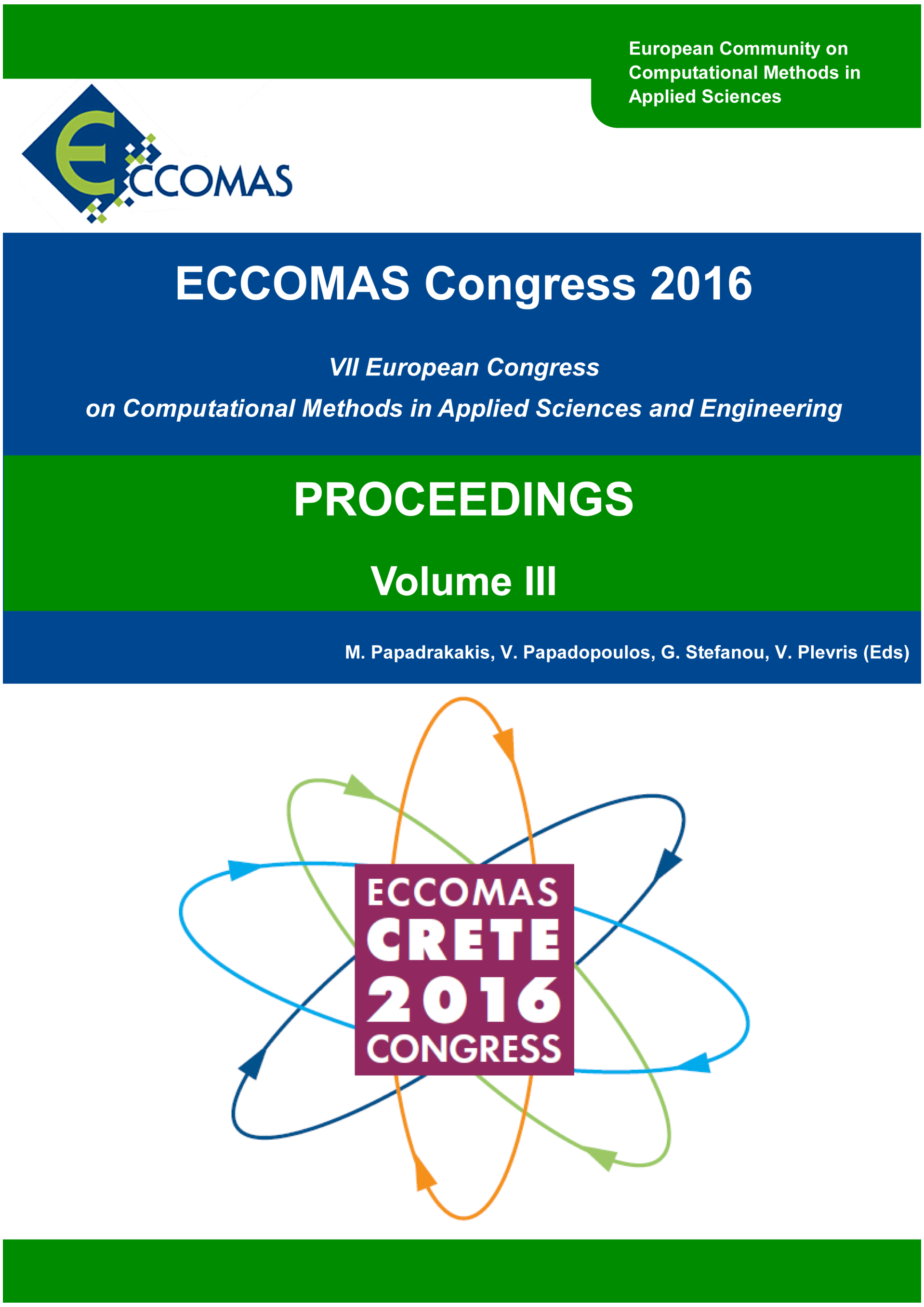

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NUMERICAL DYNAMIC ANALYSIS OF BEAMS ON NONLINEAR ELASTIC FOUNDATIONS UNDER HARMONIC MOVING LOAD

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Keywords: Dynamic response of beams and railways; Nonlinear Winkler elastic foundation; Harmonic moving load; Numerical Finite Element Method implementation; Critical velocities.

Abstract. *As a main background practical context of the present numerical investigation, the appropriate description of track vibrations induced by high-speed trains looks crucial in contemporary railway engineering. The present paper is concerned with the modelization of the transient dynamic response of a simply-supported Euler-Bernoulli beam resting on a homogeneous in space Winkler elastic foundation, under the action of a transverse concentrated load with harmonic-varying magnitude, moving at constant velocity along the beam. Two types of constitutive laws are considered for the foundation subgrade reaction: (a) a linear law and (b) a nonlinear, cubic law. The governing linear/non-linear partial differential equation of motion is first semi-discretized in space with a Finite Element Method approach, by using cubic Hermitian polynomials as interpolation functions for the unknown deflection. Then, the dynamic solution is obtained numerically by a direct integration method, with focus on determining several characteristic response features, such as the critical velocities of the moving load, leading to high transverse deflections. Extensive numerical analyses are finally performed, with the following two main goals: (1) to demonstrate the reliability, consistency and accuracy of the present implementation, specifically by the comparison of the obtained numerical critical velocities with previously-published analytical and numerical results; (2) to investigate how the frequency of the harmonic moving load as well as its velocity do influence the response of the whole beam-foundation system, with or without taking viscous damping into account. Results show that such goals have been consistently achieved and outline new interesting trends, like the appearance of two critical velocities also for the nonlinear foundation, the first of which gets close to zero as the frequency of the load approaches the first natural frequency of the beam. The present outcomes reveal potential implications in practical terms, especially in lowering the ranges of admissible train speeds, as for structural requirement or for preventing passenger discomfort.*

1 INTRODUCTION

The observed ground, beam and track vibrations, induced by rapidly-growing high-speed railway networks and the considerable evolution of train vehicles capable to travel at more than 500 km/h may degrade rolling equipment and track and raise questions about vehicle stability, maintenance costs and possible passenger discomfort. For these reasons many studies have been carried out in the last few decades towards the prediction of the dynamic behavior of railway beams and tracks. Amongst the prevailing models, the most diffused one considers the train vehicle as a single load traveling at constant velocity on a beam supported by an elastic foundation (see Kerr [13]). Comprehensive literature reviews about the problem of moving loads acting on beams may be found in Beskou and Theodorakopoulos [2], Frýba [11], Kerr [13] and Wang et al. [20].

The main scope of the elastic foundation model is that of providing a simplified description of the contact between the rail and the sleepers-ballast-ground system. Based on the well-known Winkler model, the foundation is represented as a uniform layer of infinitely closely-spaced springs, which, according to its original formulation, push up/pull down the beam with a force linearly-proportional to the beam deflection (see e.g. Froio and Rizzi [9] and wide state of the art review, with historical perspective, presented therein).

