UNIVERSITY OF BERGAMO

DOCTORAL THESIS

Stochastic Programming Models for Optimal Risk Control with Financial Derivatives

Author:
Vivek VARUN

Supervisor:
Prof. Giorgio CONSIGLI
Prof. Diana BARRO

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

in the

Analytics for Business and Economics
Department of Management, Economics and Quantitative Methods

March 15, 2018

Declaration of Authorship

I, Vivek Varun, declare that this thesis ('**Stochastic Programming Models for Optimal Risk Control with Financial Derivatives**') and the work presented in it are my own and has been generated by me as the result of my own original research carried out during three years of PhD.

I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University;
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- Where I have consulted the published work of others, this is always clearly attributed;
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- Either none of this work has been published before submission, or parts of this work have been published which are cited in the Bibliography.

Signed: Vivek Varun

Date: 15-March-2018

UNIVERSITY OF BERGAMO

Abstract

Department of Management, Economics and Quantitative Methods

Doctor of Philosophy

Stochastic Programming Models for Optimal Risk Control with Financial Derivatives

by Vivek VARUN

Stochastic optimization models have been extensively applied to financial portfolios and have proven their effectiveness in asset and asset-liability management; hardly however they have been applied to decision universe including not only financial but also derivatives written on the underlying, derivatives such as options or futures with their dedicated risk profiles and associated modelling complexities. Including options in the portfolio gives us the opportunity to hedge the underlying and to speculate on them to increase the profit potential given a certain risk level.

The modelling of options in multi-stage stochastic programming framework would have many advantages, for instance, a put option can be used for insuring a portfolio against any downside movement in the market, high volatility in options prices is risky but rewarding if captured accurately, in-the-money call/put options at maturity can be used to buy/sell the underlying security at a price lower/higher than the market price, optimally increasing and reducing inventory. We present here multi-stage models to include cash and physical settled call and put options in a portfolio along with other asset classes.

Acknowledgements

There are a number of people without whose help this thesis would not have been possible.

Giorgio, Diana, Kalindi, Krishna, Anirudh, Sudhanshu. Thank You!

Contents

List of Figures

List of Tables

This thesis is dedicated to my grandmother

Chapter 1

Derivatives and Stochastic Programming

Stochastic optimization models have been extensively applied to financial portfolios and have proven their effectiveness in asset and asset-liability management; hardly however they have been applied to decision universe including not only financial but also derivatives written on the underlying, derivatives such as options or futures with their dedicated risk profiles and associated modelling complexities. The modelling of European type options in multi-stage stochastic programming framework is a multi-utilitarian approach, such as a put option can be used for portfolio insurance against any unfavourable outcome, a call option if exercised at maturity can be used to buy an underlying at a lower price, this could be helpful in increasing or reducing inventory. Options can be bought or sold before expiry, options premiums are very volatile and so a risky but rewarding trading opportunity exists. We discuss other benefits and state-of-the-art of options in multi-stage programming in this chapter.

1.1 Introduction

Financial derivatives have been used for mitigating downside risk and enhancing upside potential in portfolio management. Their different payoff structures give investors an appropriate instrument to model their risk-reward choice. Different types of derivatives instruments can help in meeting different goals. For instance, when buying/selling an asset a future/forward contract can be used to offset price fluctuation risk. Options can be used to multiple purposes, a put option can be used to provide hedge against a downside movement in the underlying price, whereas a call option can be used to improve the upside potential of a long underlying position. Other common financial derivatives are swaps, collateralized debt obligations and credit default swaps. However, in this research we limit ourselves to option contracts only.

Option contracts can either be cash settled contracts or physical settled contracts. Most of the options in the market are physical settled contracts.

Physical settled contracts are options contracts whereby settlement requires the actual physical delivery of the underlying asset. The most common physical settled options are stock options, since, the delivery of underlying shares is easier due to their liquidity. For instance, if an investor buys a call option on Google with strike USD 500, at expiry of the option if the price of the Google share is above USD 500, say USD 510, then the investor has the right to buy Google shares at USD 500 instead of USD 510, thereby, saving USD 10 compared to the market price on expiry. On the other hand, if the call option expires out-of-the-money, i.e. price of the underlying falls below USD 500 at expiry, then the investment in the option goes to zero. Similarly, a put option can be used to sell the underlying at a predefined strike price if the put option expires in-the-money. These physical settled options contracts allow investors to bring down the average buying cost when the market is booming and to bring up the average selling cost when the market is bearish. In the later part of this thesis, we develop a model that utilizes options contracts to optimally reduce or increase inventory when the market is bearish or bullish respectively.

The other type of options contracts are the cash settled contracts. These are contracts where settlement happens via cash. These types of options are required where physical delivery of the underlying assets is not possible or inconvenient or costly. For instance, consider the index options Standard and Poor's index (SPX) or the volatility index (VIX). These type of options do not exist physically, we cannot buy indexes, therefore, options on these types of financial instruments are cash settled.

The benefits of options are not limited to buy or sell the underlying at a certain price, options contracts can be used to develop hedging strategies where the loss is limited. Option contracts can also be used for speculation, out-of-the-money options cost much less as compared to the at-the-money options, small movement in the underlying price reflects big relative changes in the option price. From hedging to speculation, options meet the demand of all type of investors and to fit into their portfolios despite the complexities of the market.

In this research, we consider the use of both physical and cash settled options contracts. Multi-fold benefits of options contracts make this a hot topic for researchers. Now, the questions arise how to fit options into a portfolio for a desired shape of the portfolio returns; how to mitigate the overall risk of a portfolio using options; under what mathematical settings to model options in a portfolio; can dynamic approach be more efficient than the static ones; is it possible to buy/sell options before their expiry; can there be an optimal amount allocated for speculation using options and what are the implications of short-selling options contracts etc. We answer these questions in our research. We develop models that give both mathematical and financial meaning to the use of options contracts in multi-stage setting.

There have been many successful applications of dynamic stochastic programming applied to portfolio optimization answering asset-liability management challenges. However, this research considers only the use of broad asset classes, such as fixed income, equity, real estate etc. Very limited research has been developed where options are treated as an asset class. At this stage, it is important to go through some studies to understand the benefits of including options in a portfolio, some successful applications of multi-stage stochastic programming and then finally their combination to study options in multi-stage setting.

1.2 Literature Review of Options in Portfolio Optimization

Over the last four decades, many studies have been conducted on studying options in portfolio management, some researchers have focused on the hedging side of the options, some have focused on the speculative aspects and some researchers have worked on combining different types of options to mimic certain investment policies. We present here some relevant contributions that motivate our research, we start from the earliest contributions by Merton to all the way to the most recent studies.

Merton, Scholes, and Gladstein, 1978 studied on options investments in put options as term insurance to insure the portfolio against any possible loss, it was proved that no put strategy or call strategy can dominate any other strategy if options are priced correctly. It was found that investors can use uncovered put option writing and covered put option buying to produce patterns for returns on investments that cannot be proxied by any combination of equity and fixed income securities. Harrison and Pliska, 1981, talked about general hedging methods for options in complete market. Follmer and Sondermann, 1986, discussed hedging strategies in incomplete market.

Brennan and Cao, 1996, found that options/derivatives improve the Pareto efficiency as the trading gets continuous as with multiple trading sessions uninformed investors behave as rational trend followers. Aliprantis, Monteiro, and Tourky, 2004, presented minimum cost portfolio insurance investment strategy, when derivative markets are complete then holding a put option in conjunction with the reference portfolio provides minimum cost insurance at arbitrary arbitrage free security prices. It was analysed that if the asset span is a lattice-subspace, then the minimum-cost portfolio insurance can be easily calculated as a portfolio that replicates the targeted portfolio in a subset of states which is the same for every reference portfolio.

Haugh and Lo, 2001, showed that under certain conditions, a portfolio of a few number of options can be a good proxy for more complex dynamic investment policies due to the fact the derivative securities are equivalent to specific dynamic trading strategies in complete market and this is the motivation behind constructing buy-hold portfolios of options that mimic certain dynamic investment policies.

Liu and Pan, 2003, solved investment strategies in closed form given that investor has access to options along with stocks and bonds. It was shown that due the volatile nature of derivatives they enable non-myopic investors to disentangle the simultaneous exposure to diffusive and jump risks in the stock market.

Similarly, Muck, 2010, analysed trading strategies with derivatives when investor has full or partial access to the derivatives market, it's the case when options are not available on all the stocks in the portfolio. Potential benefits of adding derivatives to the market are studied, it was found that diffusion correlation and volatility or jump sizes may have a significant impact on the benefit of a new derivative product even if the market price of risk remains unchanged. Increasing or decreasing utility gains of the different types of options can be exploited for a more diversified portfolio.

Driessen and Maenhout, 2007, studied the economic benefits of giving investors access to index options in the standard portfolio problem, analysing both expected-utility and nonexpected-utility investors in order to understand who optimally buys and sells options.

These studies focused on hedging aspects of the options or on replication of some dynamic investment policy. Options have also been studied under condition of risk-neutrality on the Greeks to obtain an arbitrage free profit.

Gondzio, Kouwenberg, and Vorst, 2003, presented stochastic optimization hedging (SOH) model to consider transaction costs, stochastic volatility and trading restrictions by introducing a dynamic trading strategy. The goal of the strategy is to minimize the hedging errors at the first few trading dates. Traditional hedging strategies like delta hedging or delta-vega hedging are not appropriate in this context. The main drawback of the model is that the number of constraints grow exponentially with the number of trading dates.

Papahristodoulou, 2004, formulated a linear programming model to select the optimal hedging strategies unlike other approaches where hedging strategies are defined in advance. The main advantage of this approach is that it makes investment planning more rational and independent of market beliefs of the investor. Horasanlı, 2008, extended the work presented by Papahristodoulou, 2004 from a single asset to a multi-asset portfolio with options on them and considered all the hedging strategies using delta, gamma, theta, rho and vega. The model has many disadvantages in dealing with the options, linear constraints are forced on options while their payoff is nonlinear. Despite limitations, model through linear programming provides many advantages to the investor.

