



Cosmological constant from a deformation of the Wheeler–DeWitt equation

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Abstract

In this paper, we consider the Wheeler–DeWitt equation modified by a deformation of the second quantized canonical commutation relations. Such modified commutation relations are induced by a Generalized Uncertainty Principle. Since the Wheeler–DeWitt equation can be related to a Sturm–Liouville problem where the associated eigenvalue can be interpreted as the cosmological constant, it is possible to explicitly relate such an eigenvalue to the deformation parameter of the corresponding Wheeler–DeWitt equation. The analysis is performed in a Mini-Superspace approach where the scale factor appears as the only degree of freedom. The deformation of the Wheeler–DeWitt equation gives rise to a Cosmological Constant even in absence of matter fields. As a Cosmological Constant cannot exist in absence of the matter fields in the undeformed Mini-Superspace approach, so the existence of a non-vanishing Cosmological Constant is a direct consequence of the deformation by the Generalized Uncertainty Principle. In fact, we are able to demonstrate that a non-vanishing Cosmological Constant exists even in the deformed flat space. We also discuss the consequences of this deformation on the big bang singularity.

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1. Introduction

It is expected that the geometry of space–time cannot be measured below a minimum length scale, which is usually taken to be the Planck scale [1,2]. At this scale it is likely that quantum fluctuations of the space–time itself come into play, breaking therefore its description as a smooth manifold [3,4]. For instance, in string theory, the minimum length scale is the string length itself. This means that, in perturbative string theory, it is not possible to probe the space–time below the string length scale [5–8]. The appearance of a minimum measurable length scale has also been studied in the context of loop quantum gravity [9–12], in noncommutative field theories [13,14] and also in black hole physics [15,16]. Even though the existence of such a minimum length scale is predicted from various different approaches, it is not consistent with the usual Heisenberg uncertainty principle, which states that the position of a particle can be measured with arbitrary precision, if its momentum is not measured. This means that there is no minimum measurable length scale compatible with the usual Heisenberg uncertainty principle. To accommodate this mismatch, we need to introduce a Generalized Uncertainty Principle (GUP) [17,18]. As the uncertainty principle is closely related to the Heisenberg algebra, the generalization of the usual Heisenberg uncertainty principle to GUP deforms the Heisenberg algebra [19,20]. This in turn modifies the coordinate representation of the momentum operator, and this new representation for the momentum operator produces correction terms for all quantum mechanical phenomena [21,22]. A more general deformation of GUP which incorporates the effect of double special relativity [23,24], has also been studied [25,26]. This deformed Heisenberg algebra also has terms proportional to linear powers of momentum. Motivated from GUP, the full four momentum of a field theory has also been modified, and the gauge theory corresponding to this deformation of field theory has been constructed [27–30]. However, it is also possible to deform the second quantized commutator between the fields in a similar way. This has been done for the Wheeler–DeWitt (WDW) equation [31–34]. This deformed WDW equation has also been used for analyzing quantum black holes¹ [37]. The third quantization of this deformed WDW equation has also been studied [38]. In this analysis, the deformation parameter was analyzed perturbatively.

Motivated by the deformation of Heisenberg algebra by linear terms in momentum [25,26], a similar deformation of the second quantized commutator has been studied [39]. It may be noted that in the deformation of the first quantized theories, the GUP parameter can be related to the existence of an intrinsic measurable length scale in space. Such a relation between a physical phenomena and GUP deformation has not been studied for the second quantized theories. A remarkable feature of the WDW equation deformed by GUP is to avoid singularities in space–time [39]. This is principally due to the introduction of a minimum limit to the field resolution. Therefore, it is quite obvious to try to extend this interesting feature to other contexts, for example, the cosmological constant. Indeed, it may be noted that the WDW equation is equivalent to a Sturm–Liouville problem and the related eigenvalue can be interpreted as a cosmological constant [40–44]. In this context the cosmological constant is a measure of the degeneracy of the only energy eigenvalue of the WDW equation without matter fields which obeys the following equation, $\mathcal{H}\Psi = E\Psi$, with $E = 0$. It is true that an exact solution has been found by Vilenkin in ordinary GR with a factor ordering equal to $q = -1$ [45]. However, except this special case, no other exact solutions have been found in this context. It is for this reason that one promising

