L^p and Weak$-L^p$ estimates for the number of integer points in translated domains

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Abstract

Revisiting and extending a recent result of M.Huxley, we estimate the $L^p\left(\mathbb{T}^d\right)$ *and* Weak$-L^p\left(\mathbb{T}^d\right)$ *norms of the discrepancy between the volume and the number of integer points in translated domains.*

In this paper we estimate different norms of the discrepancy between the volume and the number of integer points in dilated and translated copies $R\Omega-x$ of a bounded convex domain $\Omega\subset\mathbb{R}^d$ having positive measure. The above number of integer points is a periodic function of the translation variable x, with Fourier expansion

$$\sum_{k\in\mathbb{Z}^d}\chi_{R\Omega-x}(k)=\sum_{n\in\mathbb{Z}^d}\left(\int_{\mathbb{T}^d}\sum_{k\in\mathbb{Z}^d}\chi_{R\Omega-y}(k)\exp(-2\pi iny)dy\right)\exp(2\pi inx)$$

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$$= \sum_{n\in\mathbb{Z}^d} \left(\int_{\mathbb{R}^d} \chi_{R\Omega}(y) \exp(-2\pi i n y) dy \right) \exp(2\pi i n x)$$
$$= \sum_{n\in\mathbb{Z}^d} R^d \widehat{\chi_\Omega}(Rn) \exp(2\pi i n x).$$

These equalities are in the L^2 sense. It follows that the discrepancy function

$$\mathcal{D}\left(R\Omega - x\right) = \sum_{k\in\mathbb{Z}^d} \chi_{R\Omega - x}(k) - R^d \left|\Omega\right|$$

has Fourier expansion

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}} R^d \widehat{\chi_\Omega}(Rn) \exp(2\pi i n x).$$

If Ω is a bounded convex domain in $\mathbb{R}^d$ with smooth boundary having positive Gaussian curvature then

$$|\widehat{\chi_\Omega}(\xi)| \leqslant C \left|\xi\right|^{-(d+1)/2}.$$

See [**16**, Chapter 8]. Kendall [**11**] observed that the Fourier expansion of the discrepancy and the above estimate for the Fourier transform of a convex domain give

$$\left\{ \int_{\mathbb{T}^d} \left|\mathcal{D}\left(R\Omega - x\right)\right|^2 dx \right\}^{1/2} \leqslant CR^{(d-1)/2}.$$

Using a smoothing argument and the Poisson summation formula, Herz [**8**] and Hlawka [**9**] (see also [**18**]) proved that

$$\sup_{x\in\mathbb{T}^d} \left\{ \left|\mathcal{D}\left(R\Omega - x\right)\right| \right\} \leqslant CR^{d(d-1)/(d+1)}.$$

Interpolating the above two upper bounds between L^2 and L^∞ gives a poor estimate. Indeed when $d = 2$ interpolation gives

$$\left\{ \int_{\mathbb{T}^2} \left|\mathcal{D}\left(R\Omega - x\right)\right|^p dx \right\}^{1/p} \leqslant CR^{(2p-1)/(3p)},$$

while M.Huxley [**10**] has recently showed a more interesting estimate: If Ω is a planar convex body having boundary with continuous and positive curvature then

$$\left\{ \int_{\mathbb{T}^2} \left|\mathcal{D}\left(R\Omega - x\right)\right|^4 dx \right\}^{1/4} \leqslant CR^{1/2} \log^{1/4}(R).$$

That is, the upper estimate for the L^2 discrepancy extends, up to a logarithm, to L^4.

Huxley's proof seems to be tailored for the planar case and for the exponent $p = 4$, where one can apply Parseval equality to the square of the discrepancy function. Huxley also asked for an analog of his result for $d > 2$. Here we will give a possible answer and our approach will be to obtain L^p results through Weak$-L^p$ techniques.