Gao, 2009, presented a general linear programming model with bounds on each Greek letter and then performs a new post-optimality analysis of the model where risks are adjusted by the investor to suit the market dynamics. With the model and the method proposed, one can take the options strategies in terms of one's subjective personality, and meanwhile, adjust the risks to suit the needs of the market change. Sinha and Johar, 2010, further extended this work by introducing quadratic programming to tackle the non-linear payoff of the options, they formulated a quadratic programming model and then approximated that with a linear programming model, it was found that the risk of the portfolio can be hedged by reducing its delta gamma and vega and at the same time it is possible to minimize the net premium to be paid for the creation of the hedged portfolio.

Liang, Zhang, and Li, 2008, studied a mean-variance formulation for the portfolio selection problem involving options. In particular, the portfolio in question contains a stock index and some European style options on the index. A refined mean-variance methodology is adopted in their approach to formulate this problem as multi-stage stochastic optimization. It turns out that there are two different solution techniques, both lead to explicit solutions of the problem: one is based on stochastic programming and optimality conditions, and the other one is based on stochastic control and dynamic programming.

Palma and Prigent, 2008, introduced a financial hedging model for global environmental risks using financial and environmental assets. It was found that options indeed provide a good hedge to the portfolio, however, there is a need to include new type of options that combine both equity and environmental assets contrary to current practice where two separate option markets are considered.

These types of models can solve the hedging issues in a very limited way, since these are the static approaches so they are not able to handle the fast-changing market conditions. In addition, as the trading dates are increased the number of constraints on different hedging strategies grows exponentially. Dimson and Mussavian, 1999, presented a historical review of option pricing techniques.

Scheuenstuhl and Zagst, 2008, proposed a mean-variance portfolio management approach using stocks and options. In addition, they use additional investor preferences in terms of shortfall constraints to allow a more detailed portfolio specification. Also, they utilize an approximation of the return distribution and develop economically meaningful conditions to transform the complex optimization problem into a linear problem.

Zymler, Rustem, and Kuhn, 2011, proposed a novel robust optimization model for designing portfolios including European-style options. Their proposed model, which is on the basis of second-order cone programming, trades off weak and strong guarantees on the worst-case portfolio return. Fonseca and Rustem, 2012, proposed a robust formulation for the international portfolio management problem that maximizes the portfolio return for the worst possible outcome of returns. They further incorporate forward contracts and quanto options to mitigate currency and market risks. Since their proposed model is not linear, they reformulate both the uncertainty set and the objective function as a semi-definite problem. Fonseca, Wiesemann, and Rustem, 2012, used a similar robust approach to cope with the international portfolio management. Instead of using currency forward contracts, they utilize equity options for domestic assets.

In the literature, there has been some studies available where disadvantages of including options in portfolio are discussed, Neuberger and Hodges, 2002, have questioned the benefits of including options in a portfolio. They model an economy with a single risky asset, the model enables them to examine the benefits to investor of using options to optimize their investments, it was found that including options makes minor improvement in the portfolio performance but only when it comes hedge the volatility. The risk-reward in holding options is not limited to volatility risk, options may be a costly hedge to the volatility.

1.3 Literature Review of Stochastic Programming and its Applications

So far, we have discussed the advantages and successful applications of including options in the portfolio. The approaches we have discussed are the static approaches, hence, there is a need to integrate the use options in multi-stage setting so that dynamic and robust hedging or trading policies can be formulated. For this we use multi-stage stochastic programming approach.

The approach is a mathematical framework for modelling optimization problem that involves uncertainty. Unlike deterministic optimization problems where all the parameters are known at the beginning of the optimization process. In stochastic programming uncertainties are revealed with time and are not known at the beginning of the optimization problem. As an example, here the decision maker takes some action in the first stage, after which a random event occurs affecting the outcome of the first-stage decision. A recourse decision can then be made in the second stage that compensates for any bad effects that might have been experienced because of the first-stage decision. The optimal policy from such a model is a single first-stage policy and a collection of recourse decisions (a decision rule) defining which second-stage action should be taken in response to each random outcome.

This ability of stochastic programming makes it useful for solving finance and economic related problems where the future prices are not known at the start of the optimization process. As the prices are revealed, new decisions are taken to achieve an objective. Another advantage of this approach is that it could be applied to any model problems with any time horizon, for instance, its application to pension fund planning, where a long-term horizon problem is solved with typical time horizon expanding to as long as 30 years, on the other hand, it can also be applied to a portfolio of few securities/assets that aim to achieve a target at the end of a year or six months, stochastic optimization approach has proven to be helpful in many aspects.

It also has some assumptions, first, underlying stochastic processes are not influenced by the values of the decision variables, second, decisions adapt to available information at the time they are made, but do not depend (invariant wrt) on specific projected future outcomes (no hindsight).

Fundamental components of a multi-stage stochastic program are: (Vladimirou, SPXI tutorial 2007)

1. the description of the underlying (multivariate) discrete stochastic process for the uncertain parameters, dynamic information structure (Scenario-tree Generation).

2. discrete time dynamic stochastic optimization program capturing the structure of the decision process.

3. Mapping (1) & (2) in a logical conformable way.

4. Defining appropriate performance and risk measure for the problem under uncertainty.

Figure 1.1 shows a sample scenario tree with discrete time stages, $t = 0, 1, 2$ & 3 are the time steps where a decision is made based on information realized. Each atom in this scenario tree is called a node and each node is linked to a previous node or parent node, this defines the basis of sequential decision process with the information process capturing their connection. This makes easier to model cash flow equations, asset balance equations and wealth equations in each scenario along the planning horizon . Key advantages of stochastic programs are they can handle multi-asset problem by determining optimal asset value at individual level, modelling uncertainties irrespective of the type of distribution, can handle regulatory policies as constraints and can alternatively use flexible risk measures or performance objectives (coherent risk measures or utility functions)

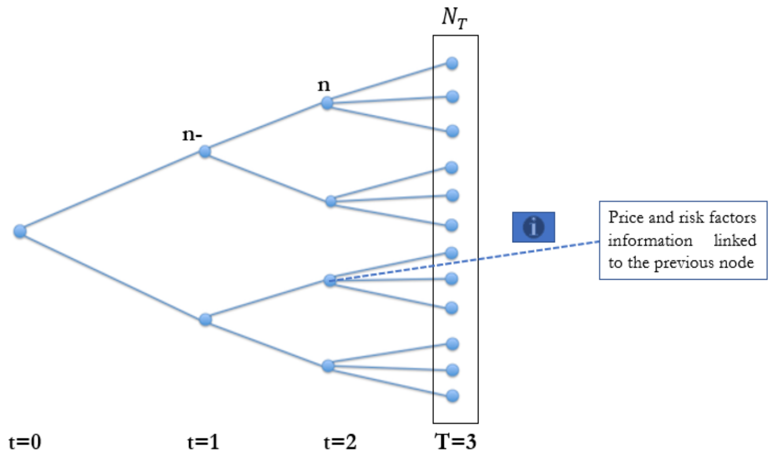

FIGURE 1.1: Scenario Tree Representation

Use of multi-stage stochastic programs started in early seventies, Ho and Manne, 1974. The first few applications were to optimize portfolio of multiple fixed income securities, Bradley and Crane, 1975. Gradually, the stochastic optimization technique was applied in many areas in finance and economics. We cite a few of them. The first successful application of the stochastic programming in finance was the famous Russell-Yasuda Kasai model, Carino et al., 1994 where an asset-liability management model was developed using multi-stage stochastic techniques. It determined an optimal investment strategy that incorporated a multi-period approach and allowed decision maker to define risk in tangible operational terms. Regulatory restrictions were implemented as constraints. The technique used yielded extra 42 bps in the fiscal year 1991 and 1992.

The approach then was further explored by the academic researchers and many successful applications and advancements came. Nielsen and Zenios, 1996, developed a dynamic stochastic optimization model to tackle the uncertainties in liability in insurance business. The model considered explicitly the uncertainties inherent in this problem due to both interest rate volatility and the behaviour of individual investors. Dert, 1995 applied this technique to analyse the investment policy and funding policy of a pension fund and proved that probability of underfunding can be reduced significantly.

Consigli and Dempster, 1998, developed a CALM model (Computer Aided Asset-Liability Management) to deal with uncertainties affecting both assets (in either portfolio or the market) and liabilities (in the form of scenario dependant payments or borrowing costs). This randomness in the assets/liabilities demanded thorough investigation on the reliability of the scenario or event trees used to solve these multistage programs. Poor scenarios can lead to bad investment decisions. Kouwenberg, 2001, talked about the reliability of scenario trees in this context. He used both randomly sampled event trees and event trees fitting the mean and the covariance of the return distribution for generating the coefficients of the stochastic program. Hence, allowing to investigate the performance of the model and the scenario generations conducted on rolling horizons. It was found that the performance of the model can be improved drastically if the right model is adopted to generate scenario trees. Klaassen, 1998, talked about the use of arbitrage-free scenario trees in the optimization procedure.

Since, policies are implemented as constraints in this programming framework, so it is possible to study the effectiveness of the regulations under which a company is working. Høyland and Wallace, 2001, developed a multi-stage stochastic optimization model for an insurance company where they showed how legal regulations put by the government are in the interest of insurance holder or not.

It was intuitive and evident that bad scenarios would lead to bad outcomes and poor investment decisions. Hence, there was need of good scenarios and sufficient number of scenarios. Researchers then came up with arbitrage-free event trees and with sufficient number of scenario trees to reduce the computational complexity of the optimization problem. Mulvey, 1996 and Gaivoronski and De Lange, 2000 talked about the outcome of the optimization problem in context of economic projection model. Bertocchi, Moriggia, and Dupačová, 2000, Pflug, 2001, Dupačová, 2002, Römisch, 2009, Casey and Sen, 2005, Dupačová, Gröwe-Kuska, and Römisch, 2003, Heitsch and Römisch, 2007, Heitsch and Römisch, 2009, and Kuhn, 2008 have discussed scenario reduction techniques.

