Choices based on asymptotic approximation

Sergio Ortobelli, Filomena Petronio [1]

Abstract

In this paper, we deal with portfolio selection decisions when the portfolio returns are approximated by stable Paretian distributions. Therefore, we examine some dominance rules to determine the optimal choices of non-satiable risk averse investors. In particular, we first preselect a subclass of assets which are not dominated by the point of view of non-satiable and risk-averse investors. Then, we optimize a multi-parametric portfolio optimization problem that takes into account the asymptotic stochastic dominance rule. Finally, we compare the ex-post wealth obtained by optimal portfolios with different levels of asymptotic skewness and stability index.

Key words

Portfolio optimization, asymptotic stochastic dominance rules, large-scale portfolio problem

JEL Classification: C16, C44, G11

1. Introduction

The Gaussian distribution has been largely used in financial analysis. However, as shown in many studies on this topic, usually the asset returns are not normally distributed. For instance, Mandelbrot (see [6], [7], [8], [9]) and Fama (see [1], [2], [3]) detected an excess of kurtosis and non-zero skewness in the empirical distributions of financial assets, which often leads to the rejection of the assumption of normality. These studies, supported by numerous empirical investigations (see [14] and [15]), suggest the use of stable Paretian distribution as an alternative model for asset returns.

The aim of this paper is to order (stochastically) the stable distributed random variables, considering some financial applications of the stable Paretian distributions. In financial contexts, the stochastic orderings are used to define an order of preferences for investors whose utility functions share certain characteristics [17]. Specifically, the stochastic dominance rules are generally aimed to lead the investors towards the best choices in terms of expected gain and risk (see, among others, [5], [12], [13]). In the literature it is well known that we can obtain the second order stochastic dominance between stable distributions using a mean-dispersion comparison (similar to the Gaussian case), however only when the stable distributions present the same skewness parameter and index of stability ([11], [14]).

In this paper, we recall some recent general results [10], taking in to account two features of stable Paretian distributions: the tail behavior and the asymmetry. Thus, using the asymptotic stochastic dominance rules, we propose a portfolio optimization analysis, referring to non-satiable risk averse investors in the US stock market and comparing the ex-post final wealth obtained for different levels of skewness and index of stability. In this framework, we have to reduce the dimensionality of the problem and we need a sufficiently robust approximations of the stable Paretian parameters, in a reasonable computational time.

[1] Sergio Ortobelli, Associate professor, University of Bergamo, via dei Caniana, 2, Bergamo, Italy; and VŠB -TU Ostrava, Sokolská třída 33, Ostrava, Czech Republic; e-mail: sergio.ortobelli@unibg.it. Filomena Petronio, PhD, VŠB-Technical University Ostrava, Sokolská třída 33, Ostrava, Czech Republic; e-mail: filomena.petronio@vsb.cz.

In Section 2, we discuss the use of stable Paretian distributions and their orderings. In Section 3, we examine the multi-parametric portfolio selection problem for different non satiable risk averse investors. Some brief conclusions are outlined in Section 4.

2. Asymptotic stochastic dominance rules

In this section, we recall the definitions of some classical stochastic orders and their possible applications to non-satiable risk averse investors when the returns are approximated by stable Paretian distributions.

Let us recall some classical definitions of orderings. Any non-satiable investor prefers a portfolio X to another one Y if and only if X dominates Y with respect to the first stochastic dominance order (in symbols X FSD Y) or, equivalently X FSD Y if and only if $\mathrm{E}(g(X)) \geq \mathrm{E}(g(Y))$ for any non-decreasing function g. Similarly, any non-satiable risk averse investor prefers a portfolio X to another one Y if and only if X dominates Y with respect to the second stochastic dominance order (in symbols X SSD Y) or, equivalently X SSD Y if and only if $\mathrm{E}(g(X)) \geq \mathrm{E}(g(Y))$ for any non-decreasing and concave function g. Finally, X dominates Y with respect to the Rothschild-Stiglitz order (in symbols X RS Y) when X SSD Y and $\mathrm{E}(X) = \mathrm{E}(Y)$ (or $Y \geq_{icx} X$ and $\mathrm{E}(X) = \mathrm{E}(Y)$) or equivalently X RS Y if and only if $\mathrm{E}(g(X)) \geq \mathrm{E}(g(Y))$ for any concave function g. See [4], for more details about the use of non-decreasing function in this framework.