We recall that the spaces $L^p(\mathbb{X}, \mu)$ and Weak$-L^p(\mathbb{X}, \mu)$, $0 < p < +\infty$, are defined by the quasi norms

$$\|f\|_{L^p(\mathbb{X},\mu)} = \left\{ \int_{\mathbb{X}} |f(x)|^p d\mu(x) \right\}^{1/p},$$
$$\|f\|_{\text{Weak}-L^p(\mathbb{X},\mu)} = \sup_{t>0} \left\{ t^p \mu\left\{ x \in \mathbb{X},\ |f(x)| > t \right\} \right\}^{1/p}.$$

The space Weak$-L^p(\mathbb{X},\mu)$ is the case $q=+\infty$ of the Lorentz spaces $L^{p,q}(\mathbb{X},\mu)$ (see e.g. [**1**, Chapter 1, §3] or [**17**, Chapter 5, §3]). Finally, the space $L^\infty(\mathbb{X},\mu)$ is defined by the norm

$$\|f\|_{L^\infty(\mathbb{X},\mu)} = \inf\{t>0:\ \mu\{x\in\mathbb{X}:\ |f(x)|>t\}=0\}.$$

In what follows $(\mathbb{X},\mu)$ will be the torus $\mathbb{T}^d$ or the integers $\mathbb{Z}^d$ with the respective translation invariant measures.

If $\mathbb{X}$ has finite measure and $p<s$, then both $L^p(\mathbb{X},\mu)$ and $L^s(\mathbb{X},\mu)$ are intermediate between $L^\infty(\mathbb{X},\mu)$ and Weak$-L^p(\mathbb{X},\mu)$:

$$L^\infty(\mathbb{X},\mu)\subseteq L^s(\mathbb{X},\mu)\subseteq L^p(\mathbb{X},\mu)\subseteq \text{Weak}-L^p(\mathbb{X},\mu).$$

The following is a sharp quantitative counterpart of these inclusions.

Lemma 1. *(1) If $\mathbb{X}$ has finite measure, then*

$$\|f\|^p_{L^p(\mathbb{X},\mu)} \leqslant \|f\|^p_{\text{Weak}-L^p(\mathbb{X},\mu)}\left\{1+\log\left(\frac{\mu(\mathbb{X})\|f\|^p_{L^\infty(\mathbb{X},\mu)}}{\|f\|^p_{\text{Weak}-L^p(\mathbb{X},\mu)}}\right)\right\}.$$

(2) If $p<s<+\infty$, then

$$\|f\|^s_{L^s(\mathbb{X},\mu)} \leqslant \frac{s}{s-p}\|f\|^p_{\text{Weak}-L^p(\mathbb{X},\mu)}\|f\|^{s-p}_{L^\infty(\mathbb{X},\mu)}.$$

Proof. (1) Observe that

$$\begin{aligned}
&\mu\{x\in\mathbb{X},\ |f(x)|>t\}\leqslant\mu\{\mathbb{X}\},\\
&\mu\{x\in\mathbb{X},\ |f(x)|>t\}\leqslant\|f\|^p_{\text{Weak}-L^p(\mathbb{X},\mu)}t^{-p},\\
&\mu\{x\in\mathbb{X},\ |f(x)|>t\}=0 \quad \text{if } t\geqslant\|f\|_{L^\infty(\mathbb{X},\mu)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|f\|^p_{L^p(\mathbb{X},\mu)} &= \int_0^{+\infty} pt^{p-1}\mu\{x\in\mathbb{X},\ |f(x)|>t\}\,dt\\
&\leqslant p\mu(\mathbb{X})\int_0^{\mu(\mathbb{X})^{-1/p}\|f\|_{\text{Weak}-L^p(\mathbb{X},\mu)}} t^{p-1}dt\\
&\quad + p\|f\|^p_{\text{Weak}-L^p(\mathbb{X},\mu)}\int_{\mu(\mathbb{X})^{-1/p}\|f\|_{\text{Weak}-L^p(\mathbb{X},\mu)}}^{\|f\|_{L^\infty(\mathbb{X},\mu)}}\frac{dt}{t}\\
&= \|f\|^p_{\text{Weak}-L^p(\mathbb{X},\mu)}\left\{1+\log\left(\frac{\mu(\mathbb{X})\|f\|^p_{L^\infty(\mathbb{X},\mu)}}{\|f\|^p_{\text{Weak}-L^p(\mathbb{X},\mu)}}\right)\right\}.
\end{aligned}$$