1.4 Derivatives in Stochastic Programming Framework

So far, we have discussed the benefits of including options in a portfolio and stochastic programming technique that allows us to come up with dynamic investment policies. It's now time to review the combination of two in the literature. Over the last two decades, researchers have applied stochastic programming techniques to options. Most of the applications are about hedging options, sine, SP techniques can efficiently handle inclusion of multiple risk factors and at the same time it can avoid myopic decisions, it becomes practical to use SP to apply to options hedging. Wu and Sen, 2000, presented a stochastic programming model to hedge currency options, American type options were studied where an importer wants to hedge currency risk on a fixed amount of US dollars at some time in future. The model includes some realistic features like sensitivity to delta and gamma, the objective function incorporates delta and gamma tracking error with some other risk factors to rebalance portfolio using different options contracts at the decision stages. It was shown that the SP based hedging model can have significant advantages over traditional approaches for currency hedging. It was reported that modelling and solution approach proposed can be applied to a very broad spectrum of the hedging problems related to contingent claims, like mortgage backed securities and some exotic fixed-income derivatives, where the returns or payoffs are path-dependent and the Monte Carlo simulation is widely adopted. King, 2002, analysed the hedging of contingent claims in the discrete time, discrete state case from the perspective of modelling the hedging problem as a stochastic program. The model was extended to the analysis of options pricing when modelling risk management concerns and the impact of spreads and margin requirements for writers of contingent claims. It was found that arbitrage pricing in incomplete markets failed to model incentives to buy or sell options. An extension of the model to incorporate pre-existing liabilities and endowments revealed the reasons why buyers and sellers trade in options. The model also indicated the importance of financial equilibrium analysis for the understanding of options prices in incomplete markets.

Villaverde, 2004, studied hedging European and Barrier options in a discrete time and discrete space setting by using stochastic optimization to minimize the mean downside hedge error under transaction costs. Scenario trees were generated using a method that ensured the absence of arbitrage and which matched the mean and variance of the underlying asset price in the sampled scenarios to those of a given distribution. It was shown that SP based method produced a lower mean downside hedge error for both types of options for a range of transaction costs. The methodology was then implemented for the case where the underlying price was driven by a discretized Varriance-Gamma process in which case delta hedging methods were not readily available. The results were found to be similar to the case where the underlying asset follows a discretized Geometric-Brownian motion. Barkhagen and Blomvall, 2016, developed a more realistic model where they considered buying and selling at observed bid-ask prices.The SP model developed relied on a realistic modelling of the important risk factors for the application, the price of the underlying security and the volatility surface. Volatility surface was estimated from a cross section of observed option quotes that contain noise and possibly arbitrage. Non-parametric estimation approach was used to produce arbitrage free volatility surfaces. By using a simple dynamic model of the squared LVS (local volatility surface) based on PCA (principal component analysis), they built an SP model that captures the most important joint dynamics of a collection of option prices. It was shown that the model presented is able to come up with a hedging strategy that performs better than both delta and delta-vega hedging in terms of producing lower realized risk and costs.

Most of these applications discuss hedging of different types of options. There are some studies that talk about profiting from including options in a portfolio in multi-stage setting using SP techniques. Blomvall and Lindberg, 2003, presented options in a portfolio of stock index and risk-free asset. They use stochastic programming to analyse the performance of different portfolios, portfolio with stock index, portfolio with stock index and a risk-free asset and a portfolio of stock index, a risk-free asset and call options on stock index. It was found that portfolio with options contracts outperform the other portfolios in terms of mean and variance. They develop a two-stage model, where second stage is the option expiry and therefore, the horizon of the planning problem is never longer than a month. Portfolios are rebalanced at daily frequency and only those options are considered which are near the expiry.

Model presented is a very good starting point to study options in multi-stage settings, however, model has some limitations. Options contracts premiums are very lucrative, when options are out-of-the money, their premium is low, when they are in-the-money their premium goes up. For pricing reasons Blomvall and Lindberg, 2003, did not consider extending the horizon of the problem, they used Black-Scholes model to price the options. Formulation of a multi-stage model that tackles buying and selling of options requires a proper pricing approach on scenario tree. The model considers only call options in the portfolio, it would be interesting to include also put option, given the protective features of put options which can serve as a low cost insurance to hedge a portfolio Aliprantis, Monteiro, and Tourky, 2004. However, return of a put option can be replicated using underlying and call option. Hence, including put options from hedging perspective is more relevant than including options for speculating high portfolio returns. Blomvall and Lindberg, 2003, ran the program for the period Feb 1990-June 1999 rebalancing the portfolio daily and running 80 scenarios each time. However, in-sample stability of the model has not been discussed.

Second important contribution along these research lines was by Topaloglou, Vladimirou, and Zenios, 2011, that became the basis of developing mathematical models in multi-stage setting where options are considered. However, they consider a single stage model where options expiry is equal to the planning horizon of the problem. Unlike, Blomvall and Lindberg, 2003, where only call options are considered in the portfolio, Topaloglou, Vladimirou, and Zenios, 2011, considered both call and put options. As a result, it allows to formulate different strategies on options contracts, such as long straddle, long strangle, strip and strap strategies. Options have non-linear payoffs, when we model them in a single stage framework (with the assumption that span of the planning problem is same as the maturity of the options in the portfolio) it becomes simpler to model their payoffs, since the value of option at the expiry depends solely on the value of the underlying. This fact was beautifully exploited by the Topaloglou, Vladimirou, and Zenios, 2011, so at the final stage either the option is in-the-money with a positive value or its worth nothing if its out-of-the-money. This actually simplifies the model equations.

Topaloglou, Vladimirou, and Zenios, 2011, developed this model in order to optimize an international portfolio with options and forwards. The main objective of the model is control the overall risk exposure of the portfolio and to achieve a balance between risk and reward of the portfolio. CVaR is identified as the risk and expected return as reward. Options are included in the portfolio to hedge market risk in the long underlying in the portfolio, whereas, forwards are included to hedge currency risk since the portfolio has securities in multiple currencies. The single stage model developed was able to achieve the risk-reward balance. However, it opened many other questions about using options or other derivatives in a multi-stage setting and specially looking into the case when maturity of the derivative is not same as the span of the planning horizon. The successful implementation of the options hedging strategies motivated the researchers to look into the viability of such strategies to generate profit in the short-run while keeping positions in the underlying for a longer time period. Different options strategies are profitable in different market scenarios. For instance, when you are on the underlying asset, a put option may be useful when the market is bearish. A straddle may be useful when the market is bullish. Each strategy implemented can be an answer to the market belief of the investor. A dynamic policy where different types of hedging strategies are used with changing market conditions could yield better performance.

Topaloglou, Vladimirou, and Zenios, 2011, studied totally unhedged portfolios where no derivatives are considered, currency risk protection using forward contracts, control of market risk using options contracts, joint protection against market risk and currency risk using forwards and options and finally the use of quantos to protect the position in stock indices. It was reported that unhedged portfolio exhibited the worst performance (lowest cumulative return and high volatility), whereas the introduction of derivatives improves the portfolio performance in terms of expected return and volatility. These results attracted other researchers to extend this work to a multi-stage setting. The work by Topaloglou, Vladimirou, and Zenios, 2011 was extended by the Yin and Han, 2013b, they developed a multi-stage stochastic model to include options in an integrated view. They moved from single-stage model to a multi-stage model where options are available at each stage. This allows investor to build options position at each stage along with the underlying, however, the model considers options that expiring at the subsequent stage.

Multi-stage model can capture evolution of underlying series in a better way, availability of call and put options would allow to effectively adopt options as per the market scenario. Including options in the portfolio in multi-stage setting gives more degree of freedom to the decision maker to dynamically rebalance his positions. The model developed has two main advantages over the model developed by Topaloglou, Vladimirou, and Zenios, 2011, first, multi-stage extension with time-varying investment opportunities and dynamic adjustment, second, they incorporate the overall risk management on five time varying Greek letters.

It was reported that overall risk management scheme improved when all the five Greek letters are considered in multi-stage setting, empirical analyses validated the effectiveness of the multi-stage model and the optimal solution of the model dominated traditional hedging strategy. Though, the model developed by Yin and Han, 2013b, is a multi-stage model with options available for purchase at each decision stage, it however, considered options that are expiring at the subsequent stage. Hence, at each stage options are available that are expiring at the very next stage. Therefore, at any decision stage the only options are available are the contracts expiring at the next stage. This simply the inventory equations for options. At each stage, we have new options, options that were bought in the previous stage do not exist anymore and neither the options are available for a decision stage later than the subsequent stage. Still, the question of holding options at any decision stage is wide open.

Another work on the similar lines is presented by Davari-Ardakani, Aminnayeri, and Seifi, 2016, they developed a multi-stage model where options are not expiring at the subsequent stage. They exploit the fact that options can be traded before their expiry that indeed could be a very lucrative thing. They extended the work by the Yin and Han, 2013b, of multi-stage stochastic model for portfolio with options expiring at the subsequent stage to a generic model where an option can be traded before its expiry. Whilst, the authors have tried to answer a very interesting question through their model, we do not agree with the model they have presented. The equations presented in the model are confusing and are not correct according to us. The authors talked about including European type call and put options in the portfolio and then introduced decision variables on the number of positions on call and put options to be exercised at a node in the scenario tree. If the options are in-the-money then all the options positions should account to the profit, if not, then options expire worthless. There is no point of having a decision variable on how many positions of options to exercise (for European options). Also, the model presented could be redundant, they have talked about decision variables on amount of call/put options and number of positions on call/put options. This is confusing, it is not clear whether the authors are referring to a nominal amount model or a monetary model. They however, do not consider short-selling of options. Their research presented positive outcomes of including options in the portfolio.

