-integrable. Therefore, heavy tailed and skewed distributions have often been considered a more realistic distributional assumption for a wide range of natural and man-made phenomena, such as natural sciences (see e.g. [2]), social sciences (see e.g. [3]) or econometrics (see [4] and the references therein). For this reason, the stable Paretian distribution has been proposed as an alternative model to the Gaussian distribution in many different frameworks.

In the financial literature, it is well known that asset returns are not normally distributed, as several studies by Mandelbrot (see [5],[6],[7],[8]) and Fama (see [9],[10],[11]) recognized an excess of kurtosis and non-zero skewness in the empirical distributions of financial assets, which often lead to the rejection of the assumption of normality and proposition of the stable Paretian distribution as an alternative model for asset returns. The Fama and Mandelbrot's conjecture was supported by numerous empirical investigations in the subsequent years, (see [12],[13]).

In view of the several financial applications of the stable Paretian distribution, the aim of this paper is to (stochastically) order stable distributed random variables. Stochastic dominance rules quantifies the concept of one random variable being "preferable" to another, by establishing a partial order in the space of distribution functions. For instance, in a financial context, stochastic orderings are used to establish an order of preferences for investors whose utility functions share certain characteristics [14]. Indeed, it is well known that stochastic dominance rules are generally aimed at addressing investors and institutions towards the best choices in terms of expected gain and risk (see, among others, [15],[16],[17],[18],[19]).

Actually, the financial interpretation of stochastic dominance is straightforward, when the order of preferences can be summarized by maximum expected gain and minimum risk. According to the literature [20], for the expected gain we generally use the first moment (expectation) and for the risk we generally use the variance: this is especially suitable in case of normality. To justify the choices based on the so-called *mean-variance* rule, we need that the return distributions are elliptical and asymptotically approximated by a Gaussian law. Under these assumptions, the mean-variance rule is consistent with the choices of non-satiable and risk averse investors.

However, in this paper, we do not rely on these non-realistic assumptions. In fact, we study the conditions for ordering choices of non satiable risk averse investors in the general case that i) the r.v. does not necessarily have finite variance; ii) the distribution is asymptotically approximated by a stable Paretian law. It is well known in literature that we can obtain the second order stochastic dominance between stable distributions by a mean-dispersion comparison (similar to the Gaussian case), but only when the stable distributions present the same skewness parameter and index of stability ([12], [21]). In this paper, we present more general results, by considering two fundamental aspects of the stable Paretian distributions, namely: the tail behaviour and the asymmetry. In particular, we show that the tail parameter (i.e. index of stability) is crucial for establishing a dominance, in that a distribution with heavier tails cannot dominate (with respect to the discussed preference order) a distribution with lighter tails. Moreover, we define a new stochastic order, weaker then the second order stochastic dominance, and prove that it holds if some conditions on the skewness parameters are verified.

In Section 2, we also introduce a stochastic order of asymmetry which is based on the absolute moments of appropriately standardized random variables. This definition generalizes the traditional definition of asymmetry, based on the Pearson's moment coefficient, and is especially suitable for dealing with heavy tailed r.v.'s, whose moments of order 2 (and 3) do not exist finite. In Section 3 we prove that the skewness parameter of the stable Paretian distribution is indeed coherent with the stochastic order of asymmetry. Finally, in Section 4 we analyze the empirical distributions of a set of asset returns from the U.S. equity market, and show the validity and usefulness of our theoretical results.

2. Stochastic dominance rules for Dispersion and Asymmetry order

In this section, we provide some general results which hold for any kind of random variable. In particular, we introduce new stochastic orderings which will be useful when dealing with stable distributions. We first recall the definitions of some classical stochastic orders.

Definition 1.

- *First order stochastic dominance (FSD): we say that X dominates Y with respect to the first stochastic dominance order (in symbols X FSD Y) if and only if $F_X(t) \leq F_Y(t), \forall\ t \in \mathbb{R}$, or, equivalently X FSD Y if and only if $E(g(X)) \geq E(g(Y))$ for any increasing function g.*

- *Second order stochastic dominance (SSD or increasing concave order): we say that X dominates Y with respect to the second stochastic dominance order (in symbols X SSD Y or $X \geq_{icv} Y$) if and only if $\int_{-\infty}^{t} F_X(u)\,du \leq \int_{-\infty}^{t} F_Y(u)\,du, \quad \forall t \in \mathbb{R}$ or, equivalently X SSD Y if and only if $E(g(X)) \geq E(g(Y))$ for any increasing and concave function g. Obviously X FSD Y implies also X SSD Y.*
- *Increasing and convex order: we say that X dominates Y with respect to the increasing convex order (in symbols $X \geq_{icx} Y$) if and only if $\int_{t}^{+\infty} 1 - F_X(u)\,du \geq \int_{t}^{+\infty} 1 - F_Y(u)\,du, \quad \forall t \in \mathbb{R}$ or, equivalently $X \geq_{icx} Y$ if and only if $E(g(X)) \geq E(g(Y))$ for any increasing and convex function g. Obviously X FSD Y implies also $X \geq_{icx} Y$.*
- *Rothschild-Stiglitz stochastic dominance (RS or concave order): we say that X dominates Y with respect to the Rothschild-Stiglitz order (in symbols X RSY) when X SSD Y and $E(X) = E(Y)$ (or $Y \geq_{icx} X$ and $E(X) = E(Y)$) or equivalently X RS Y if and only if $E(g(X)) \geq E(g(Y))$ for any concave function g.*

The following theorem extends the well-known result of Hanoch and Levy [22] which determines a sufficient condition for SSD. Similarly, we obtain a sufficient condition which allows to deny the SSD and the $\geq_{icx}$ dominance. Both conditions are based on the number of crossing points between distributions, and will be really useful in Section 3. In particular, we establish that if two distributions have an even number of crossing points, then the SSD and the $\geq_{icx}$ ordering cannot hold.

Theorem 1. *Let X and Y be two random variables.*

a) *Let F_X and F_Y have a single crossing point say t_1 such that $F_X(t) \leq F_Y(t)$ for $t < t_1$ ($F_X(t) < F_Y(t)$ for some $t < t_1$) and $F_X(t) \geq F_Y(t)$ for $t \geq t_1$. If $E(X) \geq E(Y)$, then X SSD Y. If $E(X) \leq E(Y)$, then $Y \geq_{icx} X$.*

b) *Assume $E(X) = E(Y) < \infty$. Let F_X and F_Y have an even number k of crossing points, namely $t_1, \ldots, t_k$ (where $t_1 < t_2 \ldots < t_k$) such that,* *if we let* $t_0 = -\infty$ *and* $t_{k+1} = \infty$ *we have:* $(-1)^i F_X(t) \leq (-1)^i F_Y(t)$ *for* $t_i \leq t < t_{i+1}$ *and* $i = 0, \ldots, k$ *and* $(-1)^i F_X(t) < (-1)^i F_Y(t)$ *for some* $t_i \leq t < t_{i+1}$. *Then there exists at least one point say* $t'' < \infty$ *such that* $\int_{-\infty}^{t''} F_X(z)\,dz > \int_{-\infty}^{t''} F_Y(z)dz$ *(thus the condition* X *SSD* Y *cannot be true). Moreover there exist a point* s' *such that* $\int_{s'}^{+\infty} \{1 - F_X(z)\,dz < \int_{s'}^{+\infty} 1 - F_Y(z)dz\}$ *(thus the condition* $X \geq_{icx} Y$ *cannot be true).*

Generally, stochastic dominance should be aimed at establishing an order of preference under conditions of maximum uncertainty, as the preference should hold a fortiori under sharper conditions. For this reason, it is especially worth to study the RS order, which corresponds to the condition of high uncertainty $E(X) = E(Y)$. It is known that the expected value of a random variable $X \in (-\infty, \infty)$ can be formulated as follows:

$$E(X) = -\int_{-\infty}^{0} F(x)dx + \int_{0}^{\infty} 1 - F(x)dx. \tag{1}$$

Moreover,

$$E|X| = \int_{-\infty}^{0} F(x)dx + \int_{0}^{\infty} 1 - F(x)dx. \tag{2}$$

We define the positive and negative parts of a r.v. X respectively by:

$$X_+ = \begin{cases} X & X \geq 0 \\ 0 & X < 0 \end{cases} \text{ and } X_- = \begin{cases} -X & X \leq 0 \\ 0 & X > 0 \end{cases}. \tag{3}$$

The following results show that the RS stochastic order implies some weaker orders which involve the absolute values of the random variables.

Proposition 1. *Let* X *and* Y *be random variables with finite expected values. The following implications hold.*

1 *If* X *RS* Y *then* $-(X_-)$ *SSD* $-(Y_-)$ *and* $-(X_+)$ *SSD* $-(Y_+)$

2 *If* $-(X_-)$ *SSD* $-(Y_-)$ *and* $-(X_+)$ *SSD* $-(Y_+)$ *then* $-|X|$ *SSD* $-|Y|$

3 $-|X|$ *SSD* $-|Y|$ *if and only if* $|Y| \geq_{icx} |X|$

4 *If* X *RS* Y *and* $E|X| = E|Y|$ *then* $|X|$ *RS* $|Y|$

Proposition 2. *Let X and Y be random variables with $E(X) = E(Y) = 0$. The following implications hold.*

1 *If X RS Y and X and Y are symmetric, then $|Y|$ SSD $|X|$ ($|Y| \geq_{icv} |X|$).*

2 *If X RS Y, X and Y are symmetric and $E|X| = E|Y|$, then $X =_d Y$.*

Proposition 1 is especially useful for the analysis proposed in this paper, because it provides a necessary condition for SSD when $E(X) = E(Y)$ (or equivalently, for RS). Indeed, Propositoin 1 (point 3) states that the RS order implies an inverse stochastic dominance between the absolute values of the random variables. Therefore, the order $|Y| \geq_{icx} |X|$, identified by the conditions $E|Y| \geq E|X|$ and $E(h(|X|)) \leq E(h(|Y|))$ (for h increasing and convex), reflects the major "risk" or dispersion of Y compared to X, and can be especially useful under condition of maximum uncertainty. In other words, when comparing distribution with equal means, if $|Y| \geq_{icx} |X|$ holds we can argue that X is preferable than Y in terms of risk aversion. Furthermore, in the particular condition when $E(X) = E(Y) = 0$, Propositoin 2 states that, if X RS Y and X, Y are symmetric, then $|Y| \geq_{icx} |X|$ and $|Y| \geq_{icv} |X|$, and thereby Y presents, a fortiori, a higher dispersion, or risk, compared to X. Observe that this situation might suggest a stronger condition, i.e. the first order stochastic dominance between $|Y|$ and $|X|$, meaning that $|Y|$ is stochastically "larger" than $|X|$. However, it is worth noting that these two conditions generally do not imply $|Y|$ FSD $|X|$: this can be shown by a straightforward counter-example.