Concerning the analysis of beams lying on a damped or undamped Winkler elastic foundation, excited by a moving load, different approaches have been adopted so far. One of the most common modelizations is to consider the steady-state response of infinite beams subjected to a constant magnitude moving load (see e.g. Frýba [11]), as well as to a moving load with harmonically-varying amplitude, as reported by Bogacz et al. [3]. Similar results were also obtained by Chen et al. [6], by using the dynamic stiffness matrix method in the description of the influence of the structural parameters and of the load frequency on the critical velocities of the beam. A further extension of this work was then reported by Chen and Huang [7], for the case of a beam of finite length.

Besides analytical solutions and other numerical methods, in the past two decades the Finite Element Method (FEM) has been widely used for solving structural dynamic problems involving moving loads. Combined with the Finite Element Method, the numerical direct integration in time appears one of the most common approaches for the solution of the equations of motion in the time domain. For instance, FEM and Newmark method were applied successfully by Thambiratnam and Zhuge [19] for a constant amplitude moving load and by Kien and Hai [14] for a harmonic moving load. Following a FEM approach, Andersen et al. [1] treated the steady-state response of an infinite Euler-Bernoulli beam lying on a viscoelastic foundation, subjected to a harmonic point load moving with uniform velocity.

Regarding beams of finite length, simply-supported beams on nonlinear viscoelastic foundations were tackled by Castro et al. [4, 5]. Critical velocities were determined and the effects of load intensity and foundation stiffness on both beam displacements and critical velocity were investigated. The results they presented were in agreement to those shown earlier by Dimitrovová and Rodrigues [8] for a linear elastic foundation.

The present paper is concerned with the transient dynamic response of a simply-supported Euler-Bernoulli beam resting on a homogeneous in space Winkler elastic foundation under the action of a transverse concentrated load with harmonic-varying magnitude, moving at constant velocity along the beam. Two types of constitutive laws are considered for the foundation subgrade reaction: (a) a linear law and (b) a nonlinear, cubic law. The dynamic response is obtained numerically by using a FEM implementation and a HHT-α algorithm for the time integration. Then, critical velocities are determined for both types of foundation, by considering different values of the frequency of the moving load.

The purpose of the present paper is two-fold. Firstly, this work aims at demonstrating the reliability, consistency and accuracy of the present implementation, by the comparison of the obtained numerical critical velocities to results reported by earlier Castro et al. [4] and by Chen et al. [6]. Secondly, extensive numerical analyses are performed to study the effects of the frequency of the moving load, of its velocity and of the foundation behavior, on the response of the whole beam-foundation system, with or without viscous damping, outlining here new results for the nonlinear foundation case.

The paper is organized as follows. Section 2 presents the governing boundary value problem for the linear/nonlinear partial differential equation of motion of a simply-supported beam on elastic foundation and its semi-discretization in space through a FEM formulation and implementation. In Section 3 the outcomes of a series of independent numerical analyses relative to the case of a constant amplitude moving load are presented and consistently compared to results reported in the literature. Section 4 provides the outcomes of a vast numerical investigation on the dynamic response of beams lying on visco-elastic foundations, subjected to a force which moves along the beam axis at constant velocity and simultaneously varies its amplitude in time on the basis of a harmonic law. In particular, the relationship between the amplitude frequency of the moving load and the critical velocity of the beam-foundation system is explicitly depicted in appropriate bifurcation curves, for both linear and nonlinear elastic foundations. Finally, main conclusions are outlined in closing Section 5.

2 MODEL EQUATION AND FINITE ELEMENT FORMULATION

Consider the idealized system shown in Fig. 1, consisting of a simply-supported finite beam lying on a Winkler elastic foundation under the action of a concentrated force of magnitude F (either constant or variable in time), moving with constant velocity v. The force is assumed positive if directed upward. The following assumptions are adopted throughout the formulation:

1. Euler-Bernoulli beam model with Young's modulus (E), cross section area (A), moment of inertia (I) and mass density (ρ), assumed constant along the beam axis;
2. Viscous damping smeared onto both the beam and the foundation, with constant damping coefficient (c);
3. Beam at rest with zero initial deflection and velocity at the instant when the force leaves from the left support of the beam.

From these assumptions, the equation of motion describing the transverse deflection of the beam is (see Frýba [11]):

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\rho A\frac{\partial^2 w(x,t)}{\partial t^2}+c\frac{\partial w(x,t)}{\partial t}+r\big(w(x,t)\big)=F(t)\delta(x-vt);\;\; 0<x<l,\; t>0; \quad (1)$$

where x is the axial coordinate, with the origin fixed on the left end of the beam, t is the time coordinate, with the origin at the instant on which the force starts its motion from the left extreme of the beam ($x=0$), $w(x,t)$ is the vertical deflection of the beam (positive if upward), measured from the static equilibrium position of the beam subjected to its self-weight only, acting downward, and $r(w)$ is the reaction force per unit length supplied by the foundation. The right hand side of Eq. (1) represents, by means of the Dirac delta function δ, the unit

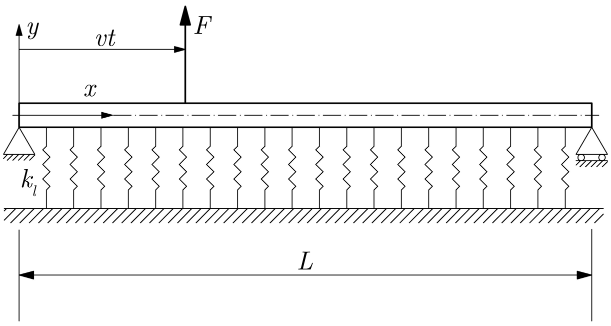


Figure 1: *Simply-supported finite beam lying on an elastic foundation subjected to a moving force.*

concentrated moving load acting at time t at relative position vt, with either constant $F(t) = F$ or variable amplitude $F(t) = F\cos(\Omega t)$, where F is its reference amplitude and Ω is the angular frequency of the harmonic amplitude variation in time.

According to Castro et al. [4] and many other authors the adopted force-displacement relation for the foundation may be described by the following polynomial cubic law:

$$r\big(w(x,t)\big) = k_l\, w(x) + k_{nl}\, w(x)^3; \tag{2}$$

where k_l is a classical linear Winkler coefficient and k_{nl} describes an additional nonlinear stiffness coefficient attached to the cubic term.