(2) As before,

$$\begin{aligned}
\|f\|^s_{L^s(\mathbb{X},\mu)} &= \int_0^{+\infty} st^{s-1}\mu\{x\in\mathbb{X},\ |f(x)|>t\}\,dt\\
&\leqslant s\|f\|^p_{\text{Weak}-L^p(\mathbb{X},\mu)}\int_0^{\|f\|_{L^\infty(\mathbb{X},\mu)}} t^{s-1-p}dt\\
&= \frac{s}{s-p}\|f\|^p_{\text{Weak}-L^p(\mathbb{X},\mu)}\|f\|^{s-p}_{L^\infty(\mathbb{X},\mu)}.
\end{aligned}$$

□

The first inequality in the above lemma is sharp. Indeed, if the measure is not atomic, one can always choose a function with distribution function that turns the above inequalities into equalities. Also the second inequality is sharp if the measure is not atomic and infinite, but it can be slightly improved when the measure is finite.

Our first result is a simple application of the Hausdorff-Young inequality.

THEOREM 2. *Let Ω be a bounded open set in $\mathbb{R}^d$.*
(1) *If $2 \leqslant p < +\infty$ and $1/p + 1/q = 1$, then*

$$\|\mathcal{D}(R\Omega - x)\|_{L^p(\mathbb{T}^d)} \leqslant R^d \left\| \{\widehat{\chi_\Omega}(Rn)\}_{n\neq 0} \right\|_{L^q(\mathbb{Z}^d)}.$$

(2) *If $2 < p < +\infty$ and $1/p + 1/q = 1$, then*

$$\|\mathcal{D}(R\Omega - x)\|_{\mathrm{Weak}-L^p(\mathbb{T}^d)} \leqslant CR^d \left\| \{\widehat{\chi_\Omega}(Rn)\}_{n\neq 0} \right\|_{\mathrm{Weak}-L^q(\mathbb{Z}^d)}.$$

(3) *If $2 < p < +\infty$ and $1/p + 1/q = 1$, then*

$$\|\mathcal{D}(R\Omega - x)\|_{L^p(\mathbb{T}^d)} \leqslant CR^d \log^{1/p}(2+R) \left\| \{\widehat{\chi_\Omega}(Rn)\}_{n\neq 0} \right\|_{\mathrm{Weak}-L^q(\mathbb{Z}^d)}.$$

(4) *If $1 \leqslant p \leqslant +\infty$, then*

$$\|\mathcal{D}(R\Omega - x)\|_{L^p(\mathbb{T}^d)} \geqslant \sup_{n\neq 0} \left\{ \left| R^d \widehat{\chi_\Omega}(Rn) \right| \right\}.$$

Proof. Point (1) readily follows from the Fourier expansion of the discrepancy and the Hausdorff-Young inequality: If $2 \leqslant p \leqslant +\infty$ and $1/p + 1/q = 1$, then

$$\|f\|_{L^p(\mathbb{T}^d)} \leqslant \left\| \widehat{f} \right\|_{L^q(\mathbb{Z}^d)}.$$

The case $(p,q) = (2,2)$ is Parseval's identity. The case $(p,q) = (+\infty, 1)$ is immediate. The intermediate cases follow by the Riesz-Thorin interpolation theorem. See [**1**, Theorem 1.1.1] or [**17**, Chapter V, §1]. Similarly, point (2) follows from the Hausdorff-Young inequality for Lorentz spaces: If $2 < p < +\infty$ and if $1/p + 1/q = 1$, then

$$\|f\|_{\mathrm{Weak}-L^p(\mathbb{T}^d)} \leqslant C \left\| \widehat{f} \right\|_{\mathrm{Weak}-L^q(\mathbb{Z}^d)}.$$

The proof of this inequality is by real interpolation between the extreme cases $L^2 \to L^2$ and $L^1 \to L^\infty$. See the general Marcinkiewicz interpolation theorem [**1**, Theorem 5.3.2] or [**17**, Chapter V, §3]. Point (3) follows from point (2), Lemma 1, and the trivial estimate $|\mathcal{D}(R\Omega - x)| \leqslant CR^d$. Finally, a Fourier coefficient is dominated by the norm of the function, and point (4) follows. □

The above theorem is quite abstract. In order to obtain explicit results, one has to estimate the norms of the sequences $\{\widehat{\chi_\Omega}(Rn)\}_{n\neq 0}$. The interest in case (3) is when the $L^q(\mathbb{Z}^d)$ norm is infinite and the Weak$-L^q(\mathbb{Z}^d)$ norm is finite.