The stable Paretian distributions are the limit distributions of all normalized sums of i.i.d. random variables (see [14], [16]). Recall that we name $X' \sim S_{\alpha_X}(\sigma_X, \beta_X, m_X)$ when X is an α-stable Paretian random variable, where $0 < \alpha_X \leq 2$ is the so-called stability index, which specifies the asymptotic behavior of the tails, $\sigma_X > 0$ is the dispersion parameter, $\beta_X \in [-1,1]$ is the skewness parameter and $m_X \in \mathbb{R}$ is the location parameter. Stable distributions do not generally have finite variance, which happens only when $\alpha = 2$ (i.e. Gaussian distribution, $E(|X|^p) < \infty$ for any p). Unfortunately, except in few cases, we do not have a closed form expression for the density of stable Paretian distribution, which is identified by its characteristic function, given by:

$$E(\exp\{itX\}) = \begin{cases} \exp\left\{\mathrm{i}t\mu - |t\sigma|^{\alpha}\left(1 - \mathrm{i}\beta \mathrm{sign}(t)\tan\left(\frac{\pi\alpha}{2}\right)\right)\right\} & \alpha \neq 1 \\ \exp\ \{\mathrm{i}t(\mu + 2\beta\sigma\ln(\sigma)\,/\pi) \\ -|t\sigma|(1 + \ 2\mathrm{i}\beta\ln|t\sigma|\,\mathrm{sign}(t)/\pi)\} & \alpha = 1 \end{cases} \tag{1}$$

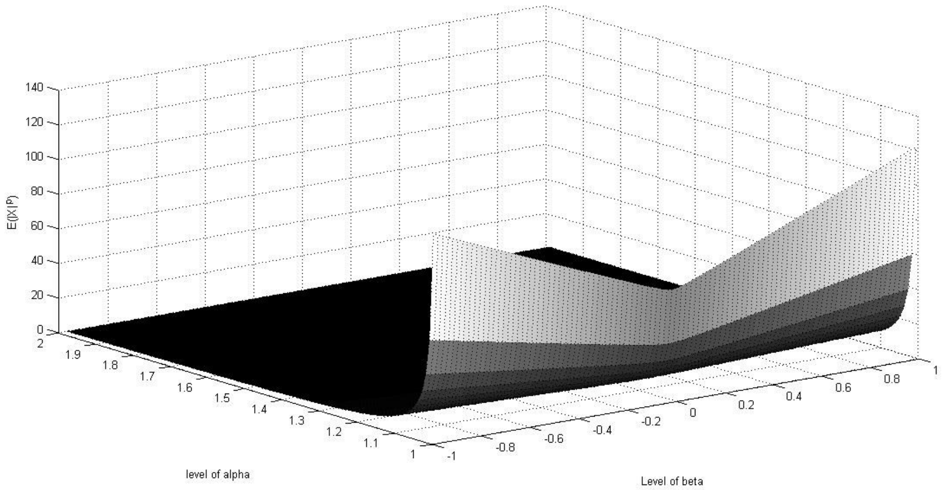
It is worth noting that, since the density and distribution functions of the stable Paretian distribution cannot be expressed with elementary functions, it is not possible to verify the integral conditions for the RS and SSD orders. In particular, from [10], we know that even if RS order is not verifiable, it is possible to analyze a weaker order of "risk" by studying the distributions of the absolute values. Let Z be a random variable and define:

$$\varphi_Z(p) = \mathrm{sign}(p)E(|Z|^p),\, p \in \mathbb{R}. \tag{2}$$

We are now able to define a new stochastic order of risk, expressed in terms of the absolute centered moments of order p.

Definition 1 *Let X and Y be random variables belonging to $L^q = \{X|\ E(|X|^q) < \infty\}$, for $q \geq 1$. We say that X dominates Y with respect to the moment dispersion (MD) order (in symbols $X \geq_{md} Y$) if and only if $\varphi_{X-E(X)}(p) \leq \varphi_{Y-E(Y)}(p)$, $\forall p \geq 1$.*

Figure 1: Different levels of alpha and beta in the p=1 frontier



Generally, the moment function $\varphi_{X-E(X)}(p)$ is well known for $Z = X - E(X)$ alpha stable distributed (see [16]). In particular, $E(|Z|^p)$ assumes the same values for β_Z and $-\beta_Z$ and $E(|Z|^p)$ is also decreasing respect to alpha when p>=1. The Figure 1 shows this behavior for p=1. Observe that in [10] is proved that if X RS Y, then $X \geq_{md} Y$. Therefore (MD) order is consistent with the choices of risk averse investors. Moreover given two stable distributed random variables $X_1 \sim S_{\alpha_1}(\sigma_1, \beta_1, \mu_1)$ and $X_2 \sim S_{\alpha_2}(\sigma_2, \beta_2, \mu_2)$ then if $\alpha_1 \geq \alpha_2 > 1$, $|\beta_1| < |\beta_2|, \sigma_1 \leq \sigma_2$, and $\hat{\mu}_1 \geq \hat{\mu}_2$ then there exists a stable random variable X3 with parameters $\alpha_2, \sigma_2, \beta_1, \mu_2$ such that $X_1 SSD\, X_3 \geq_{md} X_2$.