By means of cubic Hermitian polynomials as interpolation functions and by the application of Galerkin Finite Element Method, equation of motion (1) may be rewritten for an arbitrary finite element in semi-discretized form. Then, by assembling the contributions from all of the finite elements and imposing the boundary conditions of zero transverse displacements at the two extreme nodes of the beam, the global equations of motion are obtained as:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{Q}_{nl}(\mathbf{q}) = F\boldsymbol{\Psi}(x_c); \tag{3}$$

where $\mathbf{M}$ and $\mathbf{K}$ are the global structural mass and stiffness matrices, $\mathbf{Q}_{nl}$ is the global vector of the nonlinear forces, $\boldsymbol{\Psi}(x_c)$ is the vector deriving from the assembly of the element-wise vectors of equivalent external forces produced by the Dirac delta function, $\mathbf{q}$, $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ are the global vectors of the generalized displacements, velocities and accelerations, respectively. Both mass and stiffness matrices are symmetric and positive definite. The previously-defined matrices and vectors have been obtained according to the work of Castro et al. [4, 5].

The viscous damping term $\mathbf{C}\dot{\mathbf{q}}$ has been added to Eq. (3) to account for smeared damping. Rayleigh-type damping is assumed, e.g. the damping matrix $\mathbf{C}$ is defined as a linear combination of the mass and stiffness matrices:

$$\mathbf{C} = a_0\mathbf{M} + a_1\mathbf{K}. \tag{4}$$

The values of the coefficients a_0 and a_1 have been chosen accordingly to Dimitrovová and Rodrigues [8] as follows:

$$a_0 = 2\xi\sqrt{\frac{2k_l}{\rho A}}; \qquad a_1 = 0;$$

where ξ is the damping factor. Coefficient a_1 is taken null, meaning that a mass-proportional viscous damping is considered.

Hence, the semi-discrete equations of motion (3) represent a coupled system of linear second-order differential equations with constant coefficients, whose solution, given the initial conditions, represents the transient response of the beam. In this work, the following homogeneous initial conditions are assumed:

$$\mathbf{q}(0) = \mathbf{0}; \quad \dot{\mathbf{q}}(0) = \mathbf{0}. \tag{5}$$

The numerical solution of the initial-value problem satisfying Eqs. (3), (5) has been achieved through a HHT-α implementation (see Hilber et al. [12]). The main methodologies and computational details about the implementation will be reported elsewhere (Froio et al. [10]).

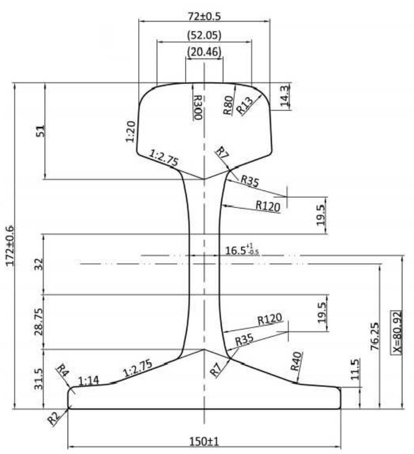
3 CONSTANT AMPLITUDE MOVING LOAD: VALIDATION OF THE FEM FORMULATION

In this section, a concentrated load of constant magnitude $F(t) = F$, namely $\Omega = 0$, moving along the beam axis with constant velocity v, is considered. A consistent validation comparison between the present numerical analyses, in terms of maximum upward and downward beam displacements versus load velocity, and analogous studies proposed in the literature is performed. In particular, results obtained by Castro et al. [4] for both linear and nonlinear foundation behaviors have been taken as reference outcomes, since their work has been lying at the basis of the present investigation.

The type of analyzed beam is a UIC60 rail (see Fig. 2a), one of the most diffused steel profiles in railway tracks. Its mechanical properties are reported in Fig. 2b. A beam length (L) of 200 m has been selected in order to reasonably represent the limit case of a beam of an infinite length. The assumed load magnitude is 83.4 kN, corresponding to a locomotive of the Thalys high-speed train (EU), which has a total axle mass of about 17000 kg (see Castro et al. [4]). The number of adopted finite elements is 200, i.e the spatial discretization consists of finite elements with a length of 1 m.

Computations are performed for velocities of the moving load varying between 50 m/s and 300 m/s with a step variation of 1 m/s. For each simulation performed at a certain value of the moving load velocity, the maximum upward (positive) and downward (negative) displace-

(a)



(b)

Mechanical properties:			
Young's modulus	E	210	GPa
Cross-sectional area	A	7684×10^{-6}	m^2
Area moment of inertia	I	3055×10^{-8}	m^4
Mass per unit length	μ	60	kg/m

Figure 2: *UIC60 rail profile (taken from the internet), quotes in millimeters (a) and mechanical properties of UIC60 rail (b).*

ments of the beam have been recorded. Then, these values have been plotted as a function of the moving load velocity. From such curves the critical velocity for a finite beam may be detected as the velocity of the moving load at which maximum displacements are attained (see Dimitrovová and Rodrigues [8]).

Regarding aspects of numerical integration, the time span taken throughout the integration process corresponds to the amount of time along which the moving load is really acting along the beam, that is $\tau = L/v$. The adopted time step corresponds to the time taken by the load to travel a distance of 0.2 m, namely a fifth of the finite element length. The HHT-α parameter expressing the numerical dissipation rate is chosen equal to -0.1. The finite element program has been implemented within a MatLab environment [16].

3.1 Beams lying on a linear elastic foundation

A uniform linear elastic foundation is considered, with two different values of Winkler's elastic coefficient equal to $k_l = 250$ kN/m^2 and $k_l = 500$ kN/m^2, respectively. Both undamped and damped behaviors are taken into account, assuming the damping factor ζ in the amount of 2%. As already noticed by Dimitrovová and Rodrigues [8], both foundation stiffness values are actually not that realistic, but they may be adopted in order to achieve a more direct inspection of the critical behavior of the system. The results obtained for the case of a beam on an undamped foundation with the two linear Winkler stiffness coefficients are shown in Fig. 3a. The same results for the damped case are reported in Fig. 3b.

From the observation of these plots, the critical velocity may be clearly detected and the corresponding maximum and minimum displacements are indicated in Tables 1 and 2. It appears that the value of the critical velocity is weakly sensitive to the damping factor, while it affects more the magnitude of the maximum and minimum displacements. From Figs. 3a and 3b, it appears clear that increasing the stiffness of the foundation causes a shift in the position of the critical velocities towards higher values, in addition to the expected effect of decreasing the deflection amplitudes.