¹ Other applications of GUP on quantum black holes can be found in Refs. [35,36].

procedure is represented by a variational approach, where the WDW equation can be cast as a vacuum expectation value (VEV). However, in ordinary GR, no cosmological constant without matter fields can be produced with the help of a VEV calculation in a Mini-superspace approach.² It is for this reason, that the same procedure has been extended to theories outside GR. Some examples are Hořava–Lifshitz gravity theory [49],³ Varying Speed of Light (VSL) cosmology [51] and Gravity’s Rainbow. However, we have to say that in all these approaches the kinetic part of the WDW equation does not get any higher order functional derivative correction. Since the deformation of the second quantized commutator produces higher order functional derivative contributions for the representation of the canonically conjugate variable to the field variable [39], it will also produce higher derivative corrections for the kinetic part of the WDW equation. Therefore, our purpose will be the analysis of the cosmological constant problem using the GUP deformed WDW equation [52–55].

The paper is organized as follows: in Section 2 we review the basic elements of the deformed WDW equation, in Section 3 we extract the corresponding WDW equation in the case of FLRW space–time and we setup the corresponding Sturm–Liouville. In Section 4 we analyze the generalized semiclassical case corresponding to the deformed WDW equation. In Section 5 we study the flat space case as an example of the generalized semiclassical case. It may be noted that even though the curvature term is absent, we still get interesting results due to the deformation of the WDW equation. In Section 6 we consider the case when the operator ordering parameters do not vanish. Finally, we summarize our results in Section 7. Throughout this manuscript we use units in which $\hbar = c = k = 1$.

2. Deformed Wheeler–DeWitt equation

In this section, we will study the deformation of the WDW equation. The standard WDW equation for a Mini-superspace approach for a homogeneous, isotropic and closed universe is obtained with the help of the Friedmann–Lemaître–Robertson–Walker (FLRW) metric

$$ds^2 = -N^2 dt^2 + a^2(t) d\Omega_3^2, \tag{1}$$

where

$$d\Omega_3^2 = \gamma_{ij} dx^i dx^j \tag{2}$$

is the line element on the three-sphere, N is the lapse function and $a(t)$ denotes the scale factor. In this background, the Ricci curvature tensor and the scalar curvature read simply

$$R_{ij} = \frac{2}{a^2(t)} \gamma_{ij} \quad \text{and} \quad R = \frac{6}{a^2(t)}, \tag{3}$$

respectively. The Einstein–Hilbert action in (3 + 1)-dim is

$$S = \frac{1}{16\pi G} \int_{\Sigma \times I} \mathcal{L} dt d^3x = \frac{1}{16\pi G} \int_{\Sigma \times I} N \sqrt{g} \left[K^{ij} K_{ij} - K^2 + R - 2\Lambda \right] dt d^3x, \tag{4}$$

with Λ the cosmological constant, K_{ij} the extrinsic curvature and K its trace. Using the line element, Eq. (1), the above written action, Eq. (4), becomes

² Note that the procedure of building a VEV associated to a Sturm–Liouville problem can be generalized to include electric and magnetic charges [46,47] and also naked singularities [48].

³ For a traditional approach to the WDW equation, see also Ref. [50].

$$S = -\frac{3\pi}{4G} \int_I \left[\dot{a}^2 a - a + \frac{\Lambda}{3} a^3 \right] dt, \tag{5}$$

where we have computed the volume associated to the three-sphere, namely $V_3 = 2\pi^2$, and set $N = 1$. The canonical momentum reads

$$\pi_a = \frac{\delta S}{\delta \dot{a}} = -\frac{3\pi}{2G} \dot{a} a, \tag{6}$$

and the resulting Hamiltonian density is

$$\begin{aligned} \mathcal{H} &= \pi_a \dot{a} - \mathcal{L} \\ &= -\frac{G}{3\pi a} \pi_a^2 - \frac{3\pi}{4G} a + \frac{3\pi}{4G} \frac{\Lambda}{3} a^3. \end{aligned} \tag{7}$$

