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Pricing and Hedging Pension Fund Liability via Portfolio Replication

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Contents

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Introduction

Asset Liability Management (ALM) is the process of finding optimal policies for long term investors which need to meet future obligations. The implemented strategy should be optimal both with respect to the financial resources and with respect to the liability of the real life problem subject to a set of management and institutional constraints. The problem is highly stochastic due to the risky nature of future demographic, actuarial and economic variables such as financial returns, liability and macroeconomic indicators. It is so necessary to find a properly right set of mathematical tools which are able to optimise the strategy considering the stochastic implications. *Multistage stochastic programming* (MSP) is an extension of mathematical programming in which some or all parameters have a stochastic nature. Traditional solutions methods for MSP required a discrete approximation of the underlying probability distribution governing the evolution of the random parameters in order to transform the problem into a deterministic equivalent representation. The deterministic representation can then be solved using traditional optimization algorithms or more specialised solvers which take advantage of the special block structure of the problem. Advancements in computer technology and in algorithms efficiency enable increasing opportunities in modelling and solving these problems with long time horizons, a large number of decision variables and constraints and with sophisticated objective functions. As a result, MSP has emerged as a fruitful technique to deal with ALM problems for its flexibility in modelling real life specific features such as, friction markets with transaction costs and taxes, complex regulatory and management constraints and multi-target objective functions.

Areas of applications where MSP has been successfully applied are asset allocation [40, 42], bank management [4, 31], fixed income portfolio management [2, 3, 16], insurance and pension fund companies [6, 7, 8, 11, 14, 17, 24] and minimum guarantee financial products [13]. Typically dynamic ALM problems are formulated to find the optimal dynamic investment strategy which fulfils a set of constraints and maximise an expected utility function over an investment horizon with a given set of portfolio rebalancing periods. In some models also policy parameters, such as the optimal contribution rate in a pension fund, are considered as variables.

MSP has also been applied to contingent claim pricing problems in incomplete markets. King [28] proposed to price a contingent claim in a discrete market environment using super-replication arguments: the price of the contingent claim is defined as the minimal initial capital which enables the implementation of a self-financing trading strategy which covers the claim without risk. Starting from this seminal work other authors have proposed different *MSP* models to price particular contingent claim as European options [39] and American options [36]. The above models differ not only in the special nature of the contingent claim, but also in the way the replication problem is defined. The same approach has been used to price a stochastic stream of payments corresponding to an insurance or a pension annuity payoff [25]: the annuity is considered as a contingent claim and its fair value is the least expensive replicating portfolio to attain such payoff. The obtained self-financing trading strategy is obviously dependent from the stochastic annuity stream of payments and the problem can be naturally considered as a particular type of ALM problem [25, 32]. This pension/insurance obligations pricing methodology is in line with the Article 75 of Directive 2009/138/EC (Solvency II-directive) which requires an economic, market-consistent approach to the valuation of assets and liabilities.

When MSP models are numerically solved using a deterministic equivalent representation, a crucial role assumes the choice of the methodology through which the original distribution is approximated by discrete scenarios. Scenarios are usually represented by an event tree in which the nodes values represent the random parameters distributions. A technique used to generate such scenario tree is called scenario tree generation method. Mostly, scenario tree generation methods rely on four types of different approaches: Monte Carlo-based sampling [5, 6, 37], the moment matching method [9, 26, 27, 38], the sequential cluster [15, 21] and optimal discretisation methods based on some probability metrics such as the Wasserstein distance [22, 23, 33, 34, 35]. In Monte Carlo-based methods a conditional discrete sample is obtained from the theoretical continuous distribution at each node and in each stage starting from the root node in a forward fashion. Different variance reduction techniques have also been implemented in order to limit the approximation error. Moment matching methods focus the attention on the error minimisation between a certain set of moments of the original distribution function with those of the discrete approximation. The problem can be formulated as a non-linear system of equations or as a non-linear minimisation problem. Optimal discretisation methods attempt to minimise some probabilistic metric such as the Wasserstein distance between the original and the approximated distributions. Another class of tree generation methods is represented by hybrid techniques where two or more of the four approaches are combined in order to take advantage of the specific features of each method. Examples are tree generation algorithms that combine the Monte Carlo-based, the sequential cluster and the moment matching methods [1, 41]. Since the computational effort of solving the MSP problem grows exponentially with the number of nodes in the scenario tree there is a trade-off between the risky parameters distribution approximation and the real problem solving capacity. This rise the issue of the extent to which the approximation error in the event tree will bias the optimal solutions of the model [12, 26, 27, 33].

Another important issue for financial problems solved by MSP is the presence of arbitrage opportunities in the returns scenario tree. An arbitrage is the opportunity to have a riskless investment with a positive return. The presence of an arbitrage strategy along the tree will be then exploited by the optimiser and the objective value of the financial planning model will increase without additional risk. In the general formulation of financial MSP models the presence of arbitrages will lead to unbounded solutions. When instead the short selling in each asset is limited, optimal solutions are obtained but they are biased. Klaassen [29, 30] was the first to show how arbitrage opportunities can bias the optimal solution of a bond selection investment problem with liability. The arbitrage opportunities issue related to the generation of a scenario tree for asset returns has been then considered by other authors. Geyer, Hanke, and Weissensteiner [19, 20] investigate the theoretical relationships between the mean vector and the covariance matrix specifications of the statistical model for asset returns and the existence of arbitrage opportunities. Consiglio, Carollo and Zenios [9] and Staino and Russo [38] proposed two similar moment matching tree generation approaches which directly consider the problem of avoiding arbitrage opportunities.

This thesis deals with two interconnected problems related to the *defined benefit* (DB) pension fund industry. In the first problem the pension fund manager seeks the actual price of the liability of the pension fund on a market valuation based approach. This can be viewed as a pricing problem in a incomplete market and it has been solved by replication arguments using MSP following the cash-flow matching method proposed by Hilli, Koivu and Pennanen [25]. The arbitrage opportunity issue related to the generation of the asset returns scenario tree has been addressed by developing an algorithm based on an hybrid approach, which combine the method proposed by Xu et al. [41] with the moment matching algorithm with additional constraints to avoid arbitrage opportunities introduced in [9, 38]. With the proposed algorithm, we are able to generate asset scenario trees which do not contain arbitrage opportunities and solve the pension fund pricing problem without imposing short selling limits. However, since the ALM pricing model must keep the economic features of the real pension fund problem, we have solved the same problem by imposing different limits, or totally exclude short selling positions. This is due to the fact that each country has its own regulation on short selling transactions. In Italy, for example, insurance and pension fund companies are forced by the legislator to avoid short positions (Decree of the treasury department 21 November 1996 n. 703). When short selling limits are imposed, the pricing problem will have an optimal solution also if computed by using an asset scenario tree with arbitrage opportunity. In this cases we are then able to compute the bias on the optimal solutions by taking the difference between the solutions obtained with trees with arbitrage and with trees without arbitrages. The trees with arbitrages are generated with the same algorithm by relaxing the constraints to avoid arbitrages. The second problem is the pricing of a longevity swap contract. Longevity swaps are financial derivative contracts recently designed in order to provide an hedge against parties that are exposed to longevity risks through their businesses, such as pension plan managers and insurers. We propose an approach from the point of view of the pension fund manager to price these contracts which is an extension of the liability valuation procedure previously proposed. The rest of the thesis is organised as follows:

1. In the first chapter we firstly present the main features of the DB pension fund business. We then present a general MSP problem which can be used as a reference model for ALM for DB pension funds and which will be used, although with some modifications, along the rest of the thesis.

2. In the second chapter the scenario generation technique to obtain arbitrage-free scenario trees is described and motivated. We then tested the proposed algorithm on a portfolio case study against a similar hybrid method proposed by Xu et al. [41] which does not directly consider the issue of arbitrage opportunity. This case study has been implemented in order to validate the proposed method by comparing it with a similar hybrid existing method in the literature.

3. In the third chapter the liability valuation problem in incomplete markets of a DB pension fund is discussed and different MSP formulations are proposed and tested on a case study based on an artificial pension fund. When short selling is limited, or totally exclude, the bias on the optimal solutions derived by arbitrages in the scenario tree is computed.

4. In the final chapter the longevity derivative market is briefly explained and a MSP problem to find the price of a longevity swap is presented and tested on the same pension fund case study developed in the third chapter.

Bibliography

[1] Beraldi P., De Simone F., Violi A., *Generating scenario trees: a parallel integrated simulation optimization approach*, Comput Appl Math, 233 (2010), pp. 2322-2331.

[2] Beraldi, P., Consigli, G., De Simone, F., Iaquinta, G. and Violi, A., *Hedging market and credit risk in corporate bond portfolios*, In Handbook on Stochastic Optimization Methods in Finance and Energy, Fred Hillier International Series in Operations Research and Management Science, edited by M. Bertocchi, G. Consigli and M. Dempster, Chapter 4, pp. 73-98, 2011 (Springer: New York).

[3] Bertocchi M., Dupaĉová J., Moriggia V., *Bond portfolio management via stochastic programming*, In Handbook of asset and liability management, vol. 1, Theory and methodology, ed. S. A. Zenios and W. T. Ziemba: 305-336, 2007.

[4] Castro J., *A stochastic programming approach to cash management in banking*, European Journal of Operational Research,192, 963-974, 2009.

[5] Chiralaksanakul A., Morton D. P., *Assessing policy quality in multi-stage stochastic programming*, Stochastic Programming E-Print Series 12, 2004.

[6] Consigli, G., Dempster, M.A.H.: *Dynamic stochastic programming for asset-liability management*, Ann. Oper. Res. 81, 131-161, 1998.

[7] Consigli G., di Tria M., Gaffo M., Iaquinta G., Moriggia V., Uristani A., *Dynamic Portfolio Management for Property and Casualty Insurance*. In

Handbook on Stochastic Optimization Methods in Finance and Energy, Fred Hillier International Series in Operations Research and Management Science, edited by M. Bertocchi, G. Consigli and M. Dempster, Chapter 5, pp. 99-124, Springer: New York, 2011 .

[8] Consiglio A., Cocco F. , Zenios S.A., *Asset and liability modelling for participating policies with guarantees*, European Journal of Operational Research 186, 380-404, 2008.

[9] Consiglio A., Carollo A., Zenios S.A., *A parsimonious model for generating arbitrage-free scenario trees*, Quantitative Finance, 01 February 2016, Vol.16(2), p.201-212.

[10] Dempster M. A. H. ,Germano M.,Medova E. A., and Villaverde M., *Global asset liability management*, British Actuarial Journal, 9 (1):137195 Part c, 2003.

[11] Dempster M.A.H., Germano M., Medova E., Murphy J., Ryan D., Sandrini F., *Risk Profiling Defined Benefit Pension Schemes*, The Journal of Portfolio Management, 07/2009, Vol.35(4), pp.76-93.

[12] Dempster M. A. H., Medova E. A., Yong Y. S., *Comparison of Sampling Methods for Dynamics Stochastic Programming*, Stochastic Optimization Methods in Finance and Energy, Ch. 16, Springer, 2011.

[13] Dempster M. A. H., Medova E. A., Rietbergen M. I., Sandrini F., Scrowston M., *Designed Minimum Guaranteed Return Funds, Stochastic Optimization Methods in Finance and Energy*, Ch. 2, Springer, 2011.

[14] Dondi G. and Herzog F., *Dynamic Asset and Liabilities Management for Swiss Pension Funds*, Handbook of Asset and Liability Management Volume 2, Ch.20, 2007.

[15] Dupaĉová J., Consigli G., Wallace S. W., *Scenarios for multistage stochastic programs*, Annals of Operations Research, 2000.

[16] Dupaĉová, J., Bertocchi, M., *From data to model and back to data: A bond portfolio management problem*, European Journal of Operational Research 134, 261-278, 2001.

[17] Dupaĉová J., Polìvka J., *Asset-liability management for Czech pension funds using stochastic programming*, Annals of Operations Research, 2009, Vol.165(1), pp.5-28.

[18] Geyer A., Ziemba W. T., *The Innovest Austrian Pension Fund Financial Planning Model InnoALM*, Operations Research, 2008, Vol.56(4), p.797-810.

[19] Geyer A., Hanke M., Weissensteiner A., *No-arbitrage bounds for financial scenarios*, European Journal of Operational Research, 2014, 236(2), 657-663.

[20] Geyer A., Hanke M., Weissensteiner A., *No-Arbitrage ROM simulation*, Journal of Economic Dynamics & Control, 45(2014)66-79.

[21] Gülpinar N., Rustem B., Settergren R., *Simulation and optimization approaches to scenario tree generation*, Journal of Economic Dynamics and Control, 2004, Vol.28(7), pp.1291-1315.

[22] Heitsch H., Romisch W., *Generation of multivariate scenario trees to model stochasticity in power management*, IEEE St. Petersburg Power Tech, 2005.

[23] Hochreiter R., Pflug G., *Financial scenario generation for stochastic multi-stage decision processes as facility location problems*, Annals of Operations Research, 2007.

[24] Hilli P., Koivu M., Pennanen T., Ranne A.,*A stochastic programming model for asset liability management of a Finnish pension company*, Annals of Operations Research, 2007.

[25] Hilli P., Koivu M., Pennanen T., *Cash-flow based valuation of pension liabilities*, Eur. Actuar. J., 2011.

[26] Høyland K.,Wallace S. W., *Generating scenario trees for multistage decision problems*, Management Science, 47(2), 295-307, 2001.

[27] Høyland K., Kaut M., Wallace S. W., *A heuristic for moment-matching scenario generation*, Computational Optimization and Applications, 2003.

[28] King A., *Duality and martingales: a mathematical programming perspective on contingent claims*, IBM Technical Report, T.J. Watson Research Center, Yorktown Heights, NY, 2000.

[29] Klaassen P., *Financial asset-pricing theory and stochastic programming model for asset/liability management: A synthesis*, Management Science, 44(1), 3148, 1998.

[30] Klaassen P., *Comment on "generating scenario trees for multistage decision problems"*, Management Science, 48(11), 1512-1516, 2002.

[31] Kusy M. I., Ziemba W. T. , *A bank asset and liability management model*, Operations Research 34(3), 356-376, 1986.

[32] Pennanen T., *Optimal investment and contingent claim valuation in illiquid markets*, Finance and Stochastics, Volume 18, Issue 4, pp 733-754, 2014.

[33] Pflug G. C., *Scenario tree generation for multiperiod financial optimization by optimal discretization*, Mathematical Programming, January 2001, Volume 89, Issue 2, pp 251-271.

[34] Pflug G. C., *Approximations for Probability Distributions and Stochastic Optimization Problems, Stochastic Optimization Methods in Finance and Energy*, Ch. 15, Springer, 2011.

[35] Romisch, *W., Stability of Stochastic Programming Problems, Chapter 8 in Stochastic Programming, Volume 10 of Handbooks in Operations Research and Management Science*, Ruszczynski A. and Shapiro A., eds. Elsevier,

Amsterdam, 2003.

[36] Pinar M., *Mixed-integer second-order cone programming for lower hedging of American contingent claims in incomplete markets*, Optimization Letters, 2013, Vol.7(1), pp.63-78.

[37] Shapiro A., *Inference of statistical bounds for multistage stochastic programming problems*, Math. Meth. Oper. Res. 58 (2003), 57-68.

[38] Staino A., Russo E. *A moment-matching method to generate arbitrage-free scenarios*, European Journal of Operational Research, 16 October 2015, Vol.246(2), pp.619-630.

[39] Topaloglou N., Vladimirou H., Zenios S. A., *Pricing options on scenario trees*, Journal of Banking and Finance, 2008, Vol.32(2), pp.283-298.

[40] Topaloglou N., Vladimirou H., Zenios S. A., *Optimizing international portfolios with options and forwards*, Journal of Banking & Finance 35, 3188-3201, 2011.

[41] Xu D., Chen Z., Yang L., *Scenario tree generation approaches using K-means and LP moment matching methods*, Journal of Computational and Applied Mathematics 236, 4561-4579, 2012.

[42] Yin L., Han L., *Risk management for international portfolios with basket options: A multi-stage stochastic programming approach*, Journal of Systems Science and Complexity, December 2015, Volume 28, Issue 6, pp 1279-1306.

Chapter 1

ALM for Pension Funds

1.1 Introduction

Modern pension systems are complex architectures of different public and private organisations designed to offer an acceptable income to individuals during their post-work residual life. Although each country has implemented a specific pension system we can follow the World Bank [28] taxonomy based on three pillars. The first pillar is usually designed as a state-managed pay-as-you-go (PAYG) with a strong redistribution purpose for avoiding old-age poverty. The second pillar is constituted as a private defined contribution DC or defined benefit DB system financed by private company sponsors that should guarantee an adequate pension in terms of the replacement rate. Finally, the third pillar should provide an opportunity for individuals to further improve their retirement income. Demographic, economic and institutional changes are putting pressure on the sustainability of state-funded and private pension systems in the European Union. The economic crisis has further deteriorate the funding position of the pension fund industry. Several pension funds have increased the retirement age, the contributory period and reformed the contributory rates. In many countries we have also assisted to a gradually shift from defined benefit (DB) to defined contribution (DC) pension schemes. In such a context the design of a proper set of risk management tools is crucial for safeguard DC and DB pension schemes business.

Increasingly over the recent years large institutional investors are moving from long established static, myopic optimization approaches to identify their optimal portfolios to dynamic models able to preserve the time structure of long term liability commitments and at the same time accommodate policy revision over long horizon. The importance of adopting the ALM approach in coupling the asset-side of the problem with liability streams has been extensively documented. It represents a critical improvement from the traditional asset-only allocation strategies, which have been shown empirically to be inadequate to jointly mange the dynamics of short and medium term evolution of asset prices, risk-factor correlations and negative future cash flow obligations. In this context MSP has been increasingly used in ALM problems designed for optimal pension fund management of both DB and DC schemes [3, 4, 7, 10, 11, 14, 15, 16]. Typically, MSP models for the pension fund industry are formulated as portfolio optimization problems in which portfolio rebalancing is allowed over a long-term horizon at discrete time points, and where liabilities, connected on the nature of the pension contract, are considered. The uncertainty affects both assets and liabilities values in the form of discrete scenario dependent realisations. The portfolio manager, given an initial wealth, looks for the maximisation of the terminal wealth at the horizon, and simultaneously, for the minimisation of some measure of the risk involved in the strategy, with investment returns modelled as discrete state random vectors. Decision vectors represent possible investments in the market and holding or selling assets in the portfolio, as well as borrowing decisions from a credit line or deposits with a bank. Starting from this general structure, different specialised models has been developed for taking into account some specific features of the pension scheme, for example DB vs. DC plans, different salary cohorts and specific national regulatory prescriptions. In some models also policy decisions are inserted as variables into the problem. As an example, Dempster et all. [7] employ MSP to determine the optimal asset allocation and the employer contribution rate that will enable the scheme to achieve a desired funding ratio within a given time horizon while respecting the trustees' risk appetite.

The rest of the chapter is organised as follow. In Section 1.2 the basic features of the pension fund activity and regulations are presented with a special attention on the valuation of the private sector liability of the EU zone. In Section 1.3 the general MSP modelling framework for manage pension fund problem is briefly depicted in its basic constitutive elements: the discrete market model formulation as a tree process, the objective function and the constraints typologies of the optimization problem.

1.2 Pension Fund Economics

Each country has developed a unique pension system as a component of its social welfare mechanism. The historical development of the social welfare has led to complex pension systems across the European countries, making the classification and the comparison of their functioning an hard task. In recent years almost every European state has undergone a significant reorganisation of its pension system due to deep changes in the demographic and economic environment. The improvement in life expectancy and the simultaneous reduction in birth rates have increased the share of retired individuals in every country of the European Union. The 2008 financial crisis has produced a series of fundamental modifications in all the sectors of the European economy and also in markets regulations. Most European countries have been forced to adopt harsh austerity measures to reduce their deficits which are perceived as the cause of the slow recovery from the crisis. As a result many states have undertaken different reforms in order to ensure the sustainability of their pension systems. A common feature of this reforms among the countries is the more weight the private pension schemes will have to guarantee a decent retirement income [17]. Private pension schemes also vary widely among states since they have been developed in order to support the public sector. These differences usually rely on the types of coverage rules, the choice of the replacement rate and the public incentives at play. In this Section we concentrate on the private sector pension fund industry which is the most prominent side of the second and the third pillars. In particular we analyse the distinction between DB and DC schemes and the major risk sources which affect the business. A pension plan is a retirement plan that requires an employer to make contributions into a pool of funds set aside for a worker's future benefit. The pool of funds is invested on the employee's behalf, and the earnings on the investments generate income to the worker upon retirement. The most important distinction of the pension systems in terms of redistribution, but also of risk sharing, is the formula translating the contributions into benefits. This is particularly the case for occupational Pillar 2 pension schemes. Generally, two types of system exist: defined benefit (DB) and defined contribution (DC) schemes.

Pension plans with defined benefits are determined by formulas taking into account number of years of contributions and the level of earnings for some part of the working career. Benefits do not necessarily have a direct link with the notional amount contributed. The amount of contributions deposited by the employee during its active life is generally used only as a condition for benefits; although some mechanisms of income adequacy can be incorporated to ensure a degree of differentiation for higher earners. However, the redistributive feature implies that higher earners will have lower replacement rates overall than lower earners [19]. In a typical DB plan a minimum number of contribution years is required to get the life annuity, which is usually a percentage of the last pre-retirement nominal earnings. Sometime the fixed guaranteed monthly benefits are indexed to inflation. The pension fund pays the annuities already matured with the contributions of the actives members, plus the eventually investment returns. Generally, both the employee and the employer contribute to the plan, and the contributions are pooled and invested by the plan sponsor. The total amount of contributions, plus the investment returns, must be adequate to cover benefit costs. If contributions from employees and employers, plus the investment returns are not adequate to cover the additional benefits earned each year, the unfunded benefit obligation increases, and the funded status of the plan deteriorates.

Defined contribution systems determine benefits in proportion to the amount contributed rather than to labour participation. Contributions are registered to an individual saving account administered by the plan sponsor. The amount in the saving account at distribution includes the contributions and investment gains or losses, minus any investment and administrative fees. Employee contributions are typically deducted directly from its salary, and frequently some portion of these contributions is matched by the employer. At retirement, the benefit can be received as a lump sum, as equal payments over a specified number of years, or it can be used to purchase an annuity for a lifetime benefit. The benefit amount at retirement is based on the ending account balance and on the assets belonging to the worker, meaning that previous contributions are portable across employers and there are no problems concerning the back-loading of accrued benefits. This in turn implies that workers are able to leave the plan assets under the administration of a previous employer, transfer the assets to a new employer plan or transfer the assets to an individual retirement savings account. In this setting it is the contribution amount, rather than the benefit, that is fixed, and the investment risk is shifted from the fund managers toward the workers. As a result, DC pension plans are always fully funded (a pension plan that has sufficient assets needed to provide for all accrued benefits). Since defined contribution plans do not guarantee a specific benefit payment amount to participants, there is no unfunded benefit obligation. As a result, DC plans do not create future cost obligations for the plan sponsor and compared to a DB plan, the accounting treatment is quite simple because each account is managed alone without pooling [6]. A hybrid of a DC system managed as a PAYG system can result in notional accounts (NDC) where the individual accounts are notional: contributions create rights of the contributor and a quantitatively determined liability of the managing institution, but the source of benefits is often tax-based. One of the main differentiating features of these two systems is the risk of investment and adequacy. Under DB, the managing institution bears the investment risk and the related risk of providing an adequate pension income. Under DC, there is no guarantee of a minimum real or even nominal income upon retirement, and therefore the risk is entirely borne by the contributor. A third class is composed by the so called hybrid schemes. Hybrid schemes share features with both DC and DB plans in order to distribute the risk between the employer and employees [1, 2, 12]. Examples of hybrid plans are career average schemes, combination hybrids, self-annuitising DC scheme, final salary lump sum schemes, underpin arrangements, cash balance schemes and fixed benefit/benefit unit schemes [26, 29].

1.2.1 Funding Ratio and Pension Fund Solvability

Pensioners receive a pension which is an annuity based on their last income at retirement age (DB case) or on their wealth accumulated by the time of retirement (DC case). Active members pay contributions during their working years in order to accumulate wealth for retirement age. The future exposure of the actual pension fund business is represented by the two concepts of obligations and liability. Obligations summarise the pension fund promised payments to pensioners and active members based on their current wealth. For obligations, we do not take into account any future contributions into the pension fund by active members. Liability consists of the present value of the pension fund promised payments to pensioners and active members taking into account current wealth and including outstanding future contributions. In order to compute the pension fund liability we need to make assumptions on the active members' projected future wages and on the evolution of the pension fund population. In particular, for a DB pension scheme, we are interested in the salary evolution of each individual whereas for a DC pension scheme we also need to project the future returns of the invested future contributions. In both the cases we have to implement a statistical population model to forecast active and passive members future dynamics. We call $\mathbf{N}^a_{\alpha,t}$ and $\mathbf{N}^p_{\alpha,t}$ the total number of active and passive members with age α at t, with $\mathcal{A}$ and $\mathcal{P}$ the two sets containing the ages α related to active members and passive members respectively. The evolution of $\mathbf{N}^a_{\alpha,t}$ and $\mathbf{N}^p_{\alpha,t}$ will depend on uncertain factors such as the mortality of each member, the number of new active members which enter in the pension at each year and also on the number of members which leave the scheme before the retirement period. The bold character is used in this Section to represent the variables with a stochastic evolution over time. We then denote by $\boldsymbol{\kappa}_{i,t}$ and $\boldsymbol{\gamma}_{i,t}$ the contribution payed and the pension received by the $i-th$ member at t. The way on which the value of $\boldsymbol{\kappa}_{i,t}$ is computed is usually a fixed proportion of the salary both in DB and DC plans, the value of $\boldsymbol{\gamma}_{i,t}$ is instead differently computed. As we have already mentioned, in a DB scheme $\boldsymbol{\gamma}_{i,t}$ is a fixed fraction of the last salary before the retirement period adjusted by some inflation benchmark. In a DC pension scheme the contributions are deposited in an individual account and then invested by the pension fund manager, the value of $\boldsymbol{\gamma}_{i,t}$ will be then determined as an annuity on the wealth accumulated by the returns gained on the individual account. We can then define the total amount of contribution $\boldsymbol{K_t}$ and pension flows $\boldsymbol{\Gamma_t}$ at time t respectively as:

$$\boldsymbol{K_t} = \sum_{\alpha \in \mathcal{A}} \sum_{i=1}^{\boldsymbol{N}^a_{\alpha,t}} \boldsymbol{\kappa}_{i,t}$$

$$\boldsymbol{\Gamma_t} = \sum_{\alpha \in \mathcal{P}} \sum_{i=1}^{\boldsymbol{N}^p_{\alpha,t}} \gamma_{i,t}$$

The pension fund net payments $\boldsymbol{l}_t$ at t are then defined as $\boldsymbol{l}_t = \boldsymbol{\Gamma}_t - \boldsymbol{K}_t$. The present value of these two future stochastic streams of flows $\{\boldsymbol{K_t}\}$ and $\{\boldsymbol{\Gamma}_t\}$ is obtained by taking their expected value, discounted by reference to a technical interest rates term structure. Formally, if we define $r^{tech}_{0,t}$ the discount interest rate between 0 and t, we have that the present values C_0 and L_0 at time $t = 0$ will be:

$$C_0 = \sum_{t=1}^{T} \left(1 + r^{tech}_{0,t}\right)^{-t} E\left[\boldsymbol{K_t}\right],$$

$$L_0 = \sum_{t=1}^{T} \left(1 + r^{tech}_{0,t}\right)^{-t} E\left[\boldsymbol{\Gamma}_t\right].$$

The net asset value V_0 of the pension fund at $t = 0$ is defined as the difference between the asset value A_0 plus the present value of the future contributions and the present value of the future pension payment:

$$V_0 = A_0 + C_0 - L_0.$$

The funding ratio Λ_0 at $t = 0$ is instead computed as the ratio between the asset value A_0 plus the present value of the future contributions and the present value of the future pension payment:

$$\Lambda_0 = \frac{A_0 + C_0}{L_0}.$$

The net asset value V, and the funding ratio Λ, are two control variables used by the pension fund manger and the authority to assess the risk exposure of the pension fund and the sustainability of its management policy: a funding ratio level below the unity indicates the pension fund inability to cover the future obligations. The methodology used to derive the technical interest rates term structure depends on the regulation adopted in each country. In the US, for instance, the Department of the Treasury publishes each month a spot yield curve computed on the basis of high-quality corporate bond (rated A or better) yields. The single-employer pension plan can then use either this spot yield curve or the 24 month average of three maturity segments of the curve (0-5 years, 5-20 years, and 20+ years), as required by the Pension Protection Act of 2006. For what concern the state members of the European Union, the European Insurance and Occupational Pensions Authority (EIOPA) publishes each month a set of discount interest rates, called risk-free interest rates, from one year maturity onwards. Risk free interest rates are extrapolated with the Smith-Wilson method and then modified by a volatility and credit spread adjustment using swap rates as inputs. In the absence of financial swap markets, or where information of such transactions is not sufficiently reliable, the risk-free interest rate is based on the government bond rates of the country. The exact methodology is describe in a technical document on EIOPA's website [13]. The choice of discount rate can make a large difference to the measured value of accrued liabilities. A decrease of one percent in the discount rate can lead to as much as a 30 percent increase in the liability [24, 30].

1.2.2 Risk Analysis

All pension schemes and their pillars face a number of risks depending on their exact design. Five fundamental types of risk has been identified: financial, longevity, inflation, behavioural and regulatory risks. Financial risks refer to the fact that the returns to the underlying financial assets are uncertain and variable. In DB schemes the financial risk is entirely faced by the pension plan sponsor, since the pension will just depend on inflation and salary dynamics, whereas in DC pension schemes the financial risk is all faced by the members. As an example, DC plan members that started their retirement period during the market downturn of the Internet bubble had retired with a much smaller plan balance than individuals with similar historical contributions flows who retired during the stock market boom of the late 1990s.

Inflation risk is linked to the inflation rate evolution: DB plans which guarantee an yearly inflation adjustment of their passive member pensions take the risk of an higher pension outflow in the case of an unexpected increase in the inflation rate. In DC pension scheme the employees bear the inflation risk and they must be able to calculate the amount of savings needed to retire and choose a complex investment decision in order to administrate their assets until their death to maintain their living standard. Although many DC plans offer a large flexibility on investment decisions like contribution amounts, portfolio allocations, and, in some countries, the timing of withdrawals, it seems from empirical evidence that the majority of the clients use standard contracts without having an active control over the asset mix, probably for the lack of basic financial literacy. Empirical evidences also suggest that there are behavioural biases such as considerable inertia and myopia regarding retirement decisions, which may ultimately threaten the capacity of DC plans to provide retirement security.

Longevity risk is the risk attached to the increasing life expectancy of pension plan participants, which can eventually translate into higher than expected pay-out-ratios for many pension funds. Individual lifespan is uncertain and, unless provisions are taken to avoid this, there is a risk that one may outlive pension means. The longevity risk is faced by the plan sponsor in DB schemes and by the pensioners in DC plans (a DC plan member needs to accumulate enough capital in order to keep its life standard until his death).

Behavioural risk denotes the risk associated with individual non-professional portfolio management now amply demonstrated in the behavioural economics literature [19, 27]. This includes a tendency to trade too often (thereby incurring excessive trading costs), to under-diversify portfolios, and to fail to regularly balance the risk profile as retirement gets closer. It is related to financial literacy for individually managed pension savings.

Finally, regulatory risks are those related with the governance of the pension fund. Key issues here are the transparency of management fees and the ability to change pension providers or fund managers. Large differences exist in management fees for second and third-pillar pension products, not necessarily related to performance. Over a long horizon, fees can have a large impact on the pension outcome at retirement.

1.3 ALM models for Pension Fund Industry

1.3.1 Scenario Tree Market Model

Multistage stochastic programming models are usually defined on a discrete filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^{T}, \mathbb{P}\right)$. The atoms of Ω are sequences of real-valued vectors at discrete time periods $t \in \mathcal{T}$, with $\mathcal{T} = \{0, 1, .., T\}$. We define $\{\mathcal{N}_t\}_{t=0}^{T}$ as a sequence of partitions of Ω such that $\mathcal{N}_0 = \Omega$, $\mathcal{N}_T = \{\{\omega_1\}, ..., \{\omega_T\}\}$ and where each element $n \in \mathcal{N}_t$ is equal to the union of some elements in $\mathcal{N}_{t+1}$ for every $t < T$. This succession of partitions of Ω defines uniquely the information structure of the probability space and each σ-algebras $\mathcal{F}_t$ is generated by the partition $\mathcal{N}_t$ and the usual properties on the filtration hold: $\mathcal{F} = \{\phi, \Omega\}, \mathcal{F}_t \subset \mathcal{F}_{t+1}, \forall t \in \mathcal{T}$ and $\mathcal{F}_T = \mathcal{F}$. At the first time $t = 0$ every state $\omega \in \Omega$ is possible whereas at the final time period $t = T$ we know exactly which $\omega \in \Omega$ is the real state of the world. At each intermediate stage $0 < t < T$ the investors know that for some subset A_t of $\mathcal{N}_t$ the true state is some $\omega \in A_t$, but they are not sure which one it is. The probability space can be viewed as a non-recombinant scenario tree and the elements n of each partition $\mathcal{N}_t$ are called nodes. Every node $n \in \mathcal{N}_t$ for $t = 1, ..., T$ has an unique parent denoted $a(n) \in \mathcal{N}_{t+1}$ and every node $n \in \mathcal{N}_t$ for $t = 0, ..., T-1$ has a non-empty set of child nodes $\mathcal{C}(n) \subset \mathcal{N}_{t+1}$. The probability distribution $\mathbb{P}$ is such that $\sum_{n \in \mathcal{N}_T} p_n = 1$ for the terminal stage and $p_n = \sum_{m \in \mathcal{C}(n)} p_m$, $\forall n \in \mathcal{N}_t$, $t = T-1, ..., 0$. The conditional probability that the node m occurs, given that the parent value is $n = a(m)$ has occurred, is defined by $p_{m|n} = \frac{p_m}{p_n}$, with $m \in \mathcal{C}(n)$. We call the sub-tree associated to the node n at the stage t the one-period tree composed by the node n and by its child nodes $\mathcal{C}(n)$.