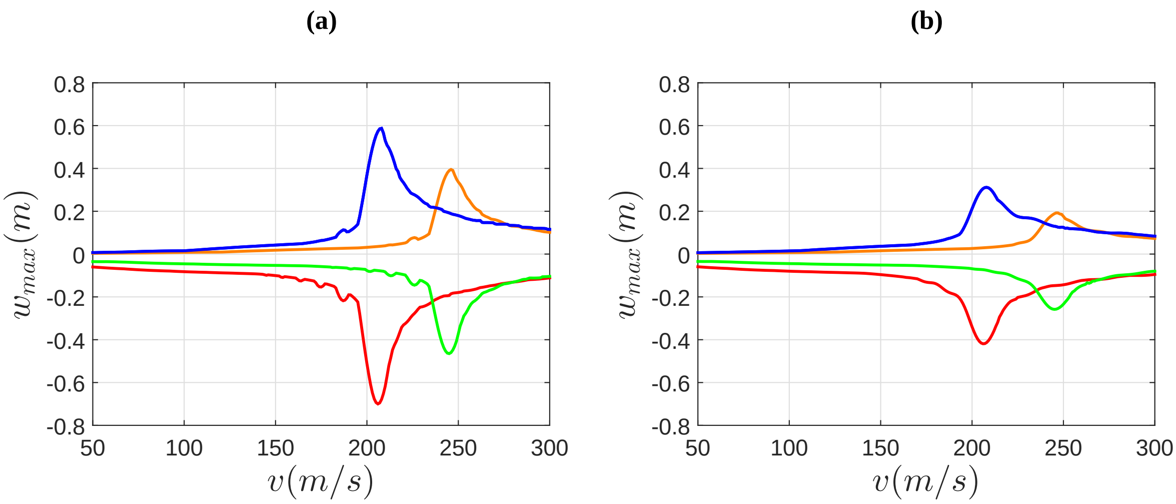


Figure 3: *Representation of beam maximum displacements as a function of load velocity for linear elastic foundations with stiffnesses of* $k_l = 250\,$kN/m^2 *(— upward and — downward displacements) and* $k_l = 500\,$kN/m^2 *(— upward and — downward displacements), undamped case* $\zeta = 0\%$ *(a) and damped case* $\zeta = 2\%$ *(b).*

	v_{cr} (m/s)			w_{max} (m)		
ζ	Present work	Ref. [4]	Error	Present work	Ref. [4]	Error
0%	206	206	$0.00\,\%$	-0.6999	-0.700	$-0.01\,\%$
	208	208	$0.00\,\%$	0.5873	0.587	$0.05\,\%$
2%	206	206	$0.00\,\%$	-0.4189	-0.419	$-0.02\,\%$
	208	208	$0.00\,\%$	0.3117	0.312	$-0.01\,\%$

Table 1: *Maximum displacements and critical velocities for a linear foundation with stiffness* $k_l = 250\,\text{kN/m}^2$. *Percentage relative error with respect to Castro et al. [4].*

	v_{cr} (m/s)			w_{max} (m)		
ζ	Present work	Ref. [4]	Error	Present work	Ref. [4]	Error
0%	245	245	$0.00\,\%$	-0.4649	-0.465	$-0.02\,\%$
	246	246	$0.00\,\%$	0.3950	0.395	$0.00\,\%$
2%	245	245	$0.00\,\%$	-0.2582	-0.258	$0.08\,\%$
	246	246	$0.00\,\%$	0.1922	0.192	$0.10\,\%$

Table 2: *Maximum displacements and critical velocities for a linear foundation with stiffness* $k_l = 500\,\text{kN/m}^2$. *Percentage relative error with respect to Castro et al. [4].*

The comparisons between the outcomes depicted in Figs. 3a and 3b and gathered in Tables 1 and 2, if compared to those reported in Castro et al. [4], reveal a very good agreement, for both damped and undamped cases. Furthermore, in view of the results above, it may be noticed that a good degree of consistency has been also achieved with respect to the analytical solution proposed by Dimitrovová and Rodrigues [8].

3.2 Beams lying on a nonlinear elastic foundation

Two examples of nonlinear foundations are examined in this section: one with a nonlinear component of foundation stiffness $k_{nl} = 2500\,\text{kN/m}^4$ and another with $k_{nl} = 25000\,\text{kN/m}^4$. The assumed linear component of foundation stiffness is $k_l = 250\,\text{kN/m}^2$ for both examples. The computed results are shown in Fig. 4a and 4b for both undamped and damped ($\zeta = 2\%$) cases.

Comparing these plots to those depicted for a linear foundation, it can be seen that the addition of the nonlinear contribution to the foundation stiffness results in an increment of the critical velocities. Moreover, a decrease of the maximum upward and downward displacements is detected. The inclusion of damping brings a further decrease of the maximum displacements entity, together with a small decrease of the critical velocities.

For the nonlinear type of foundation the values of the critical velocities are not explicitly indicated by Castro et al. [4]. Nonetheless, the results obtained with the implemented method are matching the deflection curves and the maximum values of displacements reported in that work. The values of critical velocities, maximum displacements and percentage relative errors upon the latter are reported in Table 3, for a foundation with nonlinear stiffness $k_{nl} = 2500\,\text{kN/m}^4$, and in Table 4, for a foundation with nonlinear stiffness $k_{nl} = 25000\,\text{kN/m}^4$.

The results reported in this section are preliminary to the analysis with a harmonic moving load, which represents the core of this work and will be presented in Section 4. In fact, first the concept of critical velocity of a constant amplitude moving load for a finite simply-supported beam has been again outlined and its dependence on the mechanical parameters of the systems, already reported in the literature, has been confirmed. Second, thanks to the previously-reported results, the present finite element implementation reveals to be fully reliable with respect to the outcomes presented by Castro et al. [4] and by Dimitrovová and Rodrigues [8], proving its correctness.

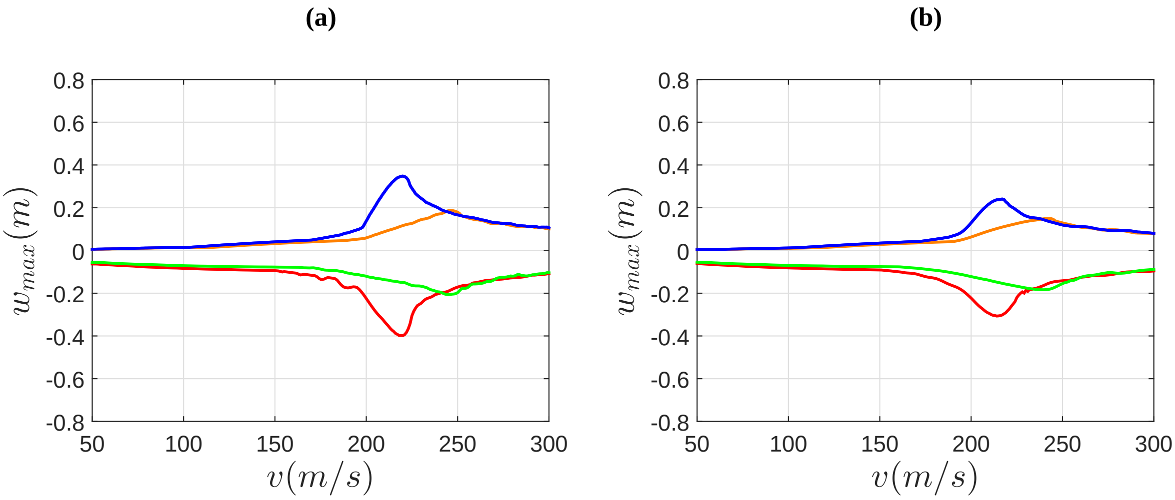


Figure 4: *Representation of beam maximum displacements as a function of load velocity for nonlinear elastic foundations with linear stiffness coefficient* $k_l = 250\,\mathrm{kN/m^2}$ *and nonlinear stiffness coefficients* $k_{nl} = 2500\,\mathrm{kN/m^4}$ *(— upward and — downward displacements) and* $k_{nl} = 25000\,\mathrm{kN/m^4}$ *(— upward and — downward displacements), undamped case* $\zeta = 0\%$ *(a) and damped case* $\zeta = 2\%$ *(b).*

