Inequality measures for intersecting Lorenz curves: an alternative weak ordering

Tommaso Lando, Lucio Bertoli-Barsotti[1]

Abstract

The Lorenz ordering is probably the most logical tool for comparison of the dispersion of non negative random variables. Indeed, many income inequality metrics are order preserving (or isotonic) with the aforementioned preorder. However, in some situations Lorenz curves may intersect and the ordering is not fulfilled. Hence, some weaker criteria need to be to introduced: we present a new different preorder, weaker than the Lorenz ordering, and propose a possible class of functionals that preserve it. This method could be especially useful under conditions of maximum uncertainty.

Key words

Lorenz ordering, inequality, disparity, majorization, stochastic dominance

JEL Classification: D31, D63, I32

1. Introduction

The Lorenz curve, which has been introduced as a representation of inequality (e.g. income inequality), is generally used to rank probability distributions in terms of an order of preference, that is, the Lorenz ordering (LO). In fact, the LO is conform to the idea that the higher of two non-intersecting Lorenz curves (as well as the corresponding distribution) has to be preferred, in that it shows less inequality compared to the lower one. As is well known, in an economic framework, the LO is coherent with the Pigou-Dalton condition ("principle of transfers"), that is, the higher of two non-intersecting Lorenz curves can be obtained from the lower one by an iteration of income transfers from "richer" to "poorer" individuals (the so called *elementary* transfers or *T-transforms,* Marshall et al., 2009, p. 32, also called *progressive* transfers, Shorrocks and Foster, 1987). For this reason, the "coherence" with the LO represents a basic property of many inequality (or concentration) measures. Nevertheless, it may happen that Lorenz curves intersect or, equivalently, the LO is not verified, which implies that the Lorenz-preserving indices disagree: in this case we can rank the distributions by relying on weaker orders of inequality. In the literature, this idea has been analyzed in several works, related to the concept of third-order stochastic dominance (see e.g. Atkinson, 2008), which emphasizes the left tail of the distribution. Indeed, many authors agree that an elementary transfer should be more equalizing the "lower" it occurs in the distribution (see e.g. Shorrocks and Foster, 1987; Dardanoni and Lambert, 1988; Atkinson, 2008). This concept has been defined as *aversion to downside inequality* or *ADI* (Davies and Hoy, 1995). On the other hand, one may be interested in what happens in the right tail of the distribution. For instance, in an economic context, a lot of attention is recently given to those variations occurring at the top of the income distribution (Makdissi and Yazbeck, 2014). Logically, it is possible to define a preorder which takes into account of this concept by emphasizing the

[1] Tommaso Lando, University of Bergamo (Italy) and VSB Technical University of Ostrava (Czech Republic), email: tommaso.lando@unibg.it
Lucio Bertoli-Barsotti, University of Bergamo (Italy), email: lucio.bertoli-barsotti@unibg.it

right tail of the distribution: in particular, the *second-degree downward Lorenz dominance* has been recently introduced by Aaberge (2009).
In this paper, we consider both these two different approaches and study the possibility of defining a class of inequality measures which is generally sensitive to transfers occurring in one or both of the tails of the distribution (i.e. the "lower" or the "higher", speaking in terms of income). More specifically, we present two different preorders and combine them in a new preorder. Then, we present a class of functionals that are isotonic with the new preorder. As for the notation, we make use of the definition of weak majorization (Marshall et al. 2009).

2. Lorenz ordering and majorization

In this first section, we define the Lorenz ordering and analyze its relation with the majorization preorder. We recall that a preorder is a binary relation $\leq$ over a set S that is reflexive and transitive. In particular, observe that a preorder $\leq$ does not generally satisfy the *antisymmetry* property (that is, $a \leq b$ and $b \leq a$ does not necessarily imply $a = b$) and it is generally not *total* (that is, each pair a, b in S is not necessarily related by $\leq$).