The financial market is described by a finite set of I liquid assets indexed by the set $\mathcal{I} = \{0, 1, 2, ..., I\}$ with price $S_{i,t}$ at t that can be traded at $t = 0, ..., T$. The stochastic riskless security price is denoted by the index $i = 0$ so that the discount factor of the economy is $\frac{1}{S_{0,n}}$ and the discounted asset price of the $i - th$ security is

$$Z_{i,m} = \frac{S_{i,m}}{S_{0,m}}.$$

We assume that $\{S_{i,t}\}_{t=0}^{T}$ is a stochastic process adapted to the filtration process $\{\mathcal{F}_t\}_{t=0}^{T}$. This implies that also the return process $\{r_i\}_{t=0}^{T}$ defined as $r_{i,t} = (S_{i,t} - S_{i,t-1})/S_{i,t-1}$ is an adapted process with respect to the same filtration process. We can uniquely define the expectation of $S_{i,t}$:

$$E_{\mathbb{P}}[S_{i,t}] = \sum_{n \in \mathcal{N}_t} p_n S_{n,i}$$

and the conditional expectation:

$$E_{\mathbb{P}}[S_{i,t+1}|\mathcal{F}_t] = \sum_{m \in \mathcal{C}(n)} \frac{p_m}{p_n} S_{i,m}$$

We define a dynamic trading strategy $x = \{x_t\}_{t=0}^{T}$ as a vector random process defining the portfolio $x_t = [x_{0,t}, x_{1,t}, ..., x_{I,t}]^T$ held in t, where $x_{i,t}$ represents the amount invested in the i-th asset. The portfolio value in each period will be then defined as $X_t = \sum_{i=0}^{I} x_{i,t}$. The pension fund net cash outflow is defined by the process $\{l_t\}_{t=0}^{T}$ adapted to the filtration $\mathcal{F}_t$. We define a stochastic vector $\xi_t = [r_t, l_t]$ as the vector containing all the stochastic processes considered in the model.

1.3.2 Optimization Model

The formulation of an optimal ALM planning problem for the pension fund is defined as an optimal control of a dynamic stochastic system on the discrete market model previously defined [5]. The state of the system is represented by a $\mathcal{F}_t$-measurable random loss process $Y = \{Y_t\}_{t=1}^{T}$ whose evolution in time is described by a set of transition functions g_t, for $t \in \mathcal{T}$:

$$\begin{aligned} Y_0 &= g_0(x_0), \\ Y_t &= g_t(Y_{t-1}, x_{t-1}, \xi_t), \ t = 1, \dots, T \end{aligned}$$

The quality of the adopted strategy at each stage t, $\forall t \in \mathcal{T}$, is measured by a risk measure $\rho_t(Y_t)$ of the random loss process, where ρ_t is a convex function on the space of real-valued random variables. The dynamic trading strategy x is assumed to belong to a control space $\mathcal{X}$ which defines the feasible region of the problem. The space $\mathcal{X} = \mathcal{X}_0 \times ... \times \mathcal{X}_T$ is normally assumed to be convex for solvability problems and it represent constraints on the trading strategy which arise from regulatory, business or specific problem issues.

Optimal Stochastic Control Problem:

$$\min_{x} \sum_{t=0}^{T} \rho_t \left(Y_t\right) \tag{1.1}$$

s.t.:

$$Y_0 = g_0\left(x_0\right),$$

$$Y_t = g_t\left(Y_{t-1}, x_{t-1}, \xi_t\right), \ t = 1, ..., T$$

$$x_t \in \mathcal{X}_t$$

In real pension fund applications the random process $\{Y_t\}_{t=0}^{T}$, the risk measures $\rho_t\left(Y_t\right)$, for $t \in \mathcal{T}$, and the feasibility state space $\mathcal{X}$ can assume different specifications. In the next two sections some specific formulations for the objective functions and the constraints pension fund problems are presented.

1.3.3 Constraints

The feasibility space $\mathcal{X}$ is defined by a set of constraints which can be divided into two main categories. The first class contains the structural constraints that ensure the time consistency of the variables that enter in all the dynamic stochastic programming ALM models. These are essentially the initialisation values of the variables and the equations describing the evolution of such values originated by the strategy implemented between two periods. It is a common practice to introduce the non-negative variables $x_{i,t}^{+}$ and $x_{i,t}^{-}$ for $i \in \mathcal{I}$ and $t \in \mathcal{T}$ which represent the amount purchase and sold respectively in each asset at t. The parameters $\bar{x}_{i,t}$ for $i \in \mathcal{I}$ are the initial values of the portfolio assets and cash at $t = 0$ before the optimization procedure, and represent input of the model. The inventory constraints are used to update the trading strategy $x_{i,t-1}$ from $t-1$ to t for each asset given the financial returns realised and the choice of sales $x_{i,t}^{-}$ and purchases and $x_{i,t}^{+}$. The cash balance constraints are instead set to update the cash account value from $t-1$ to t given the trading strategy implemented in t, taking into account the transaction costs θ_i^{-} for sales and the transaction costs θ_i^{+} for purchases, and the exogenous expenditure to cover the liability process l.

Structural Constraints

- Initial cash balance constraints :

$$x_{0,0} = \bar{x}_{0,0} + \sum_{i=1}^{I} \left(1 - \theta_i^{-}\right) x_{i,0}^{-} - \sum_{i=1}^{I} \left(1 + \theta_i^{+}\right) x_{i,0}^{+}.$$

- Initial assets inventory constraint:

$$x_{i,0} = \bar{x}_{i,0} + x^{+}_{i,0} - x^{-}_{i,0}, \text{ for } i = 1, \ldots, I.$$

- Cash balance constraints:

$$x_{0,t} = (1 + r_{0,t})\, x_{0,t-1} + \sum_{i=1}^{I} \left(1 - \theta^{-}_{i}\right) x^{-}_{i,t} - \sum_{i=1}^{I} \left(1 + \theta^{+}_{i}\right) x^{+}_{i,t} - l_t.$$

- Inventory balance constraints:

$$x_{i,t} = (1 + r_{i,t})\, x_{i,t-1} + x^{+}_{i,t} - x^{-}_{i,t}, \text{ for } i = 1, \ldots, I.$$

The second class contains particular functional constraints that are essential to represent the environment in which the company operates such as institutional (law, regulations, ...) or market rules (specific contracts for some asset). Cash borrowing and short selling constraints are defined to fix a limit on the negative value which each asset's invested amount can assume. Position limits constraints limit the amount invested in an asset to be less than some proportion $\phi < 1$ of the fund wealth, whereas the turnover constraints limit the approximate change in the fraction of total wealth invested in some equity or bond asset i from one time to the next to be less than some proportion of the fund wealth $\upsilon_i < 1$. Chance constraints require the probability of an event to be less then a pre specified parameter β. Typically the chance constraint is implemented to control the tail risk and it is imposed to the event in which the difference between some target parameter $\tilde{X}_t$ and the portfolio value X_t is negative. Using this type of constraint generally transforms the optimization procedure into a non-linear programming problem. Although different relaxing and linearisation techniques can be applied, these methods need a large number of auxiliary variables which often increase substantially the problem size.

Functional Constraints

- Solvency constraints: $X_t \geq 0$.

- Cash borrowing limits: $x_{0,t} \geq \delta_0$, with $\delta_0 \leq 0$.

- Short selling constraints: $x_{i,t} \geq \delta_i$, $\forall i = 1, \ldots, I$, with $\delta_0 \leq 0$.
- Position Limits: $x_{i,t} \leq \phi_i X_t$, $\forall i = 1, ..., I$.
- Turnover constraints: $|x_{i,t} - x_{i,t-1}| \leq \upsilon_i X_{a(n)}$, $\forall i = 1, \ldots, I$.
- Chance Constraints: $P\left(\tilde{X}_t - X_t \leq 0\right) \leq \beta$.

1.3.4 Objective function

The pension funds manager seeks the self-financing trading strategy which maximises the employers wealth and, at the same time, ensures an adequate level of risk. The risk-return trade-off faced by the investor is incorporated in the model by the choice of specific functionals for the set of risk measures ρ_t, $\forall t \in \mathcal{T}$, previously introduced. In many models each risk mapping ρ_t is composed by a convex combination of different sub risk measures, each of whom is applied on a different loss process Y and controls a particular aspect of the problem. The typical approach is to consider a convex combination between the opposite of the expected portfolio value $E_{\mathbb{P}}[-X_t]$ and a down-side risk measure on the loss process Y. Downside-risk measures penalise only the events in which the loss process jumps above a prespecified maximum risk level $\tilde{Y}_t$. Example of down-side risk measures are:

- *Expected Target Shortfall*
  $\rho_t[Y_t] = E_{\mathbb{P}}\left[max\left(0, Y_t - \tilde{Y}_t\right)\right]$.
- *Entropic risk measure*
  $\rho_t[Y_t] = \frac{1}{\gamma} E_{\mathbb{P}}\left[\exp^{\gamma\left(Y_t - \tilde{Y}_t\right)}\right]$.
- *Conditional Value at Risk* :
  $\rho_t[Y_t] = CVaR_\alpha\left(\left[Y_t - \tilde{Y}_t\right]\right)$.

As an example, let's consider the situation in which the risk at period s is computed as the negative deviation of the portfolio value X_s from a given target level $\tilde{X}_s$. The objective function can be then defined as a convex combination between the expected value and the expected target shortfall at each period:

$$\beta \sum_{s=t}^{T} E_{\mathbb{P}}[-X_s] + (1-\beta) \sum_{s=t}^{T} E_{\mathbb{P}}\left[max\left(0, \tilde{X}_s - X_s\right)\right], \tag{1.2}$$

where $\beta \in [0,1]$. In this case the parameter β is a control risk parameter. When it is set equal to zero the optimiser only penalises the events in which the portfolio value is below the specified target. Increasing the value of β also the expected portfolio value is considered and the negative deviations from the target are less weighted.

In general, the loss function and the associated target parameter $\tilde{Y}_t$ can be set according to risks related to different economic and financial criteria. In some applications for example the loss process is set equal to the opposite of the portfolio value and the target is a benchmark portfolio return [14, 11]. The definition of the risk measures is generally also dependent on the type of plan (DB vs DC) considered.

In the DB case for instance, the liabilities structure depends strongly on the salary evolution and on life expectations. It is so common to introduce a downside risk measure on the funding ratio in the objective function in order to control the liabilities process. A common approach is to set a target of 1 for the funding ratio to avoid the situation in which the degradation of the funding ratio at some time period can lead to a highly exposure to possible under funding occurrences [7, 10, 15]. In this case the target is a linear combination of the funding ratio at the beginning stage and the funding ratio at the last stage.

In the DC case instead it is more important to try to guarantee a minimal rate of return or at least to try to minimise the possible losses. This is due by two main reasons. Firstly, in this case the liabilities structure is mainly dependent on the investment return, and a great loss in one period could lead to the necessity of sell some assets to cover the gap between cash and current payments. Secondly, in many countries pension funds operate in competitive markets and they have to offer a best product with respect to the competitors. In some models also the employers contribution rate process enters in the objective function as a sub risk measure and the pension fund manager also seeks for the optimal financial strategy which maximises the wealth for a minimal contribution [7, 11, 15].

1.3.5 Deterministic Equivalent Representation

In a discrete model with a finite number of stages and a discrete partition such that presented in section 1.3.1, where the uncertain parameters are described by a non-recombinant scenario tree, the *Optimal stochastic control problem* 1.1 is usually solved by means of its equivalent deterministic representation form. We now present a specific formulation of the MSP problem which will be used in the next chapters of the thesis, although with some variations. We define as $\mathcal{N}$ the set of all nodes at every stage t, $\forall t \in \mathcal{T}$, and with $x_n = [x_{0,n}, x_{1,n} \ldots, x_{I,n}]$, $x_n^+ = \left[x_{0,n}^+, x_{1,n}^+ \ldots, x_{I,n}^+\right]$, $x_n^- = \left[x_{0,n}^-, x_{1,n}^- \ldots, x_{I,n}^-\right]$ the vectors containing the decisions for all the assets at a given node n. The equivalent deterministic formulation will be:

ALM Problem

$$\min_{x_n, x_n^+, x_n^-; \forall n \in \mathcal{N}} \sum_{t=0}^{T} \sum_{n \in \mathcal{N}_t} p_n \left(-\beta X_n + (1-\beta) \left[\tilde{X}_n - X_n \right]^+ \right) \tag{1.3}$$

s.t.:

$$x_{i,0} = \bar{x}_{i,0} + x_{i,0}^+ - x_{i,0}^- \qquad i \in \mathcal{I} \backslash \{0\} \tag{1.4}$$

$$x_{0,0} = \bar{x}_{0,0} + \sum_{i=1}^{I} x_{i,0}^- - \sum_{i=1}^{I} x_{i,0}^+ \tag{1.5}$$

$$x_{i,n} = (1 + r_{i,n}) \cdot x_{i,a(n)} + x_{i,n}^+ - x_{i,n}^-, \qquad i \in \mathcal{I} \backslash \{0\}, n \in \mathcal{N}_t, t \geq 1, \tag{1.6}$$

$$x_{0,n} = (1 + r_{0,n}) \cdot x_{0,a(n)} + \sum_{i=1}^{I} x_{i,n}^- - \sum_{i=1}^{I} x_{i,n}^+ - l_n, \qquad n \in \mathcal{N}_t, t \geq 1, \tag{1.7}$$

$$X_n = \sum_{i=0}^{I} x_{i,n}, \qquad n \in \mathcal{N}_t, t \geq 0, \tag{1.8}$$

$$x_{i,n}^+ \geq 0, \qquad i \in \mathcal{I} \backslash \{0\}, n \in \mathcal{N}_t, t \geq 0, \tag{1.9}$$

$$x_{i,n}^- \geq 0, \qquad i \in \mathcal{I} \backslash \{0\}, n \in \mathcal{N}_t, t \geq 0, \tag{1.10}$$

$$x_{i,n} \geq 0, \qquad i \in \mathcal{I}, n \in \mathcal{N}_t, t \geq 0. \tag{1.11}$$

Where $\beta \in (0,1)$ and $\left[\tilde{X}_s - X_s\right]^+ = \max\left(0, \tilde{X}_s - X_s\right)$. The above problem is a large deterministic convex problem with linear constraints. The convex linear function can be linearised introducing the auxiliary variables η. To do so we replace the objective function with: $\sum_{t=0}^{T}\sum_{n\in\mathcal{N}_t} p_n\left(-\beta X_n + (1-\beta)\,\eta_n\right)$ and we add the constraints: $\tilde{X}_n - X_n \leq \eta_n$, for $n \in \mathcal{N}_t, t \geq 0$.

1.3.6 Scenario Tree Generation

Until now we have presented an optimal control problem designed on a discrete market model framework for a pension fund manager who seeks for an optimal policy over an investment time horizon T . We have also showed that this optimal control problem can be reformulated as a large scale deterministic problem when the uncertain evolution of the risk factors driving the variables involved in the problem is represented by a non-recombinant scenario tree process. The equivalent deterministic representation is in this case possible because we have completely described the future uncertainty with a finite set of possible realisations by means of the scenario tree process $\{\xi_t\}_{t=0}^{T}$, with $\xi_t = [r_t, l_t]$. A crucial issue for a successful implementation of multistage stochastic programming models is the specification of the mass points r_n and l_n for $n \in \mathcal{N}_t$ and $t = 0, ..., T$.
As a first step, an econometric model for economic, actuarial and financial variables must be designed and calibrated. This procedure can be quite complicated, because many risky factors affect the evolution of assets and liabilities of a large pension fund scheme. The econometric model can be defined both in discrete and in continuous time. The second step is to find an efficient technique to specify the scenario tree $\{\xi_t\}_{t=0}^{T}$ which well represent the possible evolution of the estimated econometric model, and which will be used as an input in the equivalent deterministic representation problem. The grater the number of nodes in the scenario tree, the more accurate is the approximation. However, increasing the number of nodes also increases the computational effort to solve the problem. This consideration implies that we face a trade-off between the accuracy of the risk representation and the practical problem solvability. An important question is the extent to which the approximation error in the event tree will bias the optimal solutions of the model. Different approaches to specify the input parameter r_n and l_n for $n \in \mathcal{N}_t$ and $t = 0, ..., T$ have been proposed, see [8] and the references therein for a quite recent critical overview.

Another important feature a scenario tree for asset returns should satisfy, when applied to financial planning problems designed with MSP, is the absence of arbitrage opportunities. An arbitrage opportunity is a self financing strategy which guarantees to have a profit from nothing. If there is an arbitrage opportunity in the event tree, then the optimal solution of the stochastic programming model will exploit it. An arbitrage strategy creates profits with no risk, decreasing the objective value of the risk measure associated with the financial problem. A cautious design of a long term ALM model should consider scenario trees that do not allow for arbitrage. It has been pointed out that if arbitrage opportunities do arise in practice, then professional arbitrageurs will exploit them on a very short notice, while the focus of ALM modelling is on long term decisions [22]. In the next chapter of this thesis we will propose a scenario tree generation method which considers directly the problem of avoiding arbitrage opportunity. In the next Section we formally define an arbitrage opportunity and some important relations with the discounted price process Z that will be used in the tree generation method to prevent arbitrages.

1.3.7 Martingale and Arbitrage Conditions

Arbitrage opportunity

Modern financial theory is strictly linked to the notion of arbitrage. Ingersoll [18] distinguishes two types of arbitrage opportunities. We define the arbitrage opportunities on a sub-tree emanating from the node n with a number of child nodes equal to $\mathbf{card}(\mathcal{C}(n))$, where $\mathbf{card}(S)$ states the cardinality of the set S. An arbitrage opportunity of the first type exists if there is an investment strategy $x_n = [x_{0,n}, ..., x_{I,n}]$ such that:

$$\begin{aligned}
&\textit{First type Arbitrage Opportunity}\\
&\sum_{i=0}^{I} x_{i,n} = 0\\
&\sum_{i=0}^{I} (1 + r_{i,m})\, x_{i,n} \geq 0, \forall m \in \mathcal{C}(n)\\
&\sum_{i=0}^{I} (1 + r_{i,m})\, x_{i,n} > 0, \text{for at least one } m \in \mathcal{C}(n)
\end{aligned}$$

In this case we start with a zero value portfolio and we obtain in the next period a portfolio with a positive value in at least one state of the world. An arbitrage opportunity of the second type exists if there is an investment strategy $x_n = [x_{0,n}, ..., x_{I,n}]$ such that:

$$\begin{aligned}
&\textit{Second type Arbitrage Opportunity}\\
&\sum_{i=0}^{I} x_{i,n} < 0\\
&\sum_{i=0}^{I} (1 + r_{i,m})\, x_{i,n} \geq 0, \forall m \in \mathcal{C}(n)
\end{aligned}$$

In this case we start with a strictly negative investment value and we end with a portfolio with non-negative value in every state of the world. Since the existence of one type of arbitrage opportunity does not imply in general the existence of the other type of arbitrage they must be checked separately. Klaassen [21] has showed how to check the existence of this two types of opportunity.

$$\begin{aligned}
&\textit{Checking the First type Arbitrage Opportunity} \\
&\max_{x_n} \sum_{m\in\mathcal{C}(n)} \sum_{i=1}^{I} x_{i,n}\left(1+r_{i,m}\right) \\
&\text{s.t}: \\
&\sum_{i=0}^{I} x_{i,n} = 0 \\
&\sum_{i=0}^{I} x_{i,n}\left(1+r_{i,m}\right) \geq 0, \forall \in \mathcal{C}\left(n\right)
\end{aligned} \tag{1.12}$$

If there is a feasible solution to this linear program with a positive objective value, then there is at least one node m in which the asset allocation x_n yields a strictly positive return. In this case the problem is unbounded since it is possible to increase the objective value by multiplying the vector x_n by a positive constant without lose the feasibility.

$$\begin{aligned}
&\textit{Checking the Second type Arbitrage Opportunity} \\
&\min_{x_n} \sum_{i=1}^{I} x_{i,n} \\
&\text{s.t}: \\
&\sum_{i=0}^{I} x_{i,n}\left(1+r_{i,m}\right) \geq 0, \forall m \in \mathcal{C}\left(n\right)
\end{aligned} \tag{1.13}$$

If this linear program has a feasible solution with a negative objective value, then there will be an arbitrage opportunity of the second type. In this case, indeed, we can decrease the objective value by multiplying the vector x_n by a positive constant without lose the feasibility. It has been proven that if all the sub-trees do not have arbitrage opportunities also the entire tree does not have arbitrage opportunities [25].

Martingales and Arbitrage opportunities

The discounted price vector Z_t is called a martingale under the probability measure $\mathbb{Q}$ if:

$$Z_t = E_{\mathbb{Q}}\left[Z_{t+1}|\mathcal{F}_t\right], \text{ for } 0 \leq t \leq T-1.$$

In this case $\mathbb{Q}$ is called a risk neutral probability measure for the process Z_t. In case $Z_t \geq E_{\mathbb{Q}}\left[Z_{t+1}|\mathcal{F}_t\right]$, for $0 \leq t \leq T-1$ the process is called a supermartingale under $\mathbb{Q}$. In case $Z_t \leq E_{\mathbb{Q}}\left[Z_{t+1}|\mathcal{F}_t\right]$, for $0 \leq t \leq T-1$, the process is called a submartingale under $\mathbb{Q}$. One of the fundamental theorems in mathematical finance states that the discounted stochastic price process Z_t is an arbitrage-free market price process iff there is at least one probability measure Q equivalent to P under which Z_t is a martingale. The proof of the theorem in a discrete market framework, such as that presented in the paragraph 1.3.1, can be found in [20, 25]. The result can be expressed in term of returns [25]. The return tree process $\{r_t\}_{t=0}^{T}$ is an arbitrage free process iff there is at least one risk neutral probability measure $\mathbb{Q}$ such that:

$$\sum_{m\in\mathcal{C}(n)} q_{m|n}\frac{r_{i,m}-r_{0,m}}{1+r_{0,m}} = 0,\ \ i=1,...,I,\ \ n\in\mathcal{N}_t,\ \ t=1,...,T-1, \tag{1.14}$$

where the risk neutral probability distribution $\mathbb{Q}$ is such that $\sum_{n\in\mathcal{N}_T} q_n = 1$ for the terminal stage and $q_n = \sum_{m\in\mathcal{C}(n)} q_m$, $\forall n \in \mathcal{N}_t$, $t = T-1, ..., 0$. The conditional probability that the node m occurs given that the parent value $n = a(m)$ has occurred is defined by $q_{m|n} = \frac{q_m}{q_n}$, with $m \in \mathcal{C}(n)$.
When the risk free interest rate process $\{r_{0,t}\}_{t=0}^{T}$ is deterministic and equal to $r_{0,t}$ for all the nodes in $\mathcal{N}_t$, the formulas 1.14 become:

$$\sum_{m\in\mathcal{C}(n)} q_{m|n} r_{i,m} = r_{0,m},\ \ i=1,...,I,\ \ n\in\mathcal{N}_t,\ \ t=1,...,T-1. \tag{1.15}$$

Let's now assume that we have chosen the values $r_{i,n}$, for $i = 1, ..., I$, $n \in \mathcal{N}_t$ and $t = 1, ..., T$ of the returns scenario tree. The formulas 1.14, for a given choice of the stage t, and of the node $n \in \mathcal{N}_t$, can be then used to define a system of linear equations in the unknown variables $q_{m|n}$, for $m = 1, ..., N_t$. The system presents N_t variables and I constraints. This implies that if $I = N_t$ the system of linear equation is fully determined and an unique solution is guaranteed. When $I < N_t$ the system is underdetermined and infinitely many solutions exist. Finally, if $I > N_t$ the system is overdetermined and no solution exists. This means that the necessary condition to obtain an unique risk measure probability $\mathbb{Q}$ is that $N_t = I$ for all $t = 1, ..., T$. The condition is not sufficient since the probabilities $q_{m|n}$, for $m = 1, ..., N_t$ must be non negative. This implies that we are interested only in the non-negative solutions of the system.

Bibliography

[1] Barr N., *Reforming pensions: myths, truths, and policy choices*, International Social Security Review, Vol. 55, No. 2, pp. 336, 2002.

[2] Barr N., *Pensions: overview of the issues*, Oxford Review of Economic Policy, Vol.22 (Spring), No. 1, pp. 114, 2006.

[3] Chong S. F., *Building a Simplified Stochastic Asset Liability Model (ALM) for a Malaysian Participating Annuity Fund*, FSA, 14th East Asian Actuarial Conference, June 2007.

[4] Consigli, G., Dempster, M.A.H.: *Dynamic stochastic programming for asset-liability management*, Ann. Oper. Res. 81, 131-161, 1998.

[5] Consigli G., Kuhn D., Brandimarte P., *Optimal Financial Decision Making Under Uncertainty*, Optimal Financial Decision Making under Uncertainty, Volume 245 of the series International Series in Operations Research & Management Science, pp 255-290, 2016.

[6] Broadbent, Palumbo, Woodman - *The Shift from Defined Benefit to Defined Contribution Pension Plans -Implications for Asset Allocation and Risk Management*, Prepared for a Working Group on Institutional Investors, Global Savings and Asset Allocation established by the Committee on the Global Financial System, 2006.

[7] Dempster M.A.H., Germano M., Medova E., Murphy J., Ryan D., Sandrini F., *Risk Profiling Defined Benefit Pension Schemes*, The Journal of Portfolio Management, Vol.35(4), pp.76-93, 2009.

[8] Dempster M. A. H., Medova E. A., Yong Y. S., *Comparison of Sampling Methods for Dynamics Stochastic Programming*, Stochastic Optimization Methods in Finance and Energy, Ch. 16, Springer, 2011.

[9] Dempster M. A. H., Medova E. A., Rietbergen M. I., Sandrini F., Scrowston M., *Designed Minimum Guaranteed Return Funds, Stochastic Optimization Methods in Finance and Energy*, Ch. 2, Springer, 2011.

[10] Dondi G., Herzog F., *Dynamic Asset and Liabilities Management for Swiss Pension Funds*, Handbook of Asset and Liability Management Volume 2,Ch.20, 2007.

[11] Dupaĉová J., Polìvka J., *Asset-liability management for Czech pension funds using stochastic programming*, Annals of Operations Research, Vol.165(1), pp.5-28, 2009.

[12] Eichhorst W., Gerard M., Kendizia M. J., Mayrhuber C., Nielsen C., Runstler G., Url T., *Pension Systems in the EU Contingent Liabilities and Assets in the Public and Private Sector*, EP Study P/A/ECON/ST/2010, 26, European Parliament, Brussels, 2011.

[13] EIOPA, *Technical documentation of the methodology to derive EIOPA's risk-free interest rate term structures*, EIOPA-BoS-15/035, 7 December 2015.

[14] Geyer A., Ziemba W. T., *The Innovest Austrian Pension Fund Financial Planning Model InnoALM*, Operations Research, 2008, Vol.56(4), p.797-810.

[15] Haneveld W. K. K., Streutker M.H., van der Vlerk M. H., *An ALM model for pension funds using integrated chance constraints*, Annals of Operations Research, 2010, Vol.177(1), pp.47-62.

[16] Hilli P., Koivu M., Pennanen T., Ranne A.,*A stochastic programming model for asset liability management of a Finnish pension company*, Annals of Operations Research, 2007.

[17] Horstmann S., *Pensions, Health Care and Long-term Care, Synthesis Report*, Annual National Report 2012, Analytical Support on the Socio-Economic Impact of Social Protection Reforms (asisp), Cologne, 2012.

[18] Ingersoll J. E., *Theory of Financial Decision Making*, Totowa, N.J.: Rowman & Littlefield, 1987.

[19] Lannoo K., Barslund M., Chmelar A., von Werder M., *Pension Schemes*, Policy Department A, Economic and scientific policy, 2014.

[20] King A., *Duality and martingales: a mathematical programming perspective on contingent claims*, IBM Technical Report, T.J. Watson Research Center, Yorktown Heights, NY, 2000.

[21] Klaassen P.,*Comment on "generating scenario trees for multistage decision problems"*, Management Science, 48(11),1512-1516, 2002.

[22] Kouwenberg R., Zenios S. A., *Stochastic programming models for asset liability management*, Handbook of Asset and Liability Management 2008, Volume 1, Chapter 6, Pages 253-303.

[23] *OECD - Pension Markets in Focus*, 2015

[24] Pelsser A., Salahnejhad A., van den Akker R., *Market-consistent valuation of pension liabilities*, Design paper 63, Netspar industry paper series, 2016.

[25] Pliska S.R., *Introduction to Mathematical Finance, Discrete time models*, Blackwell, Malden, MA, 1997.

[26] Sender S., *Shifting Towards Hybrid Pension Systems: A European Perspective*, EDHEC-Risk Institute Publication, March 2012.

[27] Sweeting P. J., *Modelling and Managing Risk*, British Actuarial Journal, 2008, vol. 14, issue 01, pages 111-125.

[28] World Bank, *The Inverting Pyramid: Pension Systems Facing Demographic Challenges in Europe and Central Asia*, 2014, Washington, DC: World Bank.

[29] Wesbroom K., Reay T., Britain G., Hewitt B., *Hybrid pension plans: UK and international experience*, Corporate Document Services, Leeds. Research Report 271, Department for Work and Pensions, 2005.

[30] Yermo J., *Reforming the Valuation and Funding of Pension Promises: are Occupational pension Plans Safer?*, OECD Working Papers on Insurance and Private Pensions, No. 13, OECD Publishing, 2007.

Chapter 2

Scenario Generation Method

2.1 Introduction

Problems formulated as multistage stochastic programming with complex decisions structured by numerous constraints, or that lie in high-dimensional spaces, are in general not analytically tractable. The solutions are then obtained by means of a discrete approximation of the continuous state probability space. Typically this discretisation is performed by the construction of a discrete event tree and then the stochastic optimization problem is transformed in the deterministic equivalent reformulation. Since the discrete tree forms the input specification of the uncertain parameters for the optimization problem, the quality of the approximation affects strongly the solution error with respect to the true incalculable solution. Increasing the number of discrete points in the approximation potentially reduces the solution error with the cost of having a larger deterministic problem to solve. An optimal choice of the size of the discretisation and of the approximation technique is then a crucial part in the MSP modelling. A large body of literature has been devoted to elaborate methods to find the best approximation according to some statistical property and/or to some information criteria such as the expected value of perfect information. A commonly used criteria to evaluate the approximation error is the stability of the objective function value of the deterministic reformulation of the MSP problem. This error can be defined in terms of in-sample and out-of-sample analysis [24]. However it has been pointed out that in financial applications we are mainly interested in the root node solutions which generally include the portfolio allocation that the decision maker will implement in practice. Therefore, the in-sample stability should be measured with respect to both the objective function values and the first stage implementable decisions [9].

A basic approach to generate the scenario tree is to sample from a statistical model. At the root node a sample is generated in order to obtain the first stage sub-tree nodes; the operation is then repeated for each node at the first stage so that a set of sub-trees for the second stage is generated; the procedure is carried on until the tree is generated for the entire horizon [5, 28, 35]. Although the approach is quite simple, it requires a large sample size to well approximate the original distribution function and to produce stable results [9]. Variance reduction techniques, such as antithetic variables, importance and stratified sampling can be applied to mitigate these problems, but the sample size required to obtain stable results remains often too high for solving complex MSP problems. In order to overcome these issues, more complex tree generation methods have been proposed.

Scenario tree generation algorithms that are based on cluster analysis construct the tree starting from a large fan of independent trajectories [2, 11, 16]. These techniques rely on two main phases implemented stage by stage until the entire original large tree is reduced to a given topology: at each stage the original scenarios are grouped into different clusters and then one representative scenario to keep in the reduced tree is selected. A variant of this approach is the sequential cluster simulation method, in which the large simulation phase is performed along the cluster/selection phases at a given stage; then for each of the obtained nodes a new large simulation is generated, clustered and reduced in order to obtain the next stage tree values.

Another class of tree generation method is based on an optimization procedure which simultaneously determines the tree nodes values and the corresponding probabilities, such that the weighted square error between the theoretical and the tree set of moments and correlations are minimised. The tree structure must be prespecified in advance and it is an user input. The method, which was originally proposed by Høyland and Wallace [21], is not trivial to solve since it requires the solution of a non-linear and non-convex problem. Due to this fact, Høyland et al. [22] suggested to use an heuristic based on the cubic transformation to approximatively solve the problem. Ji et al. [23] proposed instead to minimise the absolute deviations from the target moments, so that the problem can be cast as a linear programming problem. Recently, two new moment matching techniques that include constraints to directly avoid arbitrage opportunity in the generated scenario tree has been proposed by Consiglio, Carollo and Zenios [7] and by Staino and Russo [36]. In the former paper the authors implements a set of transformation on the non-linear system of equations describing the moment matching problem, with and additional set of constraints to handle the requirement of absence of arbitrages, in order to apply a branch-and-bound type global optimization approach. In [36] the same non-linear system of equations describing the moment matching problem is approximated by a sequence of monomial approximation, and then linearised by a logarithmic transformation. The Moore-Penrose pseudo inverse of the coefficient matrix of the linear system is used to solve the problem when the system is undetermined. It has been shown that in general, two different probability measures can have all moments equal [31], therefore two distributions that share the same moments can have very different shapes. This in turn means that matching the moments do not ensure a good approximation of the original distribution.

Reduction techniques produce a smaller scenario tree by deleting and merging the scenarios of a larger scenario tree, or a large scenario fan, obtained with one of the methods mentioned above. These methods search the best discrete probability measure that approximates the large scenario tree on a smaller support in terms of an appropriate probability metric [18, 20, 31, 32]. The probability metric is chosen on the base of stability results of MSP [17, 19, 33]. The idea is to minimise the supremum of the distance between the solution of the MSP problem computed using a large scenario tree, and the solution obtained with the approximated tree with a smaller number of nodes. This minimisation problem has been proved to be equivalent to the minimisation of the Wasserstein distance between the distribution function of the large tree and the distribution function of the smaller approximated tree. The Wasserstein metric can be written as a mass transportation problem in which the cost of transporting the probability masses from the support points of the original large distribution to the new fewer support points of the smaller approximation is minimised. Find the optimal reduced tree can be then stated as the problem of finding the new fewer support points and their probabilities such that the Wasserstein distance with respect to the large discrete distribution is minimised. The theoretical problems of find such an optimal discretisation are known to be NP-hard and approximation algorithms and heuristics are usually used [18]. Dempster et al. [9] have proposed an heuristic method to solve the problem and they called it *Sequential Wasserstein Distance Minimisation* (SMWD). The SMWD is an heuristic method which attempts to find the new probabilities and the new support points by iteratively solving two problems. The idea is to firstly fix the support of the small distribution and find the best probabilities (fixed location problem) which minimise the Wasserstein distance, and secondly fix the probabilities values just obtained to find the best mass points (fixed flow problem) which again minimise the Wasserstein distance. The two problems are solved iteratively until the Wasserstein distance after an iteration does not decrease more then a pre-specified tolerance level. The method has been extensively tested on a financial case study against different tree generation methods belonging to different categories [9].