Example 1. Let U, V be discrete and positive random variables with the following distributions:

$$F_U(z) = \begin{cases} 0 & z < 1 \\ 1/3 & 1 \leq z < 3 \\ 1 & z \geq 3 \end{cases} ; \quad F_V(z) = \begin{cases} 0 & z < 2 \\ 2/3 & 2 \leq z < 4 \\ 1 & z \geq 4 \end{cases} \tag{4}$$

It is easy to verify that $\int_{-\infty}^{t} F_U(z)\,dz \leq \int_{-\infty}^{t} F_V(z)dz$ for any $t \in \mathbb{R}$ and $\int_t^{\infty} 1 - F_U(z)dz \geq \int_t^{\infty} 1 - F_V(z)dz$ for any $t \in \mathbb{R}$, thus $U \geq_{icv} V$ and $U \geq_{icx} V$, nevertheless we do not have U FSD V: hence generally $U \geq_{icv} V$ and $U \geq_{icx} V$ $\nRightarrow U$ FSD V.

Proposition 1 suggests that, when we know that $E(Y) = E(X)$ but the RS order is not verifiable (provable), it is possible to analyze a weaker order of "risk" by studying the distributions of the absolute values. In particular, we focus on a weaker order implied by $|Y| \geq_{icx} |X|$, which can be obtained by a particular class of increasing and convex functions (namely the power functions) and which will be especially useful in section 3, for the stable Paretian case. Let Z be a random variable and define:

$$\varphi_Z(p) = \mathrm{sign}(p) E(|Z|^p), \quad p \in \mathbb{R}. \tag{5}$$

Observe that, when $\varphi_X(p) \leq \varphi_Y(p)$ holds $\forall p \geq 1$, we have that $|z|^p$ is an increasing and convex function, thus X *RS* $Y \Rightarrow X - E(X)$ *RS* $Y - E(Y)$ $\Rightarrow |Y - E(Y)| >_{icx} |X - E(X)| \Rightarrow \varphi_{X-E(X)}(p) \leq \varphi_{Y-E(Y)}(p)$, $\forall p \geq 1$; from Proposition 1 we also deduce that X RS $Y \Rightarrow \varphi_X(p) \leq \varphi_Y(p)$, $\forall p \geq 1$.

Moreover, if X,Y are symmetric, $E(Y) = E(X) = 0$ and X *RS* Y then Propositions 1 and 2 yield a stronger condition, that is, $\varphi_X(p) \leq \varphi_Y(p)$ for any p. Hence, we argue that the condition $\varphi_X(p) \leq \varphi_Y(p)$, $\forall p \geq 1$, reflects the major risk of Y compared to X. We are now able to define a new stochastic order of risk, expressed in terms of the absolute centered moments of order p.

Definition 2. *Let X and Y be random variables belonging to $L^q = \{X | E(|X|^q) < \infty\}$, for $q \geq 1$. We say that X dominates Y with respect to the central moment dispersion (CMD) order (in symbols $X \geq_{cmd} Y$) if and only if*

$$\varphi_{X-E(X)}(p) \leq \varphi_{Y-E(Y)}(p), \forall p \geq 1.$$

We say that X dominate Y with respect the moment dispersion (MD) order, (in symbols $X \geq_{md} Y$), if and only if

$$\varphi_X(p) \leq \varphi_Y(p), \quad \forall p \geq 1$$

Observe that the moment dispersion order is a FORS ordering (see [15]) for the absolute value of centered random variables. Moreover, as a consequence of the Proposition 1 and the previous discussion, we obtain the following corollary.

Corollary 1. *Let X and Y be random variables with finite expected values. If X RS Y, then $X \geq_{md} Y$ and $X \geq_{cmd} Y$.*

It is well known that a r.v. is symmetric if and only if it exists a real number m such that $f_X(m-x) = f_X(m+x)$ for any x, where f_X is the probability density function of X. However, this definition cannot be easily empirically verified. Nevertheless, when the distribution of the r.v. X is more concentrated around its mean, compared to another random variable Y, then we expect X to be "more symmetric" than Y. Thus, in some sense, the moment dispersion is related to the concept of symmetry. If two random variables are not symmetric, we may be interested in determining which one is more asymmetric than the other. This comparison may be possible if the r.v.'s are properly standardized. We recall that usually, in statistics, a random variable is standardized by subtracting its expected value and dividing the difference by its standard deviation. Then, the asymmetry is generally measured by the Pearson's moment coefficient of skewness, that is, the standardized moment of order three.

However, the Pearson's coefficient presents two drawbacks. First, it cannot be defined for all random variables. Second, it is not always able to detect asymmetry. Differently, in this paper we also consider random variables that do not necessarily have finite variance (for which the traditional measures of skewness cannot be evaluated). Furthermore, we define asymmetry as a characteristic which involves the whole distribution of a properly standardized variable. In particular, in order to identify the concept of asymmetry we need to establish a stochastic order for more (or less) asymmetric distributions. For this purpose, we first define a generalized standardization procedure that can be used even for random variables that do not have finite variance. Indeed, observe that we can easily generalize the idea of standardization when the random variable is in the domain of attraction of a stable law.

Let X be a random variable in the domain of attraction of a stable law with finite expectation $E(X) = \mu_X$ and infinite variance. Let $\{X_i\}_{i\in\mathbb{N}}$ be independent and identically distributed observations of X. Thus, we know that there exists a sequence of positive real values $\{d_{X,i}\}_{i\in\mathbb{N}}$ and a sequence of real values $\{a_{X,i}\}_{i\in\mathbb{N}}$, such that, as $n \to +\infty$:

$$\frac{1}{d_{X,n}} \sum_{i=1}^{n} X_i + a_{X,n} \xrightarrow{d} X^{'}; \tag{6}$$

where $X^{'} \sim S_{\alpha_X}(\sigma_X, \beta_X, \mu_X)$ is an α-stable Paretian random variable, where $0 < \alpha_X \leq 2$ is the so-called stability index, which specifies the asymptotic behavior of the tails, $\sigma_X{>}0$ is the dispersion parameter, $\beta_X \in [-1, 1]$ is the skewness parameter and $\mu_X \in \mathbb{R}$ is the location parameter. Observe that, in this paper, we consider the parameterization for stable distributions proposed by Samorodnisky and Taqqu [23]). Our idea is that, when the sum of i.i.d. random variables (except for an affine transformation) admits a limiting distribution, we can standardize it by subtracting the location parameter μ_X (in our case $E(X) = \mu_X$) and by dividing by the scalar parameter σ_X. Hence we obtain the *generalized standardization* of X, given by $\overline{X} = \frac{X-\mu_X}{\sigma_X}$, where $\mu_{\overline{X}} = E\left(\overline{X}\right) = 0$ and $\sigma_{\overline{X}} = 1$. In particular, when the random variable X has also finite standard deviation $\sqrt{V(X)} = \sigma_X$, we obtain the classical standardization, by using $d_{X,n} = \sqrt{n}$ and $a_{X,n} = -\sqrt{n}\mu_X$ in 6. As specified above, we assume that the distributions have finite mean, because in most of the practical cases (e.g. in finance) we actually deal with (at least) integrable r.v.s. Hence, based on the definition of generalized standardization, it is possible to introduce an ordering of asymmetry, which is especially suitable for dealing with distributions that do not have finite variance. We argue that X is more right-asymmetric than Y if X presents a lighter left tail and a heavier right tail, compared to Y, that is, if $-(\overline{X}_-)$ stochastically dominates (in some general sense) $-(\overline{Y}_-)$ and similarly $-(\overline{Y}_+)$ dominates $-(\overline{X}_+)$ (see e.g.[24, 25]). Then, coherently with this principle, we propose the following ordering of asymmetry.

Definition 3. *Let X and Y be random variables with means μ_X, μ_Y and scalar parameters σ_X, σ_Y.*

- *We say that X is more right asymmetric than Y, (in symbols $X \gg_r Y$), if and* *only if*

$$\phi_X(p) \geq \phi_Y(p), \quad \forall p \geq 1$$

where

$$\phi_X(p) = E\left(\overline{X}^{<p>}\right) = E\left(\overline{X}_+{}^p\right) - E(\overline{X}_-{}^p),$$

$x^{<p>} = sign(x)\,|x|^p$ *and* $\overline{X} = \frac{X-\mu_X}{\sigma_X}$.

- *We say that* X *is more left asymmetric than* Y*, (in symbols* $X \gg_l Y$*), if and only if*

$$\phi_X(p) \leq \phi_Y(p), \quad \forall p \geq 1.$$

This asymmetry ordering is a simple FORS ordering [15] for standardized (in the sense specified above) random variables. In particular, $\gg_r$ (as well as $\gg_r$) is clearly consistent with the skewness ordering "$\geq_6$" defined by [24]. Moreover, the Pearsons coefficient of skewness, that is $\phi_X(3)$, is obviously isotonic (coherent) with the ordering defined above. Note that the order is reversed if we consider the opposites of the random variables, because $\phi_{-X}(p) = -\phi_X(p)$. Furthermore, the conditions $\overline{Y} \geq_{md} \overline{X}$ and $X \gg_r Y$ imply that $E\left(\overline{X}_-\right) = E\left(\overline{X}_+\right) \geq E(\overline{Y}_+) = E\left(\overline{Y}_-\right)$, because $\phi_X(1) = \phi_Y(1) = 0$ and $\lim_{p\to 1^+} \varphi_{\overline{X}}(p) + \phi_X(p) = 2E\left(\overline{X}_+\right)$. From these considerations, we deduce that the $\gg_r$ and $\gg_l$ orders (of right and left asymmetry) are strictly related with the moment dispersion order, as stated in the following corollary.