Our work is the answer to the open questions in this context, how and why to include and model options in a portfolio, how they should be implemented in a multi-stage setting, what are the benefits of trading an option before it expiry, is short-selling of option beneficial and risky.

We present in this study multi-stage stochastic optimization models that answer all these questions. We start with a simple model where options are considered in single stage setting and then extend the model to take into account the other complexities of expiry at any stage on the planning horizon, we develop a model where options are not necessarily expiring at the decision stage. We generalize the work done by Topaloglou, Vladimirou, and Zenios, 2011 and Yin and Han, 2013b. We develop a model that considers a separate inventory for the options contracts and is parallel to the inventory of the underlying assets. We present a general model that consider short selling of option contracts, we then finally, verify the model theoretically, how in special cases it reduces to the models presented by Topaloglou, Vladimirou, and Zenios, 2011, and Yin and Han, 2013b. We present strong empirical evidences of exploiting various options features.

We model optimization problems from the point of an investor who wishes to maximize the wealth at the end of planning horizon, investor is open to include options in the portfolio along with investments in equity index, fixed income index and commodity index. Planning horizon of the problem is six months and frequency of portfolio rebalancing is monthly. We develop models to maximize wealth at the horizon by considering both cash settled contracts and physical settled contracts. Figure 1.2 below summarizes how we extend the work in the literature and what contribution each model brings to the research area.

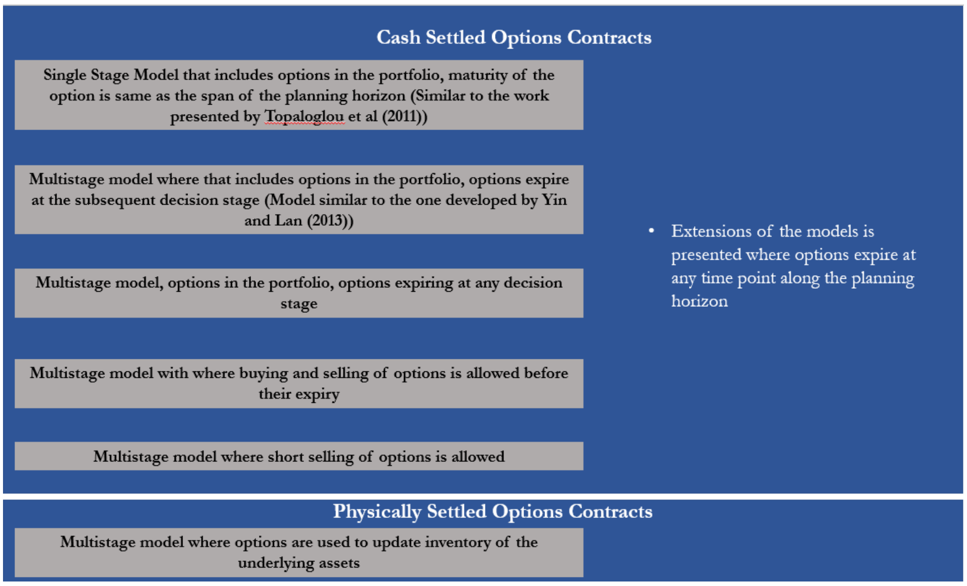

FIGURE 1.2: State-of-the-art and possible extensions

We now discuss the application of these models in brief:

- **Single stage model with options**: - We develop a single stage stochastic model to optimize a portfolio that includes European style call and put options along with the underlying assets. The goal of the options is to protect the portfolio against market risk. A protective put option is used for insuring the underlying against any bearish movement in the market. Similarly, other strategies such as long straddle, long strangle, strip and strap are implemented and tested using the combination of call and put options in the portfolio. These strategies help in tackling different market conditions. Topaloglou, Vladimirou, and Zenios, 2011 have proved that options can effectively used to contain market risk.

- **Multi-stage model with options in the model**: - Next, we develop a multi-stage model where options are available at each decision stage with the assumption that options expire at the subsequent stage. The model is similar to the model developed by Yin and Han, 2013b. The advantages of this model are; it gives more degree of freedom to the investor at each decision stage as options are available for investment along with other underlying assets, secondly, it makes the investment policy more dynamic and robust in terms of changing market scenarios. Short-term hedged strategies could be profitable here, figure 1.3.

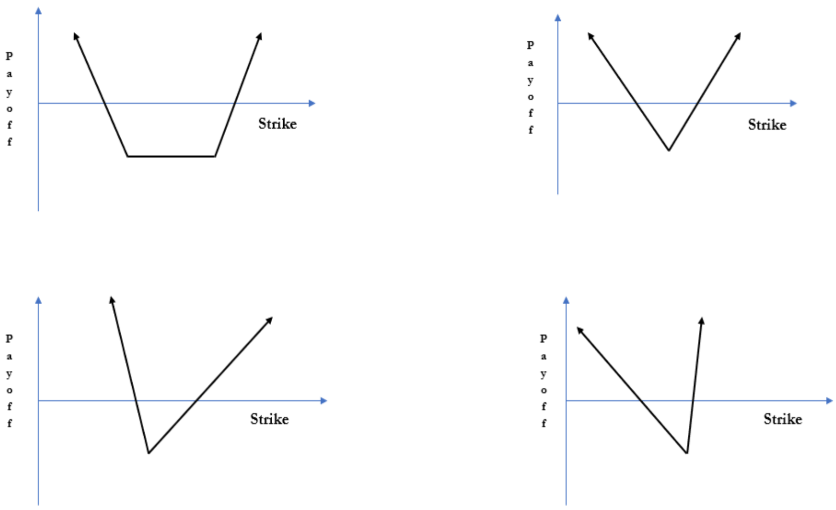

FIGURE 1.3: Options Strategies

- **Multi-stage model, options expiring at any stage**: - Next, we extend this model where we buy options that expire at any decision stage along the planning horizon. The advantage of this model is twofold: first, it allows investor to hedge his position for different time stages without affecting the short run profitability using options of shorter maturity. For example, at-the-money options with shorter maturity are cheaper than at-the-money options with longer maturity, so they leave more space for speculation than options with higher premiums. Secondly, inclusion of options with longer maturity insures the portfolio to achieve/maintain a specific wealth level at distant horizon, since long position in the options correspond to limited losses. So, this model guarantees insurance against any bearish movements in the underlying. Mathematically, this model becomes more complicated than the model introduced by Yin and Han, 2013b, in that model options expire at the subsequent stage, hence, there is no need of option inventory, each decision stage has fresh options expiring at the next stage. While in our model it is required to have inventory of the options, as all the options are not expiring on the subsequent stage, at any time it is possible that options bought in previous stage is held to the next stages. Therefore, proper pricing of option prices is also required.

- **Multi-stage model, buy/sell of long options is allowed**: - In the previous model we talk about hedging the underlying for a longer time. Now, let's consider the case where a put option is bought to hedge the underlying for six months. After, the first month it was found that the put option bought at time 0 is in-the-money, as a result its premium would be higher. The model then sells the options and buys a new put option to hedge the underlying for the remaining period. Options premiums are very volatile and thus are risk and rewarding at the same time. We develop a model to exploit this lucrative nature of option premiums. Picture below shows how volatile option premiums are, we plot S&P500 Index against call and put options expiring in December 2017 with strike price equal to 2500 for the period October 2016 to October 2017, figure 1.4. Index values are plotted on the left y-axis and option prices are plotted on the right y-axis.

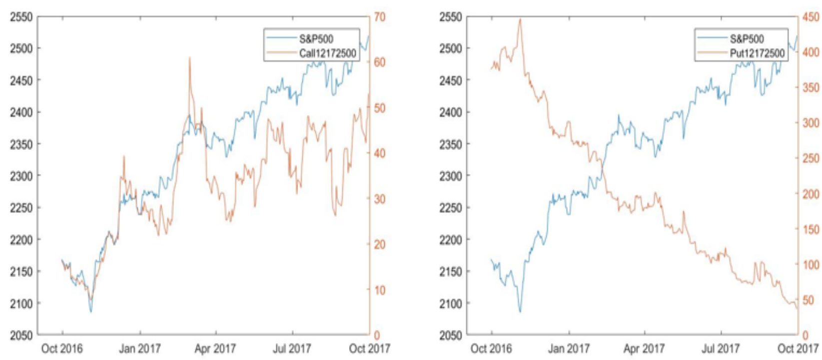

FIGURE 1.4: ATM Call and put options vs underlying index

- **Multi-stage model, buy/short-sell of options is allowed**: - We then develop a model where short selling of options is allowed. Short selling of options allow investor to model multiple hedging strategies, like bull call spread, bear call spread etc. It gives the investor maximum degree of freedom to invest compared to all the models discussed above. This sort of model is actually a generalization of the previous models and we show how it improves the chances of achieving wealth targets for the investor. Each model discussed above is a special case of this model, hence, this model is a summary and generalization of all the models in this line.

- **Multi-stage model, using options to update inventory**: - This model is different from the models discussed above and to the best of our knowledge it is first of its kind. The multi-stage model uses physical settled options contract to update the inventory of the underlying assets. The model gives flexibility to an investor who wishes to increase his inventory when the market is bullish while ensuring that the cost he pays is less than market price. In the same way, if an investor holds a stock of underlying and anticipates that market is going to be bearish then the model generates a trading strategy that helps him reducing the inventory at a price higher than the average sell price in the market. This type of model helps to accumulate inventory or reduce inventory dynamically. This model has wide applications to commodity investors, share holders or ETFs. This dynamic strategy can be applied to many areas.