Hybrid methods are structured as a combination of scenario tree generation approaches described above [1, 16, 34, 38]. Xu at al. [38], for example, proposed a hybrid algorithm in which cluster analysis and moment matching are combined. The algorithm receives as inputs a large scenario fan, obtained by numerical simulation from an econometric model, and a pre-specified branching structure defining the topology of the tree that we want construct. At the first stage they perform a K-means algorithm to cluster values of the scenario fan in a number of classes equal to the pre-specified branching structure. The mean vector of each classes is then assigned to each node at the first stage. The moments and the covariances of the scenario fan at the first stage are then computed to solve a moment matching problem, where the values of the nodes are used as an input to find the optimal real world probabilities that best match the fan scenario moments and covariances. They then used the values of the second stage scenario fan, which are associated to each cluster previously obtained, as the large fan to generate each sub-tree of the second stage. The procedure is repeated until the last stage.

In this chapter we propose a hybrid scenario tree generation technique that we have used in the next chapters of the thesis to solved different MSP problems. The algorithm generate the scenario tree, starting from a given large scenario fan and a given branching structure, in a forward fashion similar to that proposed by Xu et al. [38]. The choice of starting from a large scenario fan is due to the fact that in some cases it is difficult to exactly compute the distribution, and hence the moments, of the statistical model implemented to describe the uncertainty nature of the risky factors. The basic idea underlying the method presented in this chapter is to combine the SMWD algorithm with a moment matching approach with arbitrages constraints like those proposed in [7, 36]. The moment matching procedure is applied to overcome two relevant drawbacks of the SMWD algorithm: the variance underestimation and the lack of direct control on the absence of arbitrages. In order to apply the SMWD algorithm we are forced to start with a large scenario fan which is considered as the reference process we want to approximate by means of the scenario tree. The chapter is divided into two main parts. In Section 2.2 we describe in detail the proposed scenario tree generation algorithm. In Section 2.3 a case study on a simple three stages optimal portfolio problem is performed to test the algorithm with respect to the hybrid method of Xu et al. [38]. The letter algorithm indeed shares the main features of the tree construction procedure with the algorithm proposed and used in this thesis: a large scenario fan as input, a reduction phase functional to solve a moment matching problem and a cluster procedure used to replicate the time-conditional structure of the large scenario fan. The algorithm proposed in this thesis has the additional positive property of directly controlling the absence of arbitrage opportunities.

2.2 Scenario Tree Generation for Financial Returns with No-arbitrage Opportunity

2.2.1 Scenario Tree Construction

We present the scenario generation technique we have implemented to obtain an arbitrage free asset return scenario tree $\{r_t\}_{t=0}^{T}$, where $r_t = [r_{1,t}, ..., r_{I,t}]$, with features depicted in Section 1.3.1. We suppose to have an econometric model, describing the return dynamic of each asset, from which we can simulate a large number Sc of independent trajectories. The number Sc of trajectories must be decided in order to have a realistic representation of the uncertain nature of the chosen econometric model. We defined as $\bar{r}^s = [\bar{r}_0^s, \dots, \bar{r}_T^s]$, for $s = 1, \dots, Sc$, the trajectories of the fan scenario tree, where $\bar{r}_t^s$ is the multidimensional vector of I asset returns. The scenario tree $\{r_t\}_{t=0}^{T}$ will be constructed in a forward fashion using as inputs the scenario fan trajectories $\bar{r}^s$, $s = 1, \dots, Sc$, and a given symmetric branching structure $[N_1, \dots, N_T]$ describing the number of nodes in each sub-tree at each stages t. The branching structure is defined by the users but it must be such that $N_t \geq I + 1$ in order to satisfy the necessary condition for the absence of arbitrages. The method relies on the application of the SMWD algorithm along with a particular type of moment matching problem, which also considers a set of constraints to directly avoid arbitrage opportunities [7, 36], to generate each sub-tree starting from the root node. Since the moment matching problem is highly non-linear we have used a set of transformations and an approximation procedure in order to solve it more efficiently. The moment matching problem and the approximation procedure will be presented in section 2.2.2. The whole tree construction methodology can be summarised as follows:

1. Compute the first four moments and the covariances of the large fan tree trajectories $\bar{r}_1^s$, for $s = 1, ..., Sc$, at the first stage.

2. Run the SMWD algorithm on the first stage values of the scenario fan. In this way we obtain N_1 new points value and the related conditional probabilities. The first guess for the SMWD algorithm solution is obtained by a Cholesky decomposition approach to reduce the variance underestimation.

3. The values and the probabilities just obtained are used as the first guess solution for an approximated moment matching problem in order to match the moments and covariances of the first stage scenario fan. Solving the problem we then obtain the N_1 scenario tree values for each of the I asset returns and the corresponding N_1 real world and neutral probabilities.

4. Compute the euclidean distance between the original Sc scenarios and the

new N_1 points. We associated to each of the N_1 nodes, the nodes of the large fan with the minimal distance (with respect to all the N_1 points). In this way we obtain N_1 groups of the original Sc trajectories.

5. Move forward in time at the second stage and compute the first four moments and the covariances for each of these N_1 groups of the fan scenario tree at the second stage $t = 2$.

6. Run the SMWD algorithm and the approximated moment matching problem to each of the N_1 groups obtaining $N_1 \cdot N_2$ new nodes for the second stage and the corresponding probabilities.

7. Repeat the euclidean distance procedure to each of the N_1 subgroup obtaining N_2 sub groups for each of the N_1 groups.

8. Repeat the procedure until the end of the horizon.

According to the above structure the scenario tree is constructed in a forward fashion starting from the root node. The operation of clustering the trajectories of the large scenario fan, in order to define the moments that have to be matched in the successive stage, has been developed with the aim of taking into account conditional variance, or in general, conditional correlations processes. In order to better clarify this issue let's consider the situation in which we have obtained the first stage nodes of the scenario tree by matching the moments of the trajectories of the large fan at the first stage. Now we have to construct a sub-tree describing the second stage uncertainty for each of the nodes that we have just obtained for the first stage and so we have to decide the moments that have to be matched for each of these second period sub-trees that we want to generate with the moment matching algorithm. To do that, we associate to a given first stage tree node the trajectories of the large fan at the first stage which are closer to this node. The moments of the second stage values of these trajectories will be then computed and they will form the inputs of the moment matching problem to generate the sub-tree departing from the given node. In this way we can try to consider the time dependency, although under an heuristic approach, which is embedded in the asset returns conditional process.

2.2.2 Moment Matching via Geometric Programming

We define N_t as the total number of nodes in the sub-tree with ancestor node n at $t-1$, $\left[p_{1|n}, ..., p_{N_t|n}\right]$ as the vector of real world conditional probabilities given that the ancestor node is n, $\left[q_{1|n}, ..., q_{N_t|n}\right]$ as the vector of conditional risk neutral probabilities, μ_i as the mean , σ_i as the variance, ζ_i as the skewness, κ_i as the kurtosis of the i-th asset, $\sigma_{i,j}$ as the covariance between the $i-th$ asset and the

$j-th$ asset that we want to match. We recall that $r_{i,m}$ is the return of the i-th asset in the m-th node with $r_{0,m}$ the riskless return. The moment matching can be stated as the problem of finding the value of the variables $p_{m|n}$, $q_{m|n}$ and $r_{i,m}$, $m = 1, ..., N_t$, $i = 1, ..., I$ such that:

$$\sum_{m=1}^{N_t} p_{m|n} \cdot r_{i,m} = \mu_i, \qquad i = 1, \dots, I \tag{2.1}$$

$$\sum_{m=1}^{N_t} p_{m|n} \cdot (r_{i,m} - \mu_i)^2 = \sigma_i, \qquad i = 1, \dots, I \tag{2.2}$$

$$\sum_{m=1}^{N_t} p_{m|n} \cdot (r_{i,m} - \mu_i)^3 = \zeta_i \cdot \sigma^3, \qquad i = 1, \dots, I \tag{2.3}$$

$$\sum_{m=1}^{N_t} p_{m|n} \cdot (r_{i,m} - \mu_i)^4 = \kappa_i \cdot \sigma^4 \qquad i = 1, \dots, I \tag{2.4}$$

$$\sum_{m=1}^{N_t} p_{m|n} \cdot (r_{i,m} - \mu_i) \cdot (r_{j,m} - \mu_j) = \sigma_{i,j}, \quad i = 1, \dots, I-1, \qquad j = i+1, \dots, I \tag{2.5}$$

$$\sum_{m=1}^{N_t} q_{m|n} \cdot r_{i,m} = r_0, \qquad i = 1, \dots, I \tag{2.6}$$

$$\sum_{m=1}^{N_t} p_{m|n} = 1 \tag{2.7}$$

$$\sum_{m=1}^{N_t} q_{m|n} = 1 \tag{2.8}$$

$$p_{m|n} > 0, \qquad m = 1, ..., N_t \tag{2.9}$$

$$q_{m|n} > 0, \qquad m = 1, ..., N_t. \tag{2.10}$$

In the above formulation we have assumed a constant risk free interest rate, i.e. $r_{0,m} = r_0$ for all the nodes m, with $m = 1, \dots, N_t$. The set of constraints (2.6) has been used in order to guarantee the absence of arbitrage opportunities; it has been derived from the formula (1.15), which states the relationship between the existence of the conditional risk neutral probabilities vector $\left[q_{1|n}, ..., q_{N_t|n}\right]$ and the absence of arbitrage opportunities. In the general case of a stochastic interest rate we have to use the formula (1.14) and the set of constraints (2.6) is replaced by $\sum_{m=1}^{N_t} q_{m|n} \frac{r_{i,m} - r_{0,m}}{1 + r_{0,m}} = 0$, $i = 1, \dots, I$. The choice of using a constant risk free interest rate has been motivated in order to simplify the exposition of the method. All the passages that will be showed in this chapter hold also in the case of a stochastic risk free interest rate. The constraints (2.7) and (2.8) are defined to ensure that the probability values will be positive and that the two probabilities measure $\mathbb{P}$ and $\mathbb{Q}$ will be equivalent.

The non-linear system (2.1)-(2.10) has been reformulated as a *Signomial Geometric Programming* (SGP) problem and then solved applying the global optimization approach proposed by Xu [37]. As a first step we add a positive constant c to the normalised returns in order to ensure that all the variables are positive:

$$z_{i,m} = \frac{r_{i,m} - \mu_i}{\sigma_i} + c, \ m = 1, ..., N_t$$

We call u the vector containing all the strictly positive variables p_s, q_s, $z_{i,s}$, for $s = 1, ..., N_t$ and $i = 1, ..., I$. The new target moments and covariances are now $\tilde{\mu}_{1,i} = c$, $\tilde{\mu}_{2,i} = 1+c^2$, $\tilde{\mu}_{3,i} = \zeta_i + c\left(c^2+3\right)$, $\tilde{\mu}_{4,i} = \kappa_i + c\left(4\zeta + c^3 + 6c\right)$, $\tilde{\sigma}_{i,j} = \sigma_{i,j} + c^2$ for $i = 1, \ldots, I$ and $j = i+1, \ldots, I$. The new risk free rate target is instead $\tilde{r}_0 = r_0 - \mu_i + c\sigma_i$. The nonlinear equations (2.1)-(2.6) can be rewritten as:

$$\frac{1}{\tilde{\mu}_{k,i}} \sum_{m=1}^{N_t} p_{m|n} \cdot z_{i,m}^k = 1, \qquad i = 1, \ldots, I, \qquad k = 1, \ldots, 4 \tag{2.11}$$

$$\frac{1}{\tilde{\sigma}_{i,j}} \sum_{m=1}^{N_t} p_{m|n} \cdot z_{i,m} z_{j,m} = 1, \qquad i = 1, \ldots, I, \qquad j = i+1, \ldots, I \tag{2.12}$$

$$\frac{1}{\tilde{r}_0} \sum_{m=1}^{N_t} q_{m|n} \cdot z_{i,m} = 1, \qquad i = 1, \ldots, I \tag{2.13}$$

The system is now composed by a set of $K = \frac{I^2 - 9I + 4}{2}$ equations, each of whom is described by a posynomial function $f_k(u)$, $k = 1, \ldots, K$, on the strictly positive vector u. At this point, the moment matching problem can be reformulated as a SGP optimisation problem by introducing K auxiliary variables $s = [s_1, \ldots, s_K]$ and K positive weights $w = [w_1, \ldots, w_K]$:

$$\min_{u,s} \sum_{k=1}^{K} w_k s_k \tag{2.14}$$

s.t.

$$f_k(u) \leq 1, \qquad k = 1, \ldots, K \tag{2.15}$$

$$\frac{s_k^{-1}}{f_k(u)} \leq 1, \qquad k = 1, \ldots, K \tag{2.16}$$

$$u_i > 0, \qquad i = 1, \ldots, N \tag{2.17}$$

$$s_k \geq 1, \qquad k = 1, \ldots, K \tag{2.18}$$

The optimisation problem (2.14)-(2.18) is non-linear and difficult to solve. However, Xu [37] has recently proposed an approximation scheme, based on the monomial approximation, through which SGP problems are solved via *Geometric Programming* (GP). In order to transform (2.14)-(2.18) into a GP problem, we need to approximate the posynomial $f_k(u)$ in each of the K constraints (2.16) with a monomial term. Given a posynomial function $g(u) = \sum_v f_v(u)$, where $f_v(u)$ are monomial terms, the monomial approximation $\tilde{g}(u)$ is defined by:

$$\tilde{g}(u) = \prod_v \left(\frac{f_v(u)}{\alpha_v(\tilde{u})} \right)^{\alpha_v(\tilde{u})}$$

with $\alpha_v(\tilde{u}) = \frac{f_v(u)}{g(\tilde{u})}$, $\forall v$.
The vector $\tilde{u} > 0$ is a fixed point, or in other words, the starting guess for the solution. For the arithmetic-geometric mean inequality we have [3, 38]: $g(u) \geq \tilde{g}(u)$. Applying the monomial approximation to each of the posynomials $f_k(u)$ for $k = 1, \ldots, K$, at the denominator of constraints (2.16) we obtain:

$$\min \sum_{k=1}^{K} w_k s_k \tag{2.19}$$

s.t:

$$f_k(u) \leq 1, \qquad k = 1, \ldots, K \tag{2.20}$$

$$\left(s_k \cdot \tilde{f}_k(u) \right)^{-1} \leq 1, \qquad k = 1, \ldots, K \tag{2.21}$$

$$u_i > 0, \qquad i = 1, \ldots, N \tag{2.22}$$

$$s_k \geq 1, \qquad k = 1, \ldots, K \tag{2.23}$$

The above optimization problem is a standard GP that can be turned into a non-linear convex problem and it can be solved efficiently [3, 38]. Since the problem (2.19)-(2.23) is solved using the monomial approximation, which in turn depends on the first guess solutions vector $\tilde{u}$, the optimization problem is iteratively solved adjusting every time the initial guess solution. We can summarise the whole procedure proposed by Xu [38] as follow:

Step 0 Set iteration counter $h = 0$. Choose a starting value for the variables $\tilde{u}^{(0)}$, for the weights $w_k^{(0)}$ and a termination parameter ϵ.

Step 1 Compute the monomial approximation parameter $\alpha_v\left(\tilde{u}^{(h)}\right)$ to obtain $\tilde{f}_k\left(u^{(h)}\right)$ for each $k \in K$.

Step 2 Solve the the problem (2.19)-(2.23). If $\|u^{(h)} - u^{(h-1)}\| \leq \epsilon$ then stop, otherwise go to step 3.

Step 3 Update the parameter weights $w_k^{(h)} = G\left(w_k^{(h-1)}\right)$. With $G(\cdot)$ a monotonically increasing function. Set $h = h + 1$ and go to Step 1.

2.3 Case Study - Optimal Portfolio

2.3.1 Experimental Set Up

The proposed algorithm has been applied to generate scenario trees for a simple MSP portfolio optimization problem on a three years investment horizon with yearly rebalancing stages. We have compared the algorithm performance against the hybrid moment matching method proposed by Xu et al. [38] which does not directly consider the no-arbitrage problem. We label our algorithm as MMGP and the algorithm of Xu at al. as MMXU. The technique proposed by Xu et al. has the same inputs of our method: a large fan scenario tree obtained by numerical simulation from an econometric model and a pre-specified branching structure $[N_1, ..., N_T]$ defining the topology of the tree that we want construct. Starting from the first stage they perform a K-means clustering algorithm to cluster the fan scenario values $\bar{r}_1^s$, $s = 1, ..., Sc$, in N_1 classes $C_1, ..., C_{N_1}$ and they choose the mean $\bar{r}_k$ of each classes C_k, $k = 1, .., N_1$, as the tree values. They then compute the moments and the covariances of the fan scenario $\bar{r}_1^s$, $s = 1, ..., Sc$, and solve a moment matching problem where the mass point values $\bar{r}_k$, $k = 1, .., N_1$, are used as an input to find the optimal real world probabilities that best match the fan scenario moments. They then move to the next period $t = 2$ and they perform the same procedure for each of the classes C_k, $k = 1, ..., N_1$ of the fan scenario values $\bar{r}_2^s$, $s = 1, ..., Sc$. The procedure is repeated until the last stage.

We have designed the optimal portfolio problem as closed as possible to that proposed by Xu at al. in order to test the performance of our algorithm on a similar experiment to that used to validate the MMXU method. The investment universe of the portfolio optimization problem is composed by ten stocks from the NYSE market and a cash account. The scenario tree represents the stochastic returns for the stocks whereas the risk free interest rate of the cash account is set constant and equal to the one-year compounded yield on the 3-months treasury bill at the starting date of the problem. In the portfolio problem we look for the dynamic trading strategy which minimises a downside risk measure of the portfolio value with respect to a deterministic target process $\tilde{X} = (1 + g)^t \cdot x_{0,0}$, where the parameter g is a yearly target return. In particular, we have used the ALM problem (1.3 - 1.11) presented in Section 1.3.5. However, since in this test we just consider a portfolio optimization problem without considering any liability process, we have removed the l process by the set of constraints (1.7). The optimal portfolio problem is solved for two different periods that represent two different financial market conditions. The exact dates corresponding to the stages for the two problems are depicted in Table 2.1. We label as Problem A the problem solved on the first period and as Problem B the problem solved on the second period.

Table 2.1: Time structure of problem A and B

	Root Node	First Stage	Second Stage	Third Stage (final portfolio value)
Problem A	25/09/2006	24/09/2007	29/09/2008	28/09/2009
Problem B	24/09/2012	23/09/2013	29/09/2014	28/09/2015

The asset universe $\mathcal{I}$ is composed by the following stocks: General electrics (NYSE:GE), Exxon (NYSE:XOM), Johnson & Johnson(NYSE:JNJ), Wells Fargo (NYSE:WFC), General Dynamics Corp (NYSE:GD), Public Storages (NYSE:PSA), Nike Inc (NYSE:NKE), Apple Inc (NASDAQ: AAPL), The Home Depot Inc (NYSE:HD), JpMorgan Chase (NYSE:JPM).

We applied a VAR(p) model to estimate the conditional mean process for the stock returns: $\hat{r}_t = k + \sum_{i=1}^{p} \Theta_i \hat{r}_{t-i} + \epsilon_t$, where $k \in R^I$, $\Theta_i \in R^{I \cdot I}$, $i = 1,..,p$ are the coefficient vector and the coefficient matrices which we have to estimate and $\epsilon_t = (\epsilon_{1,t}, ..., \epsilon_{I,t})'$ is the white noise vector or the innovation term. The DCC-GARCH model [12] is then used to fit the conditional correlation process. The model is based on the assumption that the I innovation components of the vector ϵ_t of the VAR model follows a conditionally multivariate normal distribution with zero mean and covariance matrix H_t:

$$\epsilon_t | \mathcal{F}_{t-1} \sim N\left([0], H_t\right),$$

with:

- $H_t := D_t R_t D_t$
- D_t is a $I \cdot I$ diagonal matrix with the standard deviation $\sqrt{h_{i,t}}$ of the univariate GARCH models: $[D_t]_{i,i} = \sqrt{h_{i,t}}$.
- $h_{i,t} = \omega_i + \sum_{p=1}^{P_i} \alpha_{i,p} \epsilon_{i,t-p}^2 + \sum_{q=1}^{Q_i} \beta_{i,q} h_{i,t-p}, \quad , i = 1, ..., I.$
- R_t is the time varying correlation matrix with $[R_t]_{i,j} = \frac{q_{ij,t}}{\sqrt{q_{ii,t} q_{jj,t}}}$.

Following the notation of Engle and Sheppard [12] the dynamic correlation structure is defined by:

$$Q_t = \left(1 - \sum_{m=1}^{M} \alpha_m - \sum_{n=1}^{N} \beta_n\right) \bar{Q} + \sum_{m=1}^{M} \alpha_m \left(z_{t-m} z_{t-m}'\right) + \sum_{n=1}^{N} \beta_n Q_{t-n}$$

$$R_t = Q_t^{*-1} Q_t Q_t^{*-1}$$

where:

- $z_t \sim N\left([0], R_t\right)$ is the residual vector standardized by its conditional standard deviation

- $\bar{Q}$ is the unconditional covariance of the standardised residuals resulting from the first stage estimation.

- Q_t^* is a diagonal matrix composed of the square root of the diagonal elements of Q_t

The estimation of model parameters is performed by a three stages procedure. In the first step a VAR(p) model is estimated. In the second step the univariate GARCH parameters for each of the single residual series of the VAR(p) are estimated with maximum likelihoods. Finally in the third step the parameters of the dynamic structure of correlation are estimated using the residuals standardised by the standard deviation obtained in the second step. We use weekly data from 25/09/1994 to 25/09/2006 to estimate the model for the problem A and data from 24/09/2000 to 24/09/2012 for the model B. The likelihood ratio test and the AIC and BIC information criteria are used to choose the diagonal structure and the order p of the VAR process and the parameters P, Q, M, N of the order of the autocorrelation model. The optimal specification according to these criteria is a VAR(1) model, where all the coefficients in the matrix Θ_1 are significantly different from zero, for the conditional mean process whereas the parameters P, Q, M, N of the conditional correlation process are set equal to one. Once we have estimated the parameter of the VAR(1)/DCC(1,1)-GARCH(1,1) model, we simulate a large scenario fan of 30.000 independent trajectories for the three years horizon with weekly frequency.

In order to compare the ability of the MMGP and the MMXU algorithms to match the first four moments and the covariance matrix of the large fan scenario tree generated with MC simulation of the VAR/DCC-GARCH model, we have firstly compounded the weekly returns of the large fan to obtain a three years fan of yearly returns. From this new scenario fan we have generated 10 trees with branching structure [11 11 11], which is the structure with the minimal number of nodes which allows us to obtain trees with no arbitrage opportunities, using the MMGP and the MMXU algorithms for both the Dataset A and B. The models have been implemented on a 2,8 GHz Intel Core i7 machine, with a RAM of 16 GB 1600 MHz DDR3, running OS X Yosemite as operating system. The data preprocessing, the econometric model and the input specification for the optimization problems have been developed using the commercial software package MATLAB R2014b (The MathWorks, Inc., Natick, Massachusetts, United States). All the optimization problem are instead solved using the interior-point solver implemented in the software MOSEK 7 which has been directly linked to the MATLAB software through a MEX file. The computational time to generate a tree with the MMGP

Table 2.2: Absolute Moment Deviation for Dataset A

		Moment Metric (MAPE)					
Algorithm	**Stage**	**1**	**2**	**3**	**4**	**Cov**	**WD**
MMGP	**1**	$1,1668e^{-12}$	0,34677	3,5551	0,0584	0,7225	5.8683
	2	4,0810	0,4845	4,5093	11,5843	2,6377	4.9222
	3	3,7038	1,1584	4,3619	7,2285	2,2771	4.5954
MMXU	**1**	$1,1333e^{-14}$	11,9009	2,5557	11,8793	17,0771	6.3697
	2	2,0262	11,5440	3,48116	7,9046	20,675	5.3442
	3	5,3934	9,4871	6,6552	16,7618	19,0907	5.1728

Table 2.3: Absolute Moment Deviation for Dataset B

		Moment Metric (MAPE)					
Algorithm	**Stage**	**1**	**2**	**3**	**4**	**Cov**	**WD**
MMGP	**1**	$1,0564e^{-13}$	0,1176	3,7885	0,0708	1,7458	5.4175
	2	6,5675	1,9913	4,3451	3,33256	7,4728	5.0072
	3	7,10577	2,0409	4,1732	4,73378	5,6784	4.9932
MMXU	**1**	$6,9057e^{-09}$	18,8511	4,5901	4,66304	18,4819	5.5854
	2	6,7442	12,7421	7,3019	4,25249	23,5741	5.0849
	3	10,2651	20,4010	8,3741	20,12374	47,9844	5.0147

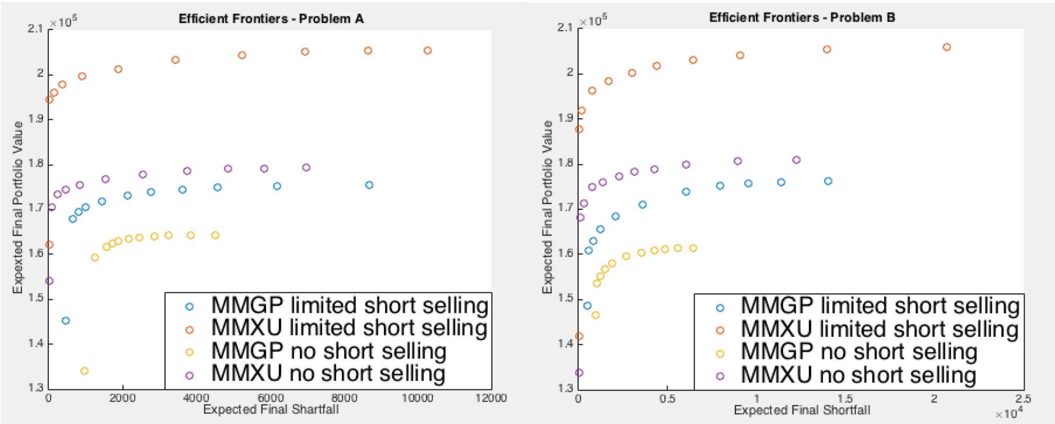
method and branching structure [11 11 11] is approximatively 418.116 s (average computational time among the ten trees generated). We have computed the *maximum absolute percentage error* (MAPE) between the moments of the large fan and the moments of the scenario tree for each of the ten assets in each stage, among the ten trees generated with one method, and we label this distance as "moment metric". For what concerns the covariance matrix we have summed the MAPE of each non diagonal element of the matrix. The results are stored in Table 2.2 and table 2.3. We also compute the Wasserstein distance (WD) with respect the large fan and each of the trees generated with the two methods. The MMGP method fits better the moments of the large fan MC simulation for both the problems A and B and it also gains a less value for the Wasserstein distance. The MMGP method significantly outperforms the MMXU method in matching the moments at the first stage. The MMGP efficiency in matching the moments, although it is always superior with respect the results of the MMXU, deteriorates in the later stages. This suggests that the clustering method designed in the MMGP technique in order to take into account the conditional correlation property of the econometric model could be improved. All trees generated with the MMGP method do not present arbitrage opportunities, whereas all scenario trees generated with the MMXU method present arbitrage opportunities. The arbitrage opportunity checking has been performed by solve the problems (1.12) and (1.13) suggested by Klaassen [26] in each sub-tree.

2.3.2 Numerical Results

The ten trees generated with each of the two methods are used to solve the optimal portfolio problem for both the periods A and B for different values of β ranging form 0 to 1 with 0.1 increments: $\beta = \{0, 0.1, ..., 0.9, 1\}$. The parameter β sets the trade-off between the choice of a better expected wealth and the risk control in terms of wealth expected negative deviation from the target. A low value of β forces the optimiser to prefer trading strategies which lead to a less average shortfall. As the value of β increases, a higher return is sought with a detriment in the risk control. The target return is set equal to the risk free interest rate plus 0.03, meaning that we are looking for a trading strategy which guarantees a yearly returns of 3% greater then the cash account return. The transaction costs $\theta^+ = \theta^-$ are set equal to 0.001. The initial portfolio is just composed by a disposable cash amount: $\bar{x}_{0,0} = 100000$ and $\bar{x}_{i,0} = 0$, $i = 1, ..., I$. The problem is also solved in the case in which a maximum amount X_0/I of short selling positions in each asset is allowed ($\delta = -X_0/I$) and in the case in which no short positions are permitted ($\delta = 0$). The limit case of infinite amount of short selling can be solved for all the levels of β only using the trees generated by the *MMGP* methods since they do not contain arbitrage opportunities. In the Appendix A (Section 2.4.1) we report in Figures 2.4, 2.5, 2.6 and 2.7 the mean of the expected wealth and the expected shortfall for the final stage obtained by solving the optimization problems A and B with the ten trees of each scenario generation method in the case of limited and no short selling. Plotting the expected shortfall on the x-axe against the expected terminal wealth on the y-axe for all the value of β we can mimic the efficient frontier obtained in the traditional mean-variance optimization framework (see Figure 2.1). In this case the risk measure is represented by the expected terminal shortfall instead of the variance. The mean of the root node implementable decisions are drawn in Figures 2.2 - 2.3 - 2.4 - 2.5. All the problems solved using the *MMXU* trees exhibit a substantially better mean-expected shortfall position, as it emerged by looking at the efficient frontiers which always lie above those obtained with the problems solved with the return parameters generated with the *MMGP* algorithm. The standard deviation, which is a measure of the stability of the tree generation algorithm, is less when we apply the *MMGP* method. The more stability in terms of standard deviation is also confirmed looking at the root node implementable decisions: the *MMGP* trees led to a more diversified portfolio with a lower standard deviations among the ten trees compared to the solutions obtained with the *MMXU* method.

Since we do not know if the much better efficient frontiers obtained with the

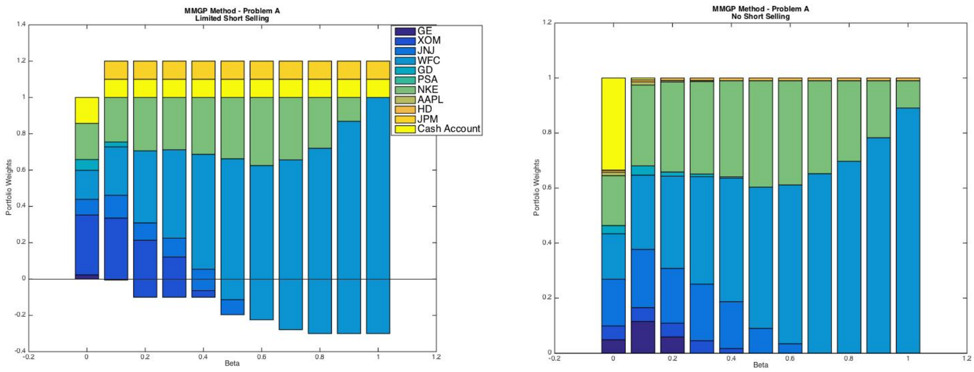
Figure 2.1: Efficient Frontiers



(a) Efficient Frontiers - Problem A

(b) Efficient Frontiers - Problem B

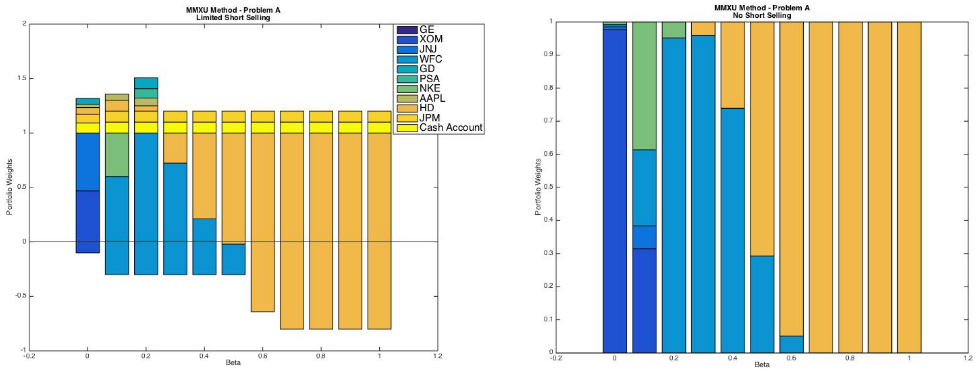
Figure 2.2: Initial Portfolio Proportion - MMGP - Problem A



(a) MMGP - Limited Short Selling

(b) MMGP - No Short Selling

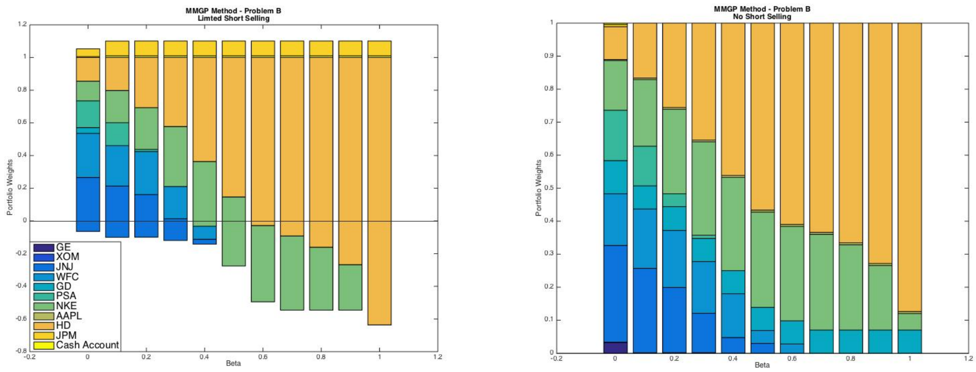
Figure 2.3: Initial Portfolio Proportion - MMXU - Problem A



(a) MMXU - Limited Short Selling

(b) MMXU - No Short Selling

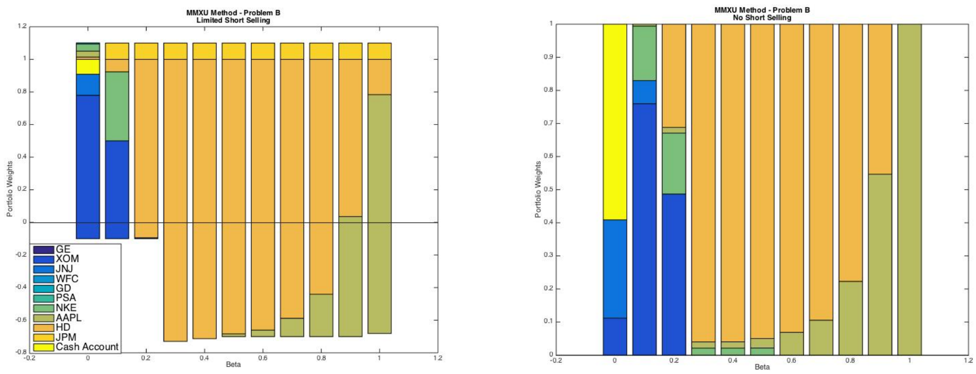
Figure 2.4: Initial Portfolio Proportion - MMGP - Problem B



(a) MMGP - Limited Short Selling

(b) MMGP - No Short Selling

Figure 2.5: Initial Portfolio Proportion - MMXU - Problem B



(a) MMXU - Limited Short Selling

(b) MMXU - No Short Selling

MMXU methods are a consequence of the presence of arbitrage opportunities in the scenario trees or of the worst moments matching and a higher WD with respect the large sample MC fan, an historical backtest with telescoping horizon has been performed to investigate the real portfolio performances if we were applying the model in the out-of-sample period. In the historical backtest with telescoping horizon the statistical models are fitted to data up to each trading time $t \in \mathcal{T}$ and the related scenario trees are generated to the horizon T. Starting from the first date corresponding to $t = 0$ the optimal root node decisions are computed and then implemented so that we can obtain the realised portfolio value at time $t = 1$ using the historical returns. Afterwards the whole procedure is rolled forward for $T - 1$ trading times. At each decision time t the parameters of the stochastic processes driving the stock return are re-calibrated using historical data up to and including time t, and the initial values of the simulated scenarios are given by the actual historical values of the variables at these times. Re-calibrating the simulator parameters at each successive initial decision time t captures information in the history of the variables up to that point.