Corollary 2. *Let* X *and* Y *be two random variables. Then the following implications hold:*

a) $X \gg_r Y$ *if and only if* $Y \gg_l X$.

b) $X \gg_r Y$ *if and only if* $-Y \gg_r -X$.

c) *If* $X \geq_{cmd} Y$ *and* $\sigma_X \leq \sigma_Y$*, then* $\overline{X} \geq_{md} \overline{Y}$*. In particular, if* $\sigma_X \leq \sigma_Y$ *and* X *RS* Y*, then* $\overline{X} \geq_{md} \overline{Y}$.

d) *If* $X \gg_r Y$ *and* $\overline{Y} \geq_{md} \overline{X}$*, then*

$$E\left(\overline{X}_-\right) = E\left(\overline{X}_+\right) \geq E\left(\overline{Y}_+\right) = E\left(\overline{Y}_-\right).$$

While, if $X \gg_r Y$ *and* $\overline{X} \geq_{md} \overline{Y}$*, then*

$$E\left(\overline{X}_-\right) = E\left(\overline{X}_+\right) \leq E\left(\overline{Y}_+\right) = E\left(\overline{Y}_-\right).$$

Observe that generally the Pearson's coefficient of skewness is improperly employed for identifying asymmetry, because it cannot be used to rank all distributions. Note that the MD ordering between standardized variables could be interpreted as an order of symmetry, in that $\overline{X} \geq_{md} \overline{Y}$ implies that X is "more symmetric" than Y (in a general sense). Moreover, it can be interpreted as an ordering of kurtosis (see [24]), in that $\overline{X} \geq_{md} \overline{Y}$ is a location and scale invariant ordering which implies that Y presents heavier tails than X (this is especially clear in the case that X and Y are symmetric) and which makes it possible to extend the well known kurtosis index (based on the central moment of order four) to the case of distributions with infinite variance.

Another important considerations is as follows. Roughly speaking, a distribution is intuitively right (or left) asymmetric if its right (or left) tail is heavier than its left (or right) tail. Thus, we can give a definition of right (left) asymmetry consistent with the asymmetry orderings.

Definition 4. *Let X be random variables with mean μ_X and scalar parameters s_X.*

- *We say that X is right asymmetric if and only if $\phi_X(p) \geq 0, \forall p \geq 1$.*
- *We say that X is left asymmetric if and only if $\phi_X(p) \leq 0, \forall p \geq 1$.*
- *We say that X is symmetric if and only if $\phi_X(p) = 0, \forall p \geq 1$.*

Clearly, a symmetric distribution according to Definition 4 is also symmetric according to the Pearson's coefficient, because $E\left(\overline{X}_+{}^p\right) = E(\overline{X}_-{}^p)\ \forall p \geq 1$ implies that $\overline{X}_- = \overline{X}_+$ in distribution. Moreover, a r.v. is right asymmetric if and only if it can be seen as the opposite of a left asymmetric r.v.. Finally, if X is right asymmetric then $E\left(\overline{X}_+{}^p\right) \geq E(\overline{X}_-{}^p)$ for any $p \geq 1$, and thereby $\overline{X}_- \geq_{md} \overline{X}_+$. Thus, the following corollary is a logic consequence of the previous discussion.

Corollary 3. *Let X and Y be two random variables. Then the following implications hold:*

i) *X is right asymmetric if and only if $\overline{X}_{-} \geq_{md} \overline{X}_{+}$.*

ii) *X is right asymmetric if and only if $-X$ is left asymmetric.*

iii) *If $\overline{X}_{-} RS\ \overline{X}_{+}$ then X is right asymmetric.*

As in some cases it would not be possible to verify the conditions for Definition 3, we introduce a weaker order of asymmetry.

Definition 5. *Let X and Y be random variables belonging to $L^q = \{X | E(|X|^q) < \infty\}$, for $q \in (1, 2)$, with means μ_X, μ_Y and scalar parameters σ_X, σ_Y.*

- *We say that X is weakly more right asymmetric than Y (in symbols $X \gg_{wr} Y$) if and only if there exists a value $s \in (1, q)$ such that*

$$\phi_X(p) \geq \phi_Y(p), \quad \forall p \geq s.$$

- *We say that X is weakly more left asymmetric than Y (in symbols $X \gg_{wl} Y$) if and only if there exists a value $s \in (1, q)$ such that*

$$\phi_X(p) \leq \phi_Y(p), \ \forall p \geq s.$$

On the one hand, the value $q \in (1, 2)$ ensures that we can always compare random variables which are in the domain of attraction of a stable law. On the other hand, Definition 5 is consistent with the Pearson's coefficient of skewness, besides being weaker than Definition 3 (i.e. $X \gg_r Y$ implies $X \gg_{wr} Y$ and $X \gg_l Y$ implies $\gg_{wl} Y$).

3. Stochastic orders in the stable Paretian case

In this section, we deal with the problem of ordering stable Paretian distributions. In particular, we prove that, in the stable Paretian case, the index of stability is crucial for establishing the SSD order, based on some results proved in Section 2. Furthermore, we prove that the skewness parameter is strictly related to the moment dispersion (MD) order, and show that it is also consistent with the stochastic order of asymmetry defined in Section 2.

Let us briefly summarize some important characteristics of the stable distribution. As discussed in the introduction, the stable Paretian law is especially appropriate for approximating the distribution of a random variable X whose tails are significantly heavier than the Gaussian law, that is, for large x

$$P(|X| > x) \approx x^{-\alpha} L(x) \tag{7}$$

where $0 < \alpha < 2$ and $L(x)$ is a slowly varying function at infinity. This tail condition implies that the random variable X is in the domain of attraction of a stable law. In this case, we obtain the convergence condition described in 6. This convergence result is a consequence of the Stable Central Limit Theorem (SCLT) for normalized sums of i.i.d. random variables (see [23], [12]) and it is the main justification for the use of stable distribution in many areas of study, such as finance and econometrics. In particular, the SCLT makes it possible to characterize the skewness and kurtosis of a wide range of phenomena in a statistically proper way.

We recall that, if $X \sim S_\alpha(\sigma, \beta, \mu)$, and $\alpha < 2$, then $E(|X|^p) < \infty$ for any $p < \alpha$ and $E(|X|^p) = \infty$ for any $p \geq \alpha$. Therefore, stable distributions do not generally have finite variance, which happens only when $\alpha = 2$ (i.e. Gaussian distribution, $E(|X|^p) < \infty$ for any p). Unfortunately, except in few cases, we do not have a closed form expression for the density of stable Paretian distribution, which is identified by its characteristic function, given by:

$$E(\exp\{itX\}) =$$

$$= \begin{cases} \exp\left\{it\mu - |t\sigma|^\alpha \left(1 - i\beta \operatorname{sign}(t) \tan\left(\frac{\pi\alpha}{2}\right)\right)\right\} & \alpha \neq 1 \\ \exp\left\{it(\mu + 2\beta\sigma \ln(\sigma)\pi) - |t\sigma| (1 + 2i\beta \ln|t\sigma| \operatorname{sign}(t)/\pi)\right\} & \alpha = 1 \end{cases} \tag{8}$$

It is worth noting that, since the density and distribution functions of the stable Paretian distribution cannot be expressed with elementary functions, it is not possible to verify the integral conditions for the RS and SSD orders. However, some simple orderings are a consequence of the stable Paretian tail behavior which is mainly determined by the stability parameter α. As a matter of fact, in the stable case, the behavior of the tails is so crucially important to determine a stochastic order between the distribution functions, for fixed values of location and scale parameter.

It is known that, if $X \sim S_\alpha(\sigma, \beta, 0)$, then, as x approaches infinity, we obtain:

$$P\left(\pm X > x\right) \approx$$

$$\approx \begin{cases} \exp\left\{-\left(\alpha-1\right)\left(\frac{x}{\alpha\sigma B_\alpha}\right)^{-\frac{\alpha}{\alpha-1}}\right\} \frac{\left(\frac{x}{\alpha\sigma B_\alpha}\right)^{-\frac{a}{2\alpha-2}}}{\sqrt{2\pi\alpha(\alpha-1)}} \\ if\ \beta = \mp 1 \wedge < \alpha < 2 \\ \exp\left\{-\frac{1}{2}\left(\frac{\pi x}{2\sigma}-1\right) - \exp\left[\left(\frac{\pi x}{2\sigma}-1\right)\right]\ \right\} / \sqrt{2\pi} \\ if\ \beta = \mp 1 \wedge \alpha = 1 \\ \\ C_\alpha\left(1 \pm \beta\right)\sigma^\alpha x^{-\alpha} \quad \text{otherwise} \end{cases} \tag{9}$$

where $B_\alpha = \left[\cos\left(\frac{\pi}{2}(2-\alpha)\right)\ \right]^{-1/\alpha}$and $C_\alpha = \frac{\Gamma(\alpha)}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)$.

Equation 9 shows that, for $\beta = \mp 1$ and $\alpha < 1$, stable distributions are negative ($\beta = -1$) or positive ($\beta = +1$).

The theorems and porpositions of this section establish under which conditions we can obtain a stochastic dominance between stable distributions.

Let $X_1 \sim S_{\alpha_1}(\sigma_1, \beta_1, \mu_1)$ and $X_2 \sim S_{\alpha_2}(\sigma_2, \beta_2, \mu_2)$. As quite logical and trivial, if $\mu_1 > \mu_2$, and the other parameters are equal, we obtain that the distribution of X_1 is right-shifted compared to the distribution of X_2, and therefore $F_{X_1} < F_{X_2}$ (X_1 FSD X_2) as it is apparent also from Fig. 1 (in the case $\alpha < 1$)[1].

[1]The figures have been obtained with the Mathematica 9 package for the numerical computation of stable distributions, see [26] and [27].

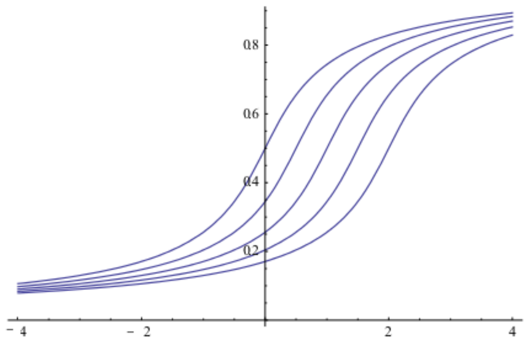

Figure 1: CDFs of stable distributed r.v.'s, with fixed $\alpha < 1, \sigma, \beta$ (respectively $0.8, 1, 0$) and different values of $\mu = 0, 0.5, 1, 1.5, 2$, (from left to right).