ζ	v_{cr} (m/s)	w_{max} (m) Present work	Ref. [4]	Error
0%	220	-0.3999	-0.400	$-0.02\,\%$
	220	0.3497	0.349	$0.20\,\%$
2%	215	-0.3064	-0.306	$0.10\,\%$
	217	0.2421	0.242	$0.04\,\%$

Table 3: *Maximum displacements and critical velocities for a nonlinear elastic foundation with* $k_l = 250\,\mathrm{kN/m^2}$ *and* $k_{nl} = 2500\,\mathrm{kN/m^4}$. *Percentage relative error with respect to Castro et al. [4].*

ζ	v_{cr} (m/s)	w_{max} (m) Present work	Ref. [4]	Error
0%	245	-0.2042	-0.204	$0.10\,\%$
	246	0.1861	0.186	$0.05\,\%$
2%	241	-0.1832	-0.183	$0.10\,\%$
	242	0.1497	0.150	$-0.20\,\%$

Table 4: *Maximum displacements and critical velocities for a nonlinear elastic foundation with* $k_l = 250\,\mathrm{kN/m^2}$ *and* $k_{nl} = 25000\,\mathrm{kN/m^4}$. *Percentage relative error with respect to Castro et al. [4].*

4 HARMONIC MOVING LOAD

In the present section the main effects of the load frequency of the harmonic-varying magnitude of the moving load on the displacements of the beam and on the critical velocities are outlined. In order to obtain values of beam displacements and critical velocities representative of real railway supports, more realistic values of foundation stiffness have been adopted for the numerical simulations presented herein, with respect to those reported in Section 3.

Regarding the load frequency range, according to Chen et al. [6] the frequency (Ω) shall be always lower than 420 rad/s in nowadays engineering problems. Consequently, the computations have been performed in the range from 0 rad/s to 440 rad/s, with intervals of 40 rad/s. For each of these frequencies, the maximum upward and downward beam displacements have been computed, for velocities of the moving load starting from 10 m/s to 600 m/s, at speed intervals of 10 m/s. Then, when the neighborhood of the position of the critical velocity has been approximately recognized, the velocity step is further refined to 1 m/s, for reaching locally a better accuracy. Although the upper bound of the selected range of moving load velocity is clearly unattainable by today railway transports, its adoption looks necessary in order to properly capture more than one critical velocity.

The number of adopted finite elements is 400 (twice as in the previous analyses), each one with a length of 0.5 m. It is worth to point out that for some of the simulations presented in this section, the results retrieved with a time step chosen accordingly to the criterion given by Castro et al. [4], namely choosing the time step as the time taken by the load to travel a fifth of a finite element length, are actually characterized by a low accuracy. Therefore, a lower time step, equal to 10^{-4} s, has been adopted for all the numerical analyses presented in this section. Further information and data are extensively reported in Moioli [17].

4.1 Beams lying on a linear elastic foundation

First, a uniform linear elastic foundation is considered, with a value of Winkler linear elastic coefficient (k_l) equal to 10^4 kN/m^2. The relationship between beam maximum upward and downward displacements and load velocities, retrieved for an undamped and a damped foundation, is shown in Fig. 5.

It is noticeable that, as the amplitude of the moving load starts oscillating with frequency Ω, the critical velocities might be either one or two within the displayed range of velocities. In particular, the higher critical velocity (v_{cr2}) increases and quickly moves towards the upper limit of the plot as the load frequency increases. Conversely, the lower critical velocity (v_{cr1}) decreases until reaching zero at a very high load frequency. Such value corresponds to the lowest natural frequency of a simply-supported beam lying on a linear elastic foundation, which is defined as (see Frýba [11]):

$$\omega_1 = \sqrt{\frac{\pi^4}{L^4}\frac{EI}{m} + \frac{k_l}{m}} = 408.5\,\text{m/s}.$$

The interpretation of this result is straightforward: in fact, it is well known that a fixed load oscillating at the same natural frequency of the beam causes the resonance of the system.

A comparison between the results for the undamped foundation and those for the damped foundation reveals that, as for the case of a moving load with constant magnitude, the maximum beam displacements of the damped foundation are much smaller and much more difficult to be defined, with respect to those computed for the undamped foundation. The latter observation may be clearly visualized in Fig. 5, where multiple little peaks appear in the neighborhood of