Following the canonical quantization prescription, we promote π_a to a momentum operator, setting

$$\pi_a^2 \rightarrow -a^{-q} \left[\frac{\partial}{\partial a} a^q \frac{\partial}{\partial a} \right], \tag{8}$$

where we have introduced a factor order ambiguity q . The generalization to $k = 0, -1$ is straightforward. The WDW equation for such a metric is

$$\begin{aligned} H\Psi(a) &= \left[-a^{-q} \left(\frac{\partial}{\partial a} a^q \frac{\partial}{\partial a} \right) + \frac{9\pi^2}{4G^2} \left(a^2 - \frac{\Lambda}{3} a^4 \right) \right] \Psi(a), \\ \left[-\frac{\partial^2}{\partial a^2} - \frac{q}{a} \frac{\partial}{\partial a} + \frac{9\pi^2}{4G^2} \left(a^2 - \frac{\Lambda}{3} a^4 \right) \right] \Psi(a) &= 0. \end{aligned} \tag{9}$$

It represents the quantum version of the invariance with respect to time reparametrization. Nevertheless, when we include higher derivative correction to the kinetic part of the WDW equation, we need to modify the second quantized commutator between the field variables and their conjugate momentum [31–34]. As the GUP can be generalized to include linear contributions in the momentum [25,26], a similar modification to the second quantized momentum has been studied [39], and this deformation of the second quantized commutators modified the representation of the momentum conjugate to field variables. Now if the original undeformed momentum conjugate to the scalar factor a is $\tilde{\pi}_a$, where

$$\tilde{\pi}_a = -i \frac{d}{da}, \tag{10}$$

then the deformed momentum can be written as [39]

$$\pi_a = \tilde{\pi}_a (1 - \alpha \|\tilde{\pi}_a\| + 2\alpha^2 \|\tilde{\pi}_a\|^2) \tag{11}$$

and the generalization of the WDW equation for a FLRW metric is deformed to

$$\left[\tilde{\pi}_a^2 - 2\alpha \tilde{\pi}_a^3 + 5\alpha^2 \tilde{\pi}_a^4 + \left(\frac{3\pi}{2l_p^2} \right)^2 a^2 \left(1 - \frac{\Lambda}{3} a^2 \right) \right] \Psi(a) = 0. \tag{12}$$

This deformation of the WDW equation prevents the existence of singularities [39]. This is because this deformation modifies the uncertainty principle as

$$\Delta a \Delta \pi_a = 1 - 2\alpha \langle \pi_a \rangle + 4\alpha^2 \langle \pi_a^2 \rangle. \tag{13}$$

Thus, we obtain a minimum value for the scale factor of the universe, $\Delta a \geq \Delta a_{min}$. This minimum value for the scale factor of the universe can prevent the existence of big bang singularity. As we will relate the deformation of the WDW equation to the existence of the cosmological constant, this means it might be possible to avoid the big bang singularity because of the existence of the cosmological constant in our universe.

The introduction of a factor ordering leads to the following form of the derivative terms

$$\begin{aligned} \tilde{\pi}_a^2 &= -a^{-s} \frac{d}{da} \left(a^s \frac{d}{da} \right), \\ \tilde{\pi}_a^3 &= -ia^{-p} \frac{d}{da} \left(a^{p-s} \frac{d}{da} \left(a^s \frac{d}{da} \right) \right), \\ \tilde{\pi}_a^4 &= a^{-u} \frac{d}{da} \left(a^{u-p} \frac{d}{da} \left(a^{p-s} \frac{d}{da} \left(a^s \frac{d}{da} \right) \right) \right). \end{aligned} \tag{14}$$

So, our final WDW equation would become,

$$\begin{aligned} &\left[\tilde{\pi}_a^2 - 2\alpha\pi_a^3 + 5\alpha^2\pi_a^4 + \left(\frac{3\pi}{2l_p^2} \right)^2 a^2 \left(1 - \frac{\Lambda}{3} a^2 \right) \right] \Psi(a) \\ &= -a^{-q} \frac{d}{da} \left(a^q \frac{d}{da} \Psi(a) \right) \\ &\quad + 2\alpha i a^{-u} \frac{d}{da} \left(a^{u-t} \frac{d}{da} \left(a^t \frac{d}{da} \right) \Psi(a) \right) + 5\alpha^2 a^{-s} \\ &\quad \times \frac{d}{da} \left(a^{s-r} \frac{d}{da} \left(a^{r-p} \frac{d}{da} \left(a^p \frac{d}{da} \Psi(a) \right) \right) \right) \\ &\quad + \left(\frac{3\pi}{2l_p^2} \right)^2 a^2 \left(1 - \frac{\Lambda}{3} a^2 \right) \Psi(a) = 0, \end{aligned} \tag{15}$$