In particular, it is worth noting that, when $\alpha_1, \alpha_2 < 1$, and $\beta_1, \beta_2 < 1$, X_1 and X_2 cannot be compared with the second-order stochastic dominance SSD, because the integral $\int_{-\infty}^{t} F_{X_i}(z)\, dz$ (for $i = 1, 2$) diverges. However, it is possible to establish the first order stochastic dominance according to the location (as observed above) and also the skewness parameters, as proved in [23]. Indeed, Fig. 2 shows that, if $\beta_1 > \beta_2$, $\alpha_1 = \alpha_2 < 1$, $\sigma_1 = \sigma_2$ and $\mu_1 = \mu_2$, then the disproportions between the left and right tails of X_1 and X_2 yield an FSD order, i.e. $F_{X_1} < F_{X_2}$. Therefore, on fixed values of $\alpha < 1$ and σ, it is sufficient that $\mu_1 \geq \mu_2$ and $\beta_1 \geq \beta_2$ with at least one strict inequality to obtain the FSD order. This is stated in the following proposition.

Proposition 3. *Consider $X_1 \sim S_{\alpha_1}(\sigma_1, \beta_1, \mu_1)$ and $X_2 \sim S_{\alpha_2}(\sigma_2, \beta_2, \mu_2)$. Suppose $\alpha_1 = \alpha_2 < 1$ and $\sigma_1 = \sigma_2$. Moreover $\mu_1 \geq \mu_2$ and $\beta_1 \geq \beta_2$ with at least one strict inequality. Then X_1 FSD X_2.*

The case $\alpha < 1$ is however less interesting for our purposes, especially because of the several empirical investigations, e.g. in the Financial literature, which have shown that generally asset returns comply to stable Paretian laws, but the tail parameter α is generally greater than 1. Hence, in what follows

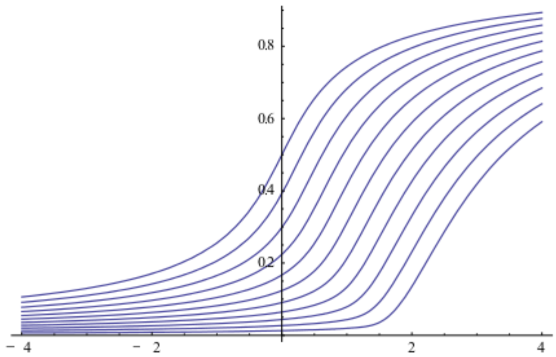

Figure 2: CDFs of stable distributed r.v.'s, with fixed $\alpha < 1, \sigma, \mu$ (respectively $0.8, 1, 0$) $S_{0.8}(1, \beta, 0)$ and different values of $\beta = 0 + 0.09i$, $i = 1, \ldots, 10$, (from left to right).

we analyze the case $\alpha > 1$, and therefore consider only distributions with finite means and for which it is possible to use the SSD order. Obviously, on fixed values of tail and skewness parameters, we might verify the SSD with a mean-dispersion approach (similar to the mean-variance approach used in the Gaussian case): the aim of this paper is to overcome this quite restrictive assumption and deal with the problem under more general conditions. For this analysis, note that, given $X \sim S_\alpha(\sigma, \beta, \mu)$, then $X = \sigma Y + \mu$, where Y is the standardized stable $S_\alpha(1, \beta, 0)$. For the standardized stable Y we know that $\int_{-\infty}^{0} F_Y(z)\, dz = E(Y_-)$, which represents the expected losses valued for a stable random variable (see [28]) and it is given by:

$$\int_{-\infty}^{0} F_Y(z)\, dz = \frac{\Gamma\left((\alpha-1)/\alpha\right)}{\pi} \frac{cos(\theta)}{\left(cos(\alpha\theta)\right)^{1/\alpha}} \tag{10}$$

where $\theta = \frac{\arctan(\beta \tan(\pi\alpha/2))}{\alpha}$. This formula is graphically represented by Fig.3 (varying alpha and beta).

Let $X_1 \sim S_{\alpha_1}(\sigma_1, \beta_1, \mu_1)$ and $X_2 \sim S_{\alpha_2}(\sigma_2, \beta_2, \mu_2)$. Clearly, Fig.3 shows that, for fixed $\alpha_1 = \alpha_2$, $\sigma_1 = \sigma_2$ and $\mu_1 = \mu_2$, we cannot have the SSD dominance except for $|\beta_1| < |\beta_2|$. However, we shall show in Theorem 3 that the SSD order is actually not verified in this case. Moreover, if $\alpha_1 < \alpha_2$, $\beta_1 = \beta_2$,

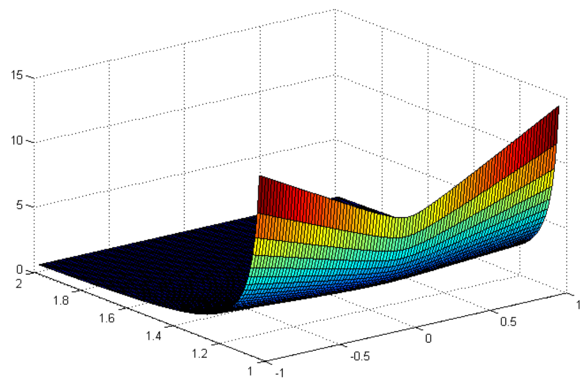

Figure 3: Expected losses $E(Y_-)$ for standardized stable distributed r.v.'s, where α varies from 1.01 to 2 and β from -1 to 1.

$\sigma_1 = \sigma_2$ and $\mu_1 = \mu_2$ then formula 9 suggests that X_1 presents heavier tails than X_2. Thus, when $\mu_1 = \mu_2$, X_1 and X_2 have equal mean, i.e.

$$-\int_{-\infty}^{0} F_{X_1}(x)\, dx + \int_{0}^{\infty} 1 - F_{X_1}(x)\, dx =$$
$$= -\int_{-\infty}^{0} F_{X_2}(x)dx + \int_{0}^{\infty} 1 - F_{X_2}(x)dx,$$

we argue that the area under the curve F_{X_2}on the left tail of X_2 (which is finite because $\alpha_2 > 1$) is sufficiently large that $\int_{-\infty}^{t} F_{X_2}(x)dx$ is always greater or equal than $\int_{-\infty}^{t} F_{X_1}(x)dx$ and that the area $\int_{-\infty}^{t} F_1(x)dx$ should approach $\int_{-\infty}^{t} F_{X_2}(x)dx$ only asymptotically (note that, as t tends to infinity, the integrals obviously diverge). We illustrate this concept in Fig. 4 and Fig. 5. In particular, Fig. 4 refers to the case when X_1 and X_2 are symmetric ($\beta_1 = \beta_2 = 0$) (observe that the distributions show some similarity with the Gaussian distribution, as α approaches 2). The effect of the stability indices is evident, especially in the case of skewed distributions (Fig. 5). From Fig. 4 we also note that, when the index of stability approaches 2, the distribution gets close to the normal distribution, regardless of the skewness parameter. The following theorem states that we can obtain the SSD and $\geq_{icx}$ orders between stable Paretian distributions by a comparison between location, scalar, and tail parameters.

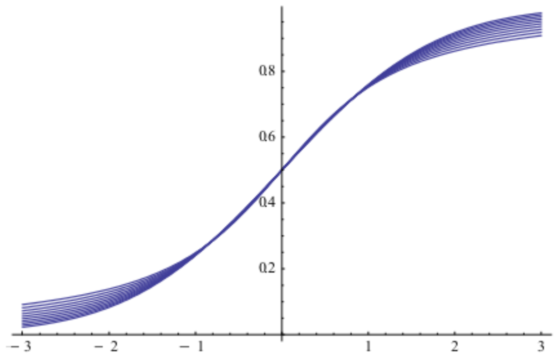

Figure 4: CDFs of stable distributed r.v.'s, with fixed β, σ, μ (respectively $0, 1, 0$) $S_\alpha(1, 0, 0)$ and different values of $\alpha = 1 + 0.9i$, $i =, 1, \ldots, 10$.

Theorem 2. *Consider* $X_1 \sim S_{\alpha_1}(\sigma_1, \beta_1, \mu_1)$ *and* $X_2 \sim S_{\alpha_2}(\sigma_2, \beta_2, \mu_2)$. *Suppose* $\alpha_1 \geq \alpha_2 > 1$, $\beta_1 = \beta_2$, $\sigma_1 \leq \sigma_2$ *with at least one strict inequality. If* $\mu_1 \geq \mu_2$, *then* X_1 *SSD* X_2. *If* $\mu_2 \geq \mu_1$, *then* $X_2 \geq_{icx} X_1$. *In particular, if* $\mu_1 = \mu_2$, *then* X_1 *RS* X_2.

As a straightforward consequence, the assumptions of thoerem 2 (regardless of the location parameter) also imply $\overline{X}_1 \geq_{md} \overline{X}_2$, which can be interpreted as an ordering of kurtosis (as discussed in section 2). Hence, as expected, we obtain the α parameter is actually consistent with kurtosis.

Now, we can investigate the effects of the skewness parameters, e.g. whether a more positively skewed variable dominates a less skewed one (like in the case $\alpha < 1$), or rather whether a more symmetric variable could be considered, in some sense, less "risky". Indeed, equation 9 shows that the asymptotic behavior of the tails depends on the skewness parameter β, besides the tail parameter α. The skewness parameter determines the disproportion between the left and the right tails, in particular, if β is positive (or negative), we obtain that the

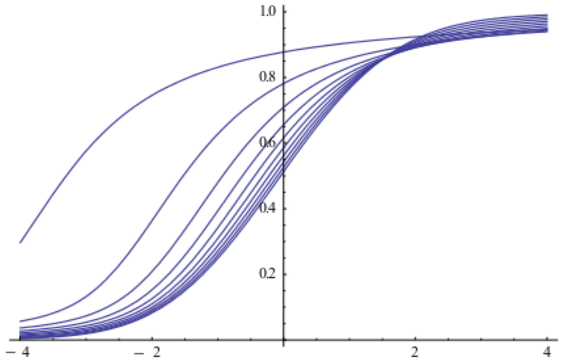

Figure 5: CDFs of stable distributed r.v.'s, with fixed β, σ, μ (respectively $0.5, 1, 0$) $S_\alpha(1, 0.5, 0)$ and different values of $\alpha = 1 + 0.9i$, $i =, 1, \ldots, 10$.

right (or left) tail is heavier than its left (or right) tail. The extreme cases when $\beta = \pm 1$ are especially explicatory. On the one hand, when $|\beta| < 1$ the tails follow a typical Paretian (power law) distribution, and therefore they can be defined "heavy", compared to those of a Gaussian distribution. On the other hand, when $\beta = +1$ (or -1) then the left (or right) tail distribution is decreasing with an exponential behavior (as the Gaussian distribution) and therefore we have that the distribution is heavy tailed, but only on its right (or left) side.