In order to introduce the next result, we recall that the modulus of continuity of a characteristic function shows that such a function does not belong to a Sobolev class $W^{\alpha,2}(\mathbb{R}^d)$ whenever $\alpha \geqslant 1/2$. See [**15**, Chapter 5, §5]. Moreover, in [**12**, Corollary 2.2] it is proved that for every set Ω with finite positive measure, without any regularity assumption, there exists a constant C such that

$$\int_{|\xi|>R} |\widehat{\chi_\Omega}(\xi)|^2 \, d\xi \geqslant CR^{-1}.$$

It follows that a uniform inequality of the kind $|\widehat{\chi_\Omega}(\xi)| \leqslant C|\xi|^{-\beta}$ cannot hold with $\beta > (d+1)/2$. On the other hand, this estimate holds with $\beta = (d+1)/2$ if Ω is a bounded convex domain with smooth boundary with non-vanishing Gaussian curvature. See [**16**]. See also [**6**] for possible generalizations to convex bodies with smooth boundary containing isolated points with vanishing Gaussian curvature. This motivates the following.

Corollary 3. *Assume that Ω is a bounded convex domain such that*

$$|\widehat{\chi_\Omega}(\xi)| \leqslant C|\xi|^{-(d+1)/2}.$$

(1) If $p < 2d/(d-1)$ and $R > 2$, then

$$\|\mathcal{D}(R\Omega - x)\|_{L^p(\mathbb{T}^d)} \leqslant CR^{(d-1)/2}.$$

(2) If $p \leqslant 2d/(d-1)$ and $R > 2$, then

$$\|\mathcal{D}(R\Omega - x)\|_{\text{Weak}-L^p(\mathbb{T}^d)} \leqslant CR^{(d-1)/2}.$$

(3) If $p = 2d/(d-1)$ and $R > 2$, then

$$\|\mathcal{D}(R\Omega - x)\|_{L^p(\mathbb{T}^d)} \leqslant CR^{(d-1)/2}\log^{(d-1)/(2d)}(R).$$

(4) If $p > 2d/(d-1)$ and $R > 2$, then

$$\|\mathcal{D}(R\Omega - x)\|_{L^p(\mathbb{T}^d)} \leqslant CR^{d(pd-p-d+1)/p(d+1)}.$$

Proof. Points (1), (2) and (3) follow from Theorem 2, and the observation that the sequence $\left\{|n|^{-\alpha}\right\}_{n\neq 0}$ is in $L^q(\mathbb{Z}^d)$ if and only if $q\alpha > d$, and it is in Weak$-L^q(\mathbb{Z}^d)$ if and only if $q\alpha \geqslant d$. Point (4) follows from point (2) with $p = 2d/(d-1)$, the pointwise estimate $|\mathcal{D}(R\Omega - x)| \leqslant CR^{d(d-1)/(d+1)}$ proved in [**8**] and [**9**], and (2) in Lemma 1. □

The estimates in the above Corollary for $p < 2d/(d-1)$ are essentially sharp. In order to show this, we first recall the following result on the Fourier transform of the characteristic function of a convex set.

Theorem 4. *Let $\Omega \subset \mathbb{R}^d$ be a convex body with smooth boundary having everywhere positive Gaussian curvature. For every $\xi \in \mathbb{R}^d \setminus \{0\}$ let $\sigma(\xi)$ be the unique point on the boundary $\partial\Omega$ with outward unit normal $\xi/|\xi|$. Also let $K(\sigma(\xi))$ be the Gaussian curvature of $\partial\Omega$ at $\sigma(\xi)$. Then, as $|\xi| \to +\infty$, the Fourier transform of $\chi_\Omega(x)$ has the asymptotic expansion*

$$\begin{aligned}&\widehat{\chi_\Omega}(\xi)\\ &= -\frac{1}{2\pi i}|\xi|^{-\frac{d+1}{2}}\left[K^{-\frac{1}{2}}(\sigma(\xi))\,e^{-2\pi i\left(\xi\cdot\sigma(\xi)-\frac{d-1}{8}\right)} - K^{-\frac{1}{2}}(\sigma(-\xi))\,e^{-2\pi i\left(\xi\cdot\sigma(-\xi)+\frac{d-1}{8}\right)}\right]\\ &\quad + O\left(|\xi|^{-\frac{d+3}{2}}\right).\end{aligned}$$

Proof. See e.g. [**7**], [**8**], or [**9**]. □

The following result partially complements Corollary 3.