where we have defined $l_p = \sqrt{G}$. Even though this is a very general deformation of the WDW equation, we will study the only the deformation of the WDW equation corresponding to quadratic terms. This is because the modification of first quantized Heisenberg algebra by linear terms is also very complicated [29,30], and the quadratic deformation are better understood [27, 28]. So, we will restrict the WDW equation to the following form

$$\left[\mathcal{H}_1 + \mathcal{H}_2 + \left(\frac{3\pi}{2l_p^2} \right)^2 a^2 \left(1 - \frac{\Lambda}{3} a^2 \right) \right] \Psi(a) = 0, \tag{16}$$

where

$$\mathcal{H}_1 = -a^{-q} \frac{d}{da} \left(a^q \frac{d}{da} \right) \tag{17}$$

and

$$\mathcal{H}_2 = 5\alpha_0^2 l_p^2 a^{-s} \frac{d}{da} \left(a^{s-r} \frac{d}{da} \left(a^{r-p} \frac{d}{da} \left(a^p \frac{d}{da} \right) \right) \right). \tag{18}$$

To further proceed, we have to transform the WDW equation (16) in a Sturm–Liouville problem. We recall to the reader that a Sturm–Liouville differential equation is defined by

$$\frac{d}{dx} \left(p(x) \frac{dy(x)}{dx} \right) + q(x) y(x) + \lambda w(x) y(x) = 0 \tag{19}$$

and the normalization is defined by

$$\int_a^b dx w(x) y^*(x) y(x). \tag{20}$$

In the case of the FLRW model we have the following correspondence

$$\begin{aligned} p(x) &\rightarrow a^q(t), \\ q(x) &\rightarrow \left(\frac{3\pi}{2l_p^2} \right)^2 a^{q+2}(t), \\ w(x) &\rightarrow a^{q+4}(t), \\ y(x) &\rightarrow \Psi(a), \\ \lambda &\rightarrow \frac{\Lambda}{3} \left(\frac{3\pi}{2l_p^2} \right)^2, \end{aligned} \tag{21}$$

and the normalization becomes

$$\int_0^\infty da a^{q+4} \Psi^*(a) \Psi(a). \tag{22}$$

It is a standard procedure, to convert the Sturm–Liouville problem (19) into a variational problem of the form

$$F[y(x)] = \frac{-\int_a^b dx y^*(x) \left[\frac{d}{dx} \left(p(x) \frac{d}{dx} \right) + q(x) \right] y(x)}{\int_a^b dx w(x) y^*(x) y(x)} \tag{23}$$

or equivalently

$$F[y(x)] = \frac{-\left[y^*(x) p(x) \frac{d}{dx} y(x) \right]_a^b + \int_a^b dx p(x) \left(\frac{d}{dx} y(x) \right)^2 - q(x) y(x)}{\int_a^b dx w(x) y^*(x) y(x)}, \tag{24}$$

with appropriate boundary conditions. If $y(x)$ is an eigenfunction of (19), then

$$\lambda = \frac{-\int_a^b dx y^*(x) \left[\frac{d}{dx} \left(p(x) \frac{d}{dx} \right) + q(x) \right] y(x)}{\int_a^b dx w(x) y^*(x) y(x)}, \tag{25}$$

is the eigenvalue, otherwise

$$\lambda_1 = \min_{y(x)} \frac{-\int_a^b dx y^*(x) \left[\frac{d}{dx} \left(p(x) \frac{d}{dx} \right) + q(x) \right] y(x)}{\int_a^b dx w(x) y^*(x) y(x)}. \tag{26}$$

It is immediate to recognize that the correspondence (21) can be applied directly for an ordinary FLRW model without GUP. In the next section the Sturm–Liouville procedure will be generalized to include also the GUP correction.