Figure 2.6: Telescoping horizon backtest schema for Problem A

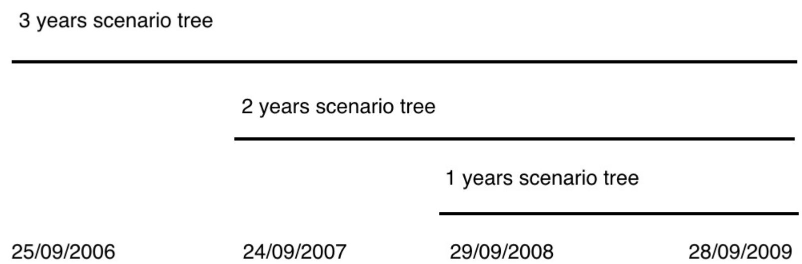
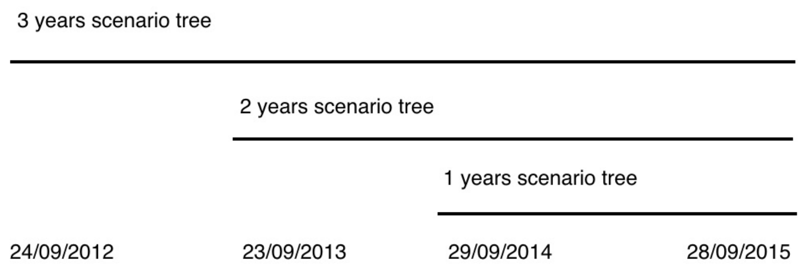
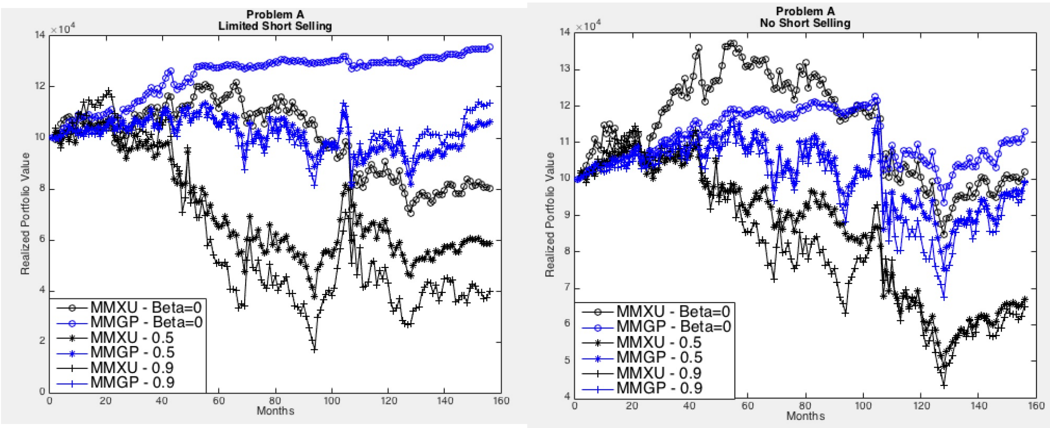


Figure 2.7: Telescoping horizon backtest schema for problem B



The average and the standard deviation of the realised portfolio values among the ten trees generated with one of the two methods is computed for the terminal horizon $T = 3$ for any choice of the risk control parameter β. The results are stored in Tables 2.8 and 2.9 in the Appendix B. We can see how in general the realised

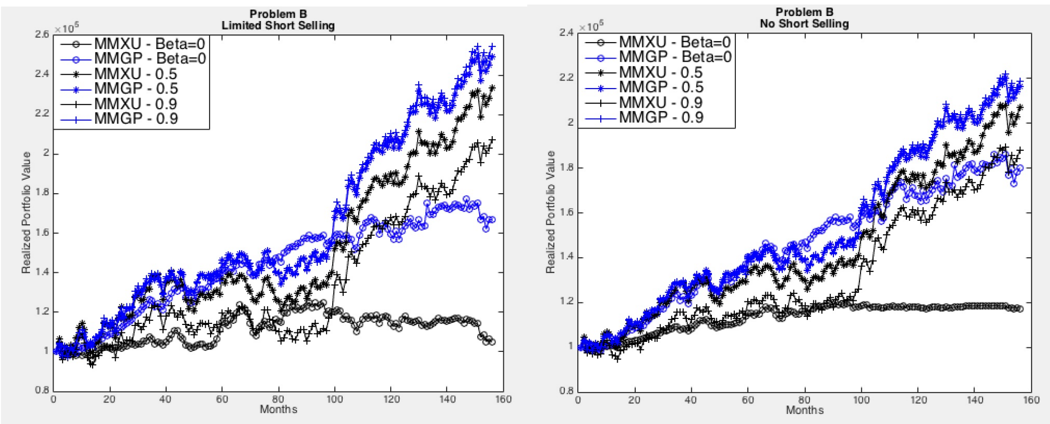
Figure 2.8: Realised Portfolio Values - Problem A



(a) Limited Short Selling

(b) No Short Selling

Figure 2.9: Realised Portfolio Values - Problem B



(a) Limited Short Selling

(b) No Short Selling

portfolio values obtained with the MMGP methods outperforms those achieved with the MMXU. This is a consequence of the less diversification in the optimal portfolio choices obtained with the MMXU trees. The standard deviation of the realised portfolio values are greater in the MMXU methods for any risk level parameters β confirming the previous results on the highest standard deviation of the trading strategy. In Figures 2.8 and 2.9 the average of the realised monthly portfolio values for the two problems A and B are plotted for three reference values of β.

2.4 Conclusions

In this chapter a new hybrid method (MMGP) to generate scenario tree for asset returns is proposed. The algorithm receives as input a large scenario fan with independent trajectories, which is assumed to represent the original multivariate distribution function, and a given branching structure defining the topology of the tree. The SMWD heuristic [9] has been combined with a moment matching method with additional constraints, which directly avoid the presence of arbitrage opportunities, to generate each sub-tree in a forward fashion from the root node. A clustering technique, based on the euclidean distance, has been used to approximate the time-conditional structure of the large fan. We have applied a monomial approximation in order to solve the moment matching problem with a sequence of geometric programming problems. This approach is very closed to that proposed by Staino and Russo [36]: in this case, the monomial approximation is used to transform the moment matching problem into a linear system of equations solved with the Moore-Penrose pseudo inverse. In order to test the efficiency of the new algorithm we have applied the scenario trees to a simple multi-period optimal portfolio selection problem for two different investment periods and we have tested the results against those obtained solving the same problem with scenario trees generated with the hybrid algorithm (MMXU) developed by Xu at al. [38]. The proposed method outperforms the MMXU algorithm in matching the moments and it also achieves a lower Wasserstein distance with respect the large scenario fan. The MMXU method achieves substantially better expected in-sample risk-return portfolios which are not confirmed in the out-of-sample tests. On the contrary the proposed method obtains more stable in-sample results and it leads to higher out-of-sample portfolio returns. It is difficult to analyse if the worst out-of-sample results of the optimal portfolio obtained with the MMXU method are a consequence of the presence of arbitrage opportunities in the scenario trees. However, since all the efficient frontiers of the portfolio problem solved with the MMXU method lie above those obtained with the MMGP method and the portfolios of the former are more concentrate we can guess that the presence of arbitrage opportunities can have a distortion impact on the choice of the optimiser.

2.4.1 Appendix A

Table 2.4: Expected Wealth and Expected Shortfall - Limited Shortfall Case - Dataset A

		MMGP				MMXU			
		EW		ES		EW		ES	
		Avg	Std	Avg	Std	Avg	Std	Avg	Std
Beta	0	145170	1167	400	112	162300	1715	0	0
	0.1	167790	1644	603	123	194370	1421	18	14
	0.2	169540	1609	802	147	196020	1315	133	42
	0.3	170430	1545	983	146	197850	1601	371	97
	0.4	171890	1343	1430	289	199680	1799	912	245
	0.5	173170	786	2130	674	201310	2132	1898	476
	0.6	173880	761	2772	662	203290	2416	3421	858
	0.7	174440	710	3618	762	204500	2406	5231	1076
	0.8	174840	670	4589	860	205100	2531	6939	1547
	0.9	175180	719	6189	678	205410	2498	8657	1596
	1	175350	722	8689	534	205490	2474	10264	1338

Table 2.5: Expected Wealth and Expected Shortfall - Limited Shortfall Case - Dataset B

		MMGP				MMXU			
		EW		ES		EW		ES	
		Avg	Std	Avg	Std	Avg	Std	Avg	Std
Beta	0	128170	1028	223	25	141960	2503	0	0
	0.1	159750	1542	444	75	187780	2296	31	32
	0.2	163010	1833	776	185	191800	2159	194	65
	0.3	165000	1756	1146	223	196400	1935	737	240
	0.4	167670	1571	2016	402	198440	2121	1676	470
	0.5	171280	1605	3978	918	200350	2172	3024	750
	0.6	174810	1035	7121	1025	201740	2362	4405	1180
	0.7	177100	976	10596	1019	203110	2063	6425	1142
	0.8	179070	1322	15727	2071	204240	1763	9050	1692
	0.9	180580	1406	23283	2386	205520	1937	13946	4169
	1	181070	1359	30870	2284	205940	1650	20690	3398

Table 2.6: Expected Wealth and Expected Shortfall -No Shortfall Case - Dataset A

		MMGP				MMXU			
		EW		ES		EW		ES	
		Avg	Std	Avg	Std	Avg	Std	Avg	Std
Beta	0	134070	940	957	152	154190	567	8	9
	0.1	159350	810	1265	169	170520	1591	77	28
	0.2	161740	973	1561	262	173320	830	235	66
	0.3	162450	625	1716	304	174310	961	459	98
	0.4	162960	639	1887	318	175350	1226	829	217
	0.5	163440	527	2153	447	176790	1215	1548	289
	0.6	163780	555	2465	422	177880	1447	2538	549
	0.7	164040	502	2856	530	178620	1453	3740	718
	0.8	164200	501	3255	547	179030	1540	4850	911
	0.9	164330	519	3830	440	179220	1524	5855	967
	1	164380	518	4504	350	179290	1590	6974	792

Table 2.7: Expected Wealth and Expected Shortfall -No Shortfall Case - Dataset B

		MMGP				MMXU			
		EW		ES		EW		ES	
		Avg	Std	Avg	Std	Avg	Std	Avg	Std
Beta	0	125480	728	557	217	133710	5014	16	148
	0.1	153820	1320	844	193	168100	1676	79	263
	0.2	156220	1433	1134	225	171200	1474	310	470
	0.3	157780	1447	1436	246	174830	1168	786	161
	0.4	159420	976	1969	291	176090	1326	1346	315
	0.5	161560	765	3071	611	177350	1457	2248	632
	0.6	163290	784	4610	601	178240	1530	3111	854
	0.7	164500	842	6422	719	178990	1407	4225	897
	0.8	165640	1000	9272	1303	179790	1302	6039	1135
	0.9	166760	997	14278	1070	180610	1401	8927	2322
	1	167080	870	18069	1758	188084	1279	12232	1604

Table 2.8: Realised Final Portfolio Value - Dataset A

		MMGP				MMXU			
		Limited		No-Short		Limited		No-Short	
		Avg	Std	Avg	Std	Avg	Std	Avg	Std
Beta	0	140226	5761	1221577	9437	80209	25414	113517	31297
	0.1	134225	4040	113909	3191	91527	33513	107813	14423
	0.2	135098	5045	114310	4876	65144	13028	84273	4182
	0.3	131348	3953	114372	6332	60217	3713	82806	2402
	0.4	124550	4235	114410	3622	61621	5054	81143	2667
	0.5	121683	5611	117291	4915	59838	5645	78476	2117
	0.6	126671	6212	120284	5345	54117	5800	78707	2016
	0.7	130062	6430	121497	6057	49552	3933	82047	4082
	0.8	132424	7203	124541	5975	45842	4300	80999	2255
	0.9	138331	7947	124064	6090	41889	3481	65510	4491
	1	141348	8517	122325	5946	43445	5611	62141	546

Table 2.9: Realised Final Portfolio Value - Dataset B

		MMGP				MMXU			
		Limited		No-Short		Limited		No-Short	
		Avg	Std	Avg	Std	Avg	Std	Avg	Std
Beta	0	175652	24153	188378	20217	103797	23344	118662	8641
	0.1	252083	17518	229566	8128	191860	37503	156938	28920
	0.2	261384	15853	232609	7912	217498	19215	200960	18495
	0.3	267974	14665	233521	6707	259895	13145	225836	6668
	0.4	270599	14833	233643	6983	258227	12763	224644	6558
	0.5	274940	15069	233883	6694	252930	14296	222118	7499
	0.6	278797	10284	233517	6840	247166	13480	214827	7997
	0.7	279290	14362	234270	8091	239327	14510	210320	11895
	0.8	276459	14523	233198	7533	223092	21643	199629	18460
	0.9	270642	14208	229495	7534	177861	43230	167873	29503
	1	251293	16316	222246	10070	112009	27915	119692	546

Bibliography

[1] Beraldi P., De Simone F., Violi A., *Generating scenario trees: a parallel integrated simulation optimization approach*, Comput. Appl. Math, 233, 2010, pp. 2322-2331.

[2] Beraldi P., Bruni M. E., *A clustering approach for scenario tree reduction: an application to a stochastic programming portfolio optimization problem*, TOP, 2014, Volume 22, Number 3, Page 934.

[3] Boyd S., Kim S., Vandenberghe L., Hassibi A., textslA tutorial on geometric programming, Optimization and Engineering, 2007, Vol.8(1), pp.67-127.

[4] Chen Z., Consigli G., Dempster M. A. H., Hicks-Pedrón N., *Towards sequential sampling algorithms for dynamic portfolio management*, New Operational Tools for the Management of Financial Risks, Zopounidis C. (ed.), Kluwer Academic Publishers: Portland, 1997, 197-211.

[5] Chen Z., Xu C., *Global convergence of a general sampling algorithm for dynamic nonlinear stochastic programs*, Numerical Functional Analysis and Optimization, 2002, 23(5):495-514.

[6] Consigli G., Iaquinta G., Moriggia V., *Path-dependent scenario trees for multistage stochastic programmes in finance*, Quantitative Finance, 2010.

[7] Consiglio A., Carollo A., Zenios S.A., *A parsimonious model for generating arbitrage-free scenario trees*, Quantitative Finance, 01 February 2016, Vol.16(2), p.201-212.

[8] Dempster M. A. H., Thompson R., *EVPI-based importance sampling solution procedures for multistage stochastic linear programmes on parallel MIMD architectures*, Annals of Operations Research 1999, 90:161-184.

[9] Dempster M. A. H., Medova E. A., Yong Y. S., *Comparison of Sampling Methods for Dynamics Stochastic Programming*, Stochastic Optimization Methods in Finance and Energy, Ch. 16, Springer, 2011.

[10] Duffin R. J., Peterson E. L., Zener C., *Geometric Programming*, John Wiley and Sons. p. 278, 1967.

[11] Dupaĉová J., Consigli G., Wallace S. W.,*Scenarios for multistage stochastic programs*, Annals of Operations Research, 2000.

[12] Engle R. F., Sheppard K., *Theoretical and empirical properties of dynamic conditional correlation multivariate GARCH*, NBER Working Paper No. W8554, Issued in October 2001.

[13] Geyer A., Hanke M., Weissensteiner A., *No-arbitrage conditions, scenario trees, and multi-asset financial optimization*, European Journal of Operational Research, 2010.

[14] Geyer A., Hanke M., Weissensteiner A., *No-arbitrage bounds for financial scenarios.*, European Journal of Operational Research, 2014, 236(2), 657-663.

[15] Geyer A., Hanke M., Weissensteiner A., *No-Arbitrage ROM simulation*, Journal of Economic Dynamics & Control, 45, 2014, 66-79.

[16] Gülpinar N., Rustem B., Settergren R., *Simulation and optimization approaches to scenario tree generation*, Journal of Economic Dynamics and Control, 2004, Vol.28(7), pp.1291-1315.

[17] Heitsch H., Römisch W., Strugarek C., *Stability of multistage stochastic programs*, SIAM Journal on Optimization, 17:511-525, 2006.

[18] Heitsch H., Römisch W., *Scenario tree reduction for multistage stochastic programs*, Computational Management Science, 6:117-133, 2009b.

[19] Heitsch H., Römisch W., *Stability and scenario trees for multistage stochastic programs*, In G. Infanger, editor, Stochastic Programming, 139-164. The State of the Art, In Honor of G.B. Dantzig, Springer, 2010.

[20] Hochreiter R., Pflug G., *Financial scenario generation for stochastic multi-stage decision processes as facility location problems*, Annals of Operations Research, 2007.

[21] Høyland K.,Wallace S.W.(2001). *Generating scenario trees for multistage decision problems.*, Management Science, 47(2), 295-307.

[22] Høyland K., Kaut M., Wallace S. W., *A heuristic for moment-matching scenario generation*, Computational Optimization and Applications, 2003.

[23] Ji X., Zhu S.,Wang S., Zhang S., textslA stochastic linear goal programming approach to multistage portfolio management based on scenario generation via linear programming, IIE Transactions 2005, 37:957-969.

[24] Kaut M., Wallace S.W., *Evaluation of scenario generation methods for stochastic programming*, Pacific Journal of Optimization, 2007, 3(2), 257-271.

[25] Klaassen P., *Discretized reality and spurious profits in stochastic programming models for asset/liability management*, European Journal of Operational Research,1997, 101(2), 374-392.

[26] Klaassen P., *Financial asset-pricing theory and stochastic programming model for asset/liability management: A synthesis*, Management Science, 1998, 44(1), 31-48.

[27] Klaassen P., *Comment on "generating scenario trees for multistage decision problems"*, Management Science, 2002, 48(11),151-1516.

[28] Kouwemberg R.P.P, *Scenario generation and stochastic programming models for asset liability management*, European Journal of Operational Research, 2001.

[29] Pennanen T., *Epi-convergent discretization of multistage stochastic programs*, Mathematics of Operations Research 2005, 30:245-256.

[30] Pennanen T., *Epi-convergent discretizations of multistage stochastic programs via integration quadratures*, Mathematical Programming, 2009, 116:461-479.

[31] Pflug G. C., *Approximations for Probability Distributions and Stochastic Optimization Problems*, Stochastic Optimization Methods in Finance and Energy, Ch. 15, Springer, 2011.

[32] Pflug G. C., Pichler A., *Dynamic generation of scenario trees*, Comput. Optim. Appl., 62, 2015, pp. 641-668.

[33] Romisch W., *Stability of Stochastic Programming Problems*, Chapter 8 in Stochastic Programming, Volume 10 of Handbooks in Operations Research and Management Science, Ruszczynski A. and Shapiro A., eds. Elsevier, Amsterdam, 2003.

[34] Rubasheuski U., Oppena J., Woodruff DL., *Multi-stage scenario generation by the combined moment matching and scenario reduction method*, Oper Res Lett., 2014, 42:374-377.

[35] Shapiro A., *Inference of statistical bounds for multistage stochastic programming problems*, Math. Meth. Oper. Res. 58 (2003), 57-68.

[36] Staino A., Russo E. *A moment-matching method to generate arbitrage-free scenarios*, European Journal of Operational Research, 16 October 2015, Vol.246(2), pp.619-630.

[37] Xu G., *Global optimization of signomial geometric programming problems*, European Journal of Operational Research, 16 March 2014, Vol.233(3),

pp.500-510.

[38] Xu D., Chen Z., Yang L., *Scenario tree generation approaches using K-means and LP moment matching methods*, Journal of Computational and Applied Mathematics 236, 2012, 4561-4579.

Chapter 3

Pension Fund Liabilities Replication

3.1 Introduction

In this chapter we consider the problem of a DB pension fund which needs to price the future stream of obligations on a market valuation based approach. Until now the most common actuarial method in the UE countries for discounting the pension fund obligations is based on a fixed actuarial interest rate term structure prescribed by the supervisory authority. The European Union is currently preparing a new set of rules for the supervision of insurance companies known as Solvency II. The quantitative funding requirements of the Solvency II framework are based on market-consistent valuation of assets and liabilities and risk management techniques similar to those implemented with the Basle II framework for banking supervision. The main motivation is that the quantitative funding requirements based on the fair price better reflect the true risk of an insurance undertaking. Although at the moment the modifications of the Directive 2003/41/EC, which states the rules governing the activities of DB pension funds in all EU countries, do not yet consider the introduction of fair price valuations prescriptions in different EU countries such as Netherlands, Sweden and Denmark, market valuation of pension fund liability has become regulatory practice and an institutional debate in the European Commission concerning the extension of the market valuation principle of Solvency II for the pension fund activity is going on. The fair value approach for valuing liabilities imposed that cashflows are discounted using observed market rates and guarantees priced consistently with asset derivatives. This should be done by discounting the liabilities with a market interest rate for loans with the same maturity and risk characteristics. In practice, this approach poses some challenges, as pension fund liabilities are typically not marketed assets and depends on risky factors which are not traded in financial markets such as salary growth and mortality rates.

An alternative approach is to treat the future pension fund obligation payments as payoffs of a general contingent claim. Traditional contingent claim pricing is performed by replication argument: the price of the claim is equal to the price of the portfolio that exactly replicates all the claim cash flows. We know that a perfect replication is possible only in the case of complete market whereas in the case of incomplete market part of the claim risk is uncorrelated with the price change of the replication assets [6]. Although the literature proposed different methods for the pricing of securities in incomplete markets in a continuous time framework (see [27] for a review of the main methods) the complexity of the liability pricing problem for a DB pension fund does not permit in general analytical solutions. Multistage stochastic programming has been extensively applied to the strategic allocation problem of a pension fund manager [1, 2, 4, 5, 7, 9, 12, 15]. This approach overtakes the traditional asset allocation strategies with no dependency on the expected stream of future obligations. Although finance applications of MSP focus mainly on the choice of the optimal asset allocation, the same framework can be applied to price a contingent claim in a discrete market environment formulating a particular type of ALM problem. King [17] was the first to propose a MSP approach to price a general contingent claim using a super hedging approach in discrete time. The idea is to define the value of a contingent claim as the minimal capital required to cover the future pay-off without risk when the capital is invested in financial instruments available in the market. Recently Koivu end Pennanen [16] have proposed to use a more general approach to price the current pension fund liability exposure: instead of looking for the minimal capital which at least replicates the obligation expenditure avoiding any risk, they use a convex risk measure which can incorporate a prudential risk exposure. The MSP liability valuation approach is market consistent by taking into account the investment opportunities available to the pension fund manager at the time of valuation, and it can be naturally integrated with an ALM model framework for the choice of the strategic asset allocation, such as presented in Section 1.3.

MSP solutions rely on a discretisation of the original distribution of the risk factors which is mostly implemented by scenario tree procedures. Several authors notice that particular attention must be paid to the presence of arbitrage opportunities in the scenario tree used to solved financial problems via MSP [10, 18]. In pricing problem in particular, since an arbitrage is the opportunity to have a riskless investment with a positive return, the presence of an arbitrage strategy along the tree should imply an unbounded solution of the optimization problem: we can start from any negative level of wealth and reach a null, or positive portfolio value at the final stage. However, the assumptions of unlimited short selling is not always suitable for the DB pension fund liability valuation: in many countries pension funds and insurance companies are forced to limit or totally exclude short selling. This in turn implies that the trading strategy adopted to replicate the future obligations payoffs should avoid short positions. When limits on short positions are imposed to each asset, the arbitrage opportunity can not be totally exploited, and the optimisation problem will have a bounded optimal solution. However, the solution will be bias. Also in the cases in which we can use different techniques that not rely on a scenario tree definition to solve multistage stochastic programming, such as the *Galërkin* method used by Koivu end Pennanen [19], a discrete approximation of the reference continuous distributions for the random parameters is needed and non considering arbitrage issues could potentially lead to bias solutions.

In this chapter we present a MSP based model to price the liability of a fictional DB pension plan, where the obligations process is assumed to be driven by inflation and mortality risks and where the asset universe is composed by a set of government bonds, corporate bonds and stock market indexes. The lack of market instruments connected with inflation and mortality risk will lead, in the case of continuous probability spaces, to market incompleteness. However, in a market model defined on a discrete probability space, a sufficient and necessary condition to the market completeness is that the number of arcs emanating from a node at each node of the tree must be equal to the number of non-redundant assets in the optimization problem. We have considered both the cases of complete and incomplete market just by choosing the branching structure of the scenario trees according to the previous statement. The liability value has been firstly computed with no limits on short selling, using a scenario tree for financial returns, obtained with the MMGP method presented in Chapter 2, which does not contains arbitrage opportunities. We then solved the same problem considering the case of limited short positions allowed and the case where the short positions are totally excluded. In these cases the liability pricing problem is also solved using trees generated with the MMGP method without the no-arbitrage set of constraints (2.6). In this way we are able to compute the bias in the optimal solution produced by the presence of arbitrages in the scenario tree. The analysis of the bias arising from arbitrage opportunities has been performed in order to better clarify the sensitivity of the optimal solution of the MSP liability pricing problem with respect the presence of arbitrage in the scenario tree. The analysis can be used to validate other modelling choices to solve the MSP pricing problem which do not guarantee the absence of arbitrage opportunities. The choice of the tree topology can be an example. We know from empirical analysis that trees with a large number of nodes in the first stage better approximate the original distribution [3].

The rest of the chapter is defined as follow. In Section 3.2 we present the formulation of a general ALM problem that can be used to price the DB pension fund liability under different assumptions. In Section 3.3 we then illustrate a numerical case study based on a fictional DB pension fund.

3.2 Pension Fund Liability Pricing with Stochastic Programming

We consider the problem of evaluating the present fair value of a pension fund future payments process in the discrete market presented in Section 1.3.1 of the first chapter. The net payment process $\{l_t\}_{t=0}^T$ is computed as the difference between future pensions and contributions flows (l has positive value but is a net expenditure for the pension fund) and it is assumed driven by two uncertain factors: the inflation and the mortality risks. The financial market is described by a finite set of I liquid assets indexed by the set $\mathcal{I} = \{0, 1, 2, ..., I\}$ that can be traded at $t = 0, ..., T$. After paying out l_t at time t, the pension fund manager chooses how to invest the remaining wealth in the I liquid assets. The process l does not depend on the realisation of the multivariate asset process. An arbitrage free market model is dynamic complete when we are able to replicate any contingent claim just by constructing a dynamic self-financing trading strategy that perfect replicates the cash flow of the contingent claim. In a discrete market model the perfect replication is possible, in general, just by linear algebra consideration, if and only if the following two conditions hold [25]:

- the number of asset returns, considering the risk free asset, is equal to the number of nodes in each of the sub-trees.
- no constraints on short selling are imposed.

When at least one of these two conditions fails to hold we are in an incomplete market case since the perfect replication is not possible. An often used approach to price a contingent claim in such a market is the super hedging replication approach. A super hedging cost is defined as the minimal amount of initial capital required to buy a riskless hedging strategy for a contingent claim process $\{l_t\}_{t=0}^T$ [17]. In this framework the portfolio investment returns may exceed the claims. We defined the residual wealth X_n, for $n \in \mathcal{N}_T$ as the unhedged part of the liabilities. Hilli, Koivu and Pennanen [16] proposed a different approach based on the risk preferences of the pension fund manager. This is a discrete time analogous of the utility approach proposed in continuous time [27]. Due to the fact that the obligations are not perfectly matched, the optimal strategy should be constructed in order to obtain a null or negative value of some risk measure of the final wealth. This is an extended approach with respect the super hedging problem for at least two different aspects: the risk measure introduces risk preferences of the investors leading to a set of prices that depend on the different risk attitudes in the market and it does not require a risk free replication strategy. The choice of the specific functional ρ for the risk measure is related to the risk preferences of the pension fund manager. Furthermore the functional ρ will depend in general on the probability values, which are a sub product of the financial and economic model choice and implementation. These two features of the risk measure valuation make it a more subjective approach to value the liability in an incomplete market setting with respect the super hedging approach. The problem of pricing the DB pension fund liability at $t = 0$, on a market based valuation criterion, is formally expressed as the problem of finding the the minimal initial capital that allows to implement a dynamic self-financing investment strategy that fits the future payments according to some risk measure ρ_T. We set the risk measure as: $\rho_{t,T}\left(X_{t,T}\right) = \beta \sum_{s=t}^{T} E_{\mathbb{P}}\left[-X_s\right] + (1-\beta) \sum_{s=t}^{T} E_{\mathbb{P}}\left[max\left(0, -X_s\right) \vert \mathcal{F}_t\right].$ The parameter β controls the optimiser risk attitude: when β is set equal to zero negative portfolio value positions are ruled out and we fall in the super hedging case. As we increase the risk parameter toward the unity value more risk the optimiser is willing to take in exchange of an higher expected final return. The problem can be generally formulated as follow:

ALM-pricing Problem

$$\min_{x_n, x_n^+, x_n^-; \forall n \in \mathcal{N}} \sum_{i=0}^{I} x_{i,0} \tag{3.1}$$

s.t.:

$$x_{i,n} = (1 + r_{i,n}) \cdot x_{i,a(n)} + x_{i,n}^+ - x_{i,n}^-, \qquad i \in \mathcal{I} \setminus \{0\}, \qquad n \in \mathcal{N}_t, \qquad t \geq 1, \tag{3.2}$$

$$x_{0,n} = (1 + r_{0,n}) \cdot x_{0,a(n)} + \sum_{i=1}^{I} x_{i,n}^- - \sum_{i=1}^{I} x_{i,n}^+ - l_n, \qquad n \in \mathcal{N}_t, \qquad t \geq 1, \tag{3.3}$$

$$X_n = \sum_{i=0}^{I} x_{i,n}, \qquad n \in \mathcal{N}_t, \qquad t \geq 0, \tag{3.4}$$

$$\sum_{n \in \mathcal{N}_T} p_n \left(-\beta X_n + (1-\beta) \left[-X_n\right]^+\right) \leq 0, \qquad n \in \mathcal{N}_T \tag{3.5}$$

$$x_{i,n}^+ \geq 0, \qquad i \in \mathcal{I} \setminus \{0\}, \qquad n \in \mathcal{N}_t, \qquad t \geq 1, \tag{3.6}$$

$$x_{i,n}^- \geq 0, \qquad i \in \mathcal{I} \setminus \{0\}, \qquad n \in \mathcal{N}_t, \qquad t \geq 1, \tag{3.7}$$

$$x_{i,n} \geq \delta_f, \qquad i \in \mathcal{I}, \qquad n \in \mathcal{N}_t, \qquad t \geq 0. \tag{3.8}$$

Let x_0^* be the vector of optimal portfolio amounts at the root node $n = 0$ obtained from the above *ALM-pricing Problem*. The minimal capital $L_{s,0} := \sum_{i=0}^{I} x_{i,0}^*$ is then the price at $t = 0$ of the process $\{l_t\}_{t=0}^{T}$ at $t = 0$. We can consider $L_{s,0}$ as the minimal price that the seller of a contingent claim, with a payoffs process described by l, would accept to enter the contract. The maximum price $L_{b,0} := -\sum_{i=0}^{I} x^*_{i,0}$ that the buyer of the contingent claim would accept to enter the contract is simply obtained by changing the sign of the process l in the set of constraints (3.3). The buyer price $L_{b,0}$ is indeed the greatest initial debt one could cover when receiving the process l. It has been proved that under perfect replicability $L_{b,0} = L_{s,0}$, whereas $L_{b,0} \leq L_{s,0}$ holds under a super hedging approach [17, 23].

The parameter δ_f is a non positive real vector used to model short-selling constraints: when it is set equal to zero we prevent the choice of short positions. In the complete market case the price of a contingent claim is just the minimal initial investment to create a dynamic self-financing trading strategy that perfect replicates the security cash flow at each stage and at each node. We can state this problem just by replacing the risk measure functional constraints (3.5) in the *ALM-pricing Problem* with the set of constraints $X_n = 0$, for $n \in \mathcal{N}_T$, and by setting $\delta_f = -\infty$. Note that in a discrete complete market the perfect replication is also possible in the case in which we have arbitrage opportunities in the scenario tree, although in this case we do not have any risk neutral measure.