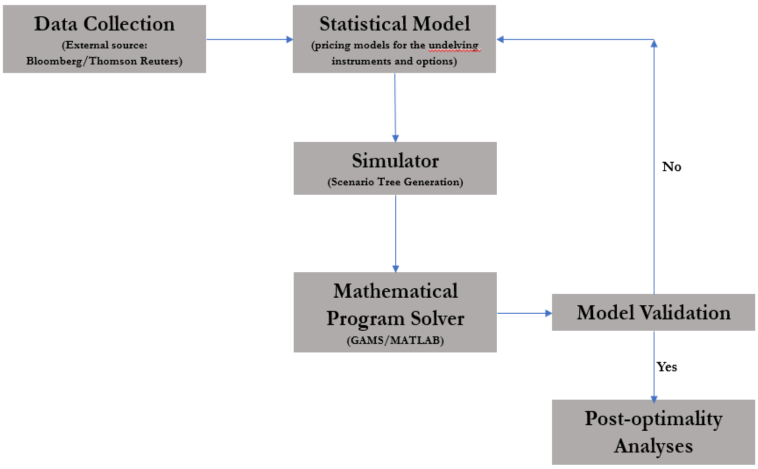

FIGURE 1.5: Methodology

The methodology that we adopt is summarized in the figure 1.5.

- **Data Collection**: - We collect data through external sources Bloomberg and Thomson Reuters Datastream.

- **Statistical Modeling**: - The data collected is then cleaned and used to develop statistical model to fit for forecasting the underlying time series and the associated risk factors. The statistical model is validated before passing it to the scenario generation part. We forecast the data at discrete time steps as per the planning horizon. The reliability of this forecasting model is tested by rolling the model one period every time and seeing if the forecasted value at the first step is within the forecasted range or not.

- **Simulator**: - Once the statistical model is validated, it is then passed to the simulator to generate scenario/event trees for the optimization model.

- **MSP Solver**: - The generated scenarios are then passed to the mathematical standard program solver, which converts the scenario tree optimization to an equivalent deterministic linear program. CPLEX solver is used to solve the mathematical program in GAMS.

- **Model Validation**: - We then validate the model we developed. We go by each constraint implemented in the model to check if we have achieved the desired outcome or not. If a constraint is supposed to put bound on investment in a particular asset then in the result that bound must be satisfied. If we implement a protective put strategy in the model on a certain underlying, then in the result its effectiveness must be reflected when the market is bearish. So, we go by all the variables and constraints to validate the model variables and model equations. Once the model is validated for the equations and constraints we then check the stability of the model. In-sample stability is checked by varying the number of scenarios passed to the optimization problem and observing how the value of the objective function is changing and how the distribution of the wealth is changing in the results. Once the model passes the stability test then we move to the post-optimality analyses.

- **Post-optimality Analyses**: - When the results are validated we then analyse them on various schemes such as wealth distribution, portfolio composition at different time stages, portfolio performance in the mean and worst scenarios, optimal trading strategy, consistency in the portfolio performance etc.

In the next chapter we are going to present the mathematical model to tackle options in a portfolio, we start with a single stage model and gradually introduce extensions to make it multi-stage model where option buying and selling is allowed before the expiry.

Chapter 2

Multi-stage Models for Portfolios with Derivatives

So far, many successful applications of stochastic programming to asset-liability management (ALM) have been published in the literature. Some famous applications of stochastic programming to asset liability management have been reported e.g. in Nielsen and Zenios, 1996, Carino and Ziemba, 1998, Carino, Myers, and Ziemba, 1998, Høyland, 1998, Consigli and Dempster, 1998 and Kouwenberg, 2001. See also the collections Ziemba and Mulvey, 1998 and Zenios and Ziemba, 2004 and the references therein. For a general introduction to stochastic programming we suggest the reader to the official (COSP) stochastic programming site: www.stoprog.org.

These studies however, consider only traditional asset classes in the portfolio, rarely, there have been any study that talks about using derivatives in multi-stage stochastic programming framework. Topaloglou, Vladimirou, and Zenios, 2011s have studied using options and forwards for managing international portfolios using stochastic programming techniques. The article introduces a single stage model with options and forwards along with conventional asset classes. The study suggests that use of derivatives improve the upside potential of the portfolio performance and it also helps in reducing downside risk. We extend the research carried by them to consider options in multi-stage setting.

The motivation for including options in a portfolio can either be hedging or speculation. Different objectives of using options would require to develop different mathematical models. For instance, a long vanilla put option can be used to hedge a position in the underlying, similarly, a call option can be used to speculate on the price movements of the underlying. Both call and put option can be used together to make a straddle or strangle. If an option is bought to hedge a position and before the planning horizon if its found to be in the money, then it can be sold and a new option can be bought to hedge the position in the underlying. Since, the option premiums are very volatile, their price movements are risky but can be lucrative at the same time. Using options in a portfolio give rise to many possibilities, we develop a mathematical model for each of them. We start with the single-stage model presented by Topaloglou, Vladimirou, and Zenios, 2011.

2.1 Introduction to the Optimization Model with Options

The following model is similar to the model presented by Topaloglou, Vladimirou, and Zenios, 2011. The optimization problem is considered for the US investor having positions in some assets and options on them.The portfolio is composed of stock index, bond index and commodity index, then we have European call and put options on these indexes. The portfolio thus is exposed to market risk. We develop scenario based approach to address risks involved here. The deterministic inputs are initial position in each asset class and in options. European call and put options available at time 0 with exactly the same expiry as the horizon.

The scenario generated data for asset prices (and its related risk factors described later) defines the option payoffs at the horizon under each scenario. The model presented here is a nominal amount model and the decision variables (buy/sell) represent the quantity bought/sold in an asset class and option in a given node.

The model is a single-stage model, all decisions are taken at time t=0 of the planning horizon [0,T]. The objective is to maximize the expected wealth minus penalized risk measure which is expected shortfall in our case.

We go by introducing sets, parameters and decision variables to formulate the optimization model. The notations used are the following.

2.1.1 Sets, Parameters and Variables

Sets

- $\mathcal{T}$, set of discrete time space indexed by t, $\mathcal{T} : t = \{0, 1, 2..., T\}$
- $\mathcal{N}$, Set of nodes in the scenario tree indexed by n, $\mathcal{N}_t$ is the set of nodes at time stage t
  (Every $n \in \mathcal{N}_t$ has a unique ancestor $n- \in \mathcal{N}_{t-1}$ and for $t \leq T-1$ there exists a non-empty set of nodes $n+ \in \mathcal{N}_{t+1}$)
- $\mathcal{I}$, set of financial assets, indexed by i
- $\mathcal{O}$, set of vanilla options
  - $\mathcal{O}^c$ & $\mathcal{O}^p$ are set of call and put options respectively
  - $\mathcal{J}_i$, set of expiries of the options $\mathcal{O}$, indexed by j, $\mathcal{O}_{ij}$ represents the set of options on asset i expiring at maturity j, $\mathcal{J}_i : j = \{J_{i1}, J_{i2}...\}$
  - $\mathcal{K}_i^j$, set of strikes of the options in $\mathcal{O}$, indexed by k, $\mathcal{O}_k$ K_i^j represents the vector of strikes at maturity j on asset i, $K_i^j : k = \{K_{i1}^j, K_{i2}^j, K_{i3}^j....\}$

Input Parameters

- $\bar{x}_i$, initial position in asset $i \in \mathcal{I}$
- C_0, initial available cash
- $\bar{C}$, is the initial available cash
- T, length of planning horizon
- χ^+ and chi^-, are the proportional transaction cost for purchase and sale in underlying
- χ_o^+, χ_o^- and χ_o, are proportional transaction costs on buying, selling and exercising option respectively.

- $\bar{\mu}$, user defined target
- v_{i0}, current price of the asset i per unit face value
- $O^c_{i0}(j,k)$, is the current price of the European call option on asset i with expiry j and strike price k
- $O^p_{i0}(j,k)$, is the current price of the European put option on the asset i with expiry j and strike price k

Now, we introduce parameters that would model the flow of information along the scenario tree.

Scenario Dependent Parameters

- $p(n)$, probability of node $n \in \mathcal{N}$ such that $\sum_{n \in \mathcal{N}_T} p(n) = 1$ and for every non-terminal node $p(n) = \sum_{m \in n+} p(m)$, $\forall n \in \mathcal{N}_t$, $t \leq T-1$.
- r_n, is the annual risk-free rate in node n.
- v_{in}, price of asset i, in node n.
- $t(n)$, $t(n-)$ and $t(n+)$, represent the time stages of node n, its predecessor node $n-$ and its successor node $n+$.
- $O^c_{in}(j,k)$, is the price of the European call option on the asset i in node n, with strike price equal to $K(j,k), \forall K(j,k) \in \mathcal{K}$ that expires at $t_j, j \in \mathcal{J}$.
- $O^p_{in}(j,k)$, is the price of the European put option on the asset i in node n, with strike price equal to $K_k, \forall K_k \in \mathcal{K}$ that expires at $t_j = J, j \in \mathcal{J}$.

Computed Parameters

Value of the initial portfolio is the sum of position in each asset.

$$W\_0 = \bar{C} + \sum_i \bar{x}_i v_{i0}, \tag{2.1}$$

Assumption: there is no initial position in the options

Decision Variables

- x^+_{in}, nominal amount of asset i purchased in node n, *buying decision*
- x^-_{in}, nominal amount of asset i sold in node n, *selling decision*
- x_{in}, nominal amount of asset i held in node n in the revised portfolio, *hold decision*
- $c_{in}(j,k)$, units purchased of a European call option on asset i with expiry j and strike price k, *buying decision in call option*
- $p_{in}(j,k)$, units purchased of a European put option on asset i with expiry j and strike price k, *buying decision in put option*

Auxiliary Variables

- W_n, value of portfolio in node n

2.1.2 The Objective Function

The objective function of a mathematical program is the key for choosing good decisions over the bad ones. It depends on the type of objective function that drives optimal solution, if the objective is to maximize wealth at the planning horizon then the procedure (sequence of decisions over stages) would try to move in the direction that maximizes wealth. If the objective is to find an optimal trade-off between risk and reward then objective function would take the optimal solution that gives a balance between risk and reward. We formulate the optimal decision problem as multi-stage optimization problem with recourse(Birge and Louveaux, 1997, Dempster and Ireland, 1988, Dupačová and Bertocchi, 2001).