3. The cosmological constant and the GUP deformation

It may be noted that the value of the cosmological constant depends on the operator ordering chosen. Thus, our hope is to find a combination of the operator ordering in such a way to obtain the observed value of the cosmological constant. Note that in this approach, it also depends on the GUP parameter which, in the first quantized quantum mechanics, is fixed by the requirement of space to have an intrinsic minimum measurable length scale. However, in this paper, we have deformed a second quantized theory. Therefore, we will relate the GUP parameter to some physical phenomena in the second quantized theory, which is here represented by the cosmological constant. For practical purposes, it is better to rescale the cosmological constant $\tilde{\Lambda} = \Lambda l_P^2$ and introduce the dimensionless variable $x = a/l_P$, then Eq. (16) in the Sturm–Liouville form becomes

$$\frac{\int dx x^{q+r+s+p} \Psi^*(x) \left[\mathcal{H}_1 + \mathcal{H}_2 + \frac{9\pi^2}{4} x^2 \right] \Psi(x)}{\int dx x^{q+r+s+p+4} \Psi^*(x) \Psi(x)} = \frac{3\pi^2 \tilde{\Lambda}}{4}, \tag{27}$$

where \mathcal{H}_1 and \mathcal{H}_2 are the dimensionless version of the operators defined in (17) and (18). A crucial point is represented by the choice of the wave function. The ordinary WDW in GR is represented by Eq. (27) without \mathcal{H}_2 . A proposal for the trial wave function could be

$$\Psi(x) = x^{-\frac{q+1}{2}} \exp\left(-\frac{\beta x^4}{2}\right). \tag{28}$$

This form has been tested in Ref. [51] and it has not produced any eigenvalue. The form (28) has been considered by looking at the asymptotic behavior of the original WDW equation without GUP. Always in Ref. [51], because of the VSL distortion, the form of $\Psi(x)$ in (28) has been modified into the form

$$\Psi(a) = a^{-\frac{q+1}{2}} (\beta a)^{-3\alpha} \exp\left(-\frac{\beta a^4}{2}\right), \tag{29}$$

without a rescaling of the scale factor. Note that in (29), it has been introduced a scale factor with a power which is able to take into account the short distance behavior. When we introduce the GUP distortion, the effect of \mathcal{H}_2 introduces a similar behavior but with a power to the scale factor that we are unable to fix. For this reason, we will adopt the following trial wave function of the form

$$\Psi(x) = x^{\frac{\beta-(q+r+p+s+1)}{2}} \exp\left(-\frac{\beta x^4}{2}\right), \tag{30}$$

which is suggested by the asymptotic behavior of the WDW equation for a large scale factor and for the short range behavior we have introduced a power depending on the variational parameter. With the help of the integrals calculated in the appendix, Eq. (27) including GUP becomes

$$\begin{aligned} \tilde{\Lambda}(\beta) &= \frac{4}{3\pi^2} \left(\frac{K_1 + K_2 + P_1}{P_2} \right) = \frac{4}{3\pi^2} \left(\frac{\beta^{\frac{1}{2}} (A + 4\beta) \Gamma\left(\frac{\beta-2}{4}\right)}{4\Gamma\left(\frac{\beta}{4}\right)} \right. \\ &\quad \left. + \frac{\alpha_0^2}{\beta-4} \left(-5B_1\beta - 40B_2\beta^2 + 240\beta^3 \right) + \frac{9\pi^2 \Gamma\left(\frac{2+\beta}{4}\right)}{\Gamma\left(\frac{\beta}{4}\right) \beta^{\frac{1}{2}}} \right). \end{aligned} \tag{31}$$

We demand that

$$\frac{d\tilde{\Lambda}(\beta)}{d\beta} = 0. \tag{32}$$

It is immediate to recognize that a general analytic solution is difficult to find. We are therefore led to consider some specific cases. It may be noted that as the cosmological constant is non-zero in our universe, this means that we do have a GUP deformation of the Wheeler–DeWitt equation. However, such a deformation is known to present the existence of singularities [39], and so the existence of the cosmological constant can prevent the existence of the big bang singularity.