Thus, if we maximize the ratio between the location parameter and the scalar parameter as suggested by [11] and [14], varying properly different level of beta and alpha we are able to approximate the optimal choices of different non satiable risk averse investors.

3. An empirical ex post comparison among portfolio strategies

In this section, we examine different portfolio strategies based on the stochastic dominance rule outlined in Section 2. We use the daily observations of the S&P 500 components from January 1, 1999 to July19, 2014, and we assume that no short sales are allowed. We point out with $x = [x_1, \dots, x_{500}]'$ the vector of percentages invested in each asset and with $r = [r_1, \dots, r_{500}]'$ the vector of returns. Therefore, starting from 1 January 2000 and using 1 year of historical observations, we optimize monthly a portfolio selection that takes into account the asymptotic behavior of the data series. Since this large scale portfolio problem requires the estimation and optimization of the stable parameters $\alpha_{x'r}$, $\sigma_{x'r}$, $\beta_{x'r}$, $\mu_{x'r}$ of each portfolio x', r, we clearly need to reduce the dimensionality of the problem. Thus we preselect the first 100 assets with the highest ratio between the stable location parameter and the scalar parameter. On this preselected assets we optimize the choice for non satiable risk averse investors. Finally, we examine the ex post wealth obtained with this procedure. In particular, we optimize the following portfolio problem

$$\begin{aligned} & max_x \frac{\mu_{x,r}}{\sigma_{x,r}} \\ & s.t. \;\; \alpha_{x,r} > \alpha^*; \textstyle\sum_{i=1}^{n} x_i = 1 \\ & |\beta_{x,r}| \leq \beta^* ; \;\; x_i \geq 0 \end{aligned} \qquad (3)$$

using $\alpha^* = 1.2,\ 1.35,\ 1.5,\ 1.7,\ 1.8$ and $\beta^* = 0.7, 0.3$.

The results are reported in Tables 1, 2, and Figures 2, 3. Table 1 shows three risk measures (the standard deviation, the conditional value at risk $CVaR_p(x'r)$ at p=1%, 5%, where $CVaR_p(X) = \frac{-1}{p}\int_0^p F_X^{-1}(u)du$), of the ex-post returns obtained for the different strategies.

Table 1: Standard deviation, $CVaR_{1\%}$, and $CVaR_{5\%}$ of ex-post returns obtained for different strategies

	Beta*			Beta*			Beta*	
Std Dev	**0.7**	**0.3**	$\boldsymbol{CVaR_{5\%}}$	**0.7**	**0.3**	$\boldsymbol{CVaR_{1\%}}$	**0.7**	**0.3**
1.2	0,0238	0,0224	**1.2**	0,0554	0,0522	**1.2**	0,0902	0,0828
1.35	0,0243	0,0232	**1.35**	0,0574	0,0561	**1.35**	0,0954	0,0916
Alpha* 1.5	0,0236	0,0221	**Alpha* 1.5**	0,0571	0,0527	**Alpha* 1.5**	0,0961	0,0857
1.7	0,0217	0,0216	**1.7**	0,0515	0,0512	**1.7**	0,0818	0,0816
1.8	0,0211	0,0226	**1.8**	0,0509	0,0532	**1.8**	0,0801	0,0870

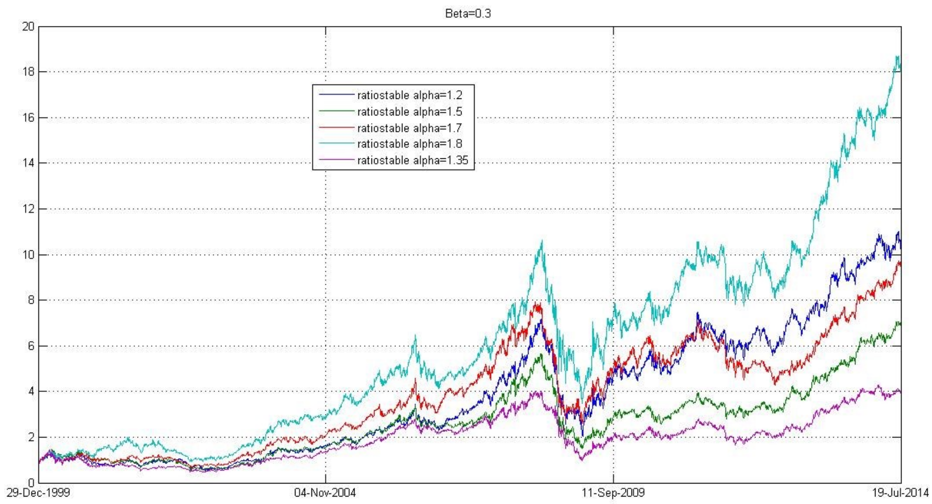
Table 2 reports three performance measures (the ex-post final wealth, the Sharpe ratio – i.e. (mean excess return)/(standard deviation) – and the STARR ratio 1% -i.e. (mean excess return)/($CVaR_{1\%}(x'r)$)) of the ex-post returns obtained for the considered strategies.