We consider the following objective function which is the convex combination of two parts *(risk and reward)*:

$$max(1-\lambda)E[W^T] - \lambda R_\zeta \tag{2.2}$$

$$A_0 X_0 = D_0 \tag{2.3}$$

$$A_n X_{n-} + G_n X_n = D_n, \forall n \in \mathcal{N} \tag{2.4}$$

The first part is the expected wealth ($E[W^T]$) (*reward*) at the planning horizon T and the second part is a risk measure R_ζ (*risk*), λ is the risk aversion coefficient that defines how risk averse the investor is. Expected wealth is defined as $\sum p_n W_n, \forall n \in \mathcal{N}_T$. We discuss risk measure in later part of this section. A, G & D are the constraint matrices and define inventory balance equations, cash balance equations and other constraints optimization model is subjected to. X_n are the control variables, this vector decides buy, sell and hold decision for each asset in each node of the tree. The idea behind choosing this type of objective function is the trade-off between risk and reward; as the future price of financial instruments is an uncertain phenomenon and so the expected value of portfolio is not well defined, this convex combination of risk and reward allows the investor to maximize their expected wealth while keeping a check on the risk measure.

It is important to identify the correct risk measure for every financial planning problem. Traditionally used tools for assessing and optimizing market risk assume that the portfolio return is normally distributed. In this way, the two statistical measures, mean and standard deviation, can be used to balance return and risk. However, in certain cases distribution of portfolio returns is far from normal distribution. Since, we are trying to achieve minimum return at the horizon, we consider *Expected Average Shortfall* (EAS) as a risk measure. It was introduced by Dempster et al., 2007, where minimum guaranteed return fund problem was solved. The advantage of such risk measure is twofold; firstly, to manage the strategies of the fund and secondly to take into account guarantees given to the investors of the fund. Dempster et al., 2007, averaged the shortfall at each decision stage, however, in our research we consider expected average shortfall only at the horizon. The mathematical formulation of the risk measure considered is the following:

$$R_\zeta = \sum [\bar{\mu} - W_n]^+ p_n, \forall n \in \mathcal{N}_T \tag{2.5}$$

Where, $\bar{\mu}$ is user defined target, any scenario that yields wealth (W_n) lower than this target would be reflected in the expected shortfall, shortfalls are then weighted by their probability (p_n) to calculate the expected average shortfall (EAS).

In this research we formulate nominal amount model (where we deal with number of assets) to calculate the optimal investments in options and other securities. The reason for choosing this particular approach is that it would help us in tracking the number of options contracts in the inventory. For instance, in a protective put strategy, where we are going to use put options to hedge the position in the underlying we need exactly same number of put options as the number of underlying contracts. Consider in other cases, where we need to buy straddle or a strip or strap, we need to have proportional number of call options and put options, nominal amount model makes calculation easier. In the case of monetary amount model where inventory equations are modelled through investment in the assets, it would bring additional complexity to the model to calculate number of options contracts to hedge the underlying position. Let us also consider the case where we buy/sell options to update underlying inventory, a nominal amount model would make it trivial to handle the inventory equations, we show this later in Chapter 3.

2.1.3 Options Payoff Modelling Approach

Option payoff is a non-linear function unlike other asset, payoff is expressed as $\max$ function. This $\max$ function if implemented in constraints in the optimization program (GAMS) would make it non-linear. To avoid this, we define $moneyness$ to track the intrinsic value of the option. This would help us in modelling problems where options expire before a decision stage.

We define *moneyness* (δ) of the call option on asset i with maturity j and strike k in node n as:

$$\delta^c_{in}(j,k) = max(v_{in} - K^j_{ik}, 0) \tag{2.6}$$

Similarly, *moneyness* for put option would be:

$$\delta^p_{in}(j,k) = max(K^j_{ik} - v_{in}, 0) \tag{2.7}$$

2.2 Problem Formulation: Single Stage Models

As described earlier, options can be used for different purposes depending on the need of the investor/trader, it can be either for hedging or speculation purposes. We now present a nominal amount model to optimize a portfolio of stock, bond and commodity indexes and European call and put options on them. Maturity of the option is same as the planning horizon for single stage model, then we extend the model to take into account additional complexities. Buy/sell decisions are made at time $t = 0$ and the option payoff depends on the price of the underlying assets evolved in the leaf nodes of the scenario tree. We first present a case where, options on indexes are cash settled contracts. So, payoff of options (positive cash flow) from option expiry adds to the cash available. All the computation here is done in the context of a scenario tree optimization framework and therefore all the variables are scenario/node dependent to channel information flow.

Aim is to optimize a portfolio with index instruments and to buy options on them to maximize profitability at the end of the planning horizon. Options expire at the horizon and are cash settled. We present here next the set of decision variables and constraints actually implemented to characterize the random constraint matrices $A_n, G_n, D_n, \forall n \in \mathcal{N}$ and solve the problem.

We consider three types of constraints, to be satisfied: the *inventory balance equations* define the portfolio evolution over time; the *cash balance constraints* include in each node all cash inflows and outflows generated by the current strategy; the *upper and lower bounds* on the decision vector which define policy constraints on the adoptable strategy.

For each node n of the scenario tree and the asset/derivative i, the optimal strategy is defined through the following possible decisions, x_{in} is the nominal amount *held* in asset i in node n; $c_{in}(j,k)$ and $p_{in}(j,k)$ is the nominal amount *bought* in call and put options contract on asset i in node n with strike k and maturity j respectively; x_{in^+} refers to a *buying decision* in asset i in node n; while x_{in}^- refers to a *selling decision* in asset i in node n. All the decision variables are constrained to be non-negative.

At root node:

We first introduce inventory equation at root node $n = 0$, the quantity in root node is equal to the sum of quantity held initially and quantity bought in the root node less what is sold in that node.

$$x_{i0} = \bar{x}_i + x_{i0}^+ - x_{i0}^-, \quad \forall\, i \in \mathcal{I} \tag{2.8}$$

Option Inventory:

$$c_{i0}(j,k) = c_{i0}^+(j,k), \forall (j,k) \tag{2.9}$$

$$p_{i0}(j,k) = p_{i0}^+(j,k), \forall (j,k) \tag{2.10}$$

Option inventory at time 0 is due to any purchase decisions made at root node.

We impose cash constraint in the first stage decision (which is the only decision in a single stage problem)

$$\begin{aligned} C_0 = \bar{C} &+ \sum_{i\in\mathcal{I}} x_{i0}^- v_{i0}(1-\chi^-) - \sum_{i\in\mathcal{I}} x_{i0}^+ v_{i0}(1+\chi^+) \\ &- \sum_{j=T,k} [c_{i0}^+(j,k) O_{i0}^c(j,k) + p_{i0}^+(j,k) O_{i0}^p(j,k)](1+\chi_o^+) \end{aligned} \tag{2.11}$$

Cash in the root node would be the sum of cash that is held initially and value of the assets sold in root node less the amount invested in assets and options.

$$W_0 = C_0 + \sum_i x_{i0}v_{i0} + \sum_{i,j=T,k} [c_{i0}(j,k)O^c_{i0}(j,k) + p_{i0}(j,k)O^p_{i0}(j,k)] \tag{2.12}$$

Wealth in root node is the sum of cash C_0 (cash in root node after rebalancing), value of assets and value of options in root node.

At node: $n \in \mathcal{N}_T$

In the node n, the inventory would be equal to what is held in the previous node and what is bought in the current node less the amount sold in the current node. Since, no buying or selling decision is allowed in the leaf node the amount held in the parent node would be the same as amount held in the current node n.

$$x_{in} = x_{i0}, \quad \forall\, i \in \mathcal{I} \tag{2.13}$$

$$x^+_{in} \;=\; x^-_{in} \;=\; 0 \tag{2.14}$$

Option Inventory:

$$c_{in}(j,k) = 0, \forall (i,j,k) \tag{2.15}$$

$$p_{in}(j,k) = 0, \forall (i,j,k) \tag{2.16}$$

At node n the position in options will always be 0, as it the final stage and options either expire in the money or expire worthless, there are no options in the portfolio in the leaf nodes.

$$C_n = C_0 e^{r_n \Delta t} + \sum_{j,k}[c_{i0}(j,k)max(0, v_{in} - K^j_k) + p_{i0}(j,k)max(0, K^j_k - v_{in})] \tag{2.17}$$

Cash at node n is the sum of cash carried from the previous stage,(it is compounded by the annual risk free rate r_n over the time period $\Delta t = t(n) - t(n-)$, expressed in years), and cash inflows from options expiring in-the-money. Wealth in node n is the sum of cash available in that node and the value of assets held.

$$W_n = C_n + \sum_{i \in \mathcal{I}} x_{i0}v_{in}, \; \forall n \in \mathcal{N} - \{0\} \tag{2.18}$$

We now introduce policy constraints to consider bounds in the investment in various asset classes and options on them. These policy constraints also help in formulating strategies such as straddle and other options strategies, we show this in later part of this chapter. Let ϕ_L and ϕ_U be the set of lower and upper bounds on the securities, ϕ_{icL}, ϕ_{icU}, ϕ_{ipL} & ϕ_{ipU} be the upper and lower bounds on investment in call and put options on asset i respectively. All these ϕs are $card(\mathcal{I})$ X 1 vectors of values between 0 (no investment) and 1 (100% investment), this corresponds to upper or lower investment bound in the corresponding assets or derivatives, equation 2.19. These constraints can help limit investments in assets. Similarly, Equations (2.20 and 2.21) define the upper and lower bounds on investment in derivatives. Then we have non-negativity constraints (equations 2.22 & 2.23) to ensure that the decision variables are positive on all the nodes of event tree. Equation 2.24, is the cash constraint, γ_C is the fractional wealth that is allowed to be kept in the cash account. It is important to note the index 0 in all the constraints below, it corresponds to the root node.

