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Balancing Bilinearly Interfering Elements

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Abstract. Many decisions in various fields of application have to take into account the joint effects of two elements that can interfere with each other. This happens, for example, in Medicine (synergic or antagonistic drugs), Agriculture (anti-cryptogamics), Public Economics (interfering economic policies), Industrial Economics (where the demand of an asset can be influenced by the supply of another asset), Zootechnics, and so on. When it is necessary to decide about the dosage of such elements, there is sometimes a primary interest for one effect rather than another; more precisely, it may be of interest that the effects of an element are in a certain proportion with respect to the effects of the other. It may also be necessary to take into account minimum quantities that must be assigned.

In Carfì, Gambarelli and Uristani (2013), a mathematical model was proposed to solve the above problem in its exact form. In this paper, we present a solution in closed form for the case in which the function of the effects is bilinear.

Keywords: bargaining problems, game theory, antagonist elements, interfering elements, optimal dosage, synergies

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JEL Classification: C71, C72, C78

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1. INTRODUCTION

Many decisions in various fields of application have to take into account the joint effects of two elements that can interfere with each other. This happens, for example, in Medicine (synergic or antagonistic drugs), Agriculture (pesticides), Public Economics (interfering economic policies), Industrial Economics (where the demand of an asset can be influenced by the supply of another asset), Zootechnics, and so on. When it is necessary to decide about the dosage of such elements, there is sometimes a primary

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interest for one effect rather than another; more precisely, it may be of interest that the effects of an element are in a certain proportion with respect to the effects of the other. It may also be necessary to take into account the minimum quantities that should be assigned.

In Carfì, Gambarelli and Uristani (2013), a mathematical model was proposed to solve the above problem in its exact form. In this paper, we present a solution in closed form for the case in which the function of the effects is bilinear.

In the next two sections, the problem will be defined in general terms. In Sections 4 and 5, the case of bilinear interference (free and truncated) will be dealt with. In the following section, an algorithm will be presented for the direct calculation of solutions. At the end, we shall provide some examples of application, and we shall indicate some open problems.

1.1. LITERATURE REVIEW

D. Carfì (2010, 2012a) has introduced a new analytical methodology to examine differentiable normal-form games. He and various collaborators have developed the applicative aspects of the new methodology in several directions, such as Management, Finance, Microeconomics, Macroeconomic, Green Economy, Financial Markets, Industrial Organization, Project Financing and so on – see, for instance, Carfì and Fici (2012), Carfì and Lanzafame (2013), Carfì, Magaudda and Schilirò (2010), Carfì and Musolino (2015a, 2015b, 2014a, 2014b, 2013a, 2013b, 2013c, 2012a, 2012b, 2012c, 2011a, 2011b), Carfì, Patanè and Pellegrino (2011), Carfì and Perrone (2013, 2012a, 2012b, 2011a, 2011b, 2011c), Carfì and Pintaudi (2012), Carfì and Schilirò (2014a, 2014b, 2013, 2012a,2012b, 2012c, 2012d, 2011a, 2011b, 2011c), Carfì, Musolino, Ricciardello and Schilirò (2012), Carfì, Musolino, Schilirò and Strati (2013), Carfì and Trunfio (2011), Okura and Carfì (2014).

The methodology can suggest useful solutions to a specific Game Theory problem. This analytical framework enables us to incorporate solutions designed "to share the pie fairly". The basic original definition we propose and apply for this methodology is introduced also in Carfì and Schilirò (2014a, 2014b, 2013, 2012a, 2012b, 2012c, 2012d, 2011a, 2011b, 2011c) and Carfì (2012a, 2012b, 2010, 2009a, 2009b, 2009c, 2009d, 2009e, 2008). The method we use to study the payoff space of a normal-form game is devisable in Carfì and Musolino (2015a, 2015b, 2014a, 2014b, 2013a, 2013b, 2013c, 2012a, 2012b, 2012c, 2011a, 2011b), and Carfì and Schilirò (2014a, 2014b, 2013, 2012a, 2012b, 2012c, 2012d, 2011a, 2011b, 2011c). Other important applications, of the complete examination methodology, are introduced in Agreste, Carfì, and Ricciardello (2012), Arthanari, Carfì and Musolino (2015), Baglieri, Carfì, and Dagnino (2012), Carfì and Fici (2012), Carfì, Gambarelli and Uristani (2013), Carfì and Lanzafame (2013), Carfì, Patanè, and Pellegrino (2011), Carfì and Romeo (2015). A complete treatment of a normal-form game is presented and applied by Carfì (2012a, 2012b, 2010, 2009a, 2009b, 2009c, 2009e, 2008), Carfì and Musolino (2015a, 2015b, 2014a, 2014b, 2013a, 2013b, 2013c, 2012a, 2012b, 2012c, 2011a, 2011b), Carfì and Perrone (2013, 2012a, 2012b, 2011a, 2011b, 2011c), Carfì and Ricciardello (2013a, 2013b, 2012a, 2012b, 2010, 2009) and Carfì and Schilirò (2014a, 2014b, 2013, 2012a,2012b, 2012c, 2012d, 2011a, 2011b, 2011c). Carfì (2008) proposes a general definition and explains the basic properties of Pareto boundary, which constitutes a fundamental element of the complete analysis of a normal-form game.

2. DEFINITIONS

Let $N = \{1, 2\}$ be a set of labels of the considered interfering elements (i.e., drugs, commodities, and so on) and any related effects resulting from their use (e.g., curing diseases, commodity demand, and so on). From here on, if not otherwise specified, the use of the index "i" will imply "for all $i \in N$", with an analogous use of the index "j".

2.1. THE QUANTITIES

We denote the non-negative quantities of the i-th element as follows:

- Q_i is the quantity effectively used;
- $Q_i^{\max}$ is the optimal quantity if the i-th element is used alone;
- $Q_i^{\min}$ is the minimum necessary quantity if the i-th element is used alone;
- q_i and $q_i^{\min}$ are the corresponding ratios with respect to $Q_i^{\max}$:
  - $q_i = Q_i / Q_i^{\max}$,
  - $q_i^{\min} = Q_i^{\min} / Q_i^{\max}$.

We call Q, $Q^{\max}$, $Q^{\min}$, q, and $q^{\min}$ the corresponding n-vectors.