First, we recall that the (generalized) inverse of a distribution function F is given by

$$F^{-1}(p) = \inf\{z \in [0,\infty]: F(z) \geq p\}, p \in (0,1).$$

If F has finite expectation μ_F, then the Lorenz curve is defined as follows (Gastwirth, 1971):

$$L_F(p) = \frac{1}{\mu_F}\int_0^p F^{-1}(t)dt.$$

The Lorenz ordering $\leq_L$ is a pre-order defined over the space $\mathcal{F}_0$ of non-negative distributions with finite expectations ($\mathcal{F}_0 = \{F: F(z) = 0 \text{ for } z < 0 \text{ and } \int_0^\infty z dF(z) = \mu_F < \infty\}$), and it is defined as follows.

Definition. Let $F, G \in \mathcal{F}_0$: we write $F \leq_L G$ if and only if $L_F(p) \geq L_G(p), \forall\, p \in (0,1)$.

On the other hand, majorization $\prec$ is a preorder defined in the space of integrable functions, and it is aimed at comparing functions in terms of "diversity" between their values (Marshall et al., 2009). Here we focus on continuous majorization, thus we consider functions that are integrable with respect to the Lebesgue measure *m* on the set (0,1).

Definition. Let $a, b \in \mathcal{L}^1(0,1)$. We say that a *is majorized by* b and write $a \prec b$ if and only if
1) $\int_0^z a_\downarrow(u)\, du \leq \int_0^z b_\downarrow(u)\, du, \forall\, z \in (0,1)$,
2) $\int_0^1 a_\downarrow(u)\, du = \int_0^1 b_\downarrow(u)\, du$,

where $a_\downarrow(u) = (m_a(x))^{-1}$ and $m_a(x) = m(\{u: a(u) > x\})$ (note that the function $a_\downarrow$ is referred to as the *decreasing re-arrangement* of a).

When condition 2) does not hold, we rely on weaker definitions of majorization. In particular, in this paper we shall use the following one.

Definition. Let $a, b \in \mathcal{L}^1(0,1)$. We say that a *is weakly majorized by* b *from below* and write $a \prec_w b$ if and only if

$\int_0^z a_\downarrow(u)\, du \leq \int_0^z b_\downarrow(u)\, du, \forall\, z \in (0,1)$.

We say that that a *is weakly majorized by* b *from above* and write $a \prec^w b$ if and only if

$\int_0^z a_\uparrow(u)\, du \geq \int_0^z b_\uparrow(u)\, du, \forall\, z \in (0,1)$,

where, similarly to $a_\downarrow$, $a_\uparrow$ denotes the *increasing re-arrangement* of a (see for instance Lando, Bertoli-Barsotti, 2014).

If we denote the derivative of the Lorenz curve by l_F, that is $l_F = (L_F)' = F^{-1}/\mu_F,$ we can express the relation between the LO and majorization as follows (Bertoli-Barsotti, 2001):

$$\text{if } F \leq_L G \text{ then } l_F \prec l_G.$$

This equivalence relation makes it possible to define several classes of functionals that are isotonic with the LO, based on some well known results of majorization theory (Bertoli-Barsotti, 2001).

3. A new approach

As explained in the introduction, the aim of this paper is to propose a class of inequality measures that are basically sensitive to transfers occurring in the "lower" or the "higher" parts of the income distribution. We first consider two different weak preorders, defined as follows. Generally, F is preferable to G if it presents less inequality when we the comparison starts from the left tails of the distributions. By using the weak majorization definition we can express this condition with:

1) $L_F \prec^w L_G$, that is: $\int_0^t L_F(p)dp \geq \int_0^t L_G(p)dp$, for any t in [0,1].

Observe that condition 1) implies that L_F starts above L_G and presents a larger (underlying) area, that is, a lower value of the Gini index. Furthermore, note that $L_F \prec^w L_G$ is equivalent to the *second-degree upward Lorenz dominance* of Aaberge (2009).
Similarly, one can also prefer F to G if it presents less inequality when the comparison starts from the right tails of the distributions. We can formulate this condition as:

2) $L_G \prec_w L_F$, that is: $\int_t^1 L_G(p)dp \leq \int_t^1 L_F(p)dp$, for any t in [0,1],