Policy Constraints:

$$\begin{aligned}&\phi_{iL}W_0 \leq x_{i0}v_{i0} \leq \phi_{iU}W_0,\ \phi_L = \{\phi_{iL}\}',\ \phi_{iL} \in [0,1]\\ &\phi_U = \{\phi_{iU}\}', \phi_{iU} \in [0,1],\ \forall\, i\end{aligned} \tag{2.19}$$

$$\phi_{icL}W_0 \leq c_{i0}O^c_{i0}(j,k) \leq \phi_{icU}W_0, \quad \phi_{icL} \in [0,1],\ \phi_{icU} \in [0,1], \forall\, i,j,k \tag{2.20}$$

$$\phi_{ipL}W_0 \leq p_{i0}O^p_{i0} \leq \phi_{ipU}W_0, \quad \phi_{ipL} \in [0,1],\ \phi_{ipU} \in [0,1], \forall\, i,j,k \tag{2.21}$$

Non-negativity constraints:

$$x^+_{i0} \geq 0,\ x^-_{i0} \geq 0,\ x_{i0} \geq 0, \forall\, i\ \in \mathcal{I} \tag{2.22}$$

$$c_{i0}(j,k) \geq 0,\ p_{i0}(j,k) \geq 0,\ \forall\, j\ \&\ k \tag{2.23}$$

Cash Constraint:

$$0 \leq C_0 \leq \gamma_C W_0,\ \gamma_C \in [0,1] \tag{2.24}$$

This completes the single-stage model for optimizing a portfolio of multiple asset classes with options on them, the model is similar to Topaloglou, Vladimirou, and Zenios, 2011 single stage model.

Possible Extension: Single stage model with options in the portfolio that are expiring on or before the Horizon

Next, we extend the above model for a different possibility. It is possible that options expire before the decision stage, in that case, we need to take the profit (if options expire in-the-money) to the next decision stage at the risk free rate. We need to know all the expiries between two rebalancing stages and the price of the underlying at those expiries. We need to define some variables to track the moneyness of the options.

Let $\tau(j) = [\tau_1, \tau_2, \tau_3..]$ be the maturities between $[0, T]$, a scenario based notation would be $\tau_n(j)$, i.e. the maturities between $[0, T]$ in the nth scenario. Let $S_{\tau_n(j)}$ be the price of the underlying on the intermediate maturity. Cash inflow from such expiries Θ_n would be:

$$\Theta_n = \sum_{i,j\in\tau(j)\leq T,k} [c_{i0}(j,k)\delta^c_{in}(j,k) + p_{i0}(j,k)\delta^p_{in}(j,k)]e^{r_n(t-\tau_n(j))} \tag{2.25}$$

The new model that allows options expiry before horizon would have a modified cash equation 2.17, all other features including asset balance equations and policy constraints will remain the same. So, we just write the modified cash equation for this particular case instead of repeating the entire model formulation.

$$C_n = C_0 e^{r_n \Delta t} + \Theta_n, \forall n \in \mathcal{N}_T \tag{2.26}$$

We can also rewrite this equation in terms of indicator variable δ and moneyness λ defined earlier in this chapter.

$$C_n = C_0 e^{r_n \Delta t} + \sum_{i,j\in\tau(j)\leq T,k} [c_{i0}(j,k)\delta^c_{in}(j,k) + p_{i0}(j,k)\delta^p_{in}(j,k)]e^{r_n(t-\tau_n(j))} \tag{2.27}$$

This completes single stage model in this study. Single stage model presented here is just a start to make things clearer when integrating this work in multi-stage framework.

2.3 Multi-stage Model: Long Position in European Options

In this section we are going to talk about put and call options in a portfolio. There could be many possibilities with options when discussing them in multi-stage setting depending on the expiry of the options and whether selling of options is allowed before expiry or not.

2.3.1 Multi-stage Model (Buying options expiring at the next decision stage)

Now, we aim to optimize a portfolio where investor/trader along with assets has European call and put options available on them, however, to avoid any volatility risk investor/trader wants only those options that are expiring at the next stage, so that any rebalancing stage no options are held in the portfolio and only buying decisions in options contracts are made. We are going to extend the single-stage model to a multi-stage model. At each node, new options are available that expire at subsequent nodes (children nodes), Yin and Han, 2013b. The equations developed in the previous model would change because of increased number of decision stages, equations for multi-stage model are below:

The asset inventory equation at root node would remain the same as in the single stage model.

Asset Inventory Balance Equation

$$x_{i0} = \bar{x}_i + x_{i0}^{+} - x_{i0}^{-},\ \forall\ i \in \mathcal{I} \tag{2.28}$$

We keep the option inventory separate from the underlying assets' inventory, making it easier to track the position in options contracts.

Option Inventory Balance Equation

$$c_{i0}(j,k) = c_{i0}^{+}(j,k), \forall (j,k) \tag{2.29}$$

$$p_{i0}(j,k) = p_{i0}^{+}(j,k), \forall (j,k) \tag{2.30}$$

The cash balance constraint is imposed for the first stage takes into account the number of options contracts bought in the root node n.

$$\begin{aligned} C_0 = \bar{c} &+ \sum_{i \in \mathcal{I}} x_{i0}^{-} v_{i0}(1-\chi^{-}) - \sum_{i \in \mathcal{I}} x_{i0}^{+} v_{i0}(1+\chi^{+}) \\ &- \sum_{i,j=t+1,k} [c_{i0}^{+}(j,k) O_{i0}^{c}(j,k) + p_{i0}^{+}(j,k) O_{i0}^{p}(j,k)](1+\chi_o^{+}) \end{aligned} \tag{2.31}$$

We now write the equations and above constraints at node n (later than first stage). The inventory balance constraints reflect the decision problem Markovian structure: as time evolves, along each scenario, the portfolio evolution will be fully specified in nominal value through holding, buying and selling decisions. Each such decision generates, jointly with other commitments, cash flows in each node resulting in cash surpluses or deficits to be compounded to the following stage.

Asset Inventory Balance Equation: At node $n \in \mathcal{N} - \{0\}$]

$$x_{in} = x_{in-} + x_{in}^{+} - x_{in}^{-},\ \forall\ i \in \mathcal{I}, \tag{2.32}$$

Amount held in the current node is equal to amount held in the previous node and the amount bought less the amount sold in the current node.

Option Inventory:

$$c_{in}(j,k) = c_{in}^{+}(j,k), \forall (i,j,k) \tag{2.33}$$

$$p_{in}(j,k) = p_{in}^{+}(j,k), \forall (i,j,k) \tag{2.34}$$

The number of options held in a node are exactly same as the number of options bought in that node. Any options held previously would be expiring on the current decision stage, since, the model assumes option expiry same as the time length between two consecutive decision stages which is constant for the entire planning horizon.

Cash Balance Constraint in node n

$$\begin{aligned} C_n = C_{n-}e^{r_n \Delta t} &+ \sum_{i\in\mathcal{I}} x^-_{in} v_{in}(1-\chi^-) - \sum_{i\in\mathcal{I}} x^+_{in} v_{in}(1+\chi^+) \\ &- \sum_{i,j=t(n^+),k} [c^+_{in}(j,k)O^c_{in}(j,k) + p^+_{in}(j,k)O^p_{in}(j,k)](1+\chi^+_o) \\ &+ \sum_{i,j=t(n),k} [c_{in-}(j,k)max(0, v_{in} - K^j_k) + p_{in-}(j,k)max(0, K^j_k - v_{in})](1-\chi_o) \end{aligned} \tag{2.35}$$

Cash at node n would be the sum of cash carried from the previous stage,(it is compounded by the annual risk free rate r_n over the time period $\Delta t = t(n) - t(n-)$, expressed in years), and cash inflows from options expiring in-the-money. The above equation can be re-written in terms of δ.

$$\begin{aligned} C_n = C_{n-}e^{r_n \Delta t} &+ \sum_{i\in\mathcal{I}} x^-_{in} v_{in}(1-\chi^-) - \sum_{i\in\mathcal{I}} x^+_{in} v_{in}(1+\chi^+) \\ &- \sum_{i,j=t(n^+),k} [c_{in}(j,k)O^c_{in}(j,k) + p_{in}(j,k)O^p_{in}(j,k)](1+\chi^+_o) \\ &+ \sum_{i,j=t(n),k} [c_{in-}(j,k)\delta^c_{in}(j,k) + p_{in-}(j,k)\delta^p_{in}(j,k)](1-\chi_o) \end{aligned} \tag{2.36}$$

Wealth at node n is the sum of the cash available in that node, value of asset investments and value of options held in that node.

$$W_n = C_n + \sum_{i\in\mathcal{I}} x_{in}v_{in} + \sum_{i,j=t(n^+),k} [c_{in}(j,k)O^c_{in}(j,k) + p_{in}(j,k)O^p_{in}(j,k)],\ \forall\, n \in \mathcal{N} \tag{2.37}$$

The policy and non-negativity constraints remain same as in the previous model with the exception that constraints now must hold for all the nodes on which a decision is taken i.e. $n \in \mathcal{N}_{0,T-1}$. We add one more constraint 2.46 to make sure that no decision is taken on the leaf nodes, $n \in \mathcal{N}_T$.