Theorem 5. *Let $\Omega \subset \mathbb{R}^d$ be a convex body with smooth boundary having everywhere positive Gaussian curvature.*

(1) If Ω is not symmetric about a point, or if the dimension $d \not\equiv 1 \,(\mathrm{mod}\, 4)$, then for every $p \geqslant 1$ there exists $C > 0$ such that for every $R > 2$,

$$\|\mathcal{D}(R\Omega - x)\|_{L^p(\mathbb{T}^d)} \geqslant CR^{\frac{d-1}{2}}.$$

(2) If Ω is symmetric about a point and if $d \equiv 1 \ (\mathrm{mod}\, 4)$ then

$$\limsup_{R\to+\infty} \left\{R^{-\frac{d-1}{2}} \|\mathcal{D}(R\Omega - x)\|_{L^p(\mathbb{T}^d)}\right\} > 0 \qquad \text{if } p \geqslant 1,$$

$$\liminf_{R\to+\infty} \left\{R^{-\frac{d-1}{2}} \|\mathcal{D}(R\Omega - x)\|_{L^p(\mathbb{T}^d)}\right\} = 0 \qquad \text{if } p < \frac{2d}{d-1}.$$

More precisely, if $p < 2d/(d-1)$ there exist $C > 0$, and a sequence $R_j \to +\infty$, such that

$$\|\mathcal{D}(R_j\Omega - x)\|_{L^p(\mathbb{T}^d)} \leqslant CR_j^{\frac{d-1}{2}} \left(\frac{\log(R_j)}{\log(\log(R_j))}\right)^{\frac{d-1}{2d}-\frac{1}{p}}.$$

Proof. In order to prove point (1) observe that, by Theorem 2,

$$\|\mathcal{D}(R\Omega - x)\|_{L^p(\mathbb{T}^d)} \geqslant \sup_{n\neq 0} \left\{\left|R^d \widehat{\chi_\Omega}(Rn)\right|\right\}.$$

Moreover, by Theorem 4, for $n \neq 0$,

$$R^d \widehat{\chi_\Omega}(Rn) = \frac{-1}{2\pi i} R^{\frac{d-1}{2}} |n|^{-\frac{d+1}{2}} \left[K^{-\frac{1}{2}}(\sigma(n))\, e^{-2\pi i\left(Rn\cdot\sigma(n) - \frac{d-1}{8}\right)} - K^{-\frac{1}{2}}(\sigma(-n))\, e^{-2\pi i\left(Rn\cdot\sigma(-n) + \frac{d-1}{8}\right)}\right] + O\left(R^{\frac{d-3}{2}} |n|^{-\frac{d+3}{2}}\right).$$

If Ω is not symmetric, then also $K(\sigma(u))$ is not symmetric (see [**2**, §14, p. 133]). Since the set $\left\{\frac{n}{|n|} : n \in \mathbb{Z}^d \setminus \{0\}\right\}$ is dense in the unit sphere, by continuity there exists $m \in \mathbb{Z}^d$ such that $K(\sigma(m)) \neq K(\sigma(-m))$. Then, for this m and R large enough,

$$\begin{aligned}
&\left|R^d \widehat{\chi_\Omega}(Rm)\right| \\
&\geqslant \frac{1}{2\pi} R^{\frac{d-1}{2}} |m|^{-\frac{d+1}{2}} \left|K^{-\frac{1}{2}}(\sigma(m)) - K^{-\frac{1}{2}}(\sigma(-m))\right| + O\left(R^{\frac{d-3}{2}} |m|^{-\frac{d+3}{2}}\right) \\
&\geqslant CR^{\frac{d-1}{2}}.
\end{aligned}$$