(note that $\int_t^1 L_F(p)dp = \int_0^{1-t} (L_F)_\downarrow(p)dp$).
According to Aaberge (2009), the ordering $L_G \prec_w L_F$ can be equivalently referred to as the *second-degree downward Lorenz* dominance.
Observe that $F \leq_L G$ implies $L_F \prec^w L_G$ as well as $L_G \prec_w L_F$.
In what follows, we attempt to combine the preorders 1) and 2) into a single preorder, weaker than the LO, which emphasizes inequality in both the tails of the distribution.
Let us define:

$$\bar{L}_F(t) = (1 - L_F(t))_\uparrow = 1 - L_F(1-t)$$

(note that if $a(t)$ is a decreasing function in [0,1] then $a_\uparrow(t) = a(1-t)$). $\bar{L}_F(t)$ can be interpreted as a "complementary" Lorenz curve (see Eliazar, 2015). Actually, for a given percentage t, $L_F(t)$ represents the percentage of "total" possessed by the low $100t\%$ part of the distribution, while $\bar{L}_F(t)$ represents the percentage of "total" corresponding to the top $100t\%$ part of the distribution. Hence, as $\bar{L}_F(t) \geq L_F(t)$ for any t, the difference between the Lorenz curves

$$\Delta_F(t) = \bar{L}_F(t) - L_F(t)$$

expresses the *disparity* between the "higher" and the "lower" parts of the distribution. In terms of income distributions, Δ_F equals the difference between the proportion of the society's overall wealth that is held by the society's top (rich) $100t\%$, and the proportion of the society's overall wealth that is held by the society's low (poor) $100t\%$.

It can be easily shown that Δ_F is increasing for $t \in [0,1/2]$ and decreasing for $t \in (1/2,1]$, thus it attains its maximum at $t = 1/2$. Moreover, for any t in [0,1/2], $\Delta_F(t) = -\Delta_F(1-t)$, therefore, in order to analyze the behavior of Δ_F, it is sufficient to focus on the interval [0,1/2].
Obviously, we wish that Δ_F is (uniformly) as small as possible: this concept can be expressed in terms of weak majorization. Indeed, for any couple of distributions F, G, we prefer F to G when:

$$\Delta_G \prec^w \Delta_F,$$

which is equivalent to

$$\int_0^t \Delta_F(p)dp \leq \int_0^t \Delta_G(p)dp \text{ for any } t \text{ in } [0,1/2].$$

Observe that the preorder $\Delta_G \prec^w \Delta_F$ is weaker than the LO and combines both the orderings 1) and 2) presented in this section. Moreover, $\Delta_G \prec^w \Delta_F$ implies the condition $\int_0^{1/2} \Delta_F(p)dp \leq \int_0^{1/2} \Delta_G(p)dp$, which is equivalent to saying that G presents a higher concentration than F according to the Gini index (indeed $\int_0^{1/2} \Delta_F(p)dp = \int_0^1 (p - L_F(p))dp$). Hence $\Delta_G \prec^w \Delta_F$ expresses concentration as well as inequality in the higher and lower parts of the distribution.
From these considerations, we propose to measure inequality with a functional that preserves the ordering $\Delta_G \prec^w \Delta_F$. In particular, we can rely on a well known result of majorization theory and use the following measure of inequality:

$$\Upsilon(F) = \int_0^{1/2} \varphi(\Delta_F(p))dp$$

where φ is increasing and concave (see e.g. Marshall et al., 2009). Obviously, it is also easy to derive a normalized version of the inequality index Υ. Note that the formula for Υ determines a class of inequality measures that share certain characteristics. Indeed, we obtain that Υ is isotonic with the newly introduced ordering, which is weaker than the LO. Therefore these implications hold

$$F \leq_L G \Rightarrow \Delta_G \prec^w \Delta_F \Rightarrow \Upsilon(F) \leq \Upsilon(G).$$

The usefulness of this new approach can be shown by a simple example.