Table 2: Ex-post final wealth, Sharpe ratio and $STARR_{1\%}$ ratio of ex-post returns obtained for different strategies.

	Beta*			Beta*			Beta*	
Final Wealth	**0.7**	**0.3**	**Sharpe ratio**	**0.7**	**0.3**	$\boldsymbol{STARR_{1\%}}$	**0.7**	**0.3**
1.2	15,8160	10,2482	**1.2**	0,0317	0,0284	**1.2**	0,0084	0,0077
1.35	11,2740	3,9431	**1.35**	0,0273	0,0162	**1.35**	0,0070	0,0041
Alpha* 1.5	12,6204	6,8899	**Alpha* 1.5**	0,0294	0,0239	**Alpha* 1.5**	0,0073	0,0058
1.7	5,5461	9,4993	**1.7**	0,0216	0,0285	**1.7**	0,0049	0,0072
1.8	5,6833	17,9937	**1.8**	0,0225	0,0350	**1.8**	0,0059	0,0091

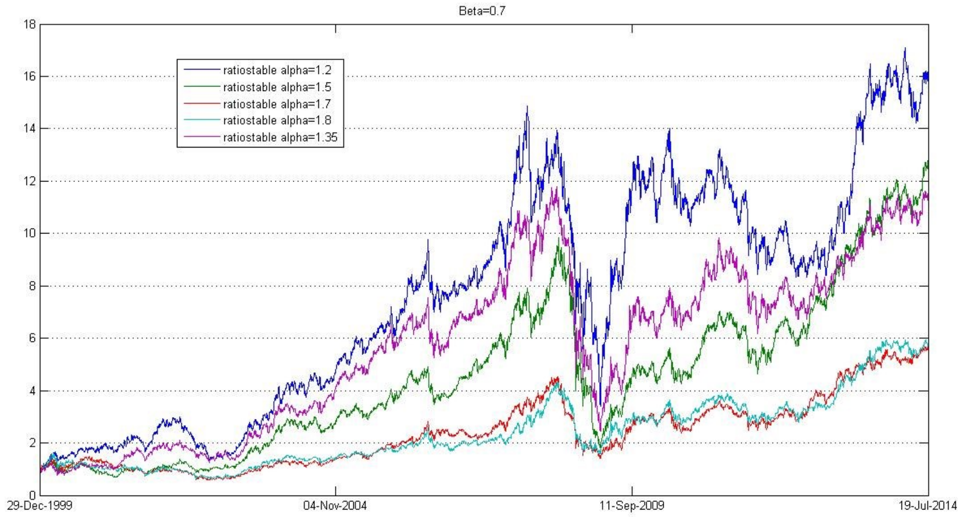
Figures 2 and 3 show the ex post wealth evolution for the strategies with skewness bounds respectively equal to 0.3 and 0.7 (i.e. $\beta^* = 0.3, 0.7$). In Table 1 we observe that the risk of the strategy decreases (except in few cases) when the skewness parameter is decreasing and the index of stability is increasing.

Figure 2: Ex post wealth obtained solving monthly problem (3) for $\beta^* = 0.3$.



From Table 2 we deduce that there is a strategy corresponding to constraints $\beta^* = 0.5$ and $\alpha^* = 1.8$ of problem (3) that presents the best ex-post performance. Moreover we do not observe a trend in the performance with respect the index of stability and the skewness parameter. Figures 2 and 3 confirm the high variability of the strategies during the 15 years of ex-post observations, in particular during financial crisis period (2008-2009).

Figure 2: Ex post wealth obtained solving monthly problem (3) for $\beta^* 0.7 =$.



4. Conclusion

In this paper we examine the ex-post wealth obtained using some portfolio strategies based on the optimization of asymptotic stochastic dominance rules. Specifically, we observe that the asymptotic skewness and tail behavior could have a crucial effect on the choices and the ex-post wealth. Moreover, we observe that the risk of the ex-post wealth generally confirms the asymptotic evaluation of the ex-ante risk. This simple observation suggests that asymptotic behavior should be considered in portfolio strategies at least for the risk valuation.

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