4. The generalized semiclassical case

In ordinary GR, the case in which the factor order parameter $q = 0$ is known as the semiclassical case. If we adopt the same fixing when the GUP is present, Eq. (16) reduces to

$$\left[-\frac{d^2}{l_P^2 dx^2} + 5\alpha_0^2 \frac{d^4}{l_P^2 dx^4} + \left(\frac{3\pi}{2l_P^2}\right)^2 l_P^2 x^2 \left(1 - \frac{\Lambda}{3} l_P^2 x^2\right) \right] \Psi(x) = 0, \tag{33}$$

where we have set $q = r = s = p = 0$. Therefore, the trial wave function (30) reduces to

$$\Psi(x) = x^\beta \exp\left(-\frac{\beta x^4}{2}\right). \tag{34}$$

If we cast Eq. (33) into the Sturm–Liouville form, we obtain

$$\frac{\int_0^{+\infty} dx x^{\beta/2} \exp\left(-\frac{\beta x^4}{2}\right) \left[-\frac{d^2}{dx^2} + 5\alpha_0^2 \frac{d^4}{dx^4} + \left(\frac{3\pi}{2}\right)^2 x^2\right] x^{\beta/2} \exp\left(-\frac{\beta x^4}{2}\right)}{\int_0^{+\infty} dx x^\beta \exp(-\beta x^4)} = \tilde{\Lambda} \frac{3\pi^2}{4} \tag{35}$$

and Eq. (31) reduces to

$$\tilde{\Lambda}_{\alpha_0}(\beta) = \frac{4}{3\pi^2} \left(\alpha_0^2 \frac{20\beta^2(16\beta^2 - 40\beta - 21)}{\pi^2(1 + \beta)(\beta - 3)} + \frac{\sqrt{\beta}(16\beta^2 + 3(3\pi^2 - 4)\beta - 9\pi^2)\Gamma(\beta/4 - 1/4)}{12\pi^2\Gamma(\beta/4 + 5/4)} \right). \tag{36}$$

To fix ideas, we can take three values of α_0 : $\alpha_0 = 1$, $\alpha_0 = 100$ and $\alpha_0 = 1000$. By demanding that

$$\frac{d\tilde{\Lambda}_{\alpha_0}(\beta)}{d\beta} = 0, \tag{37}$$

we find

$$\begin{Bmatrix} \alpha_0 & \beta_m & \tilde{\Lambda}_{\alpha_0}(\beta_m) \\ 1 & 1.053 & 32.69 \\ 10 & 1.017 & 249.69 \\ 20 & 1.012 & 484.86 \end{Bmatrix}. \tag{38}$$

We can see that the larger is α_0 , the higher is the eigenvalue $\tilde{\Lambda}_{\alpha_0}(\beta_m)$.

5. Flat space

An interesting example of the generalized semiclassical case is the flat space case. It may be noted that even when curvature is absent, due to the higher order derivatives, we expect to find non-trivial results in the procedure. Now because of the absence of the curvature term, Eq. (35) reduces to

$$\frac{\int_0^{+\infty} dx x^{\beta/2} \exp\left(-\frac{\beta x^4}{2}\right) \left[-\frac{d^2}{dx^2} + 5\alpha_0^2 \frac{d^4}{dx^4}\right] x^{\beta/2} \exp\left(-\frac{\beta x^4}{2}\right)}{\int_0^{+\infty} dx x^{\beta} \exp(-\beta x^4)} = \tilde{\Lambda} \frac{3\pi^2}{4} \tag{39}$$

and Eq. (36) simplifies to

$$\tilde{\Lambda}_{\alpha_0}(\beta) = \frac{4}{3\pi^2} \left(\alpha_0^2 \frac{20\beta^2(16\beta^2 - 40\beta - 21)}{\pi^2(1 + \beta)(\beta - 3)} + \frac{\beta^{\frac{3}{2}}(4\beta - 3)\Gamma(\beta/4 - 1/4)}{3\pi^2\Gamma(\beta/4 + 5/4)} \right). \tag{40}$$

We fix the same values of Eq. (35) for α_0 and by demanding that

$$\frac{d\tilde{\Lambda}_{\alpha_0}(\beta)}{d\beta} = 0, \tag{41}$$

we find

$$\left\{ \begin{array}{|c|c|c|} \hline \alpha_0 & \beta_m & \tilde{\Lambda}_{\alpha_0}(\beta_m) \\ \hline 1 & 1.053 & 29.24 \\ 10 & 1.017 & 246.29 \\ 20 & 1.012 & 481.46 \\ \hline \end{array} \right. . \tag{42}$$

We conclude that in a GUP distortion, the presence of the curvature term is not very relevant since the pattern of the eigenvalues is very close to the flat case.