It is assumed that $Q_i^{\min} < Q_i^{\max}$ and $Q_i^{\min} \leq Q_i \leq Q_i^{\max}$. Given such conditions, q_i and $q_i^{\min}$ belong to the interval [0,1].

2.2. THE EFFECTS

Let $e_i(q)$ be a non-negative function expressing the level of the i-th effect when percent quantities q are used. The space of the effects is the set of points $x = (x_1,..., x_n) = e(q)$ according to variations of q. This function should satisfy the conditions that follow.

If no elements are used, then all of the effects are null. If a single element is employed in the optimal dose for use alone, then the level of the relative effect is 1, while the level of the effect for the other is null. Finally, if both elements are employed in the optimal doses for use alone, the resulting effects are given by vector $\delta = (\delta_1, \delta_2)$ with real positive components. In formulae:

- if $q_1 = q_2 = 0$, then $e_1 = e_2 = 0$;
- if $q_1 = 0$ and $q_2 = 1$, then $e_1 = 0$ and $e_2 = 1$;
- if $q_1 = 1$ and $q_2 = 0$, then $e_1 = 1$ and $e_2 = 0$;
- if $q_1 = q_2 = 1$, then $e_1 = \delta_1$ and $e_2 = \delta_2$.

See Figure 1 as an example of an effect's function.

Without loss of generality, we may place the elements in order so that:

$$\delta 1 \leq \delta 2. \tag{1}$$

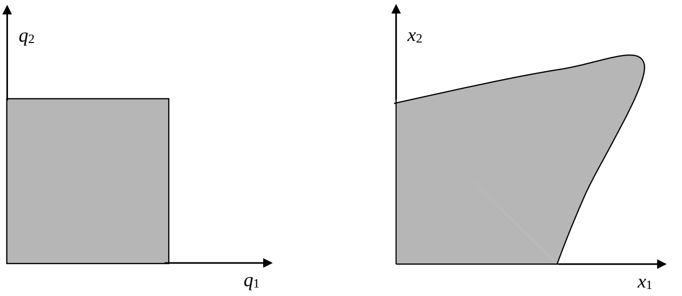

Fig. 1. *Strategy space and payoff space of the game, for $n = 2$*

The effectfunction can be defined directly, according to the faced problem, or can be constructed on the basis of the study cases, using statistical methods and applying suitable adjustments of scale, in order to respect all of the above conditions. In this paper, we study the case in which this function is bilinear: free (Section 4) or truncated (Section 5).

2.3. QUANTITIES AND MINIMUM EFFECTS

We use $e_i^{\min}$ to indicate the minimum necessary level of the i-th effect. This level is derived from the function $e_i(q)$ given $q_i = q_i^{\min}$ and $q_j = 0$ for the other component $j \neq i$. We use $e^{\min}$ to indicate the related 2-dimensional vector.

We assume the minimum necessary level of the i-th effect should not exceed 1 (if $\delta_i \leq 1$) or δ_i(elsewhere). Thus:

$$e_i^{\min} \leq \max\{1, \delta_i\} \tag{2}$$

2.4. THE REQUIRED OPTIMAL RATIOS

We use r to indicate the required optimal ratio between the effects e_1 and e_2. We call R the half-line centered on the origin, the inclination of which is r. For each point x of the feasible set, we use E to indicate the half-line centered on the origin, passing through x.

2.5. THE FEASIBLE PARETO OPTIMAL BOUNDARY

We shall call each point x of the codomain of e which is not jointly improvable a *Pareto optimal effect*, in the sense that if we move from that point in this set to improve the i-th effect, then the other effect necessarily decreases. It is easy to prove that, even here, every Pareto optimal point is a boundary point of the set of effects; we shall, therefore, call the set of Pareto optimal effects the *Pareto optimal boundary.*

The term *feasible Pareto optimal boundary P* is given to the set of the points of the Pareto optimal boundary respecting the conditions $x_i \geq e_i^{\min}$ for all $i \in N$.

3. THE OPTIMIZATION PROBLEM

3.1. THE DATA

The input data of the model is δ, $e^{\min}$, r and the option on the type of bilinear function (free or truncated).

In some applications, we do not directly know the minimal effect $e_i{}^{\min}$ for some element i, while we know the necessary minimal and optimal quantities $Q_i{}^{\min}$ and $Q_i{}^{\max}$. It is thus possible to deduce $q_i{}^{\min}$, which, introduced into the equation $e_i(q)$, gives $e_i{}^{\min}$ (as indicated in Section 2.3).

3.2. THE OBJECTIVE

The problem is to find the set of quantity-vectors q^* such that the corresponding effect vectors $e(q^*)$ belong to the feasible Pareto optimal boundary and are such that the half-lines that join them to the origin form a minimum angle with R.

3.3. EXISTENCE AND UNIQUENESS

If the necessary minimum effects are excessive as a whole, the feasible set is empty; therefore, the problem is without solution. However, for those cases where determining the minimum quantities is open to variations, we have introduced certain indications as to modifications to be used each time. Solution uniqueness is not guaranteed in general, but the various different solutions produce the same effects (payoffs).

3.4. SOLUTION METHODS

Determining the optimal combination of q depends clearly on the form of the effects function $e(q)$. Below, we shall present the solutions for free bilinear functions (Section 4) and for truncated bilinear functions (Section 5) providing closed form formulae and geometrical descriptions. For what concerns cases in which the effect functions are of different types, we refer to Carfì *et al.* (2013).

4. FREE BILINEAR CASE

In such cases, the function $e(q)$ of each effect is defined as follows:

$e_1 = q_1(1 - q_2) + q_1 q_2 q \delta_1$

$e_2 = (1 - q_1)q_2 + q_1 q_2 \delta_2$

The problem of minimizing the angle between R and E is defined as:

$$\min_{q_1, q_2} \left| \frac{e_2}{e_1} - r \right|$$

We shall examine the various types of interference separately, varying the values of δ under the constraint (1).

We shall represent such types as graphs with corresponding numbers. In each of these graphs, the grey portion indicates the area in which δ can vary, while the bold line indicates the feasible Pareto optimal boundary.

We shall then give the solutions along with the relative steps for achieving them in the corresponding tables.