Let $X_1 \sim S_{\alpha_1}(\sigma_1, \beta_1, \mu_1)$ and $X_2 \sim S_{\alpha_2}(\sigma_2, \beta_2, \mu_2)$ and suppose that $\beta_1 > \beta_2$, $\alpha_1 = \alpha_2 > 1$, $\sigma_1 = \sigma_2$ and $\mu_1 = \mu_2$. In this case, the assumptions of Thoerem 1 are not satisfied (some graphical examples below show that the FSD does not hold). Intuitively, a more right-skewed distribution transfers weight from the left to the right tail. On the one hand, when $\alpha < 1$, the tails are so heavy that basically absorb most of the body of the distribution, therefore a disproportion between right and left tail implies that a more right skewed distribution gets stochastically larger (FSD). On the other hand, when $\alpha > 1$, the weight of the tails is downsized and therefore a more right skewed distribution is anyway "heavier" on its right tail, but this is not sufficient to yield a strong (FSD) ordering between the distribution (Theorem 4 below proves indeed that $\beta_1 > \beta_2$ does not even yield the SSD). Finally, in the case $\alpha > 1$, the skewness parameter does not seem to be strictly related to a strong ordering (such as SSD), like the tail parameter does. However, we can realize that a more symmetric distribution can be preferable than a more skewed distribution, in terms of dispersion or risk-aversion. Indeed, a more symmetric distribution is generally less "spread out", so we argue that, if $|\beta_1| < |\beta_2|$, then X_1 dominates X_2 in terms of some weaker order of dispersion (risk). In particular, the following theorem states that $|\beta_1| < |\beta_2|$ yields the moment dispersion order (MD). Fig. 6 shows that different β's do not yield FSD as well as SSD (for Theorem 1, even number of crossing points), but, by contrast, the less skewed r.v.'s are less spread out around the zero value.

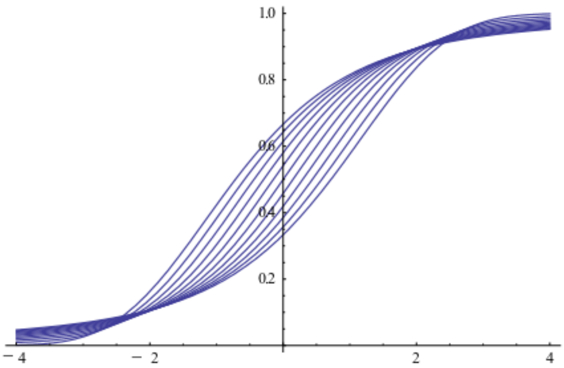

Figure 6: CDFs of stable distributed r.v.'s, with fixed α, σ, μ $(1.5, 1, \text{and } 0$ respectively), $S_{1.5}(1, \beta, 0)$ and different values of $\beta = 0 + 0.2i$, $i =, -5, \ldots, 5$.

Theorem 3. *Let $X_1 \sim S_{\alpha_1}(\sigma_1, \beta_1, \mu_1)$ and $X_2 \sim S_{\alpha_2}(\sigma_2, \beta_2, \mu_2)$. Suppose $|\beta_1| < |\beta_2|$.*

- *If $\alpha_1 = \alpha_2 > 1$, $\sigma_1 = \sigma_2$, $\mu_1 = \mu_2$ then X_1 does not dominate X_2 w.r.t. the SSD and X_1 does not dominate X_2 w.r.t. the $\geq_{icx}$ order. In addition, $\varphi_{X_1 - E(X_1)}(p) \leq \varphi_{X_2 - E(X_2)}(p)$, $\forall -1 \leq p \leq \alpha$, and thereby $X_1 \geq_{cmd} X_2$ and $\overline{X}_1 \geq_{md} \overline{X}_2$.*
- *If $\alpha_1 \geq \alpha_2 > 1$ then $\overline{X}_1 \geq_{md} \overline{X}_2$. Moreover, if $\alpha_1 \geq \alpha_2 > 1$ $\sigma_1 \leq \sigma_2$, $\mu_1 = \mu_2$ with at least a strict inequality, then $X_1 \geq_{cmd} X_2$.*

Therefore, a less skewed distribution may be preferable to a more skewed one by risk averse investors. Observe that we can also prove that the skewness parameter β determines whether a distribution is symmetric or (right-left) asymmetric, according to definition 4.

Proposition 4. *Assume $X_1 \sim S_{\alpha_1}(\sigma_1, \beta_1, \mu_1)$ and $X_2 \sim S_{\alpha_2}(\sigma_2, \beta_2, \mu_2)$ where $\alpha_1, \alpha_2 > 1$. Thus, the following implications hold.*

- *If $\beta_1 \geq 0 \geq \beta_2$, then $X_1 \gg_r X_2$*
- *$\beta_1 > 0$ if and only if X_1 is right asymmetric*
- *$\beta_1 < 0$ if and only if X_1 is left asymmetric*
- *$\beta_1 = 0$ if and only if X_1 is symmetric*

Moreover, as a consequence of what discussed above, we obtain the following orderings among the positive and negative parts of standardized stable distributions, when $\beta_1 > \beta_2 \geq 0$, or $\beta_2 < \beta_1 \leq 0$.

Theorem 4. *Let $X_1 \sim S_{\alpha_1}(\sigma_1, \beta_1, \mu_1)$, $X_2 \sim S_{\alpha_2}(\sigma_2, \beta_2, \mu_2)$ with $\alpha_1 = \alpha_2 > 1$, $\sigma_1 = \sigma_2$, $\mu_1 = \mu_2$.*

1) *If $\beta_1 > \beta_2 \geq 0$, then:*
   $X_2 \geq_{cmd} X_1$, $\overline{X}_{i-}$ *RS* $\overline{X}_{i+}$ *$(i = 1, 2)$*, $\overline{X}_{2+} \geq_{icx} \overline{X}_{1+}$ $\overline{X}_{1-}$ *SSD* $\overline{X}_{2+}$, $\overline{X}_{1-}$ *SSD* $\overline{X}_{2-}$, *and* $X_1 \gg_{wr} X_2$ *(or* $X_2 \gg_{wl} X_1$*).*
2) *If $\beta_2 < \beta_1 \leq 0$, then:*
   $X_1 \geq_{cmd} X_2$, $\overline{X}_{i+}$ *RS* $\overline{X}_{i-}$ *$(i = 1, 2)$*, $\overline{X}_{2-} \geq_{icx} \overline{X}_{1-}$ $\overline{X}_{2+}$ *SSD* $\overline{X}_{1+}$, $\overline{X}_{2+}$ *SSD* $\overline{X}_{1-}$, *and* $X_1 \gg_{wr} X_2$ *(or* $X_2 \gg_{wl} X_1$*).*

On the one hand, Theorem 4 confirms the intuition for which the skewness parameter can determine if a random variable is more or less right (or left) asymmetric than another one, according to the weak asymmetry order defined in Section 2. Hence, the skewness parameter is coherent with the defined order. On the other hand, we conjecture that under the hypothesis of Theorem 4 the strong asymmetry order of Definition 3 holds, but, unfortunately, we were not able to prove it. Moreover, the weaker orders of asymmetry are strongly influenced from the heavy tails (represented by small values of the index of stability) as proved in the following proposition.

Proposition 5. *Assume* $X_1 \sim S_{\alpha_1}(\sigma_1, \beta_1, \mu_1)$ *and* $X_2 \sim S_{\alpha_2}(\sigma_2, \beta_2, \mu_2)$. *Thus, the following implications hold.*

1. *If* $\beta_1 < 0$ *and* $1 < \alpha_1 < \alpha_2$ *then* $X_1 \gg_{wl} X_2$
2. *If* $\beta_1 > 0$ *and* $1 < \alpha_1 < \alpha_2$ *then* $X_1 \gg_{wr} X_2$

According to Theorems 2 and 4 and Proposition 5 we deduce that, generally, risk seeking non satiable decision makers (agents who optimize the increasing convex ordering) should prefer right asymmetric random variables with the lowest index of stability, highest mean and scalar parameter. Moreover, in Theorems 3 and 4 and Proposition 5 we show that, by varying the index of stability and the skewness parameter, we are not generally able to find strong orders such as SSD and $\geq_{icx}$, although we cannot exclude that some stronger orders might hold. In particular, given two random variables $X_1 \sim S_{\alpha_1}(\sigma_1, \beta_1, \mu_1)$ and $X_2 \sim S_{\alpha_2}(\sigma_2, \beta_2, \mu_2)$ with $\alpha_1 > \alpha_2 > 1$, $\sigma_1 = \sigma_2$, $\mu_1 = \mu_2$ and $\beta_1 \neq \beta_2$, then the two distributions generally present an odd number of crossing points (i.e. 1 or 3 for each different sub-case analyzed, e.g. $\beta_1 > \beta_2 \geq 0$, $\beta_2 > \beta_1 \geq 0$ etc.). Therefore, when the gap between the indexes of stability is large enough, we observe that F_{X_1} and F_{X_2} are single crossing (on the left if β_1 is positive, on the right otherwise), and therefore X_1 SSD X_2, as shown in Fig. 7.