6. Non-vanishing parameters

Now we will analyze the case where $A \neq 0, B_1 \neq 0$ and $B_2 \neq 0$. For this case, we can see what happens for arbitrary choices of the parameters. We can fix our attention on the following simple setting

$$q = 1, p = 1, r = s = 0 \implies A = -9, B_1 = 271, B_2 = 32. \tag{43}$$

Then Eq. (31) becomes

$$\tilde{\Lambda}(\beta) = \frac{4}{3\pi^2} \left(\frac{\beta^{\frac{1}{2}}(4\beta - 9)\Gamma\left(\frac{\beta-2}{4}\right)}{4\Gamma\left(\frac{\beta}{4}\right)} + \frac{\alpha_0^2}{\beta - 4} \left(-1355\beta - 1280\beta^2 + 240\beta^3 \right) + \frac{9\pi^2\Gamma\left(\frac{2+\beta}{4}\right)}{\Gamma\left(\frac{\beta}{4}\right)\beta^{\frac{1}{2}}} \right) \tag{44}$$

and Eq. (32) gives the following results

$$\left\{ \begin{array}{|c|c|c|} \hline \alpha_0 & \beta_m & \tilde{\Lambda}_{\alpha_0}(\beta_m) \\ \hline 1 & 4.271 & 988.568 \\ 10 & 4.384 & 5952.402 \\ 20 & 4.395 & 11443.529 \\ \hline \end{array} \right. . \tag{45}$$

As we can see, the pattern relating the value of α_0 with the value of $\tilde{\Lambda}_{\alpha_0}(\beta_m)$ is valid also in this case. As concern the flat case $\tilde{\Lambda}_{\alpha_0}(\beta)$ reduces to

$$\tilde{\Lambda}(\beta) = \frac{4}{3\pi^2} \left(\frac{\beta^{\frac{1}{2}}(4\beta - 9)\Gamma\left(\frac{\beta-2}{4}\right)}{4\Gamma\left(\frac{\beta}{4}\right)} + \frac{\alpha_0^2}{\beta - 4} (-1355\beta - 1280\beta^2 + 240\beta^3) \right) \quad (46)$$

and the minimization procedure gives the following results

α_0	β_m	$\tilde{\Lambda}_{\alpha_0}(\beta_m)$
1	4.408	552.721
10	4.408	5492.789
20	4.409	10981.754

(47)

7. Conclusions

In this paper, we have studied the cosmological constant problem using the deformed WDW equation. Even though we have derived a general expression for the deformed WDW equation, we have taken into account only the second and fourth powers of the momentum variable. This deformed WDW equation was obtained by deforming the second quantized canonical commutation relations between the field variable and its conjugate momentum. As a consequence, we have obtained higher order derivative correction terms for the kinetic part of the WDW equation. It has been demonstrated that these correction terms could be used to explain the existence of a cosmological constant and we have observed that the physics of this system depends on the choice of the factor ordering of the operator. This is usual also in ordinary GR. In this paper, we have also analyzed the dependence of the cosmological constant on the operator ordering parameters for various cases. We have analyzed the dependence of the cosmological constant of the deformation parameter for the generalized semiclassical case. Note that for generalized semiclassical case we mean the case in which all the parameters of the factor ordering vanish. As a further specific case, we have also analyzed flat space case as an example of this generalized semiclassical case. Finally, we also analyzed another case, where the factor ordering parameters do not vanish. Unfortunately, we have found a cosmological constant which is at Planckian scale and not at the present scale. This could look like a failure of the procedure. Actually this is not the case. Indeed, the procedure reveals a non-vanishing cosmological constant that should not be there and the most striking fact is that a cosmological constant is predicted also for flat space, namely the pure GUP distortion is able to create a VEV. However, how to drive the generated cosmological constant close to the observed value is a question that can be addressed including particular potentials like the VSL theory [51] or the Hořava–Lifshitz theory [49]. However, this goes beyond the scope of the present paper. It is also interesting to note that the deformation of the WDW equation also produced a minimum value for the scale factor of the universe. Thus, the big bang singularity can be avoided. As the deformation of the WDW equation was related to the existence of the cosmological, it was argued that the existence of the cosmological constant might prevent the existence of the big bang singularity. It may be noted that the effect of the operator ordering on the physics of the system has been studied [56,57]. In fact, it has been demonstrated that the tunneling wave function can only be consistently defined for particular choices of operator ordering, and the no-boundary wave function can be defined independently of operator ordering [58]. It would be interesting to repeat this analysis for both tunneling wave function and no-boundary wave function, by using this deformed WDW equation.