4.1. TYPE 1 (INDEPENDENT OR SYNERGIC ELEMENTS)

This type can be either $\delta_1 = \delta_2 = 1$ (independent elements) or $\delta_1 > 1$, $\delta_2 \geq 1$ (synergic elements) and is illustrated in Figure 2.

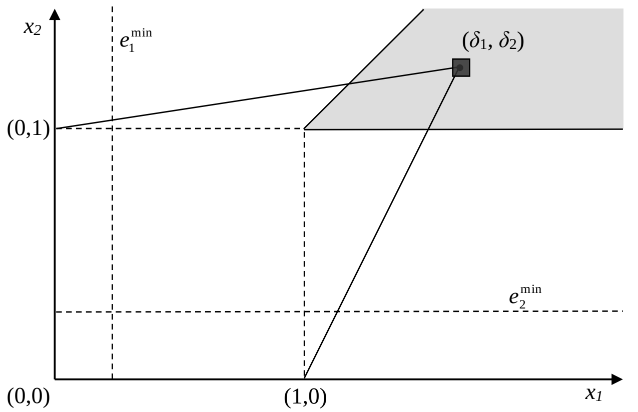

Fig. 2. *$n = 2$, case 1 (independent or synergic elements)*

The set of effects is represented by the quadrangle having vertices (0, 0), (0, 1), (1, 0), and (δ_1, δ_2). The feasible Pareto optimal boundary is made up of the single point δ. The input condition (2) guarantees the existence of the solution, given in Table 1.

Table 1. *The optimal solution in type 1*

	values
optimal effects	$x^* = (\delta_1, \delta_2)$
optimal quantities	$q_1 = 1, q_2 = 1$

4.2. TYPE 2 (PARTIALLY SYNERGIC AND PARTIALLY ANTAGONISTIC ELEMENTS)

This is the case $\delta_1 + \delta_2 > 1$, $\delta_1 \geq 1$, $\delta_2 < 1$. It is illustrated in Figure 3.

The set of effects is described by the quadrangle having vertices (0, 0), (0, 1), (1, 0), and (δ_1, δ_2).

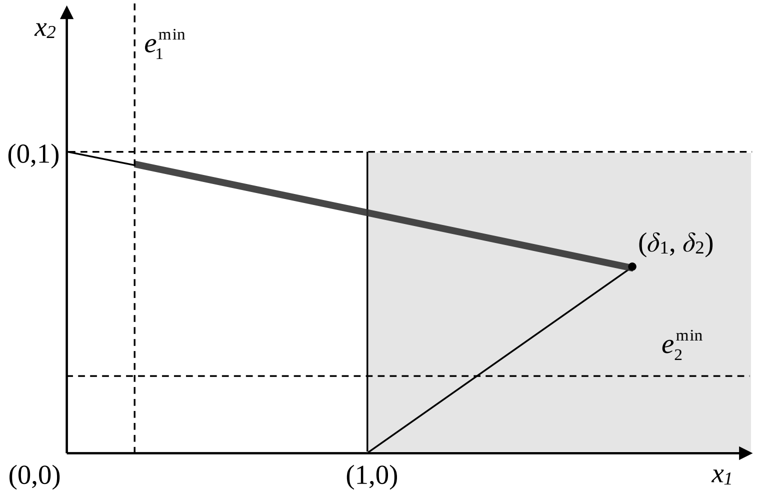

Fig. 3. *$n = 2$, case 2 (partially synergic and partially antagonistic elements)*

In order to simplify the notations, we define:

$$a_1 = \max(0,\ e_1^{\min})$$

$$b_1 = \min\left(\delta_1, \frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1)\right)$$

The existence of a solution requires, besides (2), the additional condition:

$$e_1^{\min} \leq b_1$$

This condition results in $a_1 \leq b_1$ and not-emptiness of the feasible Pareto optimal boundary. This boundary is the set of points (x_1, x_2) such that

$$x_1 \in [a_1, b_1]$$

$$x_2 = \frac{\delta_2 - 1}{\delta_1} x_1 + 1$$

In the event of no solution, the existence of one may be brought about by modifying $e_1^{\min}$ and/or $e_2^{\min}$ as follows:

– by fixing $e_2^{\min}$, we can use $e_1^{\min} = \frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1)$;

– by fixing $e_1^{\min}$, we can use $e_2^{\min} = \frac{\delta_2 - 1}{\delta_1} e_1^{\min} + 1$.

Other ways are also open, if both $e_1^{\min}$ and $e_2^{\min}$ are modified. The solution is given in the final row of Table 2.

Table 2. *The optimal solution in type 2*

<table>
<tr><td>existence condition</td><td colspan="2">$e_1^{\min} \leq \min\left(\delta_1, \frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1)\right)$</td></tr>
<tr><td>extremes of the feasible P.O. boundary</td><td colspan="2">$\mathrm{L} = (L_1, L_2) = \left(e_1^{\min}, \ \frac{\delta_2 - 1}{\delta_1} e_1^{\min} \ + \ 1\right)$
$\mathrm{R} = (R_1, R_2) = \left(\frac{\delta_1}{\delta_2 - 1}(\max\left(\delta_2, e_2^{\min}\right) - 1), \ \max\left(\delta_2, e_2^{\min}\right)\right)$</td></tr>
<tr><td rowspan="3">optimal effects</td><td>$L_2/L_1 \leq r \leq R_2/R_1$</td><td>$x^* = (w_1, \ w_2)$
$w_1 = \delta_1/(r\delta_1 - \delta_2 + 1)$
$w_2 = rw_1$</td></tr>
<tr><td>$r > L_2/L_1$</td><td>$x^* = L$</td></tr>
<tr><td>$r < R_2/R_1$</td><td>$x^* = R$</td></tr>
<tr><td rowspan="3">optimal solution</td><td>$L_2/L_1 \leq r \leq R_2/R_1$</td><td>$q_1^* = 1/(r\delta_1 - \delta_2 + 1)$
$q_2^* = 1$</td></tr>
<tr><td>$r > L_2/L_1$</td><td>$q_1^* = e_1^{\min}/\delta_1$
$q_2^* = 1$</td></tr>
<tr><td>$r < R_2/R_1$</td><td>$q_1^* = \frac{\max\left(\delta_2, e_2^{\min}\right) - 1}{\delta_2 - 1}$
$q_2^* = 1$</td></tr>
</table>

4.3. TYPE 3 (WEAKLY ANTAGONISTIC ELEMENTS)

With this type, we have $\delta_1 + \delta_2 \geq 1$, $\delta_1 < 1$, $\delta_2 < 1$. This is illustrated in Figure 4.