When $\beta = 0$ we look for the minimal initial capital on which we can construct a dynamic trading strategy that replicates the pension expenditure in each intermediate stage and which ensures at least a null portfolio value in all the leaf nodes and we are in the case of a super hedging approach. Increasing the value of the risk parameter β we accept higher negative portfolio values in some of the leaf nodes, i.e. a more risky position, and in general we are able to find solutions that are less of those obtained with the super hedging approach.

3.3 Case study and Numerical results

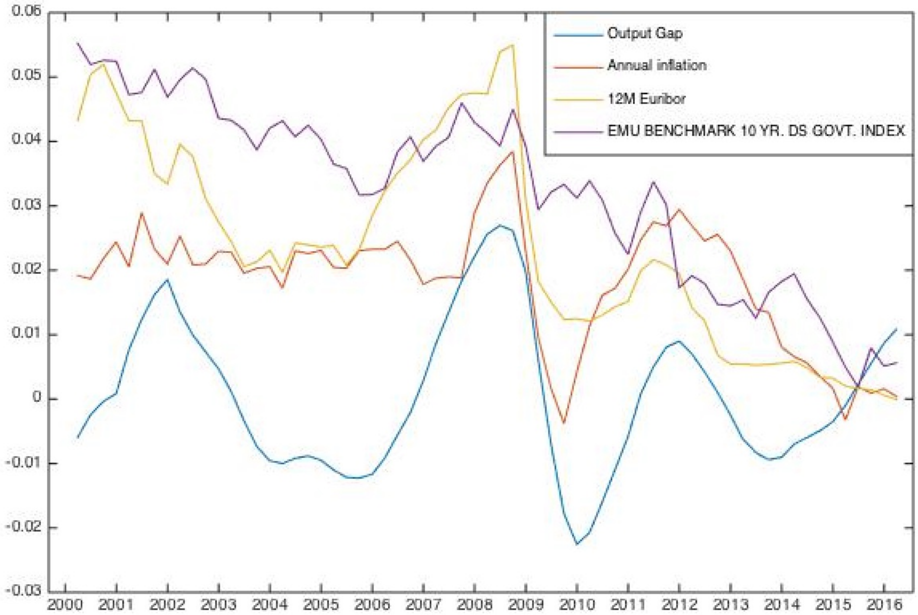
3.3.1 Experimental Set Up

We perform the liability pricing methodologies described in the second paragraph of this chapter on a case study based on an artificial pension fund problem. We consider an investment period of seven years $T = 7$ on an asset universe composed by seven risky and liquid securities plus a cash account. The portfolio rebalancing periods are defined by the time index set $\mathcal{T} = \{0, 1, 3, 5, 7\}$. The first task is to determine the values for the tree processes $\{l_t\}_{t\in\mathcal{T}}$ and $\{r_t\}_{i,t\in\mathcal{T}}$ for $i = 0, ..., I$, which will form the input parameters for the *ALM-pricing Problem.* The topology of such a tree will be uniquely determined by the set $\mathcal{N} = \{1, N_1, N_3, N_5, N_7\}$ describing the number of nodes in each sub-tree at a given decision stage. A small-scale macroeconometric model has been developed to obtain the dynamic processes describing the evolution of the 12 month euribor interest rate and the inflation rate. The former represents the risk free stochastic interest rate which will be used to model the cash account return and to discount the cash flows whereas the letter will be used to model the salary and the pension dynamics. The pension fund population evolution is then constructed on the basis of a stochastic mortality model and it is used to define the actuarial pension fund variables and their dynamics. The tree process $\{l_t\}_{t\in\mathcal{T}}$ and the risk free interest rate process $\{r_{0,t}\}_{t\in\mathcal{T}}$ are generated by random sampling from the macroeconomic and the mortality rate models. Each of the seven securities dynamic is modelled independently with an appropriate time series autoregression model and then the correlation among them is reconstructed with a copula approach. The copula time series model is used to simulate a large scenario fan with independent trajectories on the time space $\mathcal{T}$. This large scenario fan will form the input for the $MMGP$ algorithm described in the second chapter of this thesis in order to obtain the the risky returns tree process $\{r_t\}_{i,t\in\mathcal{T}}$ for $i = 1, ..., I$. We illustrate the model choice for economic, actuarial and financial variables and the relative scenario tree generation procedure in Sections 3.3.2 - 3.3.4. Finally, in Section 3.3.5, different specifications of the ALM-pricing problem are solved and the numerical solutions are showed and commented.

3.3.2 Macroeconomics Model

In order to obtain the evolution of the short interest rate and the inflation rate a parsimonious vector autoregressive (VAR) model of the economy was developed with four state variables in nominal values: the euro area output gap ($\hat{y}^{gap}$), the euro area one year consumer price index variations ($\hat{\pi}$), the 12 months euribor rate $\left(\hat{r}^f\right)$ and the ten years euro government benchmark return $\left(\hat{r}^l\right)$. We use quarterly data over the period from 1998 to 2008 to calibrate the parameters and from the last quarter of 2009 to the first quarter of 2016 for the out-of-sample analysis.

Firstly we test the stationarity of each series separately by applying the Augmented Dickey-Fuller *(ADF-Test)* test for a unit root without deterministic trend. Given the regression model $z_t = c + \alpha z_{t-1} + \beta\Delta z_{t-i} + ... + \beta\Delta z_{t-p} + \epsilon_t$, the Augmented Dickey-Fuller test without deterministic trend assesses the null hypothesis of a unit root $H_0 : \alpha = 0$, against the alternative hypothesis $H_1 : \alpha < 1$. The test has been conducted for different lags value p from one to four. We than compare the AIC and BIC information criterion to choose the more parsimonious model. The test fails to reject the null hypothesis of a unit root against the autoregressive alternative for the inflation series $\hat{\pi}$ and for the two interests rate series $\hat{r}^f$ and $\hat{r}^l$. The output gap series $\hat{y}^{gap}$ seems to be stationary. In the Table 3.1 below we report the test results. Looking at the BIC values we can conclude that the first difference is the best model for all the series. Applying a first difference on the series with



unit roots is enough to make them stationary. We have tested four different models:

Table 3.1: ***ADF-Test*** **results**

$h = 0$: the test rejects the null

			Lags			
			1	**2**	**3**	**4**
Series	**y**	**h**	1	1	1	1
		p-value	0.0010	0.0030	0.0086	0.0218
		BIC	-611.29	-599.60	-584.65	-583.27
	pi	**h**	0	0	0	0
		p-value	0.1659	0.0799	0.0683	0.7252
		BIC	-501.54	-491.02	-478.74	-474.95
	rf	**h**	0	0	0	0
		p-value	0.32	0.4792	0.3781	0.5564
		BIC	-505.22	-493.55	-483.56	-471.28
	rl	**h**	0	0	0	0
		p-value	0.32	0.4792	0.3781	0.5564
		BIC	-513.85	-506.58	-493.15	-480.97

a VAR(1) and a VAR(2) with diagonal autoregressive restrictions and the VAR(1) and VAR(2) with full parameters. The likelihood ratio test of model specification is performed with respect the two models with the same order and with respect the two full models. The results reported in Tables 3.2 - 3.3 show that the full VAR models are preferred to the diagonal models. The test does not reject the unrestricted VAR(2) model when compared to the full VAR(1) model. The two information criterion AIC and BIC are lower for the full models confirming the full

VAR structure as the best choice. The two criterions are not coherent: the AIC is less for the two orders specification whereas the BIC is less for the one one order model. The VAR(1) model was finally chosen because it shows the lower sum of square for the out-of-sample periods (see Table 3.4). The VAR(1) model is defined as follow and the estimated parameters are reported in Table 3.5:

$$\Delta\hat{y}_t^{gap} = \alpha^1 + \beta^{1,1}\cdot\Delta\hat{y}_{t-1}^{gap} + \beta^{1,2}\cdot\Delta\hat{\pi}_{t-1} + \beta^{1,3}\cdot\Delta\hat{r}_{t-1}^f + \beta^{1,4}\cdot\Delta\hat{r}_{t-1}^l + \epsilon_t^1$$

$$\Delta\hat{\pi}_t = \alpha^2 + \beta^{2,1}\cdot\Delta\hat{y}_{t-1}^{gap} + \beta^{2,2}\cdot\Delta\hat{\pi}_{t-1} + \beta^{2,3}\cdot\Delta\hat{r}_{t-1}^f + \beta^{2,4}\cdot\Delta\hat{r}_{t-1}^l + \epsilon_t^2$$

$$\Delta\hat{r}_t^f = \alpha^3 + \beta^{3,1}\cdot\Delta\hat{y}_{t-1}^{gap} + \beta^{3,2}\cdot\Delta\hat{\pi}_{t-1} + \beta^{3,3}\cdot\Delta\hat{r}_{t-1}^f + \beta^{3,4}\cdot\Delta\hat{r}_{t-1}^l + \epsilon_t^3$$

$$\Delta\hat{r}_t^l = \alpha^4 + \beta^{4,1}\cdot\Delta\hat{y}_{t-1}^{gap} + \beta^{4,2}\cdot\Delta\hat{\pi}_{t-1} + \beta^{4,3}\cdot\Delta\hat{r}_{t-1}^f + \beta^{4,4}\cdot\Delta\hat{r}_{t-1}^l + \epsilon_t^4$$

Monte Carlo simulations are performed for the out-of-sample period for each series: in figure 3.1 we report the 15000 MC simulated trajectories (the real values for the out-of-sample period have been drawn in black).

Table 3.2: Likelihood Ratio Test for different VAR model specifications

VAR(1)full -VAR(1)diag			**VAR(2)full -VAR(2)diag**			**VAR(2)full -VAR(1)full**		
h	p-value	stat	h	p-value	stat	h	p-value	stat
1	3.7740e-10	82.0294	1	7.2324e-11	108.7926	1	2.6686e-06	55.7574

Table 3.3: Information Criterion for different VAR model type specification

VAR(1)full		***VAR(1)diag***		***VAR(2)full***		***VAR(2)diag***	
AIC	BIC	AIC	BIC	AIC	BIC	AIC	BIC
-1.8327e3	-1.7852e3	-1.7867e3	-1.7677e3	-1.8565e3	-1.7836e3	-1.8077e3	-1.7823e3

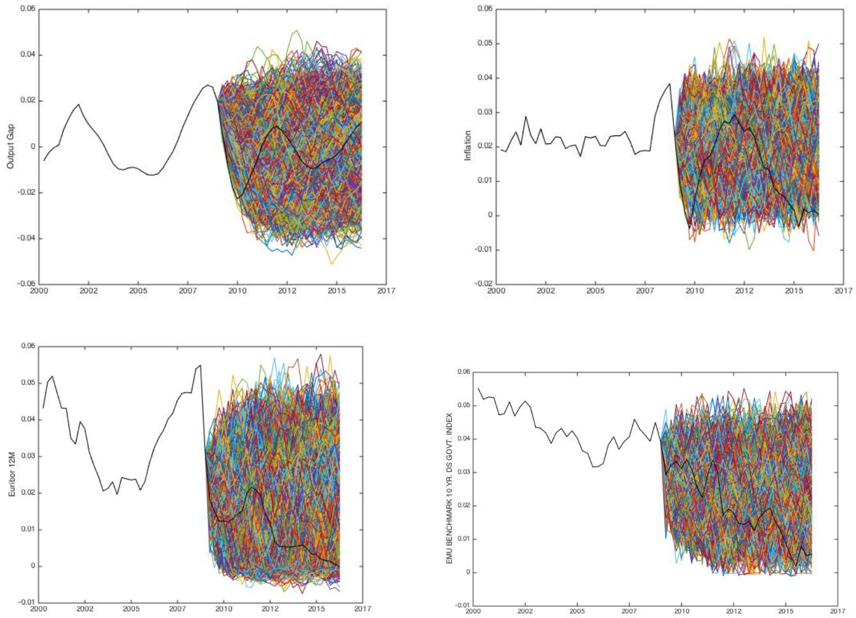
Table 3.4: sum-of-squares error between the predictions and the data for the out-of-sample period

	VAR(1)full	***VAR(1)diag***	***VAR(2)full***	***VAR(2)diag***
SSQ	0.0016	0.0014	0.0017	0.0018

Table 3.5: VAR(1) estimated parameters

α	β			
0,00017	0.6989	0.1014	0.1767	0.0712
-0,00018	0.2406	0.0764	0.2329	-0.2233
-0,00062	0.2409	-0.2740	0.5093	-0.2671
-0,00032	-0.1197	-0.1191	0.5541	-0.2120

Figure 3.1: Economic variables - MC Simulated Trajectories



3.3.3 Liability Model

The pension fund's liability consists of all current and future payments towards pensioners and active members taking into account outstanding future contributions. In a DB pension scheme the contributions payed by the active members is a fixed percentage of the salary whereas pensioners receive a pension which is an annuity based on a fixed percentage of their income at the last working year. This income typically evolves during the retirement period according to the evolution of a reference consumer price index. In order to have a detailed description of the future pension funds payment streams we model the evolution of the contributions and of the pensions of every pension fund member. Given an initial population structure the dynamics of future cash flows needs the specification of the population evolution and of the inflation rate. The liability model used in this work is based on the following assumptions:

- The initial total population of male ad female members is given as an input.
- The salary and the pension of each member at the starting period are known.
- The pension scheme is closed: no more members will enter the fund in the following period.
- The minimal age in the starting period is 19. Men retire at 66, women at 64. The maximum age considered is 100.
- Active member salaries have age and inflation related increases. Passive member pensions have just inflation related increases.
- All the payments are liquidated at the end of the year.
- The two risk factors governing the future stream of payments are the mortality risk (population dynamic) and the inflation rate.
- We do not consider the case in which the active members can leave the fund due to changing their employer nor the possibility for the pensioners to have back the accumulated contribution amount in a lump sum at the first retirement period.

Mortality Rate Model

The stochastic mortality rates model implemented in this work is a modification of the traditional LeeCarter model proposed by Mitchell et al. [21]. The model can be expressed as:

$$\mu_{\alpha,t+1} = \mu_{\alpha,t} \cdot e^{a_\alpha + \sum_{i=1}^{F} b_\alpha^i k_t^i + \epsilon_{\alpha,t}}$$

where $\mu_{\alpha,t}$ are the central mortality rates of age group α at t, a_α describes the average change in log mortality rate of the age group α, k_t^i are factors describing mortality change indexes that are the same for all age groups, b_α^i are parameters modelling the intensity of the response of each age to the mortality index , and $\epsilon_{\alpha,t}$ is the age and year error, which has expected value zero and which we assume to be uncorrelated across time and age group. The model can be expressed by a log transformation as:

$$\ln\left[\mu_{\alpha,t+1}\right] - \ln\left[\mu_{\alpha,t}\right] = a_\alpha + \sum_{i=1}^{F} b_\alpha^i k_t^i + \epsilon_{\alpha,t}.$$

Since the values of a_α, b_α^i and k_t are not unique for any representation of the model some restrictions are necessary: a_α is forced to be the mean change in log mortality rates and the parameters b_α^i must be such that: $\sum_\alpha b_\alpha^i = 1$, for $i = 1, .., F$. The estimation procedure is similar to that proposed in the Lee-Carter approach [**?**]:

- Define the matrix $M_{\alpha,t}$ of log mortality rate changes: $M_{\alpha,t} = \ln[\mu_{\alpha,t+1}] - \ln[\mu_{\alpha,t}]$
- Compute a_α as the mean of log mortality changes over time for each age.
- Obtain a demeaned matrix $\bar{M}_{\alpha,t} = M_{\alpha,t} - a_\alpha$.
- Apply a singular value decomposition to the matrix $M_{\alpha,t}$ to obtain the orthogonal matrices U and V and the non-negative diagonal matrix S such that $M_{\alpha,t} = USV'$.
- $k_t^i = U^i S_{i,i}$, where U^i is the $i-th$ column of U and $S_{i,i}$ is the element at the $i-th$ row and $i-t$ column of S.
- $b_\alpha = \frac{V^i}{S_{i,i}}$, where V^i is the $i-th$ column of V.

The model was applied to the mortality rates of male and females separately so that we can model the evolution of both sexes inside the pension fund model. The historical mortality rates for Italian males and females from 1950 to 2008 are taken from the *Human Mortality Database* in order to calibrate the model, the period from 2009 to 2014 was instead used to test the model (out-of-sample period). Since we are interested in modelling the evolution of the pension fund members, only ages from 19 to 100 are taken into account. We test both the male and female mortality rates models for different choices of the numbers of factors F ranging from 1 to 4, we then compared the square root of the sum of squared errors *(RSSE)* between the historical log mortality rate and the model prediction $RSSE = \sqrt{\sum_{\alpha,t} \epsilon_{\alpha,t}^2}$ for the four models. We have found that the four factors model is the best choice to match the males and the females historical mortality rates (see Table 3.6). The unexplained variance $UV_\alpha = \frac{Var(\epsilon_{\alpha,t})}{Var(\ln[m_{\alpha,t}])}$ (see [20] for a detailed explanation) has been computed for the different four models both for male and female as a further measure of fit. Also in this case the measure indicates that the four factors model is the best choice for modelling our data; in Table 3.7 we reported the sum $\sum_\alpha UV_\alpha$ for the different models.

Table 3.6: RSSE for different number of factors considered

	Number of Factors			
	1	2	3	4
Males	0.7931	0.7444	0.7369	0.5932
Female	0.7520	0.4599	0.4332	0.4025

Once the model is fitted, the dynamic of each factor k^i for $i = 1, .., 4$ must be modelled in order to make an out-of-sample forecast of the mortality rates. Each of

Table 3.7: Sum of the unexplained variance of different choices of the number of factors

	Number of Factors			
	1	2	3	4
Males	0.0018	0.0016	0.0016	9.5200e-04
Female	0.0010	4.0494e-04	3.5109e-04	3.0021e-04

the four factors dynamics is fitted as an independent autoregressive process which can have both a conditional mean and a conditional variance structure:

$$k^i_{s,t} = \gamma_{i,s} k^i_{s,t-1} + \epsilon_{i,s,t}, \ , \qquad i = 1, ..., 4 \qquad s \in \{m, f\} \tag{3.9}$$

$$\sigma^2_{i,s,t} = c_{i,s} + \psi_{i,s}\epsilon^2_{i,s,t-1}, \qquad i = 1, ..., 4 \qquad s \in \{m, f\}\,, \tag{3.10}$$

where the index s states if the i-th factor, with $i = 1, ..., 4$, is related to male (m) or female (f) mortality rates. Since we consider four factors for the female and four factors for the male mortality rates we have to model eight autoregressive processes (in Figure 3.2 the eight factor processes obtained from the estimation on the historical mortality rates are drawn). The *Jarque-Bera test* (JB-test) has been conducted on each series to determine the opportunity to use a gaussian distribution whereas the *Ljung-Box test* (LB-test) was used to assess the presence of autocorrelation and, when applied to the square of the series, the presence of conditional variance together with the Engle's ARCH test. The test rejects the normality assumptions for the first male factor and for the third female factor. These factors are then modelled with a t-student distribution. All the series present a first order autocorrelation component. The third male factor has an ARCH component as well the third and the four females factors. Given the following general model we show in Tables 3.8 - 3.9 the estimated parameters for the eight factor processes. If the error is assumed to follow a t-distribution we insert the value of the estimated degree of freedom (DoF).

Table 3.8: Estimated Parameters for the Male Factors

	Parameters				
Factors	**gamma**	**c**	**phi**	**DoF**	**sigma**
1	-0.33768	NaN	NaN	3.13089	0.0867501
2	0.0387247	NaN	NaN	NaN	0.0561904
3	-0.604539	0.0128465	0.210544	NaN	NaN
4	-0.523419	NaN	NaN	NaN	0.0137274

Figure 3.2: Factor Processes obtained from the in-sample estimation

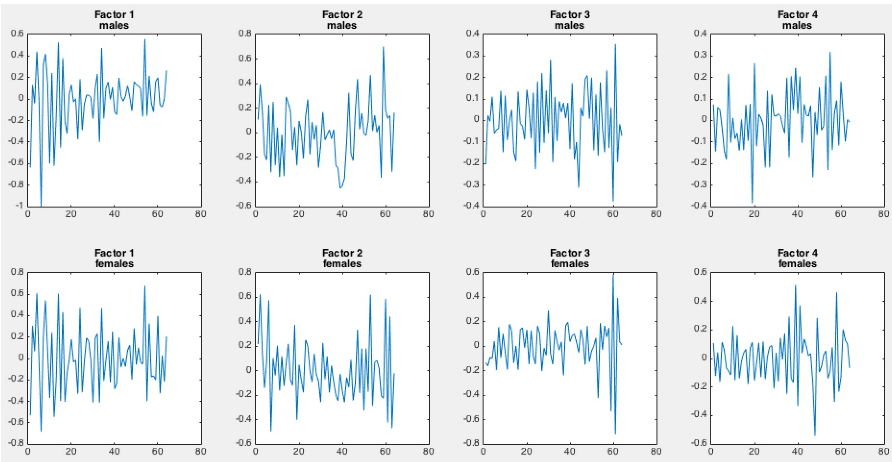


Table 3.9: Estimated Parameters for the Female Factors

	Parameters				
Factors	γ	**c**	ψ	**doF**	σ
1	-0.46967	NaN	NaN	NaN	0.0729677
2	-0.414838	NaN	NaN	NaN	0.0526764
3	-0.360976	0.0202475	0.230144	19.1028	NaN
4	-0.387619	0.0149691	0.540618	NaN	NaN

Economic and Population Risk Factors Tree Generation

Since the pension fund payments have yearly commitments, the scenarios trees for the stochastic economics and populations factors are firstly generated on a discrete time set $\tilde{\mathcal{T}} = \{1, 2, 3, 4, 5, 6, 7\}$ with yearly inter-stages steps. The number of nodes $\tilde{N}_t$ of each sub-tree at each stage is set equal to N_t if $t \in \mathcal{T}$ and equal to one if $t \in \tilde{\mathcal{T}} \setminus \mathcal{T}$. The first step is the construction of the trees $\tilde{y}^{gap}$, $\tilde{\pi}$, $\tilde{r}_0$ and $\tilde{r}_l$ for the economic factors, (output gap, inflation, short and long interest rate) on the larger time set $\tilde{\mathcal{T}}$. We do this by Conditional Monte Carlo (*CMC*) simulations from the VAR(1) process previously estimated: starting from the root node, for each node at each stage t we simulate a sample size equal to N_{t+1} using Monte Carlo sampling. The second step is the generation of the trees for the eight latent factors driving the mortality rates for the male and the female pension fund members. Since the factors are assumed independently by construction we generate these tree processes independently via *CMC* sampling. We call $\tilde{k}^i_s$ for $i = 1, .., 4$ and $s \in \{m, f\}$ these tree processes for the mortality factors. Once we have the factors trees we derive the male mortality trees for all the ages just applying the formula: $\tilde{\mu}_{m,\alpha,n} = \tilde{\mu}_{m,\alpha,a(n)} \cdot e^{a_{m,\alpha} + \sum_{i=1}^{4} b^i_{m,\alpha} \tilde{k}^i_{m,n} + \epsilon_{m,\alpha,n}}$, for $\alpha \in \mathcal{A}$, $n = 1, ..., \tilde{N}_t$ and $t = 1, 2, ..., 7$. The same holds for the female mortality trees. We have now the uncertain tree processes $\tilde{\pi}$ and $\tilde{\mu}$ defined on the time set $\tilde{\mathcal{T}}$ and with branching structure $\tilde{\mathcal{N}} = \{1, N_1, 1, N_3, 1, N_5, 1, N_7\}$. These trees are then used to construct the net pension expenditure tree process $\tilde{l}$ on the same time set $\tilde{\mathcal{T}}$. This procedure is explained step by step in the following paragraphs.

Population Model

We define $N_{f,\alpha,t}$ and $N_{m,\alpha,t}$ as the total female and male members aged α respectively at $t \in \tilde{\mathcal{T}}$, where $\alpha \in \mathcal{A} = \{19, ..., 100\}$ and $\tilde{\mathcal{T}} = \{1, 2, ..., 6, 7\}$. The total male and female active members population at t is $TA_{m,t} = \sum_{\alpha=19}^{65} N_{m,\alpha,t}$ and $TA_{f,t} = \sum_{\alpha=19}^{63} N_{f,\alpha,t}$ whereas the total male and female passive members population at t is $TP_{m,t} = \sum_{\alpha=66}^{100} N_{m,\alpha,t}$ and $TPf_t = \sum_{\alpha=64}^{100} N_{f,\alpha,t}$. The total population at t is then defined as $TN_t = \sum_{\alpha=19}^{100} (N_{f,\alpha,t} + N_{m,\alpha,t})$. The contribution rate r_c (percentage of the salary used to compute the yearly contribution of each active member) and the pension rate r_p (percentage of the last salary used to compute the pension received by each passive member) are fixed and set equal to 0.1 and 0.6 respectively. The salaries of male and female members aged α at t are $W_{m,\alpha,t}$ and $W_{f,\alpha,t}$ respectively. We start to define the total number of active and passive members of both sexes at the starting point $t = 0$: $TA_{m,0}$, $TA_{f,0}$, $TP_{m,0}$ and $TP_{f,0}$. Once these parameters are decided, the number of people in each age group is defined by applying fixed proportion rates λ's reflecting the current age distribution on the entire national population both in the active and retirement ages for males and females:

$$
\begin{aligned}
N_{m,\alpha,t} &= \lambda^A_{m,\alpha} \cdot TA_{m,0}, && \text{for } \alpha = 1, ..., 65.\\
N_{f,\alpha,t} &= \lambda^A_{f,\alpha} \cdot TA_{f,0}, && \text{for } \alpha = 1, ..., 63.\\
N_{m,\alpha,t} &= \lambda^P_{m,\alpha} \cdot TP_{m,0}, && \text{for } \alpha = 66, ..., 100.\\
N_{f,\alpha,t} &= \lambda^P_{f,\alpha} \cdot TP_{f,0}, && \text{for } \alpha = 64, ..., 100.
\end{aligned}
$$

Specifically we get the numbers of male and female aged α with $\alpha = 19, ..., 100$ on the entire national population from the *Human Mortality Database* and *ISTAT* and we call them $\bar{N}_{m,\alpha,0}$ and $\bar{N}_{f,\alpha,0}$ respectively, then the fixed proportion coefficients are set accordingly to these formulas:

$$
\begin{aligned}
\lambda^A_{m,\alpha} &= \frac{\bar{N}_{m,\alpha,0}}{\sum_{\alpha=19}^{65} \bar{N}_{m,\alpha,0}}, && \text{for } \alpha = 1, ..., 65 \\
\lambda^A_{f,\alpha} &= \frac{\bar{N}_{f,\alpha,0}}{\sum_{\alpha=19}^{63} \bar{N}_{f,\alpha,0}}, && \text{for } \alpha = 1, ..., 63 \\
\lambda^P_{m,\alpha} &= \frac{\bar{N}_{m,\alpha,0}}{\sum_{\alpha=66}^{100} \bar{N}_{m,\alpha,0}}, && \text{for } \alpha = 66, ..., 100 \\
\lambda^P_{f,\alpha} &= \frac{\bar{N}_{f,\alpha,0}}{\sum_{\alpha=64}^{100} \bar{N}_{f,\alpha,0}}, && \text{for } \alpha = 64, ..., 100
\end{aligned}
$$

From the starting values $N_{m,\alpha,t}$ and $N_{m,\alpha,t}$ we generate the yearly tree process describing the population evolution using the mortality rate trees $\tilde{\mu}_{m,\alpha,n}$ and $\tilde{\mu}_{f,\alpha,n}$ with $\alpha \in \mathcal{A}$, $n \in \bar{\mathcal{N}}_t$ and $t \in \bar{\mathcal{T}}$. The mortality rate $\tilde{\mu}_{m,\alpha,n}$ defines the probability to die during the year of a male aged α at the node n at t. According to this rates the tree processes describing the population dynamics are:

$$
\begin{aligned}
N_{m,\alpha+1,n} &= (1 - \tilde{\mu}_{m,\alpha+1,n}) \cdot N_{m,\alpha,a(n)}, && \text{for } \alpha = 19, ..., 99, \quad n \in \bar{\mathcal{N}}_t \text{ and } \quad t = 2, ..., 7 \\
N_{f,\alpha+1,n} &= (1 - \tilde{\mu}_{f,\alpha+1,n}) \cdot N_{f,\alpha,a(n)}, && \text{for } \alpha = 19, ..., 99, \quad n \in \bar{\mathcal{N}}_t \text{ and } \quad t = 2, ..., 7.
\end{aligned}
$$

Salary - Contribution - Pension Model

The salary evolution for each individual depends on the age and on the inflation rate. The age related increase is deterministic and based on a fixed yearly proportion increment whereas the inflation related increase is derived accordingly to a scenario tree for the inflation evolution $\tilde{\pi}$. We set the age related increment τ equal to 0.01, the salary at $t = 0$ for a male active member $\omega_{m,19,0}$ and for a female active member $\omega_{f,19,0}$ are set to 1200 and 1100 euro respectively. Based on these inputs we firstly model the salaries ω and pensions γ at $t = 0$ for all the ages different from 19:

$$
\begin{aligned}
\omega_{m,\alpha,0} &= \omega_{m,19,0} \cdot (1+\tau)^{(\alpha-19)} && \text{for } \alpha = 20, ..., 65 \\
\omega_{f,\alpha,0} &= \omega_{f,19,0} \cdot (1+\tau)^{(\alpha-19)} && \text{for } \alpha = 20, ..., 63 \\
\gamma_{m,\alpha,0} &= r_p \cdot \omega_{m,19,0} \cdot (1+\tau)^{(\alpha-19)} && \text{for } \alpha = 66, ..., 100 \\
\gamma_{f,\alpha,0} &= r_p \cdot \omega_{f,19,0} \cdot (1+\tau)^{(\alpha-19)} && \text{for } \alpha = 64, ..., 100
\end{aligned}
$$

From these deterministic values we model the salary ω, the contribution κ and pension γ dynamics:

$$\begin{aligned}
\omega_{m,\alpha,n} &= \omega_{m,\alpha-1,a(n)} \cdot (1+\tau) \cdot (1+\pi_n) && \text{for } \alpha = 20, ..., 65 \\
\omega_{f,\alpha,n} &= \omega_{f,\alpha-1,a(n)} \cdot (1+\tau) \cdot (1+\pi_n) && \text{for } \alpha = 20, ..., 63 \\
\kappa_{m,\alpha,n} &= r_c \cdot \omega_{f,\alpha,n} && \text{for } \alpha = 20, ..., 65 \\
\kappa_{f,\alpha,n} &= r_c \cdot \omega_{f,\alpha,n} && \text{for } \alpha = 20, ..., 63 \\
\gamma_{m,\alpha,n} &= r_p \cdot \omega_{m,\alpha-1,a(n)} \cdot (1+\pi_n) && \text{for } \alpha = 66 \\
\gamma_{f,\alpha,n} &= r_p \cdot \omega_{f,\alpha-1,a(n)} \cdot (1+\pi_n) && \text{for } \alpha = 64 \\
\gamma_{m,\alpha,n} &= r_p \cdot \gamma_{m,\alpha-1,a(n)} \cdot (1+\pi_n) && \text{for } \alpha = 67, ..., 100 \\
\gamma_{f,\alpha,n} &= r_p \cdot \gamma_{f,\alpha-1,a(n)} \cdot (1+\pi_n) && \text{for } \alpha = 65, ..., 100
\end{aligned}$$

Finally we can compute the total amount of pension fund nets payments $\tilde{l}_n$ at each node just by aggregating the total contributions K and total pensions Γ according to the number of active and passive members of both sexes at the same node:

$$\begin{aligned}
K_{m,n} &= \sum_{\alpha=19}^{65} \kappa_{m,\alpha,n} \cdot N_{m,\alpha,n} \\
K_{f,n} &= \sum_{\alpha=19}^{65} \kappa_{f,\alpha,n} \cdot N_{f,\alpha,n} \\
\Gamma_{m,n} &= \sum_{\alpha=66}^{100} \gamma_{m,\alpha,n} \cdot N_{m,\alpha,n} \\
\Gamma_{f,n} &= \sum_{\alpha=66}^{100} \gamma_{f,\alpha,n} \cdot N_{m,\alpha,n} \\
\tilde{l}_n &= \Gamma_{m,n} + \Gamma_{f,n} - K_{m,n} - K_{f,n}
\end{aligned}$$

The final step is to transform the topology of the tree processes for the short term interest rate $\tilde{r}_0$ and for the pension fund payments $\tilde{l}$ obtained until now in order to have the two trees r_0 and l with time increments and topology defined by the sets $\mathcal{T}$ and $\mathcal{N}$ which will be used as input parameters in the *ALM-pricing Problem.* The interest rate tree process r_0 is obtained by simply compound the process $\tilde{r}_0$:

$$\begin{aligned}
r_n &= \tilde{r}_{0,n}, && n \in \mathcal{N}_1 \\
r_n &= (1+\tilde{r}_{0,n}) \cdot \left(1+\tilde{r}_{0,a(n)}\right) - 1, && n \in \mathcal{N}_t && t \in \{3,5,7\}\,.
\end{aligned}$$

For what concern the pension fund payments process we just move forward in time the value of the nodes belonging to $\tilde{\mathcal{T}} \backslash \mathcal{T}$ to the child nodes, which belong to $\mathcal{T}$, by multiply them with the risk free interest rate of the same period:

$$l_n = \tilde{l}_n, \qquad n \in \mathcal{N}_1$$
$$l_n = \tilde{l}_n + (1 + \tilde{r}_{0,n})\, \tilde{l}_{a(n)}, \qquad n \in \mathcal{N}_t, \qquad t \in \{3, 5, 7\}$$

3.3.4 Asset Model

We consider the following asset universe of investment composed by four bond indexes and three stock indexes all quoted in euro currency and obtained form the Thompson Data Stream database.