Assume now that Ω is symmetric, and translate the center of symmetry to the origin, so that for every ξ we have $\sigma(-\xi) = -\sigma(\xi)$ and $K(\sigma(\xi)) = K(\sigma(-\xi))$. Choose $n \neq 0$ and observe that

$$n \cdot (\sigma(n) - \sigma(-n)) = 2n \cdot \sigma(n) \neq 0.$$

Indeed, $n \cdot \sigma(n) = 0$ would imply that the center belongs to the hyperplane tangent to $\partial\Omega$ at $\sigma(n)$, hence Ω should have measure 0. We have

$$\begin{aligned}
&\left|R^d \widehat{\chi_\Omega}(Rn)\right| \\
&\geqslant CR^{\frac{d-1}{2}} |n|^{-\frac{d+1}{2}} K^{-\frac{1}{2}}(\sigma(n)) \left|e^{2\pi i\left(2Rn\cdot\sigma(n) - \frac{d-1}{4}\right)} - 1\right| + O\left(R^{\frac{d-3}{2}} |n|^{-\frac{d+3}{2}}\right).
\end{aligned}$$

Let $\|x\|$ denote the distance of a real number x from the integers. If

$$\left\|2Rn \cdot \sigma(n) - \frac{d-1}{4}\right\| > \frac{1}{10},$$

then $\left|e^{2\pi i\left(2Rn\cdot\sigma(n)-\frac{d-1}{4}\right)}-1\right|>c$ and we have

$$\left|R^d\widehat{\chi_\Omega}(Rn)\right|\geqslant cR^{\frac{d-1}{2}}.$$

Assume now that

$$\left\|2Rn\cdot\sigma\left(n\right)-\frac{d-1}{4}\right\|\leqslant\frac{1}{10}.$$

Then

$$\left\|4Rn\cdot\sigma\left(n\right)-\frac{d-1}{2}\right\|\leqslant\frac{1}{5}.$$

Since $d\not\equiv 1\ (\mathrm{mod}\,4)$, we have

$$\left\|4Rn\cdot\sigma\left(n\right)-\frac{d-1}{4}\right\|\geqslant\frac{1}{20}.$$

Applying the previous argument with $2n$ in place of n provides the estimate

$$\left|R^d\widehat{\chi_\Omega}(R2n)\right|\geqslant cR^{\frac{d-1}{2}}.$$

In order to prove point (2), assume that Ω is symmetric and $d\equiv 1\ (\mathrm{mod}\,4)$. From the asymptotic estimate of $\widehat{\chi}_\Omega\left(\xi\right)$ we obtain

$$\left|R^d\widehat{\chi_\Omega}(Rn)\right|=\frac{1}{\pi}R^{\frac{d-1}{2}}\left|n\right|^{-\frac{d+1}{2}}K^{-\frac{1}{2}}\left(\sigma\left(n\right)\right)\left|\sin\left(2\pi Rn\cdot\sigma\left(n\right)\right)\right|+O\left(R^{\frac{d-3}{2}}\left|n\right|^{-\frac{d+3}{2}}\right).$$

Since $n\cdot\sigma\left(n\right)\neq 0$,

$$\begin{aligned}&\limsup_{R\to+\infty}\left\{R^{-\frac{d-1}{2}}\left\|\mathcal{D}\left(R\Omega-x\right)\right\|_{L^p(\mathbb{T}^d)}\right\}\\ \geqslant&\limsup_{R\to+\infty}\left\{R^{-\frac{d-1}{2}}\left|R^d\widehat{\chi_\Omega}(Rn)\right|\right\}=\frac{1}{\pi}\left|n\right|^{-\frac{d+1}{2}}K^{-\frac{1}{2}}\left(\sigma\left(n\right)\right)>0.\end{aligned}$$

The last part of the proof relies on the ideas of Parnovski and Sobolev in [**13**]. We need a variant of Dirichlet's theorem on simultaneous diophantine approximation (see [**13**]). Let $\alpha_1,\ldots,\alpha_m$ be real numbers, then for every positive integer j there exist integers $s_1,\ldots,s_m$ and r such that

$$j\leqslant r\leqslant j^{m+1}\quad\text{and}\quad\left|r\alpha_k-s_k\right|<j^{-1}\text{ for every }k=1,\ldots,m.$$