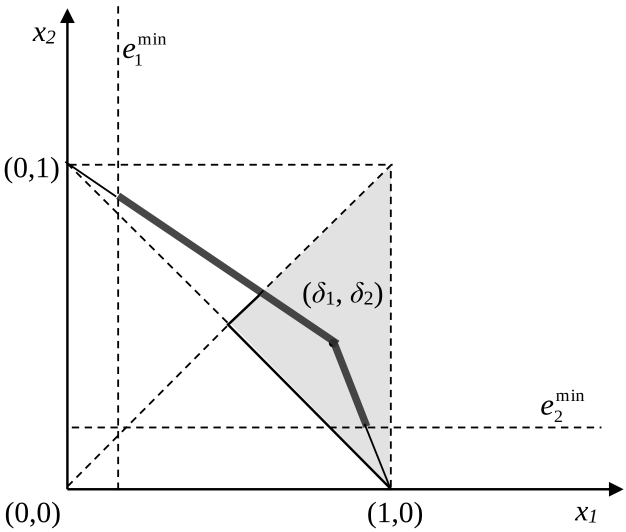

Fig. 4. *$n = 2$, case 3 (weakly antagonist elements)*

The set of effects is represented by the quadrangle having vertices (0, 0), (0, 1), (1, 0) and (δ_1, δ_2).

In order to simplify the notations, we define:

$a_1 = \max(0, e_1^{\min})$,
$b_1 = \min(\delta_1, \frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1))$
$a_2 = \max(\delta_1, e_1^{\min})$,
$b_2 = \min(1, \frac{(\delta_1 - 1)}{\delta_2} e_2^{\min} + 1)$

The existence of a solution requires, besides (2), the additional condition:

$$e_1^{\min} \leq \max(b_1, b_2)$$

This condition results in $a_1 \leq b_1$ e $a_2 \leq b_2$ and the feasible Pareto optimal boundary is not empty. This boundary is the set of points (x_1, x_2) given by $R_1 \bigcup R_2$, where:

$$R_1 = \begin{cases} \left\{ x = (x_1, x_2) \middle| \begin{array}{l} x_2 = \dfrac{(\delta_2 - 1)}{\delta_1} x_1 + 1 \\ x_1 \in [a_1, b_1] \end{array} \right\} & if\ e_1^{\min} \leq \delta_1 \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$R_2 = \begin{cases} \left\{ x = (x_1, x_2) \middle| \begin{array}{l} x_2 = \dfrac{\delta_2}{(\delta_1 - 1)} (x_1 - 1) \\ x_1 \in [a_2, b_2] \end{array} \right\} & if\ e_2^{\min} \leq \delta_2 \\ \emptyset & \text{otherwise} \end{cases}$$

In the event of no solution, the existence of one may be brought about by modifying $e_1^{\min}$ and/or $e_2^{\min}$ as follows:

– by fixing $e_2^{\min}$, we can use

$$e_1^{\min} = \max\left(\frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1), \frac{\delta_1 - 1}{\delta_2} e_2^{\min} + 1 \right);$$

– by fixing $e_1^{\min}$, we can use

$$e_2^{\min} = \min\left(\frac{\delta_2 - 1}{\delta_1} e_1^{\min} + 1, \frac{\delta_2}{\delta_1 - 1}(e_1^{\min} - 1) \right);$$

Other ways are also open, if both $e_1^{\min}$ and $e_2^{\min}$ are modified. The solution is given in the final row of Table 3.

Table 3. *The optimal solution in type 3*

existence condition	$e_1^{\min} \leq \max\left(\min\left(\delta_1, \frac{\delta_1}{(\delta_2-1)}(e_2^{\min}-1)\right), \min\left(1, \frac{(\delta_1-1)}{\delta_2}e_2^{\min}+1\right)\right)$	
extremes of the feasible P.O. boundary	$\mathrm{L}=(L_1,L_2)=\begin{pmatrix} e_1^{\min}, & \left(\frac{\delta_2-1}{\delta_1}e_1^{\min}+1\right)\chi\left(e_1^{\min}\leq\delta_1\right)+ \\ & +\left(\frac{\delta_2}{\delta_1-1}(e_1^{\min}-1)\right)\chi\left(e_1^{\min}>\delta_1\right) \end{pmatrix}$ $\mathrm{R}=(R_1,R_2)=\begin{pmatrix} \left(\frac{\delta_1-1}{\delta_2}e_2^{\min}+1\right)\chi\left(e_2^{\min}\leq\delta_2\right)+ & \\ +\left(\frac{\delta_1}{\delta_2-1}(e_2^{\min}-1)\right)\chi\left(e_2^{\min}>\delta_2\right), & e_2^{\min} \end{pmatrix}$	
optimal effects	$r > L_2/L_1$	$x^* = L$
	$r < R_2/R_1$	$x^* = R$
	$\delta_2/\delta_1 \leq r \leq L_2/L_1$	$x^* = (w_1, w_2)$ $w_1 = \delta_1/(r\delta_1 - \delta_2 + 1)$ $w_2 = rw_1$
	$R_2/R_1 \leq r \leq \delta_2/\delta_1$	$x^* = (w_1, w_2)$ $w_1 = -\delta_2/(r\delta_1 - r - \delta_2)$ $w_2 = rw_1$
optimal quantities	$r > L_2/L_1$	$q_1^* = \frac{e_1^{\min}}{\delta_1}\chi\left(e_1^{\min}\leq\delta_1\right)+\chi\left(e_1^{\min}>\delta_1\right)$ $q_2^* =$ $=\chi\left(e_1^{\min}\leq\delta_1\right)+\frac{e_1^{\min}-1}{\delta_1-1}\chi\left(e_1^{\min}>\delta_1\right)$
	$r < R_2/R_1$	$q_1^* = 1$ $q_2^* = \frac{e_2^{\min}}{\delta_2}\chi\left(e_2^{\min}\leq\delta_2\right)+\frac{e_2^{\min}}{1-\delta_1}\chi\left(e_2^{\min}>\delta_2\right)$
	$\delta_2/\delta_1 \leq r \leq L_2/L_1$	$q_1^* = \dfrac{1}{r\delta_1+1-\delta_2}$ $q_2^* = 1$
	$R_2/R_1 \leq r \leq \delta_2/\delta_1$	$q_1^* = 1$ $q_2^* = -\dfrac{r}{r\delta_1-\delta_2-r}$

4.4. TYPE 4 (STRONGLY ANTAGONISTIC ELEMENTS)

This is the case $\delta_1 + \delta_2 < 1$. This is illustrated in Figure 5.