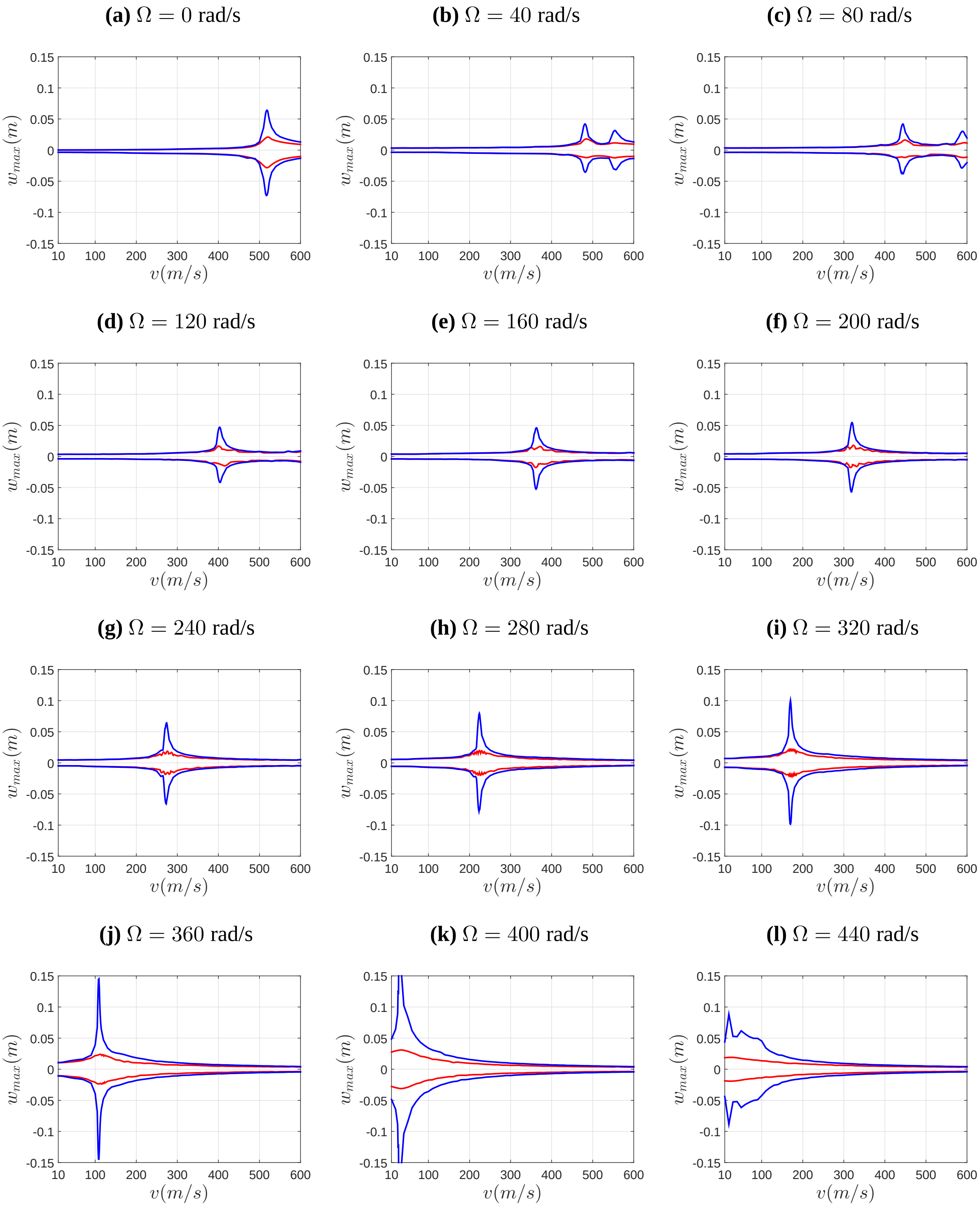


Figure 5: *Representation of beam maximum displacements as a function of load velocity and frequency for undamped (—) and damped (—) linear elastic foundations with stiffness of* $k_l = 10^4$ kN/m^2.

the critical velocity. These oscillations point out that an even more refined time step is needed for obtaining a better accuracy in the local definition of the peaks. Critical velocities are instead much less affected by damping.

The relationship between the frequency of oscillation of the moving load and the critical velocity is explicitly depicted in Figs. 6a and 6b. These bifurcation curves have been established by fitting the values of the critical velocity computed at the different frequencies of the load. The fitting is performed by adopting the following proposed models for the lower and higher critical velocities, respectively:

$$v_{cr1} = a_1 + a_2\,\Omega + a_3\,e^{a_4\Omega}; \tag{6}$$
$$v_{cr2} = b_1 + b_2\,\Omega + b_3\,\Omega^2. \tag{7}$$

The four coefficients of the first expression are retrieved with a nonlinear least squares regression method, while the three of the second expression are achieved with a linear least squares regression method, performed by using the curve fitting built-in function in MatLab. The computed values of the coefficients in Eq. (6) and Eq. (7) for the undamped system are:

$$a_1 = 521.8\ \text{m/s},\quad a_2 = -0.9206\ \text{m/rad},\quad a_3 = -2.694\ \text{m/s},\quad a_4 = 9.559\times10^{-3}\ \text{s/rad};$$
$$b_1 = 516.9\ \text{m/s},\quad b_2 = 0.9750\ \text{m/rad},\quad b_3 = -1.250\times10^{-3}\ \text{ms/rad}^2.$$

The fitted curves for the undamped linear elastic foundation are plotted in Fig. 6a. This figure shows also bifurcation curves computed in accordance with the analytical solution developed by Chen et al. [6]. The comparison between the fitted curves and the analytical result reveals a very good agreement, providing a further verification of the reliability of the present finite element implementation, even for the case of a variable load.

Even though the determination of the coefficients in Eqs. (6), (7) require a significant amount of computational time, they show explicitly the critical velocities as a function of the load frequency, differently from the implicit formulation developed by Chen et al. [6]. Furthermore, the obtained models for the critical velocity/load frequency pairs are much simpler than their analytical counterparts proposed by Chen et al. [6], which involve cumbersome complex irrational fractions of polynomials containing the mechanical parameters of the system.

It is important to notice that the solution by Chen et al. [6] presents a third branch, on the right of the first natural frequency of the beam (408 rad/s). Nevertheless, as already proven by

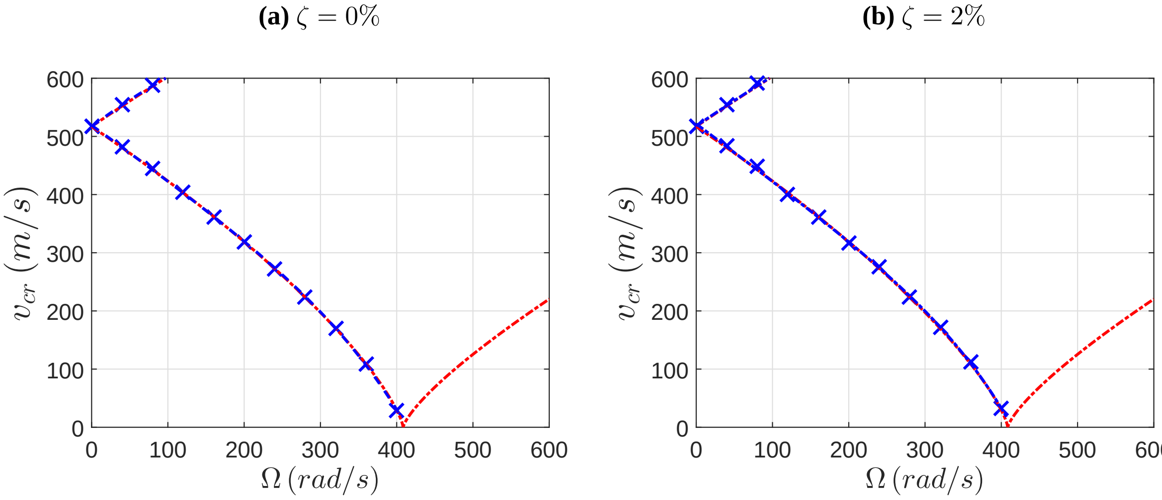


Figure 6: *Critical velocity and load frequency pairs for undamped and damped linear elastic foundations with stiffness of* $k_l = 10000\,kN/m^2$. *Results computed with: (×) finite element implementation; (-·-) fitted curves; (-·-) analytical solution by Chen et al. [6].*

Kim [15], the critical velocities represented by such a branch cannot be detected from the plots of the maximum displacements versus load velocity, because no peak of displacement corresponds to these velocities in a finite beam. Therefore, it is not necessary to perform numerical simulations for load frequencies higher than the first natural frequency of the beam.