Example
Consider the vector

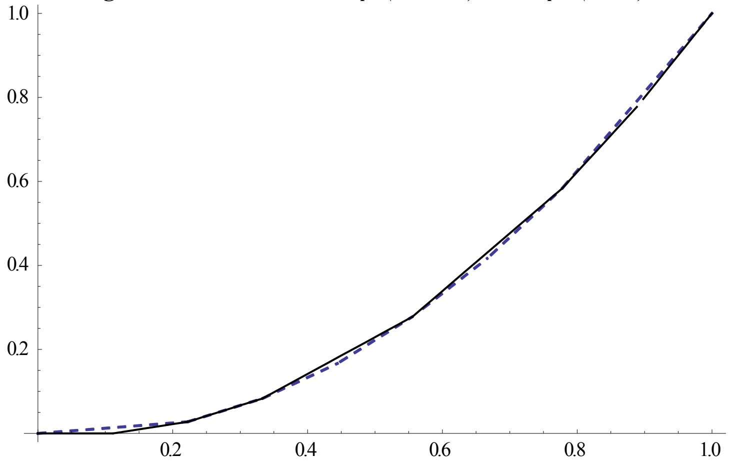
$$X = (0,10,20,30,40,50,60,70,80,90)$$

and the following two vectors, where each of them can be obtained from X by two elementary transfers, respectively in the tails and in the "core" of the distribution:

$$X' = (5,5,20,30,40,50,60,70,85,85)$$
$$X'' = (0,10,20,35,35,55,55,70,80,90).$$

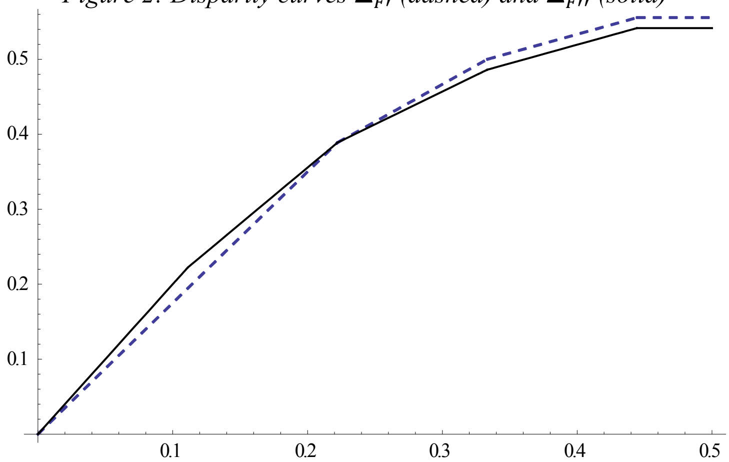
It is easy to see that X' presents less inequality in the tails but more inequality in the "core", compared to X''. Let us respectively denote with F, F' and F'' the distributions that correspond to X, X' and X''. Since any inequality measure Υ is isotonic with the LO, we know that $\Upsilon(F') \leq \Upsilon(F)$ and $\Upsilon(F'') \leq \Upsilon(F)$. However, we are interested in ranking F' and F''.
Observe that $L_{F'}$ and $L_{F''}$ intersect two times and moreover their underling area is equal (i.e. F' and F'' cannot be ranked by the LO as well as the Gini index). The Lorenz curves in Figure 1 also show that $L_{F'}$ starts and finishes above $L_{F''}$.

Figure 1: Lorenz curves $L_{F'}$ (dashed) and $L_{F''}$ (solid)



On the other hand, the curves $\Delta_{F'}$ and $\Delta_{F''}$ reveal that $\Delta_{F''} \prec^{w} \Delta_{F'}$, as is shown in Figure 2.

Figure 2: Disparity curves $\Delta_{F'}$ (dashed) and $\Delta_{F''}$ (solid)



Then, according to what is stated above, any inequality measure Υ yields $\Upsilon(F') \leq \Upsilon(F'') \leq \Upsilon(F)$. For this example we simply set $\varphi(x) = \sqrt{x}$, hence

$$\Upsilon(F) = \int_0^{1/2} (\Delta_F(p))^{1/2} dp$$

which yields: $\Upsilon(F) = 0.29$, $\Upsilon(F') = 0.288$, $\Upsilon(F'') = 0.289$.

4. Conclusion

We have proposed a new ordering of inequality/disparity, and consequently we introduced a class of functionals that are isotonic with this ordering. Future studies will be aimed at the application of this result. Since the new ordering is weaker than the LO, it can be used to rank Lorenz curves when they intersect. The ratio of the ordering, as well as the corresponding index ϒ, is that transfers should be more equalizing when they occur in the tails of the distribution.

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