Let $\beta=\frac{2p}{2d-pd+p}$, and

$$\left\{\alpha_k\right\}_{k=1}^m=\left\{n\cdot\sigma\left(n\right)\right\}_{|n|\leqslant j^\beta}.$$

Then there exist integers s_n and R_j such that

$$j\leqslant R_j\leqslant j^{cj^{\beta d}+1}\leqslant j^{Cj^{\beta d}}\quad\text{and}\quad\left|R_jn\cdot\sigma\left(n\right)-s_n\right|\leqslant j^{-1}.$$

It follows that

$$\left|\sin\left(2\pi R_jn\cdot\sigma\left(n\right)\right)\right|=\left|\sin\left(2\pi\left(R_jn\cdot\sigma\left(n\right)-s_n\right)\right)\right|\leqslant 2\pi j^{-1}.$$

By the Hausdorff-Young inequality in Theorem 2, for $p\geqslant 2$ and $1/p+1/q=1$, we have

$$\left\{\int_{\mathbb{T}^d}\left|\mathcal{D}\left(R_j\Omega-x\right)\right|^p dx\right\}^{1/p}\leqslant\left\{\sum_{0\neq n\in\mathbb{Z}^d}\left|R^d\widehat{\chi_\Omega}\left(Rn\right)\right|^q\right\}^{1/q}.$$

If $2 \leqslant p < \frac{2d}{d-1}$ then the above estimates of $\widehat{\chi_\Omega}(R_j n)$ yield

$$\begin{aligned}
&\sum_{n\neq 0} \left|R_j^d \widehat{\chi_\Omega}(R_j n)\right|^q \\
&\leqslant C R_j^{(d-1)q/2} \sum_{n\neq 0} |n|^{-(d+1)q/2} \left|\sin(2\pi R_j n\cdot\sigma(n))\right|^q + \sum_{n\neq 0} O\left(R_j^{(d-3)q/2}|n|^{-(d+3)q/2}\right) \\
&\leqslant C R_j^{(d-1)q/2}\left(j^{-q}\sum_{0<|n|\leqslant j^\beta}|n|^{-(d+1)q/2} + \sum_{|n|>j^\beta}|n|^{-(d+1)q/2}\right) + O\left(R_j^{(d-3)q/2}\right) \\
&\leqslant C R_j^{(d-1)q/2}\left(j^{-q} + j^{\beta(d-(d+1)q/2)}\right) + O\left(R_j^{(d-3)q/2}\right).
\end{aligned}$$

Since $\beta = \frac{2q}{q(d+1)-2d}$ and $R_j \geqslant j$, we obtain

$$\left\{\sum_{m\neq 0}\left|R_j^d\widehat{\chi_\Omega}(R_j n)\right|^q\right\}^{1/q} \leqslant c R_j^{(d-1)/2} j^{-1}.$$

Finally, letting $j \to +\infty$ we obtain

$$\liminf_{R\to+\infty}\left\{R^{-\frac{d-1}{2}}\left\|\mathcal{D}(R\Omega - x)\right\|_{L^p(\mathbb{T}^d)}\right\} \leqslant \liminf_{R\to+\infty}\left\{R^{-\frac{d-1}{2}}\left\{\sum_{n\neq 0}\left|R^d\widehat{\chi_\Omega}(Rn)\right|^q\right\}^{1/q}\right\} = 0.$$

More precisely, if $\varphi(t) = t^{\beta d}\log(t)$ then one can prove that, for large s,

$$\varphi^{-1}(s) \approx \left(\frac{\beta d s}{\log(s)}\right)^{1/\beta d}.$$

This implies that if $R_j \leqslant j^{Cj^{\beta d}}$ then

$$j^{-1} \leqslant C\left(\frac{\log(R_j)}{\log(\log(R_j))}\right)^{-1/\beta d}.$$

Therefore

$$\left\{\sum_{m\neq 0}\left|R_j^d\widehat{\chi_\Omega}(R_j n)\right|^q\right\}^{1/q} \leqslant C\, R_j^{(d-1)/2}\left(\frac{\log(R_j)}{\log(\log(R_j))}\right)^{-1/\beta d}.$$