It may be deduced from Carfì (2009e, pages 42–44) that the set of effects is the pseudo-triangle with vertices (0, 0), (0, 1), and (1, 0), delimited at North-East by the curve now to be defined. Having called $\delta_1' = 1-\delta_1$ and $\delta_2' = 1-\delta_2$, the resulting line is the union of:

- the segment of extremes (0, 1) and $H = (H_1,\ H_2) = (\delta_1^2/\delta_2', \delta_1')$,
- the segment of extremes (1, 0) and $K = (K_1,\ placeK_2) = (\delta_2', \delta_2^2/\delta_1')$,
- the section of the curve between H and K, having equation $x_2 = (1-\sqrt{\delta_2' x_1})^2/\delta_1'$

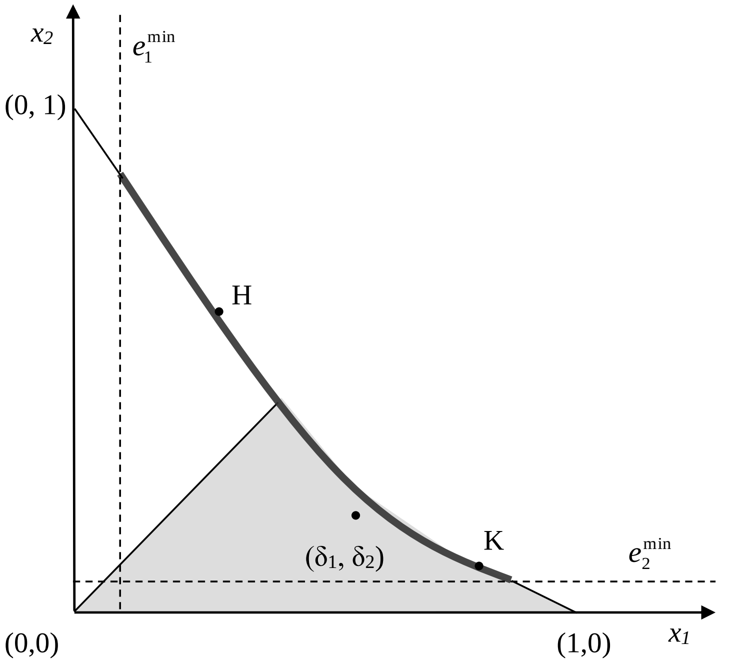

Fig. 5. $n = 2$, *case 4 (strongly antagonist elements)*

Remark. For other examples of similar calculations, we suggest to read the papers by Carfì and Schilirò (2014a, 2014b, 2013, 2012a, 2012b, 2012c, 2012d, 2011a, 2011b, 2011c) and by Carfì (2012a, 2012b, 2010, 2009a, 2009b, 2009c, 2009d, 2009e, 2008); the interested readers could also see Carfì and Musolino (2015a, 2015b, 2014a, 2014b, 2013a, 2013b, 2013c, 2012a, 2012b, 2012c, 2011a, 2011b). Other important applications, of the complete examination methodology, are shown in Agreste, Carfì, and Ricciardello (2012), Arthanari, Carfì and Musolino (2015), Baglieri, Carfì, and Dagnino (2012), Carfì and Fici (2012), Carfì, Gambarelli and Uristani (2013), Carfì and Lanzafame (2013), Carfì, Patanè, and Pellegrino (2011), Carfì and Romeo (2015).

Note that H belongs to the segment connecting (0, 1) and (δ_1, δ_2), and K belongs to the segment connecting (1, 0) and (δ_1, δ_2); then $H_1 \leq \delta_1$ and $H_2 \leq \delta_2$.

In order to simplify the notations, we define:

$$a_1 = \max(0, e_1^{\min}),$$
$$b_1 = \min\Big(H_1,\ \tfrac{\delta_1}{(\delta_2 - 1)}(e_2^{\min} - 1)\Big)$$
$$a_2 = \max(K_1, e_1^{\min}),$$
$$b_2 = \min\Big(1, \tfrac{(\delta_1 - 1)}{\delta_2} e_2^{\min} + 1\Big)$$
$$a_3 = \max(H_1, e_1^{\min}),$$
$$b_3 = \min\Big(K_1, \frac{\big(1 - \sqrt{(1-\delta_1) e_2^{\min}}\big)^2}{1 - \delta_2}\Big)$$

The existence of a solution requires, besides (2), the additional condition

$$e_1^{\min} \leq \max(b_1, b_2, b_3)$$

This condition results in $a_1 \leq b_1$, $a_2 \leq b_2$, and $a_3 \leq b_3$. In this case, the feasible Pareto optimal boundary is not empty. This boundary is the set of points (x_1, x_2) given by $R_1 \bigcup R_2 \bigcup R_3$, where:

$$R_1 = \begin{cases} \left\{ x = (x_1, x_2) \,\middle|\, \begin{array}{l} x_2 = \dfrac{(\delta_2 - 1)}{\delta_1} x_1 + 1 \\ x_1 \in [a_1, b_1] \end{array} \right\} & if\ e_1^{\min} \leq H_1 \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$R_2 = \begin{cases} \left\{ x = (x_1, x_2) \,\middle|\, \begin{array}{l} x_2 = \dfrac{\delta_2}{(\delta_1 - 1)} (x_1 - 1) \\ x_1 \in [a_2, b_2] \end{array} \right\} & if\ e_2^{\min} \leq K_2 \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$R_3 = \begin{cases} \left\{ x = (x_1, x_2) \,\middle|\, \begin{array}{l} x_2 = \dfrac{\left(1 - \sqrt{(1-\delta_2)x_1}\right)^2}{1 - \delta_1} \\ x_1 \in [a_3, b_3] \end{array} \right\} & \begin{array}{l} \text{if } K_2 \leq e_2^{\min} \leq H_2 \\ \text{and } H_1 \leq e_1^{\min} \leq K_1 \end{array} \\ \emptyset & \text{otherwise} \end{cases}$$