Bifurcation curves for critical velocity and frequency pairs have also been obtained for the case of a damped ($\zeta = 2\%$) linear elastic foundation. The adopted fitting models have been the same as those employed for the undamped foundation, i.e. those defined by Eq. (6) and Eq. (7). The coefficients computed for the damped foundation are:

$$a_1 = 522.6 \text{ m/s}, \quad a_2 = -0.9698 \text{ m/rad}, \quad a_3 = -0.9883 \text{ m/s}, \quad a_4 = 1.116 \times 10^{-2} \text{ s/rad};$$
$$b_1 = 517.9 \text{ m/s}, \quad b_2 = \ \ 0.8875 \text{ m/rad}, \quad b_3 = -3.125 \times 10^{-3} \text{ ms/rad}^2.$$

The corresponding curves are plotted in Fig. 6b. In this case the critical velocity has not been reported by Chen et al. [6]. Nonetheless, comparing the curves obtained in accordance to Chen et al. [6] for the undamped case to those fitted from the numerical results, it can be seen that the level of damping is almost ineffective on the value of the critical velocity.

4.2 Beams lying on a nonlinear elastic foundation

For the nonlinear foundation model the assumed linear and nonlinear parameters defining the foundation stiffness are $k_l = 5 \times 10^3$ kN/m^2 and $k_{nl} = 2 \times 10^6$ kN/m^4, respectively. These values are selected in such a way that to provide a foundation model that is more compliant with respect to that of the linear elastic foundation for small beam displacements and, at the same time, stiffer for large displacements, as proposed by many authors (see e.g. Nguyen and Duhamel [18]).

Fig. 7 shows the retrieved results for damped and undamped nonlinear elastic foundations. It can be seen that, as in the case of a linear foundation, a second critical velocity appears for a harmonic variation of the moving load amplitude. A comparison between the graphs in Fig. 7 and their counterparts in Fig. 5 for the linear foundation, reveals that lower critical velocities and associated smaller maximum deflections are retrieved for the case of a nonlinear elastic foundation. For a foundation with viscous damping, a decrease of the maximum displacements can be observed, together with a small decrease of the critical velocities.

Bifurcation curves describing the relationship between critical velocities and load frequency are established in an analogous way as for the case of a linear elastic foundation. The adopted models for the higher and the lower critical velocity curves are the same as those employed for the linear foundation, in Eqs. (6)-(7). The coefficients computed for the undamped nonlinear elastic foundation are:

$$a_1 = 459.6 \text{ m/s}, \quad a_2 = -1.139 \text{ m/rad}, \quad a_3 = -4.5089 \text{ m/s}, \quad a_4 = 8.007 \times^{-3} \text{ s/rad};$$
$$b_1 = 456.9 \text{ m/s}, \quad b_2 = \ \ 0.8082 \text{ m/rad}, \quad b_3 = \ \ 4.018 \times 10^{-4} \text{ ms/rad}^2;$$

and, for the damped ($\zeta = 2\%$) nonlinear elastic foundation are:

$$a_1 = 452.1 \text{ m/s}, \quad a_2 = -1.084 \text{ m/rad}, \quad a_3 = -4.991 \text{ m/s}, \quad a_4 = 0.089 \text{ s/rad};$$
$$b_1 = 443.9 \text{ m/s}, \quad b_2 = \ \ 1.078 \text{ m/rad}, \quad b_3 = -8.929 \times 10^{-4} \text{ ms/rad}^2.$$

Fig. 8 shows the curves retrieved for the undamped and the damped nonlinear foundation. Comparing these plots to those reported previously in Fig. 6, it is clear that the relationship between the critical velocities and the load frequency for linear and nonlinear elastic foundations displays similar features. Nonetheless, in contrast with the case of a linear foundation, the curves for the damped nonlinear foundation are slightly shifted downward, with respect to the undamped ones.