- Assets universe:
    1. BARCLAYS EURO AGG GOVERNMENT ALL MATS. INDEX
    2. BARCLAYS EURO AGG CORPORATE INDEX
    3. FTSE GLOBAL GOVERNMENT US ALL MATS. INDEX
    4. FTSE EURO EMERGING MARKETS ALL MATS. INDEX
    5. S&P 500 COMPOSITE - PRICE INDEX
    6. MORGAN STANLEY CAPITAL INTERNATIONAL EMU
    7. MSCI EM PRICE INDEX
- Data frequency: monthly
- Estimation period:
  From 31-December-1999 to 31-December-2008.
- Out-of-Sample period:
  From 31-January-2009 to 31-December-2015.

The choice of monthly data frequency is due to the medium horizon of the stochastic programming problem. Although the monthly frequency is convenient to deal with large Monte Carlo simulation for medium and long forecasting horizon it poses some challenges in the choice of the econometric model to represent the asset returns dynamic. Stylised fact such as fat tails, volatility clusters and distribution asymmetries that are well documented in daily asset returns are instead more difficult to detect in monthly returns. This is due both from the nature of the financial time series, stylised facts tend to gradually disappear when we use larger frequency than daily, and from the less disposable data to fit the parameters of complex models that try to capture these behaviours. In Tables 3.10 - 3.11 we store some descriptive statistics for the risky asset returns computed using the real data for the in-sample period (from 31-December-1999 to 31-December-2008). We also draw the prices and the corresponding returns on the entire data period (from 31-December-1999 to 31-December-2015) in Figures 3.3 - 3.4. The asset return econometric model has been derived with the following procedure:

1. Each series $\{\hat{r}_{i,t}\}_{t=1}^{Ts}$, $i = 1, ..., I$ (where Ts is the time dimension of the in-sample dataset) is analysed independently and the best autoregressive process with conditional variance accordingly to different criteria explained later is chosen. The general model for each series is an autoregressive process with conditional variance modelled as a Glosten, Jagannathan, and Runkle *(GJR)* model [11]:
$$\hat{r}_{i,t} = c_i + \sum_{p=1}^{P_i} \gamma_i \hat{r}_{i,t-p} + \epsilon_{i,t}$$
$$\sigma_{i,t}^2 = k_i + \omega_i \sigma_{i,t-1}^2 + \psi_i \epsilon_{i,t-1}^2 + \phi_i \left[\epsilon_{i,t-1} < 0\right] \epsilon_{i,t-1}^2$$

2. Once the autoregressive process has been chosen and calibrated the innovation process for the in-sample period is derived and the residuals $\{\epsilon_{i,t}\}_{t=1}^{Ts-P_i}$ are than standardised: $z_{i,t} = \epsilon_{i,t}/\sigma_{i,t}$

3. A density distribution function is then fitted on the standardised innovation process $\{z_{i,t}\}_{t=1}^{Ts-P_i}$. We consider just a gaussian or a t-student distribution to keep the parameter estimation stable given the small sample size.

4. The standardised residuals are transformed to uniform variates by the cumulated distribution function previously estimated.

5. All the uniform series are collected and a t-copula is estimated with an MLE algorithm.

In order to simulate Sc Monte Carlo trajectories we simulate (Sc, I) values from the t-copula and we obtaine Sc standardised innovation trajectories for each series i with $i = 1, ..., I$ using its inverse cumulative distribution function. In this way we obtain standardised innovation trajectories that are uncorrelated in time but correlated among them consistently with the t-copula previously estimated. Then we use this standardised residuals to simulate the autoregressive process for each asset return.

Estimation of the autoregressive model for each series

The first step is the definition and the estimation of the conditional mean autoregressive process for each return time series. Plotting the autocorrelation sample

Table 3.10: Main Statistic Properties of the Asset Returns - In-Sample period

	Asset Type						
	1	**2**	**3**	**4**	**5**	**6**	**7**
Mean	0.0007	0.0005	0.0005	-0.0003	0.0012	-0.0013	0.0022
Var	0.0001	0.0001	0.0007	0.0003	0.0024	0.0037	0.0042
Skewness	-0.0565	-0.3996	0.2789	-1.0846	-0.4938	-0.5833	-0.6277
Kurtosis	3.1631	4.7324	3.0028	9.2113	3.7076	3.8961	4.1152

Table 3.11: Correlation Matrix of asset Returns - In-Sample period

1	0.7361	0.3124	0.0718	-0.3110	-0.2939	-0.2649
0.7361	1	0.0802	0.2791	-0.0753	-0.0161	0.0738
0.3124	0.0802	1	-0.0416	0.2336	-0.0802	-0.0114
0.0718	0.2791	-0.0416	1	0.3278	0.3093	0.3589
-0.3110	-0.0753	0.2336	0.3278	1	0.8214	0.7640
-0.2939	-0.0161	-0.0802	0.3093	0.8214	1	0.8191
-0.2649	0.0738	-0.0114	0.3589	0.7640	0.8191	1

function we have a first indication of the autoregressive order P. The goodness-of-fit *Jarque-Bera* (JB) test is applied to test the normality assumption of the model. The only series that pass the JB test are the first and the third, the government bond indexes for the Euro Area and for the US; these series are so modelled under a gaussian hypothesis on the distribution of the errors terms, the other series residuals are modelled with a t-distribution. The *Ljung-Box* (LB) test and the *Engle's ARCH* (EA) test for residual heteroscedasticity were used to assess the presence of autocorrelation and ARCH effects in the return residuals. The choice of introducing a leverage effect for those series which present ARCH effects is firstly considered by looking at the skewness value and at the qqplot of the series. The effectiveness of this choice is then tested by applying a *Likelihood Ratio* Test (LRT) on the GJR model versus a standard GARCH model. In the following points the model for each asset return series with the main relevant passages that led to the choice are presented.

1. BARCLAYS EURO AGG GOVERNMENT ALL MATS
   The series appears to be distributed according to a gaussian distribution as suggested by the *JB-test* and by looking at the qqplot. The *LB-test* rejects the hypothesis of autocorrelation on the mean and on the variance. In order to be sure to model the returns as a white noise with mean with estimate both a white noise and an autoregressive process of order one and than we apply a *LRT* test to test the restriction on the autoregressive parameter. The *LRT* test accepts the null hypothesis of a white noise with mean with a $p-value$

Figure 3.3: Asset Prices

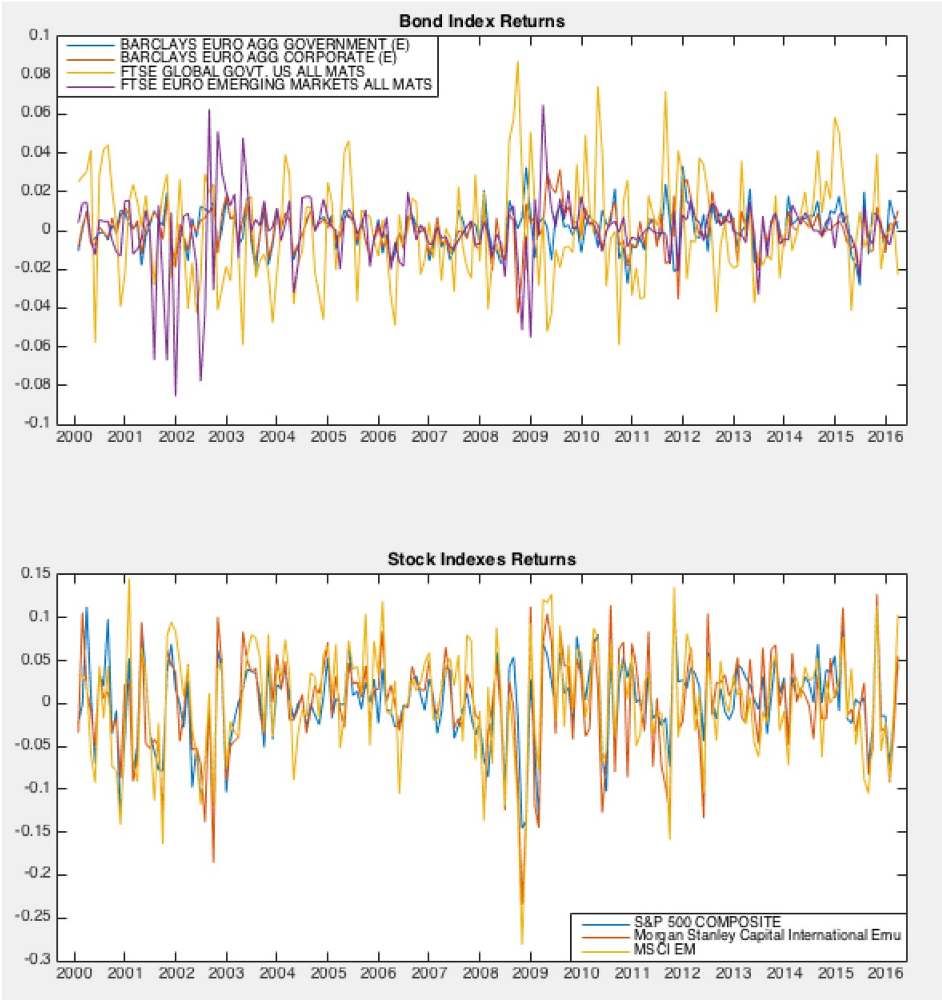
of 0.1371. The final model choice is:

$$\hat{r}_{1,t} = c_1 + \epsilon_{1,t}$$

$c_i = 0.00066$, with $\epsilon_1 \sim N(0, 0.000112)$

2. BARCLAYS EURO AGG CORPORATE ALL MATS
The series seems not to follow a Gaussian distribution both from the *JB-test* and from the qqplot graph analysis. Since the *LB-test* rejects the null hypothesis of no autocorrelation for the lag of order three we test an autoregressive process with all the three first lags against an autoregressive process with just the third order autoregression parameter: the restricted model is preferred with a $p-value$ of 0.0254 (we use a significant alpha level of 1% since we have a small sample size). The presence of an ARCH effect is detected by the *LB-test* ($p-value$ of 9.7627e-04) performed on the square returns and with a *EA-test* ($p-value$ of 1.0563e-04). The AR(3)-GARCH(0,1) is tested against the model AR(3) and the $p-value$ of the *LR* test is 5.2232e-04 confirming the necessity of incorporating an ARCH parameter. The final model choice

Figure 3.4: Asset Returns



is:

$$\hat{r}_{2,t} = c_2 + \gamma_2 \hat{r}_{2,t-3} + \epsilon_{2,t}$$

$$\sigma^2_{2,t} = k + \psi_2 \epsilon^2_{2,t-1}$$

$c_2 = 0.0004$, with $\epsilon_2/\sigma_2 \sim t$ -distribution with 8.9 degree of freedom.

3. FTSE GLOBAL GOVERNMENT US ALL MATS
   The government bond index of the USA Area presents a first order autoregression in the conditional mean and no conditional variance, the innovation distribution can be specified by a gaussian distribution.The final model choice is:

Figure 3.5: Autocorrelation Sample Functions for the Asset Returns

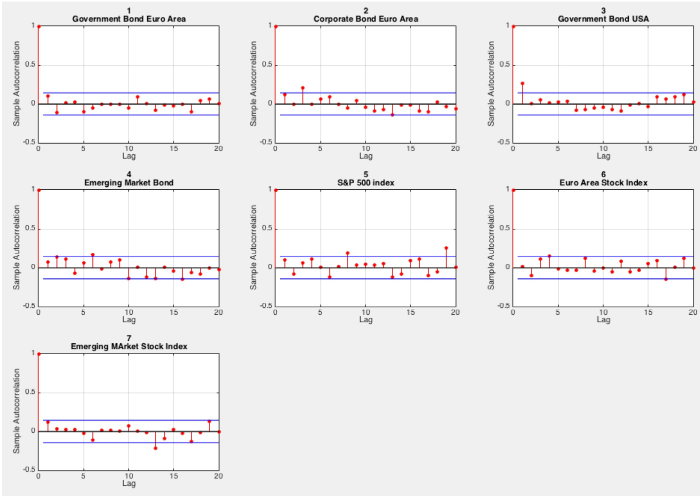
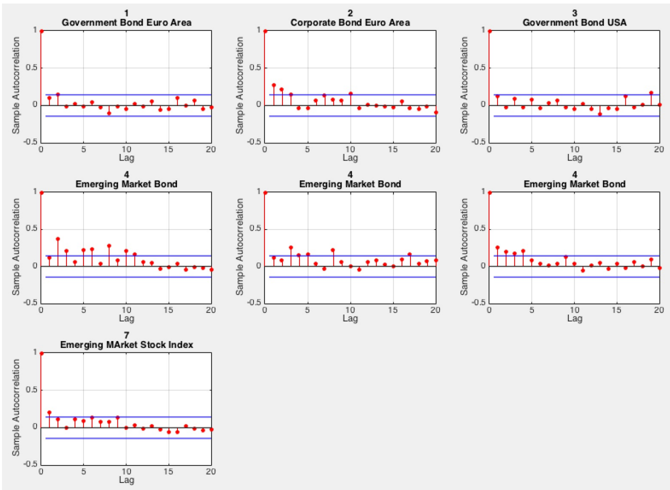


Figure 3.6: Autocorrelation Sample Functions for the Square of Asset Returns



$$\hat{r}_{3,t} = c_3 + \gamma_3 \hat{r}_{3,t-1} + \epsilon_{3,t}$$

$c_3 = 0.0003$, with $\epsilon_3 \sim N(0, 0.0007)$

4. FTSE EURO EMERGING MARKETS ALL MATS
   The bond index for the emerging market zone presents both fat tails and asymmetry. The *JB-test* fails to accept the normal distribution and the *LB-test* reject autocorrelation. We test four models: A GARCH(1,1), a GARCH(0,1), a GJR(1,1) and a GJR(0,1). We then use the *LR* test and the AIC and BIC information criterion to choose between the four proposal model. The GJR(1,1) model is preferred against the other models both in terms of the *LR*

and of the two information criteria AIC and BIC. The final model choice is:

$$\hat{r}_{4,t} = c_4 + \epsilon_{4,t}$$

$$\sigma^2_{4,t} = k_4 + \omega_4 \sigma^2_{4,t-1} + \psi_4 \epsilon^2_{4,t-1} + \phi_4 \left[\epsilon_{4,t-1} < 0\right] \epsilon^2_{4,t-1}$$

$c_4 = 1.9565e - 05$, $\omega_4 = 0.725487$, $\psi_4 = 0.362634$ and $\phi_4 = 0.427551$ with $\epsilon_4/\sigma_4 \sim t$ -distribution with 4 degree of freedom.

5. $S\&P$ 500 COMPOSITE
This series does not present autocorrelation in mean but it seems to follow a conditional variance structure. Since the asymmetry is not very high we compare a GARCH(1,1) model against the GJR(1,1) model. The LR and the information criteria AIC and BIC confirm the GARCH(1,1) model as the more parsimonious.The final model choice is:

$$\hat{r}_{5,t} = c_5 + \epsilon_{5,t}$$

$$\sigma^2_{5,t} = k_5 + \omega_5 \sigma^2_{5,t-1} + \psi_5 \epsilon^2_{5,t-1}$$

$c_5 = 0.0056$, $k_5 = 0.00015$, $\omega_5 = 0.811645$, $\psi_5 = 0.151174$ with $\epsilon_5/\sigma_5 \sim t$ -distribution with 11.4 degree of freedom.

6. MORGAN STANLEY CAPITAL INTERNATIONAL EMU
Again the series exhibits conditional variance behaviour but not autocorrelation in mean levels. In this case however the leverage effect is more pronounced than in the $S\&P$500 Index and the GJR(1,1) is preferred to the GARCH(1,1) model with respect the same criteria (LR test , AIC and BIC). The final model choice is:

$$\hat{r}_{6,t} = c_6 + \epsilon_{6,t}$$

$$\sigma^2_{6,t} = k_6 + \omega_6 \sigma^2_{6,t-1} + \psi_6 \epsilon^2_{6,t-1} + \phi_6 \left[\epsilon_{6,t-1} < 0\right] \epsilon^2_{6,t-1}$$

$c_6 = 0.003$, $k_6 = 0.00045$, $\omega_6 = 0.638642$, $\psi_6 = 0.187704$ and $\phi_6 = 0.454688$ with $\epsilon_6/\sigma_6 \sim t$ -distribution with 12.8 degree of freedom.

7. MSCI EMERGING MARKET INDEX
The series presents a very pronounced outlier that we remove in order to have a more stable parameter estimation. Similar to the other stock index models we have compared a GARCH(1,1) model and a GJR(1,1) model. The final model is:

$$\hat{r}_{7,t} = c_7 + \epsilon_{7,t}$$

$$\sigma^2_{7,t} = k_7 + \omega_7 \sigma^2_{7,t-1} + \psi_7 \epsilon^2_{7,t-1}$$

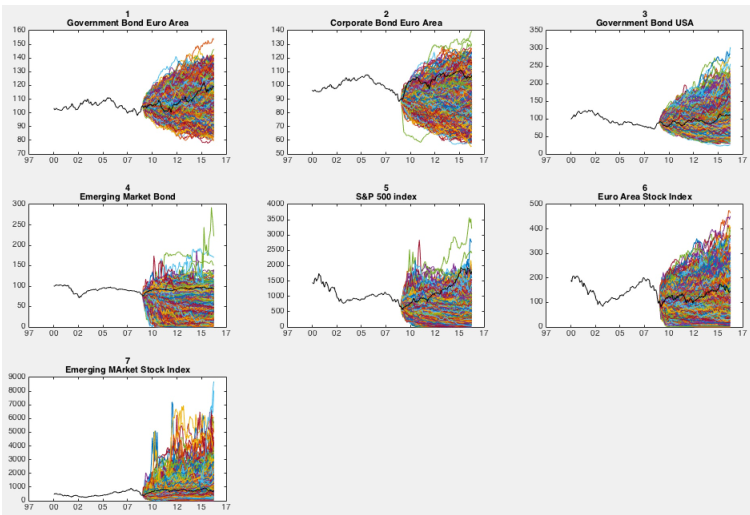
$c_7 = 0.0045$, $k_7 = 0.00053$, $\omega_7 = 0.757091$, $\psi_7 = 0.0228489$ with $\epsilon_7/\sigma_7 \sim t$ -distribution with 8 degree of freedom.

Once we have estimated the models for each series we infer the standardise residuals for the in-sample period and following the procedure describe above we fit a t-copula. The estimated degree of freedom of the t-copula is 13.4172 and the estimated correlation parameters are reported in table 3.12. In Figure **??** we plot 15000 Monte Carlo simulation of the complete model for the asset prices.

Table 3.12: Estimated correlation parameters for the fitted t-copula

1	0.8531	0.3980	0.2979	-0.2925	-0.3117	-0.2535
	1	0.3022	0.4469	-0.1353	-0.1362	-0.0500
		1	0.0861	0.2516	-0.0435	0.0182
			1	0.2777	0.3011	0.4367
				1	0.7693	0.6907
					1	0.8046
						1

Figure 3.7: Asset Returns - MC trajectories



Asset Model Tree Generation

The vector tree process representing the asset returns dynamic is directly obtained with the moment matching method presented in the second chapter of this thesis. The risk free interest rate is stochastic and it is represented by the tree process r_0. Since the probabilities associated to the risk free process are equally spaced on the interval $(0, 1)$ we modify the $MMGP$ algorithm by keeping fixed the probabilities and perform the moment minimisation on the asset returns values and on the risk neutral probabilities. The method is based on the monomial approximation and the solutions are in general dependent on the values of the starting guess solution. When we launch the algorithm to generate a scenario tree from the simulated scenario fan of 15000 MC paths using the econometric model specified in the previous paragraph and a given tree topology specified by the branching structure $\{1, N_1, N_3, N_5, N_7\}$, we solve the problem two times for each starting guess: one using the constraints to avoid arbitrages opportunities and the second relaxing these constraints. In this way we obtained two similar but different trees, one without arbitrage opportunities and the other with arbitrages. In the Appendix we collect the pictures of the trees generated in Figure 3.8. The black line represent the realised trajectory of the process and it always inside the tree.

3.3.5 Numerical Results - Complete market case

The first test has been performed on a complete market. Since the asset universe is composed by 8 securities (7 risky asset plus the cash account) we used trees with four stages and branching structure 8^4 to ensure the market completeness. In this case we can use the risk neutral probability obtained by the moment matching method with the no-arbitrage constraints and the compounded risk-free interest rate $r_{d,t} = \prod_{i=1}^{t}(1 + r_{0,i})^{-1}$ between time 0 and t to obtain the preset value of the liabilities:

$$L_0 = \sum_{t=1}^{T} E^{\mathbb{Q}}\left[(1 + r_{d,t})^{-1} l_t\right], \tag{3.11}$$

where:

$$E^{\mathbb{Q}}\left[(1 + r_{d,t})^{-1} l_t\right] = \sum_{n \in \mathcal{N}_t} q_n \left[(1 + r_{d,n})^{-1} l_n\right].$$

The value of L_0 will be used as a benchmark to compare the results achieved with the perfect replication approach.

The problem of perfect replication is feasible if and only if we allow for unbounded short selling positions and so we set $\delta_f = -\infty$. We denote by L_s^p and $L_{s,f}^p$ the arithmetic average among the solutions obtained using the ten arbitrage and no-arbitrage trees respectively whereas L_0 is the arithmetic average of the liability prices computed by applying formula 3.11 using the ten risk neutral measures $\mathbb{Q}$ derived from the arbitrage free trees. We denote as $\Delta^p := L_0 - L_s^p$ and $\Delta_f^p := L_0 - L_{s,f}^p$ the difference between the risk neutral and the perfect replication problem solutions with arbitrage and no-arbitrage trees respectively and with $\Delta_{\%}^p := \frac{L_0^p - L_s^p}{L_s^p}$ and $\Delta_{f,\%}^p := \frac{L_0^p - L_{s,f}^p}{L_{s,f}^p}$ the corresponding percentage difference. We store the result in Table 3.13. In all the cases with trees without arbitrage opportunities the error between the solution and the risk neutral expected value of the liabilities Δ_f^p is approximatively zero confirming the validity of the model implementation with respect to the mathematical financial theory. When we solve the replication problem using scenario trees with arbitrages the solution error Δ^p is greater but it is sill very low (the liability price L_s^p is just 0.15 percent lower than the liability price L_0^p on average). We can conclude that the cost of having arbitrages in the discrete market model using a perfect replication approach is relatively small. The same procedure has been also performed to obtain the liability values L_b^p and $L_{b,f}^p$ from the point of view of the buyer, i.e. by solving the *ALM-pricing Problem* by inverting the sign of l_n in all the constraints (3.3). Since we are in a complete market case, according to the theoretical results, we have: $L_b^p = L_s^p$ and $L_{b,f}^p = L_{s,f}^p$.

Table 3.13: Solutions for the Perfect Replication Problem and Comparison with the risk neutral expected value L_0

	L_0^p	L_s^p	Δ^p	$\Delta_{\%}^p$	$L_{s,f}^p$	Δ_f^p	$\Delta_{f,\%}^p$
E	132462500	132257500	203834	0.15%	132462500	-10	0,00%
Std	102103	561271	531694	0.004	102103	7	0,00

Risk Measure Valuation

When we use a risk measure valuation approach the liability value is in general different from that obtained with a perfect replication approach and it depends on the choice of the parameter β. We compute the solutions $L_{s,f}$ and $L_{b,f}$ solving the *ALM-pricing problem*, with no restrictions on short selling and using arbitrage free trees, from the point of view of the seller and of the buyer of the process l respectively. The results are computed for $\beta \in \{0, 0.02, 0.04, \ldots, 0.18, 0.2\}$ and store in Table 3.14. We represent by $\Delta_{sb} := L_{s,f} - L_{b,f}$ the spread between the two prices and by $\Delta_{sb,\%} := \frac{L_{s,f}-L_{b,f}}{L_{b,f}}$ the percentage difference. The spread Δ_{sb} is equal to zero when $\beta = 0$, i.e. in the super-hedging case, and decreases linearly with the tolerance parameter β. In order to test the impact of arbitrages in the solution of the *ALM-pricing problem* under a risk measure valuation approach we consider the cases in which we limit or totally exclude short selling positions. In the case in which we limit the short selling, the short position in each asset can not be greater than $\frac{1}{I}$ of the portfolio value at each stage. We denote by $L_{s,ls}$ and $L_{s,ns}$ the average solutions of the super replication problem using trees with arbitrage opportunities allowing for limited short selling and totally excluding short selling respectively, whereas $L_{s,f,ls}$ and $L_{s,f,ns}$ represent the same problem solutions obtained with trees with no arbitrages. In this case $\Delta_{s,i} := L_i - L_{f,i}$ and $\Delta_{s,i,\%} := \frac{L_i - L_{s,f,i}}{L_{s,f,i}}$, for $i \in \{ls, ns\}$. The average of the solutions over the ten trees for different choices of the risk parameter β are reported in Table 3.14 in the Appendix. The absolute percentage difference

$|\Delta_{s,i,\%}|$ is below the 1% both in the case of limited short selling and in the case of no short selling. Obviously, the liability price decreases gradually and proportionally as the parameter β increases. For a given choice of β the liability price increases when we reduce the share of allowed short-selling positions.

3.3.6 Numerical Results - Incomplete Market Case

In order to construct an incomplete discrete market model we generate trees with branching structure 10^4, i.e. with ten nodes in each sub tree at every stage. In this case the risk neutral probability measure associated with a non-arbitrage tree is not unique and we are not able to compare the result by solving the problems of perfect replication, so we just solve the risk measure valuation problem. We perform the same analysis of the previous section. The results for the seller/buyer price spread Δ_{sb} and the results for analysis on the arbitrage bias, in the cases of limited and totally excluded short selling, are stored in Table 3.15. Under the super-hedging approach the spread Δ_{sb} is positive, although the percentage difference is below the unity. When we increase the parameter β the optimiser accept more risk and it is able to reduce the price L_s and increase the price L_b. This in turn imply that the spread Δ_{sb} became negative and it decrease linearly in β.

In the case of limited short selling the absolute percentage spread $|\Delta_{s,ls,\%}|$ does not exceed the 1.40%. When we prevent the optimiser from entering in short positions the percentage spread $|\Delta_{s,ns,\%}|$ is still below the unity as we have found in the previous section. We can conclude that the bias on the solutions arising when the scenario tree contains arbitrage opportunities is relevant only in the case we allow for unbounded short selling positions. In this case indeed the problem solution with arbitrage trees is unbounded. The motivation to consider the possibility of limited short selling depends on the regulations in effect in the country in which the pension fund operates. If we compare the results between the complete and the incomplete market cases we note that the liability prices are always greater in the latter. This because in this discrete market setting the difference between the complete and the incomplete market case is determined by the branching structure of the scenarios trees. In this experiment we have $8^4 = 4096$ nodes at the final stages in the complete market case and $10^4 = 10000$ nodes in the incomplete market case. This means that in the incomplete market case the asset distributions are composed by more than double mass points at the final stages with a higher dispersion around the mean value, in particular for assets with leptokurtic distribution. The same hols for the obligations scenario tree process l. This in turns implies a greater value of the initial capital (the price of the liability) in order to implement a trading strategy which fulfils the constraint on the risk measure (in particular for low values of β where the risk control is tighter).

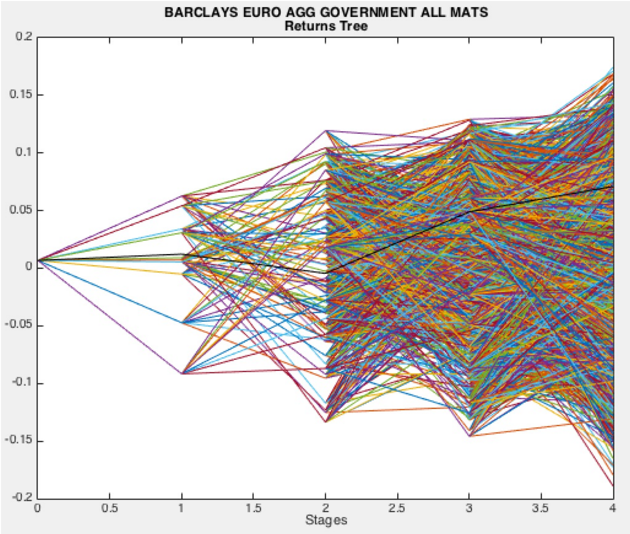
3.4 Conclusion

In this chapter we have considered the problem of pricing a stream of future obligations, linked to the inflation and the mortality risk, of a DB pension fund. We have performed the pricing according to different replication based MSP problems, following the methodology firstly proposed by King [17]. We have firstly solved the replication problems without constraints on the amount that can be invested in short positions. In order to obtain bounded and optimal solutions we are forced to used scenario trees for the asset returns that do not contain arbitrage opportunities. Since in some country, for example in Italy, pension funds are forced to limit or totally prevent the amount invested in short positions, we have solved the same set of problems introducing box constraints on the negative amount that can be invested in each asset. In these cases we can obtain bounded solutions also by using scenario trees with arbitrage opportunities, although at the cost of introducing a solution bias. We define this bias as the difference between the price obtained solving the optimization problem with scenarios with arbitrage opportunities and the price obtained with scenarios without arbitrages. We test this cost both in complete and in incomplete markets. In the complete market case, if the scenario tree does not have arbitrages, we are also able to price the liability using a martingale approach based on the risk neutral measure so that we can compare the price attained with a replication approach, both with and without arbitrages in the risky factors scenarios for the financial assets, with that based on the risk neutral measure. In this case the price is almost the same and the presence of arbitrage does not bias in a significant way the value of the solution. When we construct trees such that the discrete market is incomplete we can not use a perfect replication nor a martingale approach and we are forced to implement different methodologies such as the superhedging, [17] or the risk measure valuations [16]. We obtain clear evidence that in the cases in which the optimiser is forced to not exceed a maximum amount in short positions in each asset, for example a fixed fraction of the portfolio value, the liability value obtained with trees with arbitrage is approximatively equal to that achieved with trees without arbitrage. The results of the arbitrage bias analysis can be used to validate alternative modelling choices (for example the choice of the tree topology and the choice of the scenario generation method) concerning the scenario tree construction with respect the liability pricing problem presented in the thesis.

3.5 Appendix

Figure 3.8: Scenario Trees

The realised trajectory is drawn in black.