□

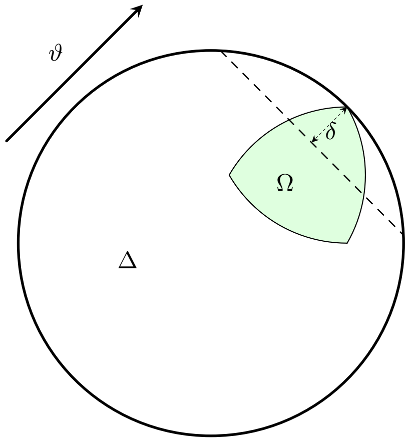
As we said, $|\widehat{\chi_\Omega}(\xi)| \leqslant C|\xi|^{-(d+1)/2}$ whenever Ω has smooth boundary with positive Gaussian curvature. However, for domains in the plane this smoothness assumption can be relaxed. Consider a convex body Ω which can roll unimpeded inside a disc Δ. This means that for any point x on the boundary $\partial\Delta$ there is a translated copy of Ω contained in Δ that touches $\partial\Delta$ in x.

THEOREM 6. *If a planar convex set Ω can roll unimpeded inside a disc, then*

$$|\widehat{\chi_\Omega}(\xi)| \leqslant C|\xi|^{-3/2}.$$

Proof. In [**14**] (see also [**5**, **18**]) it is proved that if

$$\lambda(\delta,\vartheta,\Omega) = \left|\left\{x\in\Omega : \delta + x\cdot\vartheta = \sup_{y\in\Omega}\{y\cdot\vartheta\}\right\}\right|$$



(this is the length of the chord perpendicular to the outward direction ϑ and at a small distance δ from the boundary $\partial\Omega$), then

$$|\widehat{\chi_\Omega}(\rho\vartheta)| \leqslant \frac{\text{diameter}(\Omega)}{2\rho}\left(\lambda\left((2\rho)^{-1},\vartheta,\Omega\right)+\lambda\left((2\rho)^{-1},-\vartheta,\Omega\right)\right).$$

If Ω can roll unimpeded inside a disc Δ, then $\lambda(\delta,\vartheta,\Omega) \leqslant \lambda(\delta,\vartheta,\Delta)$. This implies that the Fourier transform of Ω is dominated by the chords of a disc, and therefore $|\widehat{\chi_\Omega}(\xi)| \leqslant C\,|\xi|^{-3/2}$. □

A curve can roll unimpeded inside another curve if and only if the largest radius of curvature of the first is smaller than the smallest radius of curvature of the second. No smoothness of these curves is required, the rolling curve may also have corners. See [**2**, Chapter 17] and the references therein.

In particular, the above results give an alternative proof of the result in [**10**].

COROLLARY 7. *Let Ω be a planar convex set that can roll unimpeded inside a disc. For any $R > 2$ we have*

$$\left\{\int_{\mathbb{T}^2} |\mathcal{D}(R\Omega - x)|^4\,dx\right\}^{1/4} \leqslant CR^{1/2}\log^{1/4}(R).$$

We conclude with a remark. A spherical shell $\{x \in \mathbb{R}^d : R \leqslant |x| < R+1\}$ contains approximately R^{d-1} integer points which lie on the spherical surfaces $\{|x|^2 = n\}$ with n integer between R^2 and $(R+1)^2$. Since these integers are at most $2R+1$, one of these surfaces contains at least cR^{d-2} integer points. This implies that, if $\Omega = \{x \in \mathbb{R}^d : |x| \leqslant 1\}$, then $\|\mathcal{D}(R\Omega - x)\|_{L^\infty(\mathbb{T}^d)} \geqslant cR^{d-2}$ for a diverging sequence of R. On the other hand we proved that $\|\mathcal{D}(R\Omega - x)\|_{L^p(\mathbb{T}^d)} \leqslant cR^{(d-1)/2}$ for $p < 2d/(d-1)$. Since for $d > 3$ this L^p estimate is smaller than the L^∞ one, there exists a critical index p for which $\|\mathcal{D}(R\Omega - x)\|_{L^p(\mathbb{T}^d)} \leqslant cR^{(d-1)/2}$ starts failing. We do not know if this critical index is $2d/(d-1)$.

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