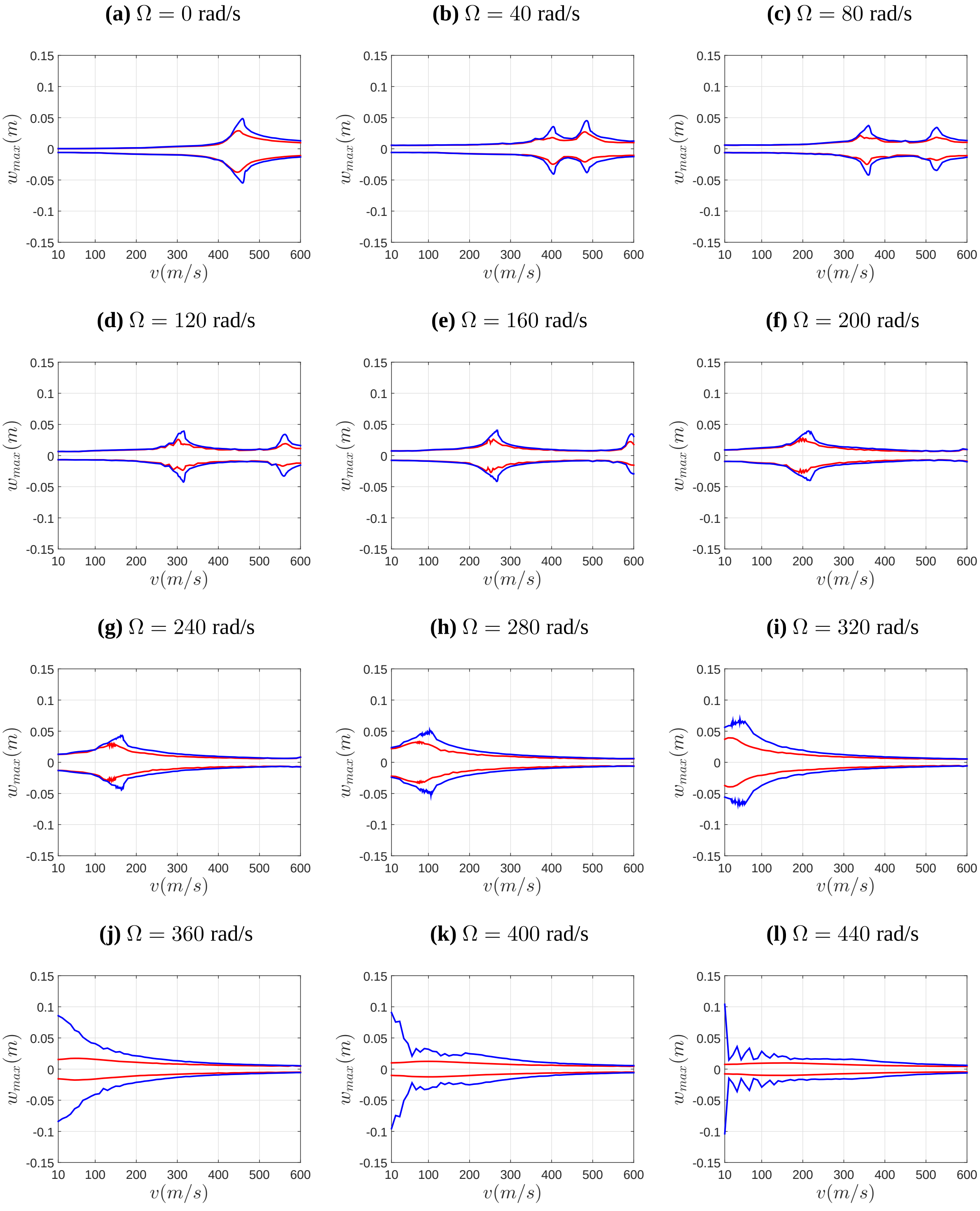


Figure 7: *Representation of beam maximum displacements as a function of load velocity and frequency for undamped (——) and damped (——) nonlinear elastic foundations with stiffnesses of* $k_l = 5 \times 10^3$ kN/m^2 *and* $k_{nl} = 2 \times 10^6$ kN/m^4.

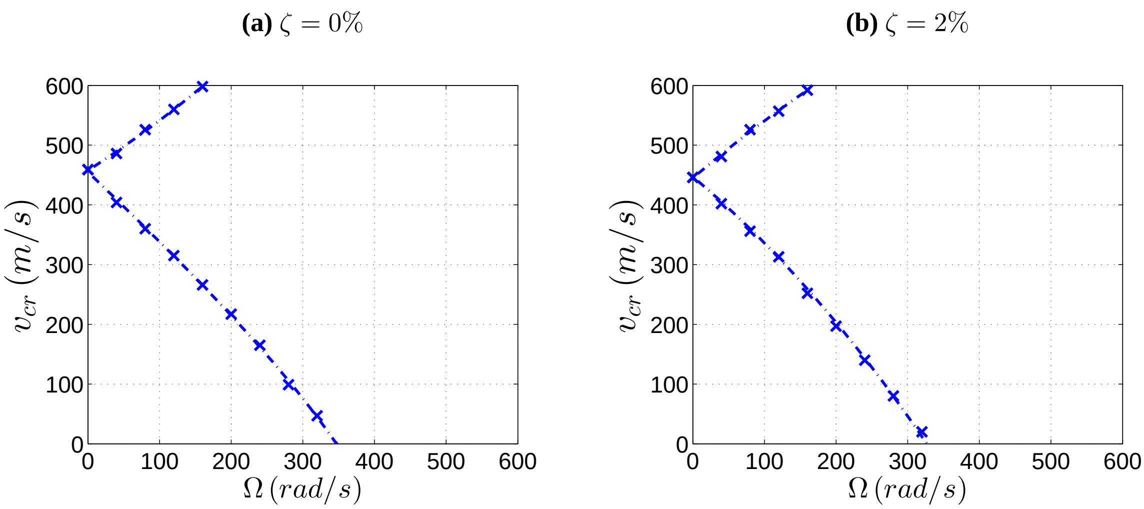


Figure 8: *Critical velocity and load frequency pairs for undamped and damped nonlinear elastic foundations with stiffnesses of* $k_l = 5 \times 10^3$ kN/m^2 *and* $k_{nl} = 2 \times 10^6$ kN/m^4. *Results computed with: (×) finite element implementation; (-·-) fitted curves.*

5 CONCLUSIONS

In this work a finite element approach has been developed within a MatLab environment to analyze the dynamic transient response of a simply-supported beam lying on linear or nonlinear Winkler elastic foundation, subjected to a concentrated load moving at constant velocity with either constant or harmonically-varying amplitude.

The extensive performed campaign of numerical simulations provided several interesting findings and new outcomes about the behavior of high-speed rail tracks. The most significant results may be summarized in the following itemized list:

- When the amplitude of the moving load is constant, the present finite element implementation has been proven to provide results that appear fully consistent to those reported by Castro et al. [4].

- For all types of foundation, the beam maximum displacements at the critical velocities initially decrease with the load amplitude frequency, before increasing sharply near the natural frequency of the beam. Furthermore, when damping is included in the analysis, no significant effects on the critical velocities are detected, while the amplitude of displacements is lowered.

- When the beam is subjected to a moving harmonic load, two critical velocities are detected, independently from the considered type of foundation behavior, either linear or nonlinear. The two critical velocities tend to separate as the loading frequency increases. The higher critical velocity v_{cr2} increases, starting from the value of the critical velocity obtained for a constant magnitude load. On the contrary, the lower critical velocity v_{cr1} decreases, until it reaches zero for a frequency of the load equal to the first natural frequency of the beam. This behavior may reveal potential implications in practical terms, especially in lowering down the ranges of admissible train speeds, when the frequency of oscillation of the moving load amplitude becomes high.

- The relationship between the moving load amplitude frequency and the critical velocity of the beam is portrayed in appropriate analytical bifurcation curves. These curves have been achieved by fitting the values of the critical velocity computed at different frequencies of the load variation, according to proposed variations with the amplitude frequency, with calibrated best-fitted coefficients.

- For the case of a linear foundation, the obtained bifurcation curves show a very good agreement to those available from the analytical approaches by Chen et al. [6]. In addition, the employed models for the description of the analytical bifurcation curves are explicit and much simpler than those exact proposed by Chen et al. [6].
- The employed models for the description of the bifurcation curves are also appropriate for the nonlinear case, with or without damping. The derived formulas are simple and possibly workable in practice; they may supply a guideline for the design of railway tracks when the magnitude of the moving load is oscillating in time.

It must be recognized that, in order to obtain a more realistic dynamic response of the system, some further improvements of the modelization could be made. Real applications usually require extensions to infinite beams; it is then necessary to eliminate the effect of the supports, to mitigate the perturbation induced by the boundary conditions and to prevent the reflection of the traveling waves. This could be achieved by using appropriate absorbing boundaries. Furthermore, a real vehicle spring-mass-damper system interacting with the rail should be considered, instead of a simple moving force. Finally, the assumption of symmetric behavior in tension and compression for the foundation stiffness should also be relaxed, leading to the analysis of a bilinear foundation. These issues will be the subject of on-going (Froio et al. [10]) and future investigations.

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