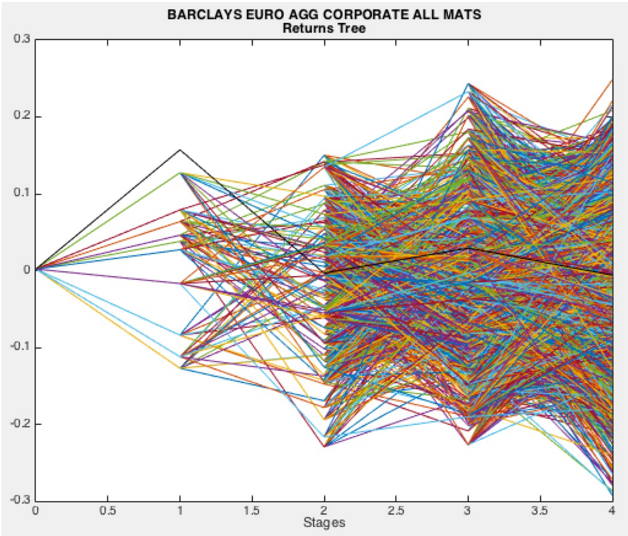


Figure 3.9: Return Scenario Trees

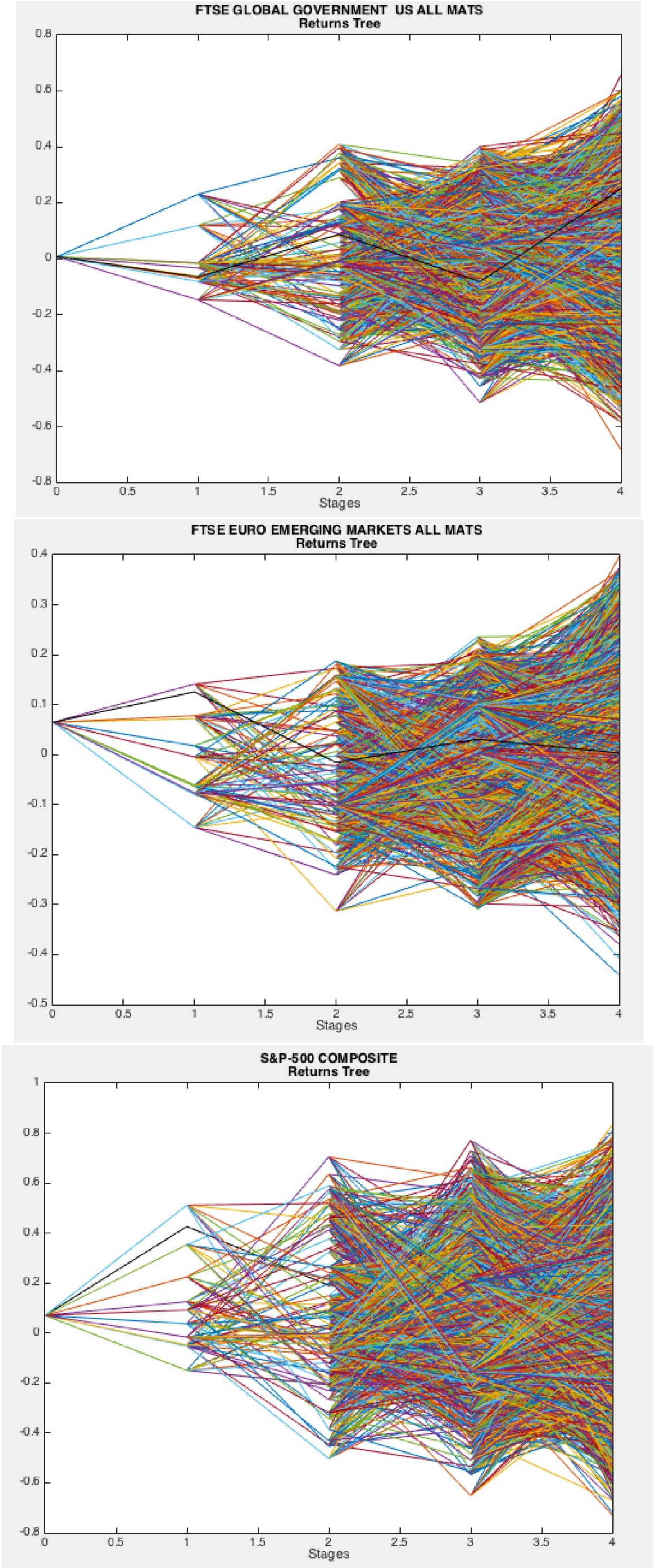
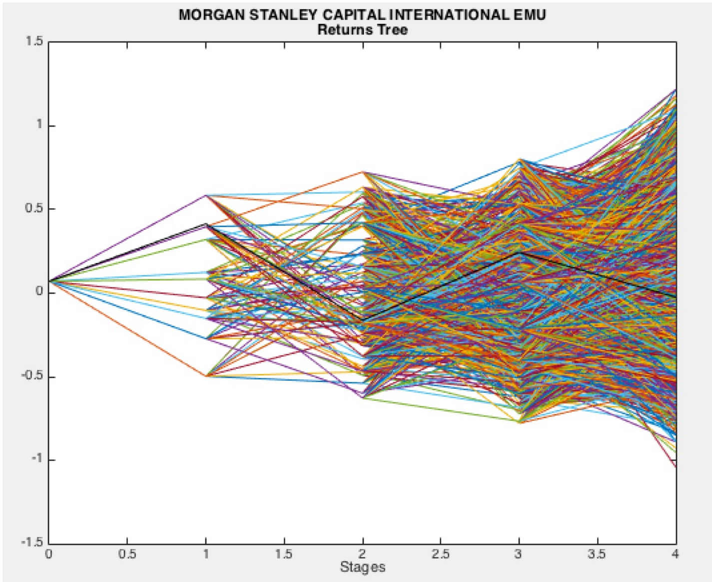


Figure 3.10: Return Scenario Trees



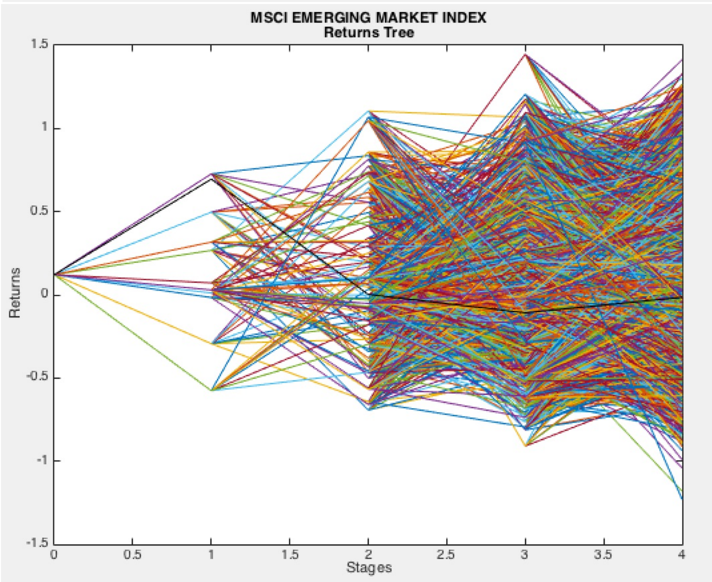


Figure 3.11: Mortality Rates Scenario Trees

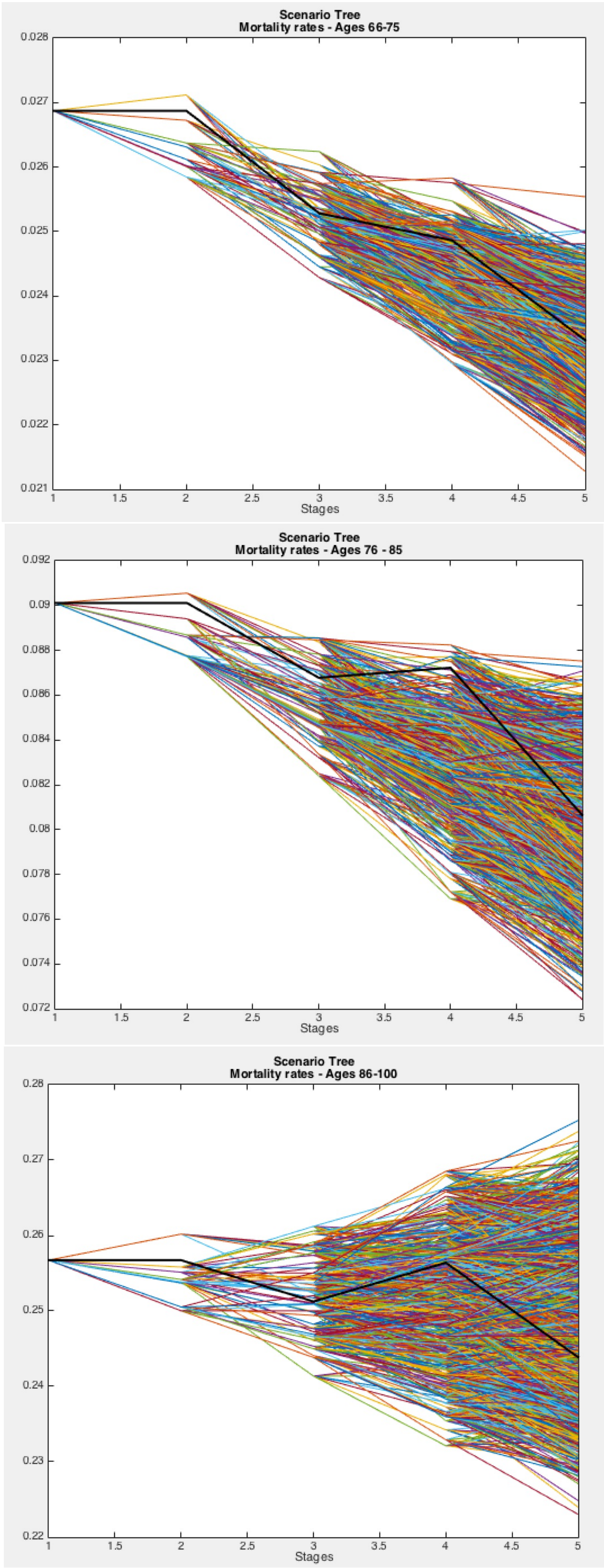


Figure 3.12: Liabilities Scenario Trees

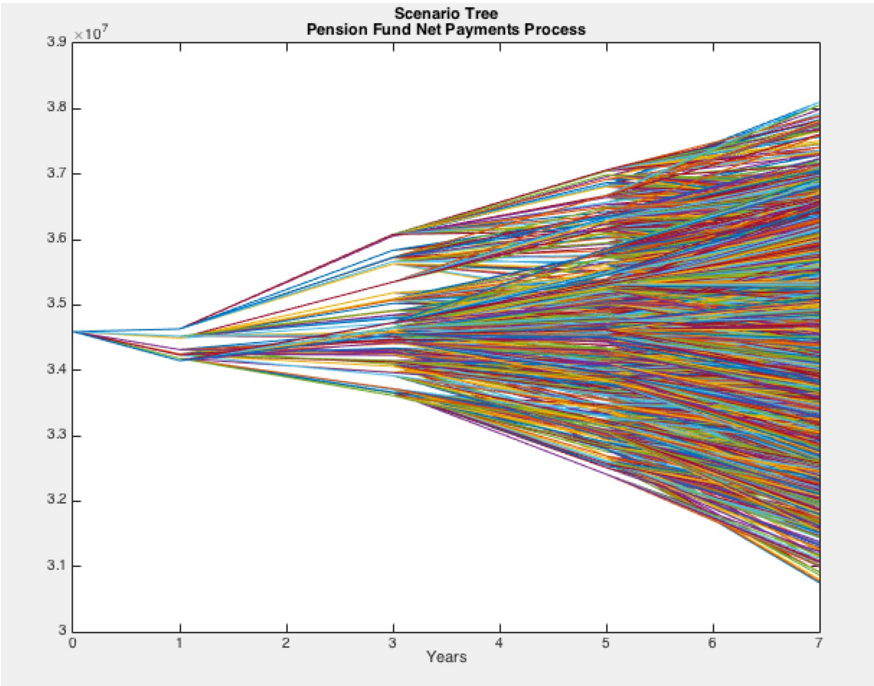


Table 3.14: Risk Measure Valuation Problem Solutions - Complete Market Case

	β										
Problem Type	0	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16	0.18	0.2
$L_{s,f}$	132.340.911	131.392.560	130.444.209	129.495.859	128.547.508	127.599.157	126.650.807	125.702.456	124.754.105	123.805.755	122.857.404
$L_{b,f}$	132.340.911	133.289.261	134.237.612	135.185.963	136.134.313	137.082.664	138.031.015	138.979.365	139.927.716	140.876.067	141.824.417
Δ_{sb}	0	-1.896.701	-3.793.403	-5.690.104	-7.586.805	-9.483.507	-11.380.208	-13.276.909	-15.173.611	-17.070.312	-18.967.014
$\Delta_{sb\%}$	0%	-1,42%	-2,83%	-4,21%	-5,57%	-6,92%	-8,24%	-9,55%	-10,84%	-12,12%	-13,37%
$L_{s,ls}$	132.917.740	131.957.905	130.998.070	130.050.405	129.179.348	128.410.307	127.767.222	127.189.037	126.647.898	126.140.963	125677334
$L_{s,f,ls}$	133.228.500	132.268.550	131.314.017	130.408.986	129.585.998	128.869.811	128.253.723	127.712.155	127.214.436	126.761.583	126342111
Δ	-310.761	-310.645	-315.947	-358.581	-406.650	-459.504	-486.502	-523.118	-566.538	-620.620	-664.777
$\Delta_{\%}$	-0,23%	-0,23%	-0,24%	-0,27%	-0,31%	-0,36%	-0,38%	-0,41%	-0,45%	-0,49%	-0,53%
$L_{s,ns}$	138.634.478	137.661.588	136.688.741	135.715.913	134.743.096	133.770.279	132.751.534	131.886.707	131.545.187	131.269.578	130996957
$L_{s,f,ns}$	138.846.696	13.7873.397	136.900.179	135.926.985	134.953.798	133.980.613	133.007.490	132.209.954	131.857.538	131.562.759	131288744
Δ	-212.219	-211.808	-211.439	-2110.72	-210.703	-210.333	-255.956	-3232.47	-312.351	-293.181	-291.787
$\Delta_{\%}$	-0,15%	-0,15%	-0,15%	-0,16%	-0,16%	-0,16%	-0,19%	-0,24%	-0,24%	-0,22%	-0,22%

Table 3.15: Market Valuation Problem Solutions - Incomplete Market Case

	β										
Problem Type	0	0.01	0.04	0.06	0.08	0.1	0.12	0.14	0.16	0.18	0.2
$L_{s,f}$	136.538.582	134.844.168	133.905.698	132.967.172	132.028.798	131.090.261	130.152.017	129.213.612	128.275.422	127.337.322	126399343
$L_{b,f}$	135.782.642	136.725.160	137.667.593	138.609.975	139.552.230	140.494.472	141.436.611	142.378.707	143.321.057	144.263.187	145205346
Δ	755.940	-1.880.992	-3.761.895	-5.642.802	-7.523.431	-9.404.211	-11.284.594	-13.165.095	-15.045.635	-16.925.866	-18.806.003
$\Delta_{\%}$	0,56%	-1,38 %	-2,73%	-4,07%	-5,39%	-6,69%	-7,98%	-9,25%	-10,50%	-11,73%	-12,95%
$L_{s,ls}$	134.742.380	133.803.388	132.864.413	131.926.287	130.992.806	130.062.476	129.201.736	128.432.055	127.765.205	127.210.359	126.726.089
$L_{s,f,ls}$	136.538.582	135.600.086	134.661.595	133.724.210	132.803.718	131.901.467	131.039.040	130.235.021	129.526.488	128.878.538	128282941
Δ	-1796.202	-1.796.698	-1.797.181	-1.797.924	-1.810.912	-1.838.991	-1.837.304	-1.802.966	-1.761.283	-1.668.179	-1.556.852
$\Delta_{\%}$	-1,32%	-1,32%	-1,33%	-1,34%	-1,36%	-1,39%	-1,40%	-1,38%	-1,36%	-1,29%	-1,21%
$L_{s,ns}$	138.406.261	137.444.015	136.482.505	135.522.365	13.4641.722	133.763.021	133.308.884	132.972.505	132.687.180	132.432.022	13.220.1387
$L_{s,f,ns}$	138.684.447	137.721.068	136.760.683	135.802.623	134.847.614	133.999.595	133.647.463	133.370.248	133.131.979	132.917.211	132.719.118
Δ	-278.186	-277.053	-278.178	-280.258	-205.892	-236.575	-338.580	-397.743	-444.798	-485.189	-517.731
$\Delta_{\%}$	-0,20%	-0,20%	-0,20%	-0,21%	-0,15%	-0,18%	-0,25%	-0,30%	-0,33%	-0,37%	-0,39%

Bibliography

[1] Chong S. F., *Building a Simplified Stochastic Asset Liability Model (ALM) for a Malaysian Participating Annuity Fund.*FSA, 14th East Asian Actuarial Conference, June 2007.

[2] Consigli, G., Dempster, M.A.H.: *Dynamic stochastic programming for asset-liability management.* Ann. Oper. Res. 81, 131-161 (1998).

[3] Dempster M. A. H., Medova E. A., Yong Y. S., *Comparison of Sampling Methods for Dynamics Stochastic Programming*, Stochastic Optimization Methods in Finance and Energy, Ch. 16, Springer, 2011.

[4] Dempster M.A.H., Germano M., Medova E., Murphy J., Ryan D., Sandrini F., *Risk Profiling Defined Benefit Pension Schemes* , The Journal of Portfolio Management, 07/2009, Vol.35(4), pp.76-93.

[5] Dondi G. and Herzog F., *Dynamic Asset and Liabilities Management for Swiss Pension Funds*, Handbook of Asset and Liability Management Volume 2,Ch.20, 2007.

[6] Duffie D., *Dynamic Asset Pricing Theory*, 1996.

[7] Dupaĉová J. and Polìvka J., *Asset-liability management for Czech pension funds using stochastic programming*, Annals of Operations Research, 2009, Vol.165(1), pp.5-28.

[8] Föllmer H., Schied A., *Stochastic Finance: An Introduction in Discrete Time*. W. De Gruyter, Berlin, 2004.

[9] Geyer, Ziemba, *The Innovest Austrian Pension Fund Financial Planning Model InnoALM*, Operations Research, 2008, Vol.56(4), p.797-810.

[10] Geyer A., Hanke M., Weissensteiner A., *No-arbitrage conditions, scenario trees, and multi-asset financial optimization*, European Journal of Operational Research, 2010.

[11] Glosten L., Jagannathan R., Runkle D., *Relationship between the expected value and volatility of the nominal excess returns on stocks*, Journal of Finance, 1993, 48, 1779-1802.

[12] Haneveld K. W., Streutker M., Vlerk M., *An ALM model for pension funds using integrated chance constraints*, Annals of Operations Research, 2010, Vol.177(1), pp.47-62.

[13] Harrison J.M., Kreps D. M., *Martingales and Arbitrage in Multiperiod Securities Markets*, Journal of Economic Theory. 1979, Vol. 20, Issue 3, Pages 381-408.

[14] Harrison J.M., Pliska S.R., *Martingales and stochastic integrals in the theory of continuous trading*, Stochastic Processes and their Applications, Volume 11, Issue 3, August 1981, Pages 215-260.

[15] Hilli P., Koivu M., Pennanen T., *A stochastic programming model for asset liability management of a Finnish pension company*, Annals of Operations Research, 2007.

[16] Hilli P., Koivu M., Pennanen T., *Cash-flow based valuation of pension liabilities*, Eur. Actuar. J., 2011.

[17] King A., *Duality and martingales: a mathematical programming perspective on contingent claims*, IBM Technical Report, T.J. Watson Research Center,

Yorktown Heights, NY, 2000.

[18] Klaassen P., *Comment on "generating scenario trees for multistage decision problems"*, Management Science, 2002, 48(11),1512-1516.

[19] Koivu M., Pennanen T., *Galerkin methods in dynamic stochastic programming*, Optimization, Vol. 00, No. 00, January 2009, 1-15.

[20] Lee, R., Carter, L., *Modeling and forecasting U.S. mortality*, Journal of the American Statistical Association 87 (419), 1992, 659-671.

[21] Mitchell, D., Brockett, P., Mendoza-Arriaga, R., Muthuraman, K., *Modeling and forecasting mortality rates.* Insurance Math. Econom., 2013, 52, 275-285.

[22] Pennanen T., King. A., *Arbitrage pricing of American contingent claims in incomplete markets - a convex optimization approach*, Technical Report, Helsinki School of Economics (2006).

[23] Pennanen T., *Optimal investment and contingent claim valuation in illiquid markets*, Finance and Stochastics, Volume 18, Issue 4, pp 733-754, 2014.

[24] Pflug G. C., *Approximations for Probability Distributions and Stochastic Optimization Problems, Stochastic Optimization Methods in Finance and Energy*, Ch. 15, Springer, 2011.

[25] Pliska S.R. , *Introduction to Mathematical Finance, Discrete time models*, Blackwell, Malden, MA, 1997.

[26] Shapiro A., *Monte Carlo Sampling Approach to Stochastic Programming*, In B.A. Peters, J.S. Smith, D.J. Medeiros, and M.W. Rohrer, editors, Proceedings of the 2001 Winter Simulation Conference, 2001, 428-431.

[27] Staum J., *Incomplete markets*, Handbooks in Operations Research and Management Science, Ch. 12, 2007, Vol.15, pp.511-563.

Chapter 4

Longevity Swap Pricing

4.1 Introduction

Longevity represents an increasingly important risk for defined benefit pension plans and annuity providers, because life expectancy is dramatically increasing in developed countries. In particular, the sponsors of DB pension plans are exposed to the risk that unexpected improvements in the survival rates of pensioners will increase the cost of pension provision. The traditional solution for dealing with unwanted longevity risk in a DB pension plan or an annuity book is to sell the liability via an insurance or reinsurance contract. This is known as a pension buy-out or a group/bulk annuity transfer. A buy-out transaction means that all pension fund liabilities are ceded to an insurer through a bulk annuity. The pension fund is fully discharged of liabilities and market risks. Increased life expectancy and high costs to transfer the full amount of risk make buy-out transactions prohibitively expensive.

To transfer longevity risk to capital markets, Blake and Burrows [2] first advocate the use of longevity bonds, whose coupon payments depend on the proportion of the population surviving to particular ages. The EIB/BNP longevity bond was the first securitisation instrument designed to transfer longevity risk but ultimately was withdrawn. The lack of success in issuing longevity bonds led to new mortality/longevity-linked derivatives where the holder's payoff is based on a function of the difference between a pre-specified rate and an expected mortality/survival rate for some group of pensioners. Several longevity derivatives, such as q-forwards and longevity swaps, are described in [3, 4, 11, 14].

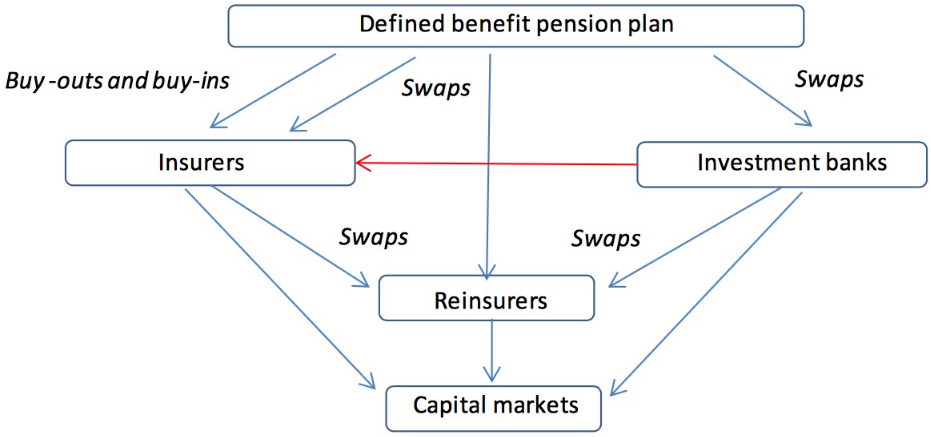
The longevity-linked securities market has recently experienced an increase in transactions for longevity swaps. A longevity swap is a scheme that makes regular payments based on agreed mortality/survival assumptions to an investment bank or insurer and, in return, the bank or insurer pays out amounts based on the scheme's actual mortality/survival rates or on a reference mortality/survival index.

The hedge provider is typically the investment bank or re-insurer, that pays variable payments based on the scheme actual survival rates, or on a reference survival index, and receives fixed payments based on agreed survival assumptions, whereas the hedge buyer is the pension fund or annuity provider, that pays fixed and receives variable payments. In this context, the valuation of longevity swaps represents an important research topic for developing capital market solutions for longevity risk. The pricing of such derivatives is complicated because mortality rates are difficult to estimate and are not themselves traded, and hence the derivative must be priced in an incomplete market setting [9, 25].
We proposed an Asset Liability Model to price longevity swap derivatives, i.e. find the contract fixed rate, on a risk measure based replication framework. The proposed methodology firstly estimates the pension fund liability market value solving a particular type of multistage stochastic programming problem as it was presented in the previous chapter of this thesis. Given an asset universe of tradable and liquid securities in which the pension fund invests the contributions and an investment horizon with discrete rebalancing portfolio periods, we look for the least expensive trading strategy to replicate the pension fund net future payments. The present cost of such portfolio, under self-financing conditions, will provide the current liability value. The pay-off structure of the longevity swap is then inserted in the stochastic programming model where the contract variable rates will be given by a discrete stochastic mortality model and the fixed rate of the contract will be set as a variable of the multistage stochastic problem. In this case we seek for the least expensive trading strategy and the maximum fixed contract rate which ensure the minimal improvement in the current liability value. This fixed rate will be the price of the longevity swap. This procedure has been then used to price different aged-related longevity swaps and a MSP problem is solved to find the optimal proportion of these contracts in an the ALM strategy of the pension fund.

4.2 Longevity Market

Life Market is the new market where longevity and mortality risks are transferred between two counterparts by the means of insurance contracts or longevity-linked financial instruments. Demographic and economic changes have caused the longevity risk to be an increasing source of uncertainty for corporations operating in insurances and pension fund businesses. Until ten years ago longevity risk could not be hedged in any capital market and it was transferred only through insurance and reinsurance contracts. The increasing outstanding global liabilities correlated to longevity/mortality variables and the size of the risk involved have put pressure on the risk premiums and on the liquidity needed to cover such insurance transactions

Figure 4.1: Example of Longevity Risk Contracts
Source: Bank for International Settlements



[5]. The first longevity derivative exchange took place in January 2008 between J. P. Morgan and the Lucida U.K. pension fund and it was designed as a q-forward mortality security. Few months later, in July 2008, the first longevity swap was executed: Canada Life transferred the longevity risk exposure over 125,000 annuitants to the investors thought J.P. Morgan which acted as the intermediary and assumed counterpart credit risk. In the following years many new longevity-linked derivative contracts, mostly in the form of q-forward and longevity swap, has been executed.

In the next two sections the most used insurance/reinsurance contracts and longevity-linked derivatives are briefly presented. This will provide a concise overview on the actual longevity risk hedging solutions implemented in the insurance market and in the new Life Market.

4.2.1 Insurance Contracts

Pension Buy-Outs

Pension Buy-Outs are contracts in which a pension scheme transfers to an insurance company the duty of paying the pensions for a subset of the scheme members. Usually the subset is defined as the pensioners in the scheme at the time in which the contract is stipulated. The pension fund pays a premium to the insurer in return for the risk transferred. This transfers all investment, inflation and longevity risks of the pension fund to the insurer and the pension liabilities are completely removed from its balance sheet. The key factors for insurance companies that go into pricing are the level of yield and return the pension fund can expect, longevity and mortality risks, capacity constraints and hedging against other risks.

Pension buy-outs have come under increasing attention since 2006 when the first contract was signed between Paternoster and Cuthbert Health Family Plan. From 2006 onwards, an increasing number of buyouts contracts have been closed. The value of these contracts also increases: in 2012 General Motors and Prudential agreed a Buy-Outs of $ 26 billion. In 2010 Mercer launched the Mercer U.S. Pension Buyout Index in order to provide pension plan managers monthly pricing information on buy-outs with insurance companies. As it emerges from the Mercer index the premium involved in such transactions is becoming increasingly high [17, 21] and longevity risks hedgers prefer the less expensive buy-in transactions [6, 8].

Pension Buy-Ins

A pension buy-in contract is an insurance policy that a pension fund buys to cover a group of its liabilities. The pension fund holds the policy as an asset and it still keeps the pension obligations. The members remain within the original scheme. This is a remarkable difference from the pension buyout where the assets and liabilities of the entire scheme are transferred to an insurer. Since not all the liability risk is transferred to the insurer, pension buy-ins have less premium but they can lock in attractive annuity rates over time [5]. The freedom in the choice of the subset of liabilities which had to be transferred to the insurance company make the buy-in a more flexible instrument with respect to the buy-out solution. As the buy-out case the first buy-in transaction took place in the UK: in January 2007 Lane Clark and Peacock & Hunting PLC agreed for a £100 million buy-in contract.

4.2.2 Longevity-Linked Derivatives

Longevity Bonds

The *Longevity Bond* (LB) was the first longevity derivative proposed in the academic literature [2, 3] and it was also the first to be proposed in the industry in 2004. Longevity Bonds are bonds whose payoffs depend on a survivor index which represents the proportion of the initial population surviving to a future time. LB pays regular floating payments that depend on the number of cohort survivors. Every period, as the population cohort dies out, the coupon value decrease. The first LB was the EIB/BNP Paribas bond issued in 2004 [10]. The bond was issued by the European Investment Bank (EIB), with commercial bank BNP Paribas as its structurer and manager, and Partner Re as the longevity risk reinsurer. The notional was £540m, the initial coupon £50m, and the maturity 25 years. The survivor index was based on the realised mortality experience of the population of English and Welsh males aged 65 in 2003. The bond was designed as an annuity bond with floating coupon payments linked to the realised mortality rates of English and Welsh males aged 65 in 2002 and with an initial coupon set at £50m. The coupon amount in pound S_t received by the pension plan in each year $t = 1, ..., 25$

was determined by the following formula: $S_t = 50 * \prod_{i=1}^{t}(1 - \mu_{64+i,2002+i})$, where $\mu_{\alpha,t}$ is the mortality rate at t of an individual aged α. The longevity risk premium built into the initial price of the EIB bond was set at 20 bp [7]. The EIB longevity bond was withdrawn after one year due to the lack of demand. Blake [5] identifies four main factors concerning the failure of this first LB: design, pricing, institutional and educational. The design issue concerns with the choice of the age (65), which was too specific, and with the fact the only male individuals were considered. The pricing issue was related to the choice of the notional. Since the instrument was the first type of LB, investors were no confident on the relation between the notional and the longevity risk the bond should cover. In particular the notional was perceived as too high, leaving no capital for other risks to be hedged. The institutional factor was that the instrument was not sufficiently liquid given the small size of the EIB longevity bond supplied. Finally, Blake considers the lack of an adequate information service on the characteristic of the new derivative as a further reason of the failure.

Longevity Notes

Longevity Notes are longevity-based insurance-linked securities based on the spread between the annualised mortality improvement in English & Welsh males ages 75-85 and the corresponding improvement in U.S. males ages 55-65 upon a period of eight years [5]. The notes were offered by the Kortis Capital, a special purpose vehicle established by Swiss Re in order to hedge its longevity risk, in 2010 and they are listed on the Cayman Islands Stock Exchange. Swiss Re is an insurance company which is involved in longevity swap derivative securities with both UK and U.S. based pension funds and life insurance companies. The longevity notes are so an effective instrument to hedge a share of the longevity exposure of Swiss Re reducing the Solvency II capital requirement [5, 26].

Mortality Forwards

A mortality forwards, often referred to as a q-forward, is the simplest instrument for transferring longevity/mortality risk [11]. A q-forward is an agreement between two subjects to exchange at a future date (the maturity of the contract) an amount proportional to the realised mortality rate of a given reference population in return for an amount proportional to a pre-specified fixed rate. More specifically, at maturity T the fixed mortality rate payer, usually an investment bank, receives $F \cdot \mu_T$ and pays $F \cdot \lambda$, where F is the notional, i.e. a given monetary amount that was decided at the date t in which the contract had been stipulated, μ_T is the realised mortality rate of the reference population and λ is the fixed rate agreed at t. The fixed mortality rate λ at which the transaction takes place is called the *forward mortality rate* for the population in question. The fixed payer is the longevity risk hedger, typically a pension fund or an annuity provider. The net settlement amount at maturity for the hedger is $F \cdot (\lambda - \mu_T)$. If $\mu_T < \lambda$ at T the bank has to pay the net settlement to the pension fund and the fund is protected from a decreasing mortality rate. The bank should be rewarded for taking the risk and it usually proposes a fixed rate that is generally lower than the expected mortality rate in order to have a positive cash flow at maturity. The q-forward performs a *value hedge* rather than a *cash flow hedge* [5]. A *value hedge* hedges the value of the hedger's liabilities at the maturity date of the contract whereas a *cash flow hedge* hedges the longevity risk in each one of the hedger's cash flows and net payments are made period by period. An example of a *cash flow hedge* is the longevity swap.

Figure 4.2: Example of a q-forward Term Sheet

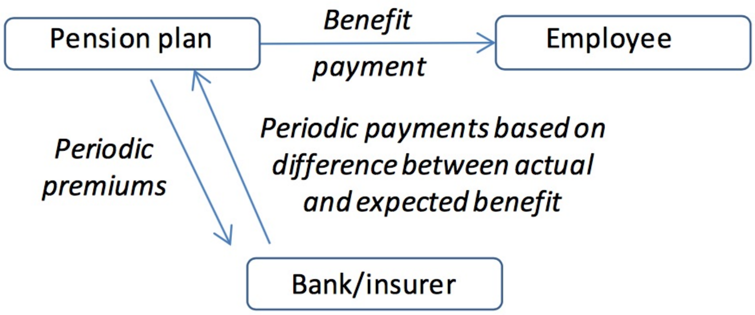
Notional amount	GBP 50,000,000
Trade date	December 31, 2008
Effective date	December 31, 2008
Maturity date	December 31, 2018
Reference year	2017
Fixed rate	1.2000%
Fixed amount payer	J.P. Morgan
Fixed amount	Notional Amount × Fixed Rate × 100
Reference rate	LifeMetrics graduated initial mortality rate for 65-year-old males in the reference year for England & Wales national population Bloomberg ticker: LMQMEW65 Index <GO>
Floating amount payer	ABC Pension Fund
Floating amount	Notional Amount × Reference Rate × 100
Settlement	Net settlement = Fixed amount − Floating amount

Source: Coughlan, Epstein, Sinha, et al. (2007, Table 1).

Longevity Swaps

A longevity swap is a contract that can be either a capital markets derivative or an insurance contract [5, 14, 13]. It is a contract in which the longevity risk hedger (a pension scheme for example) provides regular payments until a terminal date T based on agreed survival assumptions to an investment bank or insurer and, in return, the bank or insurer pays out amounts based on the pension scheme's actual survival rates of a reference population (bespoke longevity swap) or on the actual survival rates index (indexed longevity swap). The swap offers the pension plan with a long term, customised cash flow hedge of its longevity risk: the higher pension expenditure that the pension fund will face in the case in which the mortality rates of the pensioners will be less then expected will be compensated by the higher payments received from the provider of the longevity swap. At any time $t > 0$ successive to the contract stipulation date $t = 0$ the net settlement amount for the hedger is $F \cdot (\sigma_t - \lambda)$ where F is the notional, σ_t is the realised survival rate of the reference population at t and λ is the fixed rate agreed at t. The world's first publicly announced longevity swap took place in April 2007. It was between Swiss Re and Friends' Provident, a UK life assurer. It was a pure longevity risk transfer and was not tied to another financial instrument or transaction. The swap was based on Friends' Provident's £1.7 billion book of 78,000 of pension annuity contracts written between July 2001 and December 2006. Friends' Provident retains administration of policies. Swiss Re makes payments and assumes longevity risk in exchange for an undisclosed premium. However, it is important to note that this particular swap was legally constituted as an insurance contract and was not a capital market instrument. The number of stipulated longevity swap contracts is constantly increasing and it seems to be the most used longevity risk protection instrument [26, 1].

Figure 4.3: Example of a bespoke longevity swap structure
Source: Bank for International Settlements



4.3 Pricing Longevity Rate Swaps with MSP

In the previous chapter we have defined the liability pricing problem in the discrete market defined in Section 1.3.1 as the problem to find the minimum initial capital necessary to pay the future net payments almost surely in all the nodes except for the final stage, where we just want a risk measure $\rho_T(X_T) : \mathcal{L}_T(\Omega, \mathcal{F}_T, \mathbb{P}) \to R$ on the final wealth to be negative. We have set the functional for the risk measure as $\rho_{t,T}(X_{t,T}) = \bar{\beta} \sum_{s=t}^{T} E_{\mathbb{P}}[-X_s] + \left(1 - \bar{\beta}\right) \sum_{s=t}^{T} E_{\mathbb{P}}[max(0, -X_s)|\mathcal{F}_t]$, for $\bar{\beta} \in [0, 1]$. The risk measure quantifies a convex combination between the utility of a greater expected portfolio value and the disutility of expected shortfalls of the terminal portfolio values below the zero (see paragraph 1.3.4). We define with $\phi(l)$ the solution of the *ALM-pricing Problem*, described in Section 3.2, on the stream of cash flows $l \in \mathcal{M}$, where $\mathcal{M} = \left\{\{l_t\}_{t=0}^{T} | l_t \in L^0(\Omega, \mathcal{F}_t, \mathbb{P})\right\}$ is the space of $(\mathcal{F}_t)_{t=0}^{T}$-

adapted sequence of cash flows.