On the other hand, we can obtain three crossing points (1 on the left and 2 on the right if β_1 is positive, 2 on the left and 1 on the right otherwise) when α_1 and α_2 are “closer”. This is shown in Fig. 8. It should be stressed, that, in Fig. 8 we can only see two crossing points (which would exclude that X_1 SSD X_2 for Theorem 1), but actually we know from equation 9 that we have another crossing point as t approaches infinity, because $F_{X_1}(t)$ tends to 1 faster than $F_{X_2}(t)$ does. Finally, this analysis highlights that in many cases we cannot exclude the SSD when the index of stability and the skewness parameters vary at the same time. However, it is worth noting that it is also possible, in some

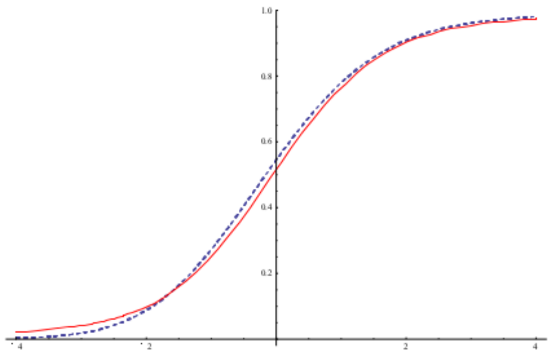

Figure 7: $X_1 \sim S_{1.8}(1, 0.8, 1)$ (dashed) and $X_2 \sim S_{1.6}(1, 0.1, 1)$ (red). 1 crossing point.

particular cases that the SSD order does not definitely hold. This can be shown by a straightforward counter-example.

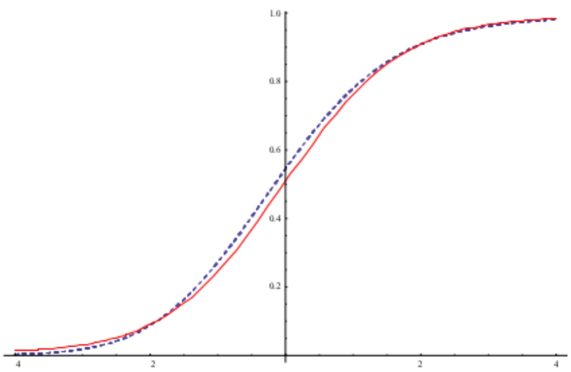

Figure 8: $X_1 \sim S_{1.8}(1, 0.8, 1)$ (dashed) and $X_2 \sim S_{1.75}(1, 0.1, 1)$ (red). 3 crossing point (the third one is not visible but we know that there exists from the behavior of the tails).

Example 2. Let $X_1 \sim S_{\alpha_1}(\sigma_1, \beta_1, \mu_1)$ and $X_2 \sim S_{\alpha_2}(\sigma_2, -1, \mu_2)$. Suppose $\alpha_1 > \alpha_2 > 1$, $\sigma_1 = \sigma_2$, ${}_1 = \mu_2$ and $\beta_1 > -1$. Consider the distributions of $-(X_{1+})$ and $-(X_{2+})$. From Theorem 1 we know that if X RS Y then $-(X_{1-})$ SSD $-(X_{2-})$ and $-(X_{1+})$ SSD $-(X_{2+})$. But equation 9 yields that the tail of $-(X_{2+})$ approaches zero faster than the tail of $-(X_{1+})$, therefore we definitely have that the condition $-(X_{1+})$ SSD $-(X_{2+})$, which is necessary for RS, does not hold.

We conclude that, in the case of $\alpha > 1$, which is surely the most interesting from a practical point of view, the index of stability is crucially important in establishing an order of preference between stable distributions. This aspect could have a strong impact for classical economical and financial choices. On the one hand, a less skewed distribution should be generally preferred than a more skewed one by risk averse decision maker that generally want to reduce the dispersion (MD order). On the one hand, a more right skewed distribution should be generally preferred than a less right skewed one by risk seeking decision maker. Therefore, we can properly order stable distributions according to the relations between the parameters. In the next section we propose one of the several possible financial applications of the stochastic order relations discussed above.

4. A financial application

The results of Section 3 have several applications in different areas of study, because of the fundamental role of the stable distribution, discussed in the introduction. In some areas of study, such as finance or econometrics, it is well known that distributions are generally heavy tailed, thus the determinantion of the tail pobabilities can play a key role (for a recent estimation method of tail probabilities for heavy tailed distributions see e.g. [29]). In particular, it is well known that the stable Paretian model is especially suitable for approximating the empirical distribution of financial assets (see, for instance, [5],[8],[9],[13],[30],[31],[32],[33]). Obviously, alternative models with explicit forms for densities have also been considered in the literature, e.g. the Student's t distribution or the generalized hyperbolic distribution [34]. Nevertheless, as specifed in the introduction, the main advantage of the stable distribution compared to others is represented by its role in the generalized central limit theorem. A possible financial application of multivariate stochastic dominance rules for symmetric stable distribution has been recently proposed by [35]. In this section, we simply apply the stochastic dominance rules stated in the previous sections to real financial data, in order to empirically verify the validity of these rules.

The dataset consists of 2242 assets that were active and sufficiently liquid[2] in the last year (from February 2014 until February 2015) in the U.S. equity market (NYSE, NASDAQ, AMEX). As it is well known, the empirical distribution of financial assets is generally conformed to the stable distribution, hence, for each asset, we estimated the unknown parameters $(\alpha, \sigma, \beta, \mu)$ with the maximum likelihood (ML) method. Note that there are mainly two approaches to the problem of ML estimation in the stable Paretian case. Modern ML estimation techniques for stable distributions either utilize i) the fast Fourier transform method for approximating the stable density function [36], [12]; or ii) the direct integration method [37]. Both approaches are comparable in terms of efficiency and the differences in performance result from different approximation algorithms. In our analysis we used the first approach. Then, we applied the stochastic dominance rules established above, based on the estimated parameters. The used methodology can be described as follows.

Let X and Y be two assets, let $(x_1, \ldots, x_n)$ and $(y_1, \ \ldots, y_n)$ be the observations of the asset returns during the considered period and $F_{n,1}$ and $F_{n,2}$ the corresponding empirical distributions. For $i = 1, \ 2$, we denote with $\hat{\alpha}_i, \hat{\sigma}_i, \hat{\beta}_i, \hat{\mu}_i$ the ML estimates of stable parameters $(\alpha, \sigma, \beta, \mu)$ and with $\widehat{F}_i$ the corresponding stable Paretian distributions. For simplicity, let us define, with a little abuse of notation, the relations "SSD" and "$\geq_{cmd}$" in the domain $\mathcal{F}$ of all distribution functions, instead of the class of random variables (hence F_X SSD F_Y is equivalent to X SSD Y and similarly $F_X \geq_{cmd} F_Y$ is equivalent to $X \geq_{cmd} Y$). From the results of the previous sections we can establish the following rules.

[2]We introduce a liquidity filter in our dataset taken from DataStream, which requires that each asset must be traded daily for at least 100000 USD on average during the period 01/02/2014 - 01/02/2015.

1) If $\hat{\alpha}_1 \geq \hat{\alpha}_2$, $\hat{\sigma}_1 \leq \hat{\sigma}_2$, $\left|\hat{\beta}_1\right| \leq \left|\hat{\beta}_2\right|$ and $\hat{\mu}_1 \geq \hat{\mu}_2$ then $\widehat{F}_1 SSD\ \widehat{F}_3 \geq_{cmd} \widehat{F}_2$ where the distribution $\widehat{F}_3$ correspond to a stable random variable with parameters $\alpha_2, \sigma_2, \beta_1, \mu_2$

2) If $F_{n,1}$ and $F_{n,2}$ are single crossing and we observe the following inequality between the sample mean of the asset returns $\frac{1}{n}\sum_{j=1}^{n} x_j = \overline{x} \geq \overline{y} = \frac{1}{n}\sum_{j=1}^{n} y_j$, then $F_{n,1}$ SSD $F_{n,2}$.

By applying rule 1 to the data, we found 329 assets (over 2242) which are dominated from some other assets with respect to two different orders between fitted distributions. As we know, the MD order is weaker than the RS order, thus we do not really know if the SSD holds, but this results give us important information on the empirical distributions, that are likely dominated at the second order from other assets. Indeed, by applying rule 2 to the empirical distribution of these 329 assets, we found that 231 assets are actually dominated at the second order, hence the strong order seems to hold at least for 2/3 of the cases. Figures 9, 10 and 11 show some particular examples of second order dominance between empirical distributions. In Figure 9 we report the estimated parameters of stable distributions and the empirical SSD dominance for a couple of asset returns. Figures 10 and 11 show left and right empirical tails of all the distributions which are SSD dominated (in the sense of rule 2) by one of the non-dominated return distributions.

In these figures, we observe that the right and left tails of the dominant distribution are much thinner than the others, which also present higher skewness (in terms of the absolute value of β): this confirms that the tail behavior, which determines the probability of big losses and/or big gains, is crucial in determining a dominance in the stable Paretian case.

From these considerations, we argue that the introduced criteria can be absolutely useful to exclude a certain set of assets from the investors' choices. Therefore, if applied properly, the same criteria can be also used for the choice of the optimal portfolio.

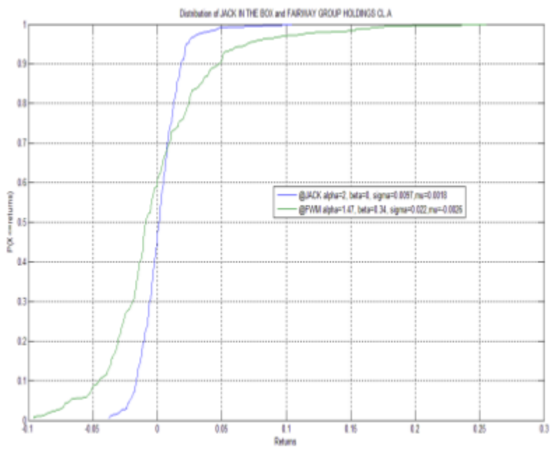

Figure 9: Example of SSD dominance between the empirical distributions of the assets: "Dollar tree" and "Inventergy global".

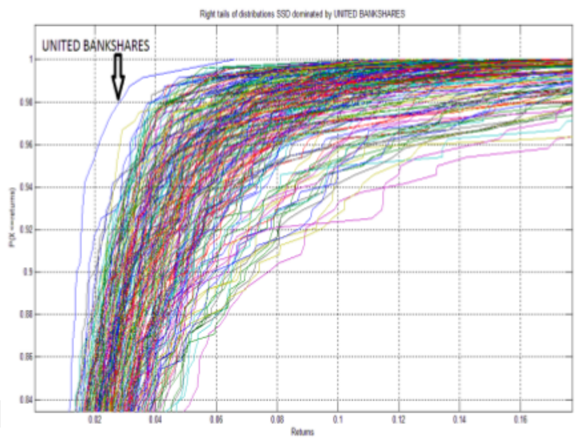

Figure 10: Example of right tails of some assets SSD empirically dominated by the "United Bankshares".