In the event of no solution, the existence of one may be brought about by modifying $e_1^{\min}$ and/or $e_2^{\min}$ in a way analogous to the previous cases:

- by fixing $e_2^{\min}$, we can use

$$e_1^{\min} = \max\left(\frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1), \frac{(\delta_1 - 1)}{\delta_2} e_2^{\min} + 1, \frac{(1 - \sqrt{(1-\delta_1)e_2^{\min}})^2}{1 - \delta_2} \right);$$

- by fixing $e_1^{\min}$, we can use

$$e_2^{\min} = \min\left(\frac{\delta_2 - 1}{\delta_1} e_1^{\min} + 1, \frac{\delta_2}{\delta_1 - 1}(e_1^{\min} - 1), \frac{(1 - \sqrt{(1-\delta_2)e_1^{\min}})^2}{1 - \delta_1} \right);$$

Intermediate solutions are also possible, in which both $e_i^{\min}$ are modified. The solution is given in the final row of Table 4.

Table 4. *The optimal solution in type 4*

existence condition	$e_1^{\min} \le \max\left(\begin{array}{l}\min\left(H_1,\ \frac{\delta_1}{(\delta_2-1)}(e_2^{\min}-1)\right),\\ \min\left(1,\ \frac{(\delta_1-1)}{\delta_2}e_2^{\min}+1\right), \min\left(K_1, \frac{\left(1-\sqrt{(1-\delta_1)e_2^{\min}}\right)^2}{1-\delta_2}\right)\end{array}\right)$	
extremes of the feasible P.O. boundary	$L=(L_1,L_2)=\left(\begin{array}{l} e_1^{\min}, \\ \left(\frac{\delta_2-1}{\delta_1}e_1^{\min}+1\right)\chi\left(e_1^{\min}\le H_1\right)\\ +\left(\frac{\delta_2}{\delta_1-1}(e_1^{\min}-1)\right)\chi\left(e_1^{\min}\ge K_1\right)\\ +\left(\frac{(1-\sqrt{(1-\delta_2)e_1^{\min}})^2}{1-\delta_1}\right)\chi\left(K_1<e_1^{\min}<H_1\right)\end{array}\right)$ $R=(R_1,R_2)=\left(\begin{array}{l} \left(\frac{\delta_1-1}{\delta_2}e_2^{\min}+1\right)\chi\left(e_2^{\min}\le K_2\right)\\ +\left(\frac{\delta_1}{\delta_2-1}(e_2^{\min}-1)\right)\chi\left(e_2^{\min}\ge H_2\right)\\ +\frac{\left(1-\sqrt{(1-\delta_1)e_2^{\min}}\right)^2}{1-\delta_2}\chi\left(K_2<e_2^{\min}<H_2\right),\\ e_2^{\min}\end{array}\right)$	
optimal effects	$r\ge L_2/L_1$	$x^*=L$
	$r\le R_2/R_1$	$x^*=R$
	$r\ge H_2/H_1$ $r<L_2/L_1$ $r>R_2/R_1$	$x^*=(w_1,w_2)$ $w_1=\delta_1/(r\delta_1-\delta_2+1)$ $w_2=rw_1$
	$H_2/H_1\le r\le K_2/K_1$ $r<L_2/L_1$ $r>R_2/R_1$	$x^*=(w_1,w_2)$ $w_1=\left(\frac{2((1-\delta_2)+r(1-\delta_1))-2\sqrt{\xi}}{2((1-\delta_2)+r(1-\delta_1))^2}\right)$ $w_2=rw_1$ where $\xi=\sqrt{r(\delta_1-1)(\delta_2-1)}$
	$r\le K_2/K_1$ $r<L_2/L_1$ $r>R_2/R_1$	$x^*=(w_1,\ w_2)$ $w_1=\left(\frac{\delta_2}{\delta_2+r(1-\delta_1)}\right)$ $w_2=rw_1$
optimal quantities	$r\ge L_2/L_1$	$q_1^*=\left(\frac{e_1^{\min}}{\delta_1}\right)\chi\left(e_1^{\min}\le H_1\right)+\chi\left(e_1^{\min}\ge K_1\right)$ $+\left(\frac{(e_1^{\min}(\delta_2-1)+\eta}{\eta(\delta_1-1)}\right)\chi\left(H_1<e_1^{\min}<K_1\right)$

Table 4. *cont.*

optimal quantities	$r \geq L_2/L_1$	$q_2^* = \chi\left(e_1^{\min} \leq H_1\right) + \left(\frac{e_1^{\min}-1}{\delta_1-1}\right)\chi\left(e_1^{\min} \geq K_1\right) + \left(\frac{\eta}{1-\delta_2}\right)\chi\left(H_1 < e_1^{\min} < K_1\right)$ where $\eta = \sqrt{e_1^{\min}(1-\delta_2)}$
	$r \leq R_2/R_1$	$q_1^* = \chi\left(e_2^{\min} \leq K_2\right) + \left(\frac{e_2^{\min}-1}{\delta_2-1}\right)\chi\left(e_2^{\min} \geq H_2\right) + - \left(\frac{\theta + e_2^{\min}(\delta_1-1)}{\theta(\delta_2-1)}\right)\chi\left(K_2 < e_2^{\min} < H_2\right)$ $q_2^* = \left(\frac{e_2^{\min}}{\delta_2}\right)\chi\left(e_2^{\min} \leq K_2\right) + \chi\left(e_2^{\min} \geq H_2\right) + + \left(\frac{\theta}{1-\delta_1}\right)\chi\left(K_2 < e_2^{\min} < H_2\right)$ where $\theta = \sqrt{e_2^{\min}(1-\delta_1)}$
	$r \geq H_2/H_1$ $r < L_2/L_1$ $r > R_2/R_1$	$q_1^* = -\frac{\delta_1-1}{2(\delta_2-1)^2\sqrt{(\delta_1-1)/(\delta_2-1)}}$ $q_2^* = -\frac{\delta_2-1}{2(\delta_1-1)^2\sqrt{(\delta_2-1)/(\delta_1-1)}}$
	$H_2/H_1 \leq r \leq K_2/K_1$ $r < L_2/L_1$ $r > R_2/R_1$	If $\delta_1 = \delta_2$ $q_1^* = -\left(\frac{1}{2}\frac{1}{\sqrt{(\delta_2-1)/(\delta_1-1)}}\frac{\delta_2-1}{(\delta_1-1)^2}\right)$ $q_2^* = -\left(\frac{1}{2}\frac{1}{\sqrt{(\delta_1-1)/(\delta_2-1)}}\frac{\delta_1-1}{(\delta_2-1)^2}\right)$ otherwise $q_1^* = -\left(\frac{\delta_1-1+\xi}{(\delta_1-1)(\delta_1-\delta_2)}\right)$ $q_2^* = -\left(\frac{\delta_2-1+\xi}{(\delta_2-1)(\delta_1-\delta_2)}\right)$ where $\xi = \sqrt{(\delta_1-1)(\delta_2-1)}$
	$r \leq K_2/K_1$ $r < L_2/L_1$ $r > R_2/R_1$	$q_1^* = 1$ $q_2^* = -\frac{r}{r\delta_1-\delta_2-r}$