Appendix A. Integrals for the wave function

If the trial wave function assumes the form

$$\Psi(x) = x^{\frac{\beta-(q+r+s+1)}{2}} \exp\left(-\frac{\beta x^4}{2}\right), \tag{A.1}$$

then the first term of the kinetic term becomes

$$\begin{aligned} K_1 &= \int_0^\infty dx x^{q+r+s+p} \Psi^*(x) \mathcal{H}_1 \Psi(x) \\ &= - \int_0^\infty dx x^{q+r+s+p} \Psi^*(x) \left[x^{-q} \frac{d}{dx} \left(x^q \frac{d}{dx} \right) \right] \Psi(x) \\ &= \frac{\beta^{\frac{2-\beta}{4}}}{16} (A + 4\beta) \Gamma\left(\frac{\beta - 2}{4}\right), \end{aligned} \tag{A.2}$$

where $\Gamma(x)$ is the gamma function and where we have defined

$$A = (q - 1)^2 - 8 - (p + r + s)^2. \tag{A.3}$$

The second term containing higher order derivatives is

$$\begin{aligned} K_2 &= \int_0^\infty dx x^{q+r+s+p} \Psi^*(x) \mathcal{H}_2 \Psi(x) \\ &= 5\alpha_0^2 \int_0^\infty dx x^{q+r+s+p} \Psi^*(x) x^{-s} \\ &\quad \times \frac{d}{dx} \left(x^{s-r} \frac{d}{dx} \left(x^{r-q} \frac{d}{dx} \left(x^q \frac{d}{dx} \right) \right) \right) \Psi(x) \\ &= 5\alpha_0^2 \left(-\frac{B_1}{64} \beta^{\frac{4-\beta}{4}} - \frac{B_2}{8} \beta^{\frac{8-\beta}{4}} + \frac{3}{4} \beta^{\frac{12-\beta}{4}} \right) \Gamma\left(\frac{\beta - 4}{4}\right), \end{aligned} \tag{A.4}$$

where we have defined

$$\begin{aligned} B_1 &= p^4 + 2p^3q - 2p^2qr - 2p^2qs - 2p^2r^2 - 2p^2s^2 \\ &\quad - 2pq^3 - 4pq^2r - 4pq^2s - 2pqr^2 - 4pqrs - 2pqs^2 \\ &\quad - q^4 - 2q^3r - 2q^3s - 4q^2rs + 2qr^3 - 2qr^2s - 2qrs^2 \\ &\quad + 2qs^3 + r^4 - 2r^2s^2 + s^4 + 2p^3 + 4p^2q + 2p^2r + 6p^2s \\ &\quad + 2pq^2 - 2pr^2 - 12prs - 2ps^2 - 2q^2r - 6q^2s - 4qr^2 - 12qs^2 \\ &\quad - 2r^3 + 6r^2s + 2rs^2 - 6s^3 - 8p^2 - 94pq - 44pr - 52ps - 86q^2 \\ &\quad - 94qr - 78qs - 8r^2 - 76rs + 8s^2 + 14p - 14r - 90s - 105 \end{aligned} \tag{A.5}$$

and

$$B_2 = 3q(q + p + r + s) + 2pr + 2ps + 2rs - p + r + 3s + 27. \tag{A.6}$$

The contribution coming from the potential term is composed by

$$P_1 = \int_0^{\infty} dx x^{1+\beta} \exp(-\beta x^4) = \frac{\Gamma\left(\frac{2+\beta}{4}\right)}{4\beta^{\frac{2+\beta}{4}}} \quad (\text{A.7})$$

and

$$P_2 = \int_0^{\infty} dx x^{3+\beta} \exp(-\beta x^4) = \frac{\Gamma\left(\frac{\beta}{4}\right)}{16\beta^{\frac{\beta}{4}}}. \quad (\text{A.8})$$

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