Consider now the situation in which the pension fund can enter into a swap contract where a fixed premium is delivered against an uncertain rate which depends on one, or more risky factors. Let $\bar{l} \in \mathcal{M}$ be a sequence $\left\{\bar{l}_t\right\}_{t=0}^{T}$ of random payments and let λ be a fixed premium rate, also called the swap rate, related to a given notional process $f \in \mathcal{M}$. The maximum premium rate λ that allows the pension fund to enter the contract without worsening his risk-return profile is given by:

$$\lambda_u\left(l, f, \bar{l}\right) = sup\left\{\lambda \in R \mid \phi\left(l + \lambda f - \bar{l}\right) \leq \phi\left(l\right)\right\}$$

The lowest swap rate the agent would accept for taking the opposite side of the trade is instead:

$$\lambda_s\left(l, f, \bar{l}\right) = inf\left\{\lambda \in R \mid \phi\left(l - \lambda f + \bar{l}\right) \leq \phi\left(l\right)\right\}$$

The swap rates $\lambda_s\left(l, p, \bar{l}\right)$ and $\lambda_u\left(l, p, \bar{l}\right)$ are called indifference swap rates [16]. In the case in which the Pension fund enters in a longevity swap as the fixed rate payer we have $f = (F, ..., F)$ and $\bar{l} = \{F \cdot \sigma_t\}_{t=1}^{T}$, where F is the notional of the contract and σ_t is the survival rate of a particular age or of a set of ages. The survival rate at t is defined just as $1 - \mu_t$, where μ_t is the mortality rate of a reference population. In $t = 0$ the contract pay-off is zero whereas in each node of the successive stages the pay-off of the swap contract is:

$$F\left(\sigma_n - \lambda\right), \text{ for } n \in \mathcal{N}_t, t \geq 1. \tag{4.1}$$

Let's now suppose that we have chosen a number value for the notional F and the fixed rate λ of the longevity swap contract and that we have constructed a tree process $\{\sigma_t\}_{t=0}^{T}$ describing the uncertain evolution of the survival rate. We can now reformulate the *ALM-pricing Problem* presented in Section 3.2 by inserting in the cash balance constraint (3.3) the pay-off of the longevity swap contract:

$$x_{0,n} = \left(1 + rf_n\right) \cdot x_{0,a(n)} + \sum_{i=1}^{I} x_{i,n}^{-} - \sum_{i=1}^{I} x_{i,n}^{+} - l_n + F\left(\sigma_n - \lambda\right), \quad n \in \mathcal{N}_t; \qquad t \geq 1, \tag{4.2}$$

In order to find the indifference swap rate λ_u we have to find the maximum value of λ such that the solution $\phi\left(l + F\left[\sigma - \lambda\right]\right)$ is less than or equal to the solution $\phi\left(l\right)$. Similarly, the indifference swap rate λ_s will be the minimal value of λ such that the solution $\phi\left(l + F\left[-\sigma + \lambda\right]\right)$ is less than or equal to the solution $\phi\left(l\right)$. The value of the indifference swap rate will depend on the future stream of payments described by the tree process l, on the survival rate process σ, on the asset return tree process and also on the probability measure $\mathbb{P}$. In the next Section we have tested the above pricing methodology on the same case study presented in the third chapter of this thesis.

4.4 Case Study

4.4.1 Pricing

The DB pension fund case study designed in the previous chapter was used to test the indifference swap rates pricing methodology. The swap pricing experiment is performed with the arbitrage free trees with branching structure 10^4 (incomplete market case) obtained in Section 3.3. We consider three swap contracts that differ just on the age group for which the variable survival rate is computed. The first age group A spans the ages from 66 to 75, the second age group B from 76 to 85 and the last group C from 86 to 100. In order to obtain the three survival rate trees we compute the mean over each node of the survival rate trees of the corresponding age group obtained in the Section 3.3.3. We call the survival rate processes for the three age groups obtained as σ^a, σ^b and σ^c. More formally we have:

$$\sigma_n^a = 1 - \sum_{\alpha=66}^{75} \frac{1}{10} \left(\mu_{m,\alpha,n} + \mu_{f,\alpha,n}\right), \qquad n \in \mathcal{N}_t, \qquad t = 1, ..., T$$

$$\sigma_n^b = 1 - \sum_{\alpha=76}^{85} \frac{1}{10} \left(\mu_{m,\alpha,n} + \mu_{f,\alpha,n}\right), \qquad n \in \mathcal{N}_t, \qquad t = 1, ..., T$$

$$\sigma_n^c = 1 - \sum_{\alpha=86}^{100} \frac{1}{15} \left(\mu_{m,\alpha,n} + \mu_{f,\alpha,n}\right), \qquad n \in \mathcal{N}_t, \qquad t = 1, ..., T$$

where $\mu_{m,\alpha,n}$ and $\mu_{f,\alpha,n}$ are the mortality rates for the age α at the node n for male and female individuals respectively. Since we want to investigate the lower and upper bounds of the indifference rates λ_u and λ_s for different levels of risk, we solve the pricing problem for values of $\bar{\beta}$ equals to [0, 0.1, 0.2, ... 0.9, 1]. We index the three swap contracts with the set $\mathcal{K} = \{a, b, c\}$ and we call as $\lambda^k_{u,\bar{\beta}}$ and $\lambda^k_{s,\bar{\beta}}$ the indifferent swap rates related to the age group $k \in \mathcal{K}$ for a given choice of $\bar{\beta}$.

In order to find $\lambda^k_{u,\bar{\beta}}$ for a given choice of the survival rate tree between σ^a, σ^b and σ^c and for all the possible choices of $\bar{\beta}$, we have modified the *ALM-pricing Problem* as follows:

- Insert the variable λ.
- Set a value for the contract notional F .
- Substitute constraints (3.3) with constraints (4.2).
- Introduce the constraint $X_0 \geq \phi(l)$
- Replace the objective function (3.1) with:

$$\min_{\lambda, x_n, x_n^+, x_n^-; \forall n \in \mathcal{N}} \sum_{i=0}^{I} x_{i,0} - d \cdot \lambda \tag{4.3}$$

We denote as *ALM-pricing Swap Problem* the problem obtained with the above modifications. The parameter d is set equal to $1e^6$ and it is introduced just to avoid numerical errors which can arise due to the small value of the optimal rate λ_u. The notional F is set equal to $1e^8$. The *ALM-pricing Swap Problem* has been performed independently for each longevity swap related to a specific age group by setting $\sigma = \sigma^i$, for a given choice of $i \in \{a, b, c\}$. We call $\lambda^a_{u,\bar{\beta}}$, $\lambda^b_{u,\bar{\beta}}$ and $\lambda^c_{u,\bar{\beta}}$ the indifference swap rates for the three age groups, for a given choice of the risk parameter $\bar{\beta}$, respectively. The indifference price λ_s can be similarly computed by reversing the sign of λ in the objective function, and by reversing the signs in the swap payoff part of the cash balance constraints 4.2. We show the obtained fixed longevity swap rates for each age group and for different value of the risk parameter $\bar{\beta}$ in Tables 4.1 and 4.2 reported in the Appendix. Increasing the risk tolerance parameter $\bar{\beta}$, the indifference swap rate λ_u increases since the optimiser is more willing to accept negative portfolio value at the final stage. Conversely, the indifference swap rate λ_s decreases as β increases, because in this case the optimiser accepts to receive less (lower risk premium). We have thus obtained a range of prices (fixed rate values) for both the long and the short position in the three contracts for different levels of risk propensity that the pension fund manager is willing to assume. When β is set equal to zero we are in a super-hedging approach and we have that $\lambda_u < \lambda_s$. When the risk tolerance parameter increase, the two indifferences prices firstly converge to an unique price and then by further increasing the parameter β we have the opposite relation: $\lambda_s < \lambda_u$. In this pricing framework we have not considered the counter-party risk and preferences of the investment bank (or insurer): the indifferent swap rates are obtained just by considering the pricing problem from the pension fund manager point of view. This means that the price of the contracts (the fixed rates λ_u) could not be suitable for the investment bank. However, this pricing methodology could be very useful in the bargaining between the pension fund and the investment bank. If, for example, the investment bank offers a longevity swap contract with a fixed rate higher than the indifferent swap rate obtained by the solution of the *ALM-Swap pricing Algorithm* the pension fund managers will not stipulate the contract. Since the solution of the optimisation problem is dependent from the the risk parameter $\bar{\beta}$, the pension fund manager has also the opportunity to evaluate the risk involved in the offered contract price. This evaluation will be consistent with both the projection of the active part (asset returns, investment strategy and contributions) and of the passive (pensions) part of the pension fund balance sheet determined by the ALM model implemented by the pension fund manager.

4.4.2 Optimal Swap Contracts Composition

Until now the *ALM-Swap pricing Algorithm* has been used in order to price the swap contracts related to different age groups. The indifferent prices $\lambda^a_{u,\bar{\beta}}, \lambda^b_{u,\bar{\beta}}, \lambda^c_{u,\bar{\beta}}$ are obtained independently by applying the *ALM-Swap pricing Algorithm* for each contract. Let's now assume that the pension fund has an amount $\bar{x}_{0,0}$ of disposable cash at the root node in $t = 0$ and that the pension fund manager looks for the optimal trading strategy which minimises a risk measure on the terminal portfolio value with also the possibility of stipulate a combination of the three swap contracts presented in the previous chapter. The combination of swaps will be determined by the choice of the notional amount F_k for each swap contract with $k \in \mathcal{K}$. We also assume that the total amount of notional $\bar{F} = \sum_{k \in \mathcal{K}} F_k$ can not exceed a real number δ_s. The problem will be dependent on the choice of the indifference swap rates $\lambda^a_{u,\bar{\beta}}, \lambda^b_{u,\bar{\beta}}, \lambda^c_{u,\bar{\beta}}$ obtained for different values of $\bar{\beta}$. Let $F = [F_a, F_b, F_c]$ be the vector containing the notional invested in each of the three contract, the problem can be formally stated as:

ALM Problem 2

$$\min_{x_n, x_n^+, x_n^-, F; \forall n \in \mathcal{N}} \sum_{n \in \mathcal{N}_T} p_n \cdot \left[-\beta X_n + (1-\beta) \cdot \left(\tilde{X} - X_n \right)^+ \right]$$

s.t.:

$$\begin{aligned}
& x_{i,0} = x^+_{i,0} - x^-_{i,0} && i = \in \mathcal{I} \setminus \{0\} \\
& x_{0,0} = \bar{x}_{0,0} + \sum_{i=1}^{I} x^-_{i,0} - \sum_{i=1}^{I} x^+_{i,0} \\
& x_{i,n} = (1 + r_{i,n}) \cdot x_{i,a(n)} + x^+_{i,n} - x^-_{i,n}, && i = \in \mathcal{I} \setminus \{0\}, \quad n \in \mathcal{N}_t, \quad t \geq 1, \\
& x_{0,n} = (1 + r_{0,n}) \cdot x_{0,a(n)} + \sum_{i=1}^{I} x^-_{i,n} - \\
& \quad - \sum_{i=1}^{I} x^+_{i,n} - l_n + \sum_{k \in \mathcal{K}} F_k \left(\sigma^k_n - \lambda^k_{u,\bar{\beta}} \right), && n \in \mathcal{N}_t, \quad t \geq 1, \\
& X_n = \sum_{i=0}^{I} x_{i,n}, && n \in \mathcal{N}_t, \quad t \geq 0, \\
& X_n \geq 0 && n \in \mathcal{N}_t, \quad t < T, \\
& x^+_{i,n} \geq 0, && \in \mathcal{I} \setminus \{0\}, \quad n \in \mathcal{N}_t, \quad t \geq 0, \\
& x^-_{i,n} \geq 0, && \in \mathcal{I} \setminus \{0\}, \quad n \in \mathcal{N}_t, \quad t \geq 0, \\
& x_{i,n} \geq 0, && i \in \mathcal{I}, \quad n \in \mathcal{N}_t, \quad t \geq 0. \\
& \sum_{k \in \mathcal{K}} F_k \leq \delta_s
\end{aligned}$$

The portfolio value target $\tilde{X}$ can be set according to the final portfolio value that the pension fund manager would like to achieve. This will also depend on the initial disposable cash amount $\bar{x}_{0,0}$. In the test that follows we consider the case of an underfunded pension fund: we choose the initial level of capital $\bar{x}_{0,0}$ equal to $1.2e^8$, which is less than the liability price we have found solving the *ALM-pricing problem* for $\beta = 0$. The aims of the experiment are to check which composition of the three swap contracts is more suited for enhancing the expected financial position of the pension fund at the last stage given a risk profile quantified by the parameter β and also to check the riskiness of this contracts. The target final portfolio value $\tilde{X}$ is set equal to zero since the pension scheme is underfunded and we are sure that there will be at least some nodes at the final stage in which the portfolio value will be negative. The experiment procedure can be summarised as follows:

- Fix the initial disposable capital $\bar{x}_{0,0}$ equal to $1.2e^8$ and the maximum amount of swap notional δ_s equal to $1e^8$.
- Choose a level $\bar{\beta}$ and the correspondent indifference swap rates $\lambda^a_{u,\bar{\beta}}$, $\lambda^b_{u,\bar{\beta}}$ and $\lambda^c_{u,\bar{\beta}}$.
- Solve the ALM Problem for increasing values of β from zero to one without swap contracts $(\delta_s = 0)$.
- Solve the ALM Problem for increasing values of β from zero to one with swap contracts $(\delta_s > 0)$.
- Compare the results.

In Figure 4.5 each subfigure shows the total notional amount for each of the three contracts for different level of β obtained solving the ALM problem for a given level of $\lambda^a_{u,\bar{\beta}}$, $\lambda^b_{u,\bar{\beta}}$ and $\lambda^c_{u,\bar{\beta}}$. The level of $\bar{\beta}$ for which the ALM problem is solved is draw as a title in each subfigure. We can see how the optimal choice of the notional amount for each longevity swap contract is highly sensitive on the choice of the two risk parameters $\bar{\beta}$ and β used respectively to price the contract and to find the optimal contracts combination. In Figure 4.4 we plot the efficient frontiers (expected wealth at the final stage on the y-axe and expected maximum shortfall at the final stage on the x-axe) for a given level of $\bar{\beta}$. As we can see the efficient frontiers obtained using indifferent prices $\lambda_{u,\bar{\beta}}$ with $\bar{\beta} \leq 0.2$ lie above the efficient frontier obtained without the possibility to subscribe any swap contracts. When the indifferent prices are instead obtained with larger value of $\bar{\beta}$ the optimiser do not choose any longevity swap contracts for levels of $\beta \leq \bar{\beta}$. This because the indifferent prices are computed with high levels of risk tolerance $\bar{\beta}$ and they are too high to be compatible for a low risk tolerance β. This point also emerges by looking at the Figure 4.6 reported in the Appendix. In Figure 4.6 the portfolio value distributions at the final stage for different value of β, reported as a title of each subfigure, without insert swap contracts in the ALM problem are drawn in the first row. In the second and in the third row the final portfolio value distributions are instead obtained using [$\lambda^a_{u,0.2}$, $\lambda^b_{u,0.2}$, $\lambda^c_{u,0.2}$] and [$\lambda^a_{u,0.5}$, $\lambda^b_{u,0.5}$, $\lambda^c_{u,0.5}$] respectively. The final portfolio distributions depicted on the second row are always better that those of the firs row. This means that the longevity swap contracts with a fixed rate obtained for $\bar{\beta} = 0.2$ allows for a better portfolio value at the final stage. The final portfolio distributions depicted on the third row are equal to those of the first row for value of $\beta \leq \bar{\beta}$ since no swap contract has been chosen. The last figure instead presents a distribution with a higher dispersion around the mean value. This because the fixed rates obtained for $\bar{\beta} = 0.5$ are such that the cumulative longevity swap pay-off has a positive mean value but it has also have more nodes in which it has a negative value with respect the case of $\bar{\beta} = 0.2$.

4.5 Conclusions

In this chapter we have presented a methodology to price a longevity swap contract from the point of view of a DB pension fund. The proposed methodology has been developed on a risk measure replication approach in a discrete time setting and solved with a numerical optimization approach. The approach needs the design of a statistical model for all the risky factors driving the pension fund asset and liability dynamic. The statistical model is then used to generate a discrete space and time representation of the risky factors dynamic by means of a scenario tree. The actual price of the pension fund liabilities will be then defined as the minimum initial capital in order to construct a self-financing trading strategy which replicates the pensions net expenditure with a certain degree. The degree in which the replication is performed is evaluated on the basis of a risk measure. Once we have obtained the present value of the pension fund liability we can define the swap price (the fixed rate) as the maximum fixed rate that allows the pension fund to enter the contract without worsening the liability present value. The methodology has been tested on an artificial pension fund for which the liability are driven by inflation and mortality risks. We have priced three longevity swaps which differ on the choice of the age groups for which the mortality rates are used to define the variable rates of the contact. We have priced each of these three contracts for different levels of the risk parameter in the risk measure functional. In this way we have obtained a range of prices for each contract which reflects different risk attitudes. In the second part of the chapter we use an optimal portfolio approach to identify the best composition of these contracts, with the price previously obtained, that allows an underfunded DB pension fund to improve its final portfolio value.

4.6 Appendix

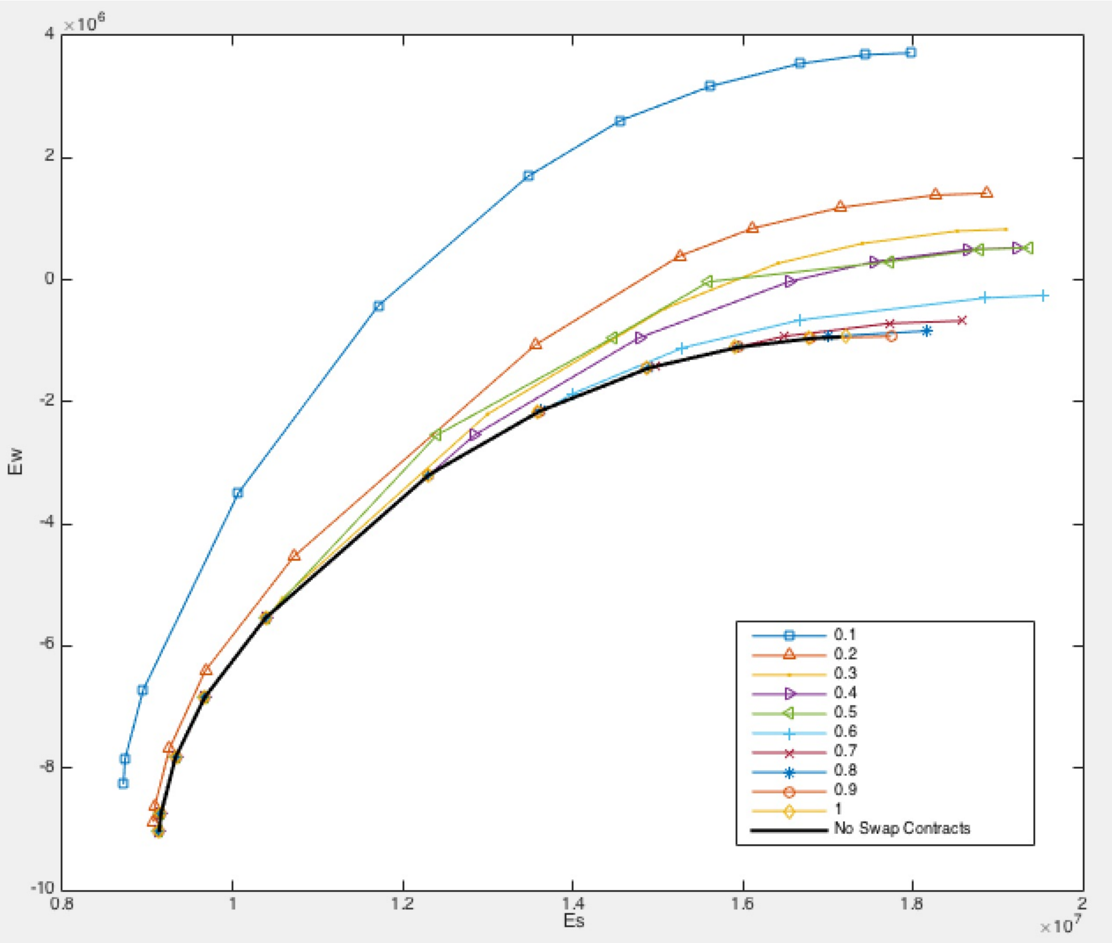
Table 4.1: Indifference prices of the Seller for three different swap contracts for different choice level of the risk parameter $\bar{\beta}$

$\bar{\beta}$	λ_u^a	λ_u^b	λ_u^b
0	0,9752096	0,9143587	0,7403400
0.1	0,9756004	0,9157502	0,7459596
0.2	0,9757551	0,9161140	0,7471159
0.3	0,9759949	0,9164440	0,7478980
0.4	0,9763637	0,9168202	0,7485299
0.5	0,9768335	0,9172702	0,7490954
0.6	0,9773350	0,9177333	0,7496230
0.7	0,9778050	0,9181817	0,7500950
0.8	0,9782387	0,9185920	0,7505084
0.9	0,9786500	0,9189700	0,7508875
0.10	0,9790305	0,9193089	0,7512250

Table 4.2: Indifference prices of the Buyer for three different swap contracts for different choice level of the risk parameter $\bar{\beta}$

$\bar{\beta}$	λ_s^a	λ_s^b	λ_s^b
0	0,9767448	0,9172085	0,7515741
0.1	0,9765494	0,9167447	0,7497009
0.2	0,9764721	0,9166234	0,7493155
0.3	0,9763522	0,9165133	0,7490548
0.4	0,9761678	0,9163880	0,7488442
0.5	0,9759329	0,9162342	0,7486557
0.6	0,9756822	0,91608361	0,7484798
0.7	0,9754472	0,91593421	0,7483225
0.8	0,9752303	0,9157974	0,7481847
0.9	0,9750247	0,9156714	0,7480583
0.10	0,9748344	0,9155584	0,7479458

Figure 4.4: Efficient Frontiers
Each curve is obtained by plotting the efficient frontiers of the ALM-Problem using the values for the vector λ obtained solving the ALM-pricing problem with the value of $\bar{\beta}$ specified in the legend.



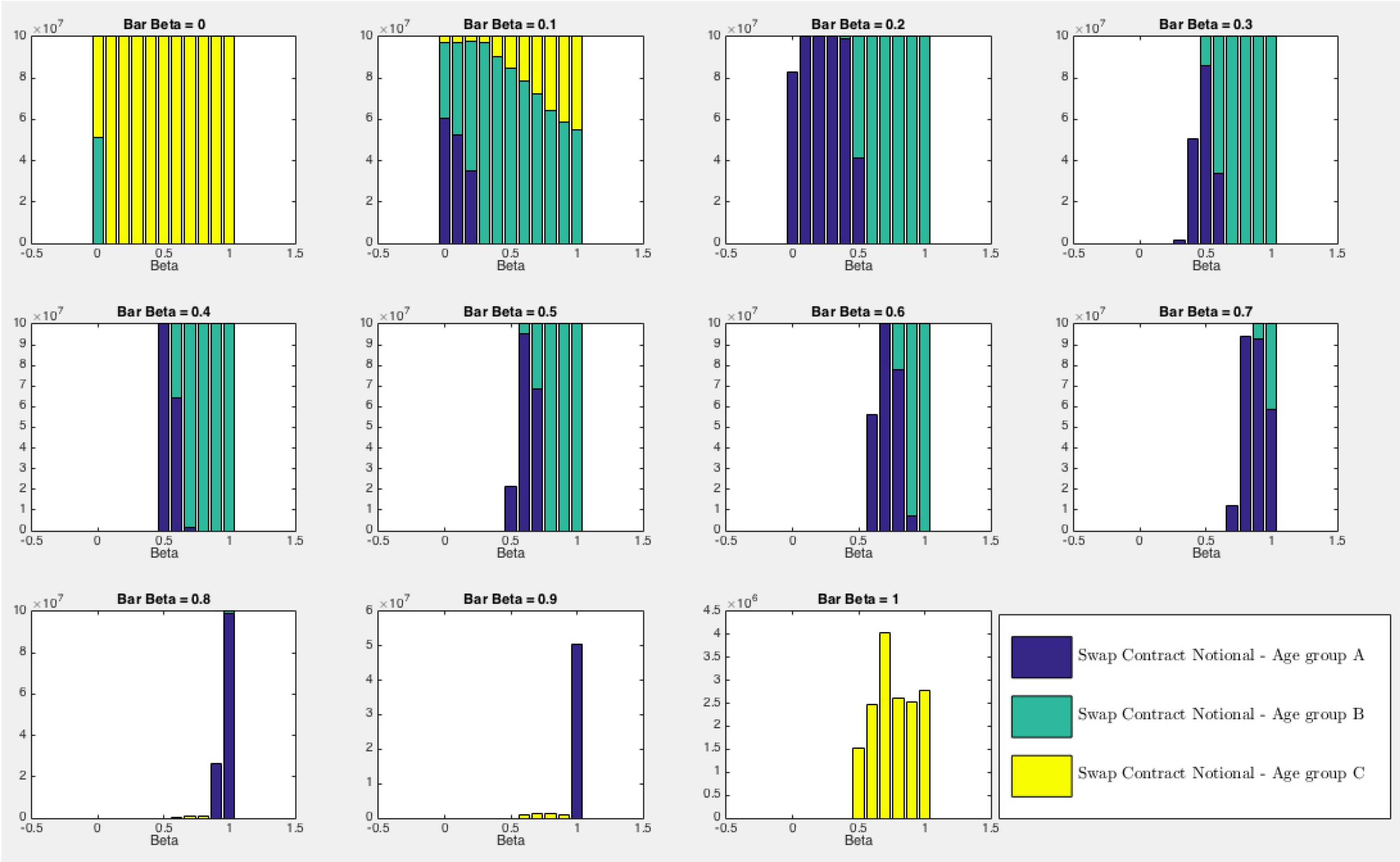


Figure 4.5: Optimal Notional Ammount

Each stack bar figure shows the allocation in the three swap for different level of β in the ALM-Problem for a given value of the vector λ obtained solving the ALM-pricing problem with the value of $\bar{\beta}$ specified in the subplot title.

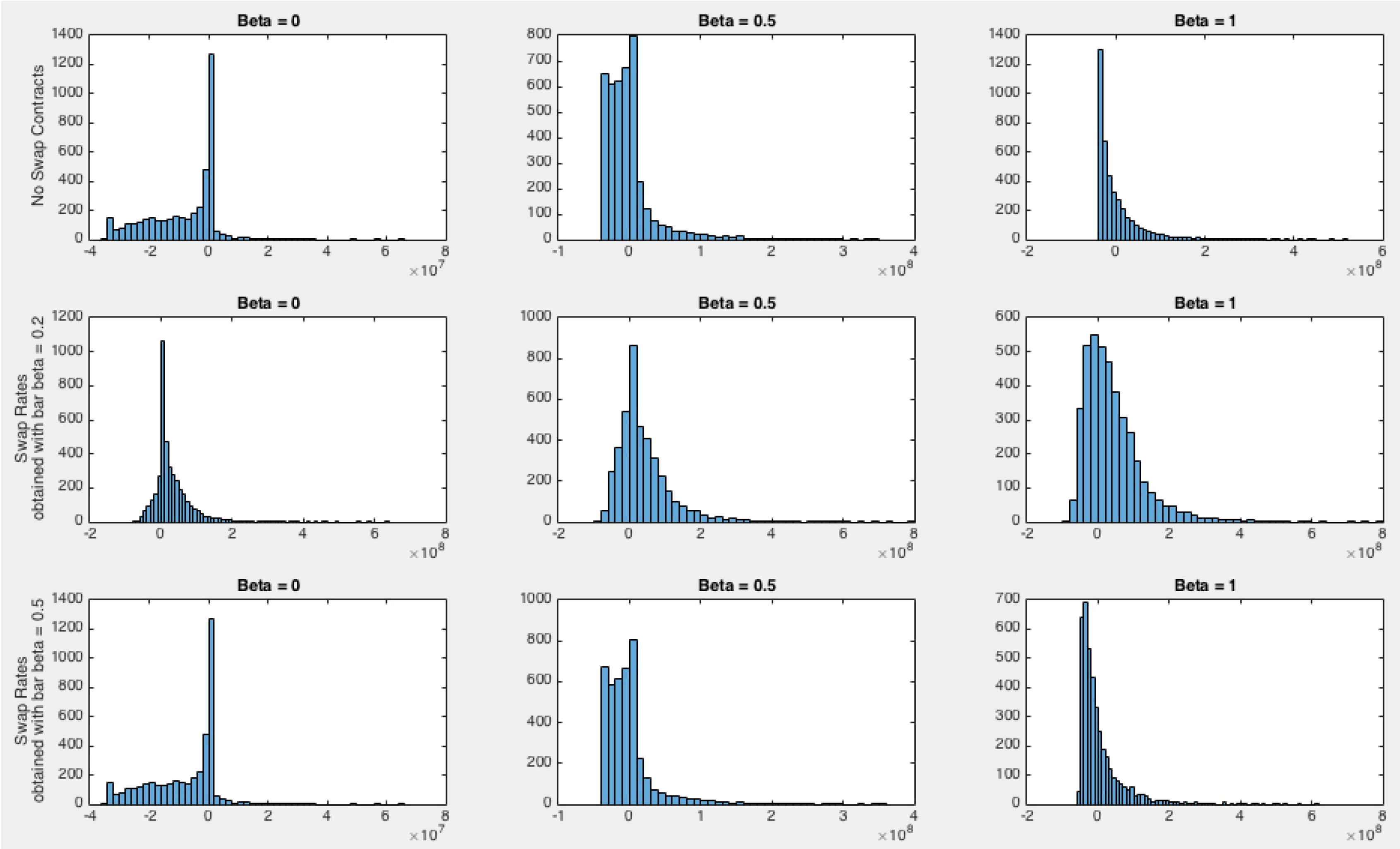


Figure 4.6: Terminal Stage Portfolio Value Distribution

Each row shows the terminal portfolio value distribution for different value of β for a given value of $\lambda^a_{u,\bar{\beta}}$, $\lambda^b_{u,\bar{\beta}}$ and $\lambda^c_{u,\bar{\beta}}$. In the first row no swap contracts are insert in the optimization. In the second and in the third row the values of $\lambda^a_{u,\bar{\beta}}$, $\lambda^b_{u,\bar{\beta}}$ and $\lambda^c_{u,\bar{\beta}}$ are obtained with $\bar{\beta} = 0.2$ and $\bar{\beta} = 0.5$ respectively.

Bibliography

[1] http://www.artemis.bm.

[2] Blake D., Burrows W., *Survivor Bonds: Helping to Hedge Mortality Risk*, October 2001 The Journal of Risk and Insurance, 2001, Vol. 68, No. 2, 339-348.

[3] Blake, D., Cairns, A., Dowd, K., 2006a. *Living with mortality: Longevity bonds and other mortality-linked securities*. British Actuar. J. 12, 153-197.

[4] Blake, D., Dowd, K., Cairns, A.J.G., 2008. *Longevity risk and the Grim Reaper's toxic tail: The survivor fan charts*. Insurance Math. Econom. 42, 1062-1068.

[5] Blake, D., Cairns, A.J.G., Coughlan, G.D., Dowd, K., MacMinn, R., 2013. *The new life market*. J. Risk Insur. 80, 501-558.

[6] Biffis E., Blake D.. *Mortality-linked securities and derivatives*. 2009, Discussion Paper PI-0901. The Pensions Institute, Case Business School, City University, U.K.

[7] Cairns, A., D. Blake, and Dowd, K., *A two-factor model for stochastic mortality with parameter uncertainty: Theory and calibration*, 2006, Journal of Risk and Insurance 73, 787-718.

[8] Chang J.-M, *Hedging longevity risk through capital markets. Capital Market Opinion*, 2011,KCMI. Available from http://www.kcmi.re.kr/Eng/cmweekly/down1.asp?num=25&seq=1&filename=(2011-0419)Weekly eng.pdf.

[9] Chuang S., L. Brockett P. L., *Modeling and Pricing Longevity Derivatives Using Stochastic Mortality Rates and the Esscher Transform*, 2014, North American Actuarial Journal, 18:1, 22-37, DOI: 10.1080/10920277.2013.873708

[10] Cipra T., *Securitization of longevity and mortality risk*, Finance a Uver 60.6 (2010): 545.

[11] Coughlan, G.D., Epstein, D., Sinha, A., Honig, P., *q-forwards: Derivatives for Transferring Longevity and Mortality Risks*. J.P. Morgan, London, 2007.

[12] Coughlan G. D., Khalaf-Allah M., Ye Y., Kumar S., Cairns A. J. G., Blake D., and Dowd K., *Longevity Hedging 101: A Framework for Longevity Basis Risk Analysis and Hedge Effectiveness, North American Actuarial Journal*, 2011, 15: 150-176.

[13] Dawson P., Blake D., Cairns A. J. G., Dowd K., *Survivor Derivatives: A Consistent Pricing Framework, Journal of Risk and Insurance*, 2010, 77: 579-596.

[14] Dowd, K., Blake, D., Cairns, A.J.G., Dawson, P., *Survivor swaps*, J. Risk Insur. 73, 1-17, 2006.

[15] Duffie D., *Dynamic Asset Pricing Theory*, 1996.

[16] Hilli P., Koivu M., Pennanen T., *Cash-flow based valuation of pension liabilities*, Eur. Actuar. J., 2011.

[17] Investment & Pensions Europe 2010, *BMW UK pension complete £ 3bn longevity swap*, 22 February 2010. Available from http://www.ipe.com/news/bmw-uk-pension-completes-3bn-longevity-deal 34134.php.

[18] http://www.istat.it

[19] King A., *Duality and martingales: a mathematical programming perspective on contingent claims*, IBM Technical Report, T.J. Watson Research Center, Yorktown Heights, NY, 2000.

[20] Loeys J., *Longevity: A market in the making. Global Market Strategy*, 2007, J.P. Morgan.

[21] Mercer Global, *Mercer pension buyout index launched to assist companies and the trustees in pension scheme de-risking*, 22 February 2010. Available from http://www.mercer.com/press-releases/1373420.

[22] Pennanen T., King. A., *Arbitrage pricing of American contingent claims in incomplete markets - a convex optimization approach*, Technical Report, Helsinki School of Economics (2006).

[23] Pennanen T., *Optimal investment and contingent claim valuation in illiquid markets*, Finance and Stochastics, Volume 18, Issue 4, pp 733-754, 2014.

[24] Pinar M., *Mixed-integer second-order cone programming for lower hedging of American contingent claims in incomplete markets*, Optimization Letters, 2013, Vol.7(1), pp.63-78.

[25] Shang Z., Goovaerts M., Dhaene J., *A recursive approach to mortality-linked derivative pricing*, Insurance Math. Econom. 49, 240-248, 2011.

[26] Tan K. S., Blake D., Macminn R., *Richard Longevity risk and capital markets: The 2013-14 update*, Insurance Mathematics and Economics, July 2015, Vol.63, pp.1-11