5. TRUNCATED BILINEAR CASE

These cases involve situations in which the effects (beyond a certain maximum level) fall to zero. The symbol χ will be used in the text to denote the indicator function; i.e.,

$$\chi\,(\text{condition}) = \begin{cases} 1 & \text{if the condition is satisfied} \\ 0 & \text{if the condition is not satisfied} \end{cases}$$

Using the above symbol, we can define the effect-function $e(q)$ of truncated bilinear cases as follows:

$$e_1 = \chi(q_1(1 - q_2) + q_1 q_2 \delta_1 \le 1)[q_1(1 - q_2) + q_1 q_2 \delta_1]$$

$$e_2 = \chi(q_2(1 - q_1) + q_1 q_2 \delta_2 \le 1)[(1 - q_1) q_2 + q_1 q_2 \delta_2]$$

5.1. TYPE 1 TRUNCATED (INDEPENDENT OR SYNERGIC ELEMENTS)

This type corresponds either to ($\delta_1 = \delta_2 = 1$) or ($\delta_1 > 1$, $\delta_2 \ge 1$). This is illustrated in Figure 6.

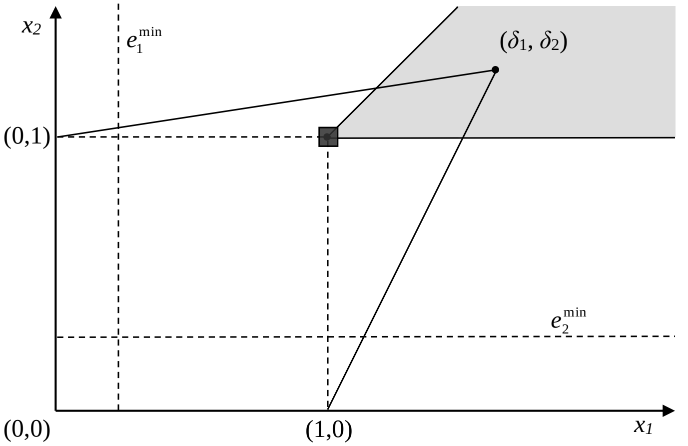

Fig. 6. $n = 2$, *case 1 (independent or synergic elements)*

The set of effects is the quadrangle having vertices (0, 0), (0, 1), (1, 0), and (δ_1, δ_2). The feasible Pareto optimal boundary is made up of the single point (1, 1). Therefore, $x_1 = x_2 = 1$.

The input condition (2) guarantees the existence of the solution, which is given in Table 5.

Table 5. *the optimal solution in type 1T*

	$\delta_1 = \delta_2 = 1$	$\delta_1 > 1$ $\delta_2 = 1$	otherwise
optimal effects	$x^* = (1,1)$	$x^* = (1,1)$	$x^* = (1,1)$
optimal quantities	$q_1 = \dfrac{1}{\delta_1}$ $q_2 = 1$	$q_1 = 1$ $q_2 = 1$	$q_1 = \dfrac{1}{1 + q_2(\delta_2 - 1)}$ $q_2 = \dfrac{\sqrt{\kappa^2 - \kappa + 4(\delta_1 - 1)}}{2(\delta_1 - 1)}$ $\kappa = (1 - (\delta_1 - 1) + (\delta_2 - 1))$

5.2. TYPE 2 TRUNCATED (PARTIALLY SYNERGIC AND PARTIALLY ANTAGONISTIC ELEMENTS)

This is the case $\delta_1 + \delta_2 > 1$, $\delta_1 \geq 1$, $\delta_2 < 1$. This is illustrated in Figure 7.

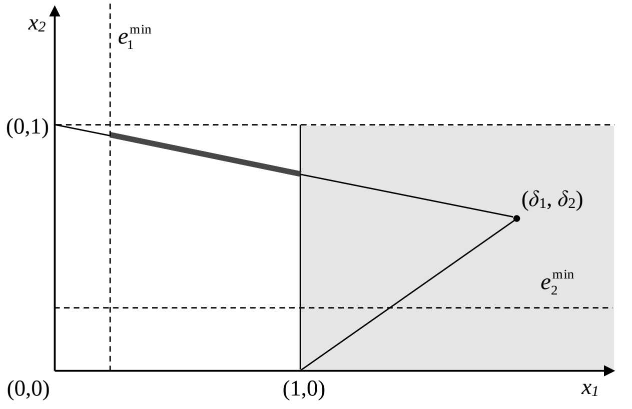

Fig. 7. $n = 2$, *case 2 (partially synergic and partially antagonistic elements)*

The set of effects is the quadrangle having vertices (0, 0), (0, 1), (1, 0), and (δ_1, δ_2). Although it is analogous to Type 2 in the case given in the previous paragraph, the effects cannot exceed the value of 1 in this case.

In order to simplify the notation, we define:

$$a_1 = \max\left(0, e_1^{\min}\right)$$
$$b_1 = \min\left(1, \frac{\delta_1}{\delta_2 - 1}\left(e_2^{\min} - 1\right)\right)$$

Using the above notations, the conditions for the existence of a solution, calculations, and all related considerations are the same as those for Section 4.2. The solution is given in the final row of Table 6.