5. Conclusion

In this paper, we determined some new rules to establish stochastic dominance between stable distributed random variables. It should be stressed that the stable Paretian distribution is justified by the generalized central limit the-

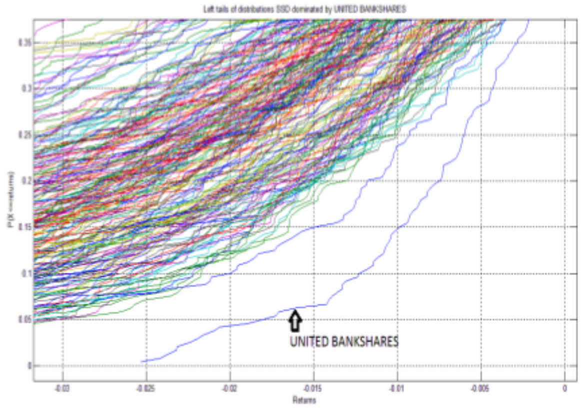

Figure 11: Example of left tails of some assets SSD empirically dominated by the "United Bankshares".

orem for sums of i.i.d. random variables. Hence, the theoretical results of this paper may have several applications in many different fields of study. In particular, we introduced two new stochastic orders, namely: i) the moment dispersion order, which expresses the major dispersion or "risk" between two variables with equal expectation; and ii) the asymmetry order, which is especially suitable for dealing with random variables that do not necessarily have finite variance.

Then, we prove and show under which conditions we can obtain these newly introduced orders in the stable Paretian case. In particular, we show that the asymptotic behavior of the tails, which is mainly determined by the index of stability, is crucially important to establish a dominance. Indeed, under some particular conditions, we are able to prove that the second order stochastic dominance is strictly related to this parameter. Moreover, we obtain that the stable skewness parameter is coherent to the asymmetry order defined in section 2.

Finally, we show the usefulness of these new results in determining dominated and dominating assets in the U.S. stock market. Our empirical findings confirm that the newly introduced order of risk is actually closely related to the stronger order, that is the SSD, and thereby it can be profitably used to compare and order financial variables, according to investors' preferences.

Appendix

A. Proof of Theorem 1.

On the one hand, Hanoch and Levy [22] proved point a) for SSD ordering. On the other hand, we know that $-X$ SSD $-Y$ if and only if Y $\geq_{icx} X$ see [14]. When. $E(X) \leq E(Y)$ then $E(-X) \geq E(-Y)$. Moreover, the crossing-point condition holds also for the opposite of the random variables, i.e. $F_{-X}(t) \leq F_{-Y}(t)$ for $t < -t_1$ ($F_{-X}(t) < F_{-Y}(t)$ for some $t < -t_1$) and $F_{-X}(t) \geq F_{-Y}(t)$ for $t \geq -t_1$.Thus, when $E(X) \leq E(Y)$ from Hanoch and Levy [21] we get that $-X$ SSD $-Y$, that implies $Y \geq_{icx} X$. As for point b), define $F_Y(z) - F_X(z) = \Delta_Y(z)$. $\int_{-\infty}^{\infty} \Delta_Y(z)dz = \int_{-\infty}^{t_k} \Delta_Y(z)dz + \int_{t_k}^{\infty} \Delta_Y(z)dz = 0$, thus $\int_{-\infty}^{t_k} \Delta_Y(z)dz = -\int_{t_k}^{\infty} \Delta_Y(z)dz$. By assumption $F_X(t) \leq F_Y(t)$ for $t > t_k$ ($F_X(t') < F_Y(t')$ for some $t' > t_k$) therefore $\int_{t_k}^{\infty} \Delta_Y(z)\,dz > 0$ and $\int_{-\infty}^{t_k} \Delta_Y(z)dz < 0$ which implies that the condition X SSD Y cannot hold. Similarly, $\int_{t_1}^{\infty} \Delta_Y(z)\,dz = -\int_{-\infty}^{t_1} \Delta_Y(z)dz$. By assumption $F_X(t) \leq F_Y(t)$ for $t < t_1$ ($F_X(t') < F_Y(t')$ for some $t' < t_1$) therefore $\int_{t_1}^{\infty} \Delta_Y(z)\,dz < 0$ and $\int_{-\infty}^{t_1} \Delta_Y(z)dz > 0$ which implies that the condition $X \geq_{icx} Y$ cannot be true.

B. Proof of Proposition 1.

1) Observe that:

$$F_{-(X_-)}(t) = \begin{cases} F_X(t) & t \leq 0 \\ 1 & t > 0 \end{cases}, \quad F_{-(X_+)}(t) = \begin{cases} 1 - F_X(-t) & t \leq 0 \\ 1 & t > 0 \end{cases}.$$

Hence, by assumption we have that $-(X_-)$ SSD $-(Y_-)$.

Nevertheless X RS Y implies $-X$ RS$-Y$ (see [14]), thus $-(X_+)$ SSD $-(Y_+)$.

2) As $[-(X_-)]+[-(X_+)] = -|X|$ and its cumulative distribution is $F_{-|X|}(t) = F_{-(X_-)}(t) + F_{-(X_+)}(t)$, then 1) yields $-|X|$ SSD $-|Y|$.

3) $-|X|$ SSD $-|Y|$ for 2), thus $-|X| \geq_{icv} -|Y|$ which is equivalent to $|Y| \geq_{icx} |X|$ (see [14]).

4) Point 1) and condition $E|X| = E|Y|$ yield $-|X|$ RS $-|Y|$, which is equivalent to $|X|$ RS $|Y|$.

C. Proof of Proposition 2.

1) Define $F_X(z) - F_Y(z) = \Delta_X(z)$ and $-\Delta_X(z) = \Delta_Y(z)$. For symmetry we have that:

$$\int_{-\infty}^{0} \Delta_Y(z)\, dz - \int_{-t}^{0} \Delta_Y(z)\, dz =$$

$$= \int_{0}^{\infty} \Delta_X(z)\, dz - \int_{0}^{t} \Delta_X(z)\, dz \geq 0$$

As $\int_{-\infty}^{0} \Delta_Y(z)dz$ is non-negative we obtain $\int_{-t}^{0} \Delta_Y(z)dz = \int_{0}^{t} \Delta_X(z)dz \geq 0$. Thus $\int_{-t}^{0} \Delta_Y(z)dz - \int_{0}^{t} \Delta_X(z)\, dz = 0$ and $\int_{-t}^{0} \Delta_Y(z)dz + \int_{0}^{t} \Delta_X(z)dz \geq 0$ which is equivalent to $|Y|$ SSD $|X|$.

2) From Proposition 1 and point 1) we respectively obtain $|X| \geq_{icv} |Y|$ and $|Y| \geq_{icv} |X|$ which yield $|X| =_d |Y|$. But X and Y are symmetric, hence $|X| =_d |Y| \;\Leftrightarrow X =_d Y$.

D. Proof of Proposition 3.

Assume $\mu_1 > \mu_2$ and let $X_3 \sim S_{\alpha_1}(\sigma_1, \beta_1, \mu_2)$. Thus $X_1 =_d X_3 + (\mu_1 - \mu_2)$ which yields X_1 FSD X_3.

Assume $\beta_1 > \beta_2$ and let $X_4 \sim S_{\alpha_1}(\sigma_1, \beta_2, \mu_2)$. We obtain X_1 FSD X_3 (in particular $X_1 =_d X_3$ if $\mu_1 = \mu_2$) and X_3 FSD X_4 as proved in [23] (Property 1.2.14).

E. Proof of Theorem 2.

When $\alpha_1 = \alpha_2$ we get the same result of Ortobelli and Rachev [21]. Now, suppose $\alpha_1 > \alpha_2$. Let $X_3 \sim S_{\alpha_1}(\sigma_2, \beta_1, \mu_2)$ and $Z_i = \overline{X}_i = \frac{X_i - \mu_i}{\sigma_i}$ (for $i = 1, 2$).

First assume that $\mu_1 \geq \mu_2$ and consider two different cases $\beta_i = 0$ and $\beta_i \neq 0$.

1) $\beta_1 = \beta_2 = 0$.

In this case, $Z_1 \sim S_{\alpha_1}(1,0,0)$ and $Z_2 \sim S_{\alpha_2}(1,0,0)$. The analytical study of the distribution functions ensures that F_{Z_1} and F_{Z_2} have a single crossing point in $(-\infty, 0)$, in that, for some $-t_0 < 0$, $F_{Z_1}(t) < F_{Z_2}(t)$ for $t < -t_0$ and $F_{Z_1}(t) > F_{Z_2}(t)$ for $0 > t > -t_0$ (see [26], [38]). Then, for symmetry, we know that F_{Z_1} and F_{Z_2} also cross in $+t_0$ and obviously in 0 ($F_{Z_1}(0) = F_{Z_2}(0) = 1\backslash 2$). Moreover, $\int_{-\infty}^{0} F_{Z_i}(z)\,dz = \Gamma\left[\frac{\alpha_i - 1}{\alpha_i}\right]/\pi$ (for $i = 1, 2$) as proved in [28], hence we derive that $\int_{-\infty}^{0} F_{Z_2}(z)\,dz \geq \int_{-\infty}^{0} F_{Z_1}(z)\,dz$ (we recall that the Gamma function is decreasing in $(0, 1/2)$). For any $t < 0$ we have $\int_{-\infty}^{t} (F_{Z_1}(z) - F_{Z_2}(z))dz < 0$, because F_{Z_1} and F_{Z_2} are single crossing functions in $(-\infty, 0)$ and $\int_{-\infty}^{0} F_{Z_1}(z)\,dz < \int_{-\infty}^{0} F_{Z_2}(z)\,dz$. Moreover, for symmetry, we obtain that, for any $t > 0$: $\int_{-\infty}^{t} (F_{Z_1}(z) - F_{Z_2}(z))dz = \int_{-\infty}^{-t} (F_{Z_1}(z) - F_{Z_2}(z))dz < 0$.

Let $X_3 \sim S_{\alpha_1}(\sigma_2, 0, \mu_2)$. We have that X_3 SSD X_2 and we also know that X_1 SSD X_3 as proved by [21]. This results yield X_1 SSD X_2.

2) $\beta_1 = \beta_2 \neq 0$

If $\beta_1 = \beta_2 \neq 0$ then X_1 SSD X_3. The analytical study of the distribution functions ensures that F_{X_3} and F_{X_2} are single crossing, i.e., for some $t_0 < \infty$, $F_{X_3}(t) < F_{X_2}(t)$ for $t < t_0$ and $F_{X_3}(t) > F_{X_2}(t)$ for $t > t_0$, where $t_0 > 0$ if $\beta_1 = \beta_2 > 0$ and $t_0 < 0$ if $\beta_1 = \beta_2 < 0$ (see [26], [38]). Thus, since X_2 and X_3 have equal mean, Theorem 1 yields X_3 SSD X_2 and we obtain the thesis.

Now, assume that $\mu_1 \leq \mu_2$.

Recall that $-X$ SSD $-Y$ if and only if $Y \geq_{icx} X$ (see [14]) and that $-X_i \sim S_{\alpha_i}(\sigma_i, -\beta_i, -\mu_i)$. Now if we consider $X_4 \sim S_{\alpha_2}(\sigma_1, -\beta_2, -\mu_2)$ we know that $-X_1$ SSD X_4 for the previous analysis and that $X_4 SSD - X_2$ as a consequence of [21]. Thus $-X_1 SSD - X_2$, which implies $X_2 \geq_{icx} X_1$.

F. Proof of Theorem 3.

Let us assume $\alpha_1 = \alpha_2 > 1$, $\sigma_1 = \sigma_2$, $\mu_1 = \mu_2$. The study of the distribution functions ensures that F_{X_1} and F_{X_2} have two crossing points, in particular, for some $t_1 < 0$ and $t_2 > 0$, $F_{X_1}(t) < F_{X_2}(t)$ for $t < t_1$, $F_{X_1}(t) > F_{X_2}(t)$ for $t_1 < t \leq t_2$ and $F_{X_1}(t) < F_{X_2}(t)$ for $t > t_2$. Thus, for Theorem 1, nor condition X_1 SSD X_2 neither condition $X_1 \geq_{icx} X_2$ are true. To prove that $X_1 \geq_{cmd} X_2$ we use the mathematical formula of $E\left(|X|^p\right)$ derived by [39], where for any stable centered random variable $X = Y - E(Y)$, we get:

$$E\left(|X|^p\right) = \sigma^p \left(1 + \beta^2 \tan^2 \frac{\alpha\pi}{2}\right)^{\frac{p}{2\alpha}} \cdot$$

$$\cdot \cos\left(\frac{p}{\alpha} \arctan\left(\beta \tan \frac{\alpha\pi}{2}\right)\right) \frac{2^p \Gamma\left(\frac{\alpha-p}{\alpha}\right) \Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}\, \Gamma\Gamma\left(\frac{2-p}{p}\right)}.$$

If we derive $E\left(|X|^p\right)$ with respect to β we obtain:

$$\frac{\partial E(|X|)^p}{\beta} = \sigma^p A(p) \left(1 + \beta^2 r^2\right)^{\frac{p}{2\alpha}-1} \frac{p}{a} r \cos y \quad [\beta r - \tan y\ ]$$

where $A(p) = \frac{2^p \Gamma\left(\frac{\alpha-p}{\alpha}\right) \Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}\ \Gamma\left(\frac{2-p}{p}\right)}$, $y = \frac{p}{\alpha} \arctan(\beta r)$ and $r = \tan\left(\frac{\pi\alpha}{2}\right)$.

It is straightforward to prove that $\sigma^p A(p) \left(1 + \beta^2 r^2\right)^{\frac{p}{2\alpha}-1} \geq 0$, hence it is sufficient to study the sign of $\frac{p}{\alpha} r \cos y \quad [\beta r - \tan y\ ]$.

Observe that $-\frac{\pi}{2} < y < \frac{\pi}{2}$ ($r = \tan \frac{\pi\alpha}{2} < 0$, therefore $y = \frac{p}{\alpha} \arctan(\beta r) \leq 0$ for $\beta \geq 0$, and $y \geq 0$ for $\beta \leq 0$), then $\cos y \geq 0$. Note that $r \cos y \leq 0$ is an even function of β, because $\cos y = cos(-y)$ and r does not depend on β. Thus the sign of $\frac{\partial E(|X|^p)}{\beta}$ depends on the sign of p and $\beta r - \tan y$. Suppose that $\beta \geq 0$: if $0 \leq p < \alpha$ then $\tan y \leq 0$ (as $\beta r \leq 0$ and $-\frac{\pi}{2} < y \leq 0$), in particular, since $0 < \frac{p}{\alpha} < 1$, the infimum of $\tan y$ is exactly equal to βr ($\tan y > \beta r$), which yields $\beta r - \tan y < 0$ and therefore $\frac{\partial E(|X|^p)}{\beta} \geq 0$. Conversely, if $-1 \leq p \leq 0$ then $\beta r \leq 0$ and $0 \leq y < \frac{\pi}{2}$, which also yields $\beta r - \tan y < 0$ and therefore $\frac{\partial E(|X|^p)}{\beta} \leq 0$ (as $p \leq 0$).

Moreover, $\arctan -x = -\arctan x$, then $\beta r - \tan y$ (and thereby $\frac{\partial E(|X|^p)}{\partial \beta}$) is an odd function of β. These results can be summarized as follows:

- If $0 \leq p < \alpha$ and $\beta \geq 0$ then $\frac{\partial E(|X|^p)}{\beta} \geq 0$;
- If $0 \leq p < \alpha$ and $\beta < 0$ then $\frac{\partial E(|X|^p)}{\beta} < 0$;
- If $-1 \leq p \leq 0$ and ≥ 0 then $\frac{\partial E(|X|^p)}{\beta} \leq 0$;
- If $-1 \leq p \leq 0$ and $\beta < 0$ then $\frac{\partial E(|X|^p)}{\beta} > 0$;

Therefore for fixed mean, scalar parameter, index of stability and $|\beta_1| < |\beta_2|$ we get the thesis, i.e., $\forall p \in (-1,\ \alpha_1)$:

$$\varphi_{X_1-E(X_1)}(p) \leq \varphi_{X_2-E(X_2)}(p).$$

Now suppose $\alpha_1 \geq \alpha_2 > 1$ and $|\beta_1| < |\beta_2|$. Consider that $\overline{X}_i \sim S_{\alpha_i}(1, \beta_i, 0)$ $(i = 1, 2)$ and assume $X_3 \sim S_{\alpha_2}(1, \beta_1, 0)$. Observe that as a consequence of the first point $X_3 \geq_{cmd} \overline{X}_2$. Moreover, if $\alpha_1 = \alpha_2$, $\overline{X}_1 = X_3$ in distribution, otherwise $\overline{X}_1 \geq_{cmd} X_3$ as a consequence of Theorem 2 and Corollary 1. Thus $\overline{X}_1 \geq_{md} \overline{X}_2$.

Now suppose that $\alpha_1 \geq \alpha_2 > 1$ $\sigma_1 \leq \sigma_2$, $\mu_1 = \mu_2$, and $|\beta_1| < |\beta_2|$ with at least an inequality strict and assume $X_4 \sim S_{\alpha_2}(\sigma_2, \beta_1, 0)$. As a consequence of Theorem 2 and Corollary 1 $X_1 \geq_{cmd} X_4$ and for the previous point $X_4 \geq_{cmd} X_2$. Thus $X_1 \geq_{cmd} X_2$.

G. Proof of Proposition 4.

It is well known that when $\beta_1 = 0$ the stable distribution X_1 is symmetric. Furthermore, observe that $-X_i \sim S_{\alpha_i}(\sigma_i, -\beta_i, -\mu_i)$. Consider the standardized versions of X_1 and X_2 given by $\overline{X}_1$ and $\overline{X}_2$. Assume $\beta_1 > 0$ and consider that $E\left(\overline{X}_{1-}\right) = E\left(\overline{X}_{1+}\right)$. Then, the analytical study of the distribution functions ensures that $F_{-\overline{X}_{1-}}$ and $F_{-\overline{X}_{1+}}$ are single crossing, i.e., for some $t_0 < 0$, $F_{-\overline{X}_{1-}}(t) < F_{-\overline{X}_{1+}}(t)$ for $t < t_0$ and $F_{-\overline{X}_{1-}}(t) > F_{-\overline{X}_{1+}}(t)$ for $t > t_0$. Thus, from Theorem 1 (or Theorem 3 in [22]) we deduce $\overline{X}_{1-}$RS $\overline{X}_{1+}$, and thus, as consequence of Corollary 3, X_1 is right asymmetric when $\beta_1 > 0$. The remainder of the Proposition is a consequence of Corollary 3.

H. Proof of Theorem 4.

To prove the theorem, we only need to prove it for $\beta_2 < \beta_1 \leq 0$ since the remainder of the theorem is a consequence of $-X_i \sim S_{\alpha_i}(\sigma_i, -\beta_i, -\mu_i)$. Observe that from property 1.2.18 in [23], we know that $\lim_{p\to\alpha_i} (\alpha_i - p)$ $E\left(\overline{X}_i^{<p>}\right) = \alpha_i 2 C_{\alpha_i} \beta_i$ where C_{α_i} is the positive value defined in 9. Therefore we have that $X_1 \gg_{wr} X_2$. From Proposition 4 we know that X_1 and X_2 are left asymmetric and from Theorem 3 we know that $X_1 \geq_{cmd} X_2$. In addition, the analytical study of the distribution functions ensures that the positive and negative parts of the standardized distributions are single crossing. Thus from formula 10 and Theorem 1 we deduce that $\overline{X}_{2-} \geq_{icx} \overline{X}_{1-}$, $\overline{X}_{2+} SSD\ \overline{X}_{1+}$ and , $\overline{X}_{2+} SSD\ \overline{X}_{1-}$.

I. Proof of Proposition 5.

From property 1.2.18 in [23], we know that $\lim_{p\to\alpha_i} (\alpha_i - p)\ \ E\left(\overline{X}_{i\pm}^{p}\right) = \alpha_i C_{\alpha_i} (1 \pm \beta_i)$ (and in particular $\lim_{p\to\alpha_i} (\alpha_i - p)\ \ E\left(\overline{X}_i^{<p>}\right) = \alpha_i 2 C_{\alpha_i} \beta_i$) where C_{α_i} is the positive value defined in 9. Therefore, we get $X_1 \gg_{wr} X_2$ anytime $\beta_1 > 0$ and $1 < \alpha_1 < \alpha_2$. Moreover, we get $X_1 \gg_{wl} X_2$ anytime $\beta_1 < 0$ and $1 < \alpha_1 < \alpha_2$.

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