Table 6. *The optimal solution in type 2T*

existence condition	$e_1^{\min} \leq \min\left(1, \frac{\delta_1}{\delta_2 - 1}\left(e_2^{\min} - 1\right)\right)$	
extremes of the feasible P.O. boundary	$\mathrm{L} = (L_1, L_2) = \left(e_1^{\min}, \frac{\delta_2 - 1}{\delta_1} e_1^{\min} + 1\right)$ $\mathrm{R} = \left(\frac{\delta_1}{\delta_2 - 1}\left(\max\left(\frac{\delta_2 - 1}{\delta_1} + 1, e_2^{\min}\right) - 1\right), \max\left(\frac{\delta_2 - 1}{\delta_1} + 1, e_2^{\min}\right)\right)$	
optimal effects	$L_2/L_1 \leq r \leq R_2/R_1$	$x^* = (w_1, w_2)$ $w_1 = \delta_1/(r\delta_1 - \delta + 1)$ $w_2 = rw_1$

Table 6. *cont.*

	$r > L_2/L_1$		$x^* = L$
	$r < R_2/R_1$		$x^* = R$
optimal solution	$L_2/L_1 \leq r \leq R_2/R_1$		$q_1^* = 1/(r\delta_1 - \delta_2 + 1)$ $q_2^* = 1$
	$r > L_2/L_1$		$q_1^* = e_1^{\min}/\delta_1$ $q_2^* = 1$
	$r < R_2/R_1$	$\delta_1 = 1$	$q_1^* = R_1$ $q_2^* = 1$
		$\delta_1 > 1$	$q_1^* = \dfrac{\frac{\delta_1}{\delta_2 - 1}\max\left(\frac{\delta_2 - 1}{\delta_1} + 1, e_2^{\min}\right) - 1}{1 + q_2(\delta_1 - 1)}$ $q_2^* = \dfrac{(\delta_1 - \vartheta - 1) + \sqrt{(\delta_1 - \vartheta - 1)^2 + 4\vartheta(\delta_1 - 1)}}{2(\delta_1 - 1)}$ $\vartheta = \max\left(\dfrac{\delta_2 - 1}{\delta_1} + 1, e_2^{\min}\right)$

5.3. TYPES 3 AND 4 TRUNCATED

Types 3 and 4 truncated are the same as those of the bilinear free case. We therefore refer the reader to the considerations given in Sections 4.3 and 4.4.

6. AN ALGORITHM

The input data is $\delta, e^{\min}$, and the option free-truncated function.

We begin by acquiring the data and by doublechecking the conditions required in Section 2.

With regard to r, it is quite possible that the user is unable to determine this *a priori*, and it is therefore useful to supply the user with an interval of variability r_int to allow this parameter to be established.

The algorithm proceeds using the tables given in Sections 4 and 5. If a feasible solution is reached, the process stops. Otherwise, the user has to be informed that ${e_1}^{\min}$ and/or ${e_2}^{\min}$ are too binding and should be modified, giving suitable indications for doing this.

A definitive calculation can now be made and the results communicated.

7. SOME APPLICATIONS

In Industrial Economics, finding the optimal quantities of goods to be produced is a well-known problem. Some goods may be complementary or substitutes; hence, their demands may influence each other. If the same firm produces such kinds of goods, it is profitable to optimally decide the production quantities of each product. This decision also depends on the willingness of the decision-maker to potentially sacrifice part of the demand of one product. This willingness to cannibalize a product depends on various factors, examples being the future market situation of the two products and a company's desire to place itself at a strategic advantage in an emerging market (for a detailed analysis of the factors influencing the willingness to cannibalize, see Chandy *et al.*, 1998; Nijssen *et al.*, 2004 and Battaggion *et al.*, 2009).

The model can be used analogously in Public Economics to calibrate two differing economic policies that are interfering with each other.

In Medicine and Veterinarian practice, the balance of interfering drugs is usually performed by successive approximations, keeping the patient monitored.

Finally, further applications can be seen in Zootechnics (to optimize diets), in Agriculture (to calculate dosages of parasiticides or additives so as to increase production), and so on.

8. SOME OPEN PROBLEMS

Figure 8 shows a graph corresponding to Figure 1 for the case $n = 3$. Working with graphic methods (as in this paper) is more difficult in the case of multilinear functions, but not impossible.

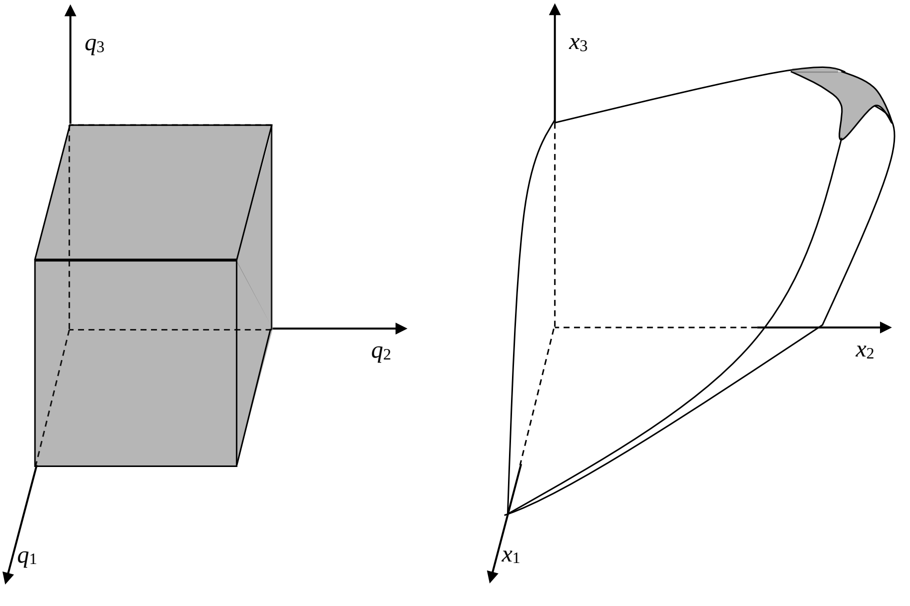

Fig. 8. $n = 3$

Further studies could apply this technique to Cooperative Game Theory, where bilinear functions are often applied (see Fragnelli and Gambarelli, 2013a, 2013b).

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