Policy Constraints:

$$\begin{aligned} &\phi_L W_n \le x_{in}v_{in} \le \phi_U W_n,\ \phi_L = \{\phi_{iL}\}',\ \phi_{iL} \in [0,1], \\ &\phi_U = \{\phi_{iU}\}',\ \phi_{iU} \in [0,1],\ \forall\, i \in \mathcal{I} \end{aligned} \tag{2.38}$$

$$\phi_{icL} W_n \le c_{in}O^c_{in}(j,k) \le \phi_{icU} W_n, \quad \phi_{icL} \in [0,1],\ \phi_{icU} \in [0,1], \forall\, i,j,k \tag{2.39}$$

$$\phi_{ipL} W_n \le p_{in} O_{in}^p(j,k) \le \phi_{ipU} W_n, \quad \phi_{ipL} \in [0,1],\ \phi_{ipU} \in [0,1], \forall\, i, j, k \tag{2.40}$$

Non-negativity constraints:

$$x_{in}^+ \ge 0,\ x_{in}^- \ge 0,\ x_{in} \ge 0,\ \forall\, i \in \mathcal{I},\ \forall\, n \in \mathcal{N}_{0,T-1} \tag{2.41}$$

$$c_{in}(j,k) \ge 0,\ p_{in}(j,k) \ge 0,\ \forall\, n \in \mathcal{N}_{0,T-1} \tag{2.42}$$

$$c_{in}^+(j,k) \ge 0,\ p_{in}^+(j,k) \ge 0,\ \forall\, n \in \mathcal{N}_{0,T-1} \tag{2.43}$$

Cash Constraint:

$$0 \le C_n \le \gamma_C W_n,\ \forall n \in \mathcal{N},\ \gamma_C \in [0,1] \tag{2.44}$$

Cash Constraint:

$$0 \le C_n \le \gamma_C W_n,\ \forall n \in \mathcal{N},\ \gamma_C \in [0,1] \tag{2.45}$$

No decision is made on the leaf nodes:

$$x_{in}^+ = 0;\ x_{in}^- = 0;\ c_{in}^+ = 0,\ p_{in}^+ = 0,\ \forall\, n \in \mathcal{N}_T,\ \forall\, i \in \mathcal{I} \tag{2.46}$$

This completes the multi-stage model when options are expiring at the subsequent decision stage.

Model Validation: We here present some observations that validate the model that we have presented. Let us consider a case where put option contracts are used to hedge the long position in the underlying. Above model then would have call option decision variable fixed at 0 and allowing purchase only in put options. We set $c_{in} = 0$ and $p_{in} \le x_{in}$ to ensure that number of put options contracts are always less than the number of assets.

TABLE 2.1: Model Validation: Protective Put Case

Month	1	2	3	4	5	6	7
Equity Index Value ($)	1870.85	1866.167	1891.48	1899.937	1958.478	1945.264	1910.727
Wealth ($)	100000	99980	101336.1	101789.1	104352.6	103648.4	103081.1
Units of Equity Index in the portfolio	0	53.57505	53.57505	53.28249	53.28249	52.99087	52.99087
Cash	100000	0	0	0	0	0	1830.01
Number of Call Options	0	0	0	0	0	0	0
Number of Put Options	0	0	0	53.28	0	52.99	0
% change in Equity	na	-0.0025	0.013564	0.004471	0.030812	-0.00675	-0.01775
% change in Wealth	na	-0.0002	0.013564	0.00447	0.025185	-0.00675	-0.00547
Cumulative Equity performance	0	-0.0025	0.011027	0.015548	0.046838	0.039776	0.021315
Cumulative Wealth performance	0	-0.0002	0.013361	0.017891	0.043526	0.036484	0.030811

To validate the model, we consider a scenario tree having 64 branches with branching structure [1 2 2 2 2 2 2], each time step is equal to one month, monthly revision of the portfolio is allowed. Equity pricing model is used to generate prices for equity index described in Chapter 4, we use Black-Scholes model to calculate option prices, historical volatility of the underlying asset is used and risk free rate is assumed to be 0. We consider only one asset in the portfolio and ATM options on it. We analyse the worst case scenario (2.1), model buys put option in stage 6 to hedge equity positions, price does fall 1.775%, but because of put options in the portfolio loss realized is only 0.5%. The constraints are implemented such that it is not obliged to have options in the portfolio in each node, depending on how the market evolves algorithm or optimization program decides when to buy and how many to buy. The objective here is to see if the put option purchased actually serve the hedging purpose or not. In our findings, it is validated that put options provide protection against a downside movement.

Let us now consider a case where trader/investor is willing to adopt a straddle strategy, strip or strap strategy. This would require to modify the constraints. We know that number of call and put options in these strategies are proportional, $c_{in} = [1, 2, 1/2]p_{in}$ for straddle, strap and strip strategies respectively. We add this constraint to the model and we see if the desired outcome is achieved or not.

TABLE 2.2: Model Validation: Straddle Strategy

Month	1	2	3	4	5	6	7
Equity Index Value ($)	1870.85	1862.765	1894.558	1877.791	1889.181	1876.274	1859.588
Wealth ($)	100000	99980.12	102215.2	101030.9	101030.9	101010.7	100112.4
Units of Equity Index in the Portfolio	0	53.06179	53.33754	0	0	53.83582	53.83582
Cash	100000	0	0	101031	101031	0	0
Number of Call Options	0	53.06	53.34	0	0	0	0
Number of Put Options	0	53.06	53.34	0	0	0	0
% change in Equity	na	-0.00432	0.017068	-0.00885	0.006065	-0.00683	-0.00889
% change in Wealth	na	-0.0002	0.022356	-0.01159	0	-0.0002	-0.00889
Cumulative Equity performance	0	-0.00432	0.012672	0.00371	0.009798	0.002899	-0.00602
Cumulative Wealth performance	0	-0.0002	0.022152	0.010309	0.010309	0.010107	0.001124

Model buys (2.2)a long straddle at the second month, equity price goes up by 1.7%, but because of call options in the straddle the realised return in wealth is about 2.2%. At the very next stage it again buys a long straddle, the price of equity goes down by 0.8% but because of put options in the straddle strategy the loss stays around 1.1% (because of the fact that a significant amount goes in buying the options).

TABLE 2.3: Model Validation: Strip Strategy

Month	1	2	3	4	5	6	7
Equity Index Value ($)	1870.85	1862.765	1894.558	1877.791	1889.181	1876.274	1859.588
Wealth ($)	100000	99980	101686.2	100807.6	100807.6	100787.5	99891.13
Units of Equity Index in the Portfolio	0	53.6729	53.21964	0	0	53.71681	53.71681
Cash	100000	0	0	100807.6	100807.6	0	0
Number of Call Options	0	0	26.61	0	0	0	0
Number of Put Options	0	0	53.22	0	0	0	0
% change in Equity	na	-0.00432	0.017068	-0.00885	0.006065	-0.00683	-0.00889
% change in Wealth	na	-0.0002	0.017065	-0.00864	0	-0.0002	-0.00889
Cumulative Equity performance	0	-0.00432	0.012672	0.00371	0.009798	0.002899	-0.00602
Cumulative Wealth performance	0	-0.0002	0.016862	0.008076	0.008076	0.007875	-0.00109

Model buys a strip anticipating that market would go down (2.3), it does. Percentage change in equity is -0.88% while in the portfolio with strip the percentage change is -0.86%. Strip strategy is more effective when market is highly volatile.

TABLE 2.4: Model Validation: Strap Strategy

Month	1	2	3	4	5	6	7
Equity Index Value ($)	1870.85	1862.765	1894.558	1877.791	1889.181	1876.274	1859.588
Wealth ($)	100000	99980.18	103294.7	101494.6	101494.6	101474.3	100571.9
Units of Equity Index in the Portfolio	0	52.75175	53.58231	0	0	54.08288	54.08288
Cash	100000	0	0	101494.6	101494.6	0	0
Number of Call Options	0	105.5	107.16	0	0	0	0
Number of Put Options	0	52.75	53.58	0	0	0	0
% change in Equity	na	-0.00432	0.017068	-0.00885	0.006065	-0.00683	-0.00889
% change in Wealth	na	-0.0002	0.033151	-0.01743	0	-0.0002	-0.00889
Cumulative Equity performance	0	-0.00432	0.012672	0.00371	0.009798	0.002899	-0.00602
Cumulative Wealth performance	0	-0.0002	0.032947	0.014946	0.014946	0.014743	0.005719

Model buys a strap in month 2 (2.4), anticipating that price would increase, it does, portfolio beats benchmark by 1.6% because of call options in the portfolio. At the month 3, price decreases this time portfolio realised more 0.8% more loss than the benchmark, as all the call options bought expired worthless. Similarly, other strategies can be modified using different constraints.

Extension: Multi-stage model (Buying the options that are expiring on or before the next rebalancing stage)

Now, we add one more complexity to the above model. It is possible that options are not expiring on the decision stages. So, we generalize cash equation for any node n to account for cash flows coming in from any option expiry before the next rebalancing stage.

$$\Theta_n = \sum_{i,j \in t(n) < \tau(j) \leq t(n^+),k} [c_{in}(j,k)\delta^c_{in}(j,k) + p_{in}(j,k)\delta^p_{in}(j,k)]e^{r_n(t(n)-\tau_n(j))} \tag{2.47}$$

The new model in this case will have some equations different from the previous model. The cash constraint at root node would be different in a way that now options are available for purchase on expiry set $[t, t+1]$, that is from $t = 0$ to $t = 1$, while in the previous model options are available for purchase at expiry time $t+1$.

$$\begin{aligned} C_0 = \bar{c} + \sum_{i \in \mathcal{I}} x^-_{i0} v_{i0}(1-\chi^-) - \sum_{i \in \mathcal{I}} x^+_{i0} v_{i0}(1+\chi^+) \\ - \sum_{i,t(n) \leq j \leq t(n+),k} [c_{i0}(j,k)O^c_{i0}(j,k) + p_{i0}(j,k)O^p_{i0}(j,k)](1+\chi^+_o), n = \{0\} \end{aligned} \tag{2.48}$$

Cash constraint in node n would be different from the cash constraint in the previous model in a sense that options are now available for purchase on the set $[t(n), t(n+)]$ and options are exercised on the interval $(t(n-), t(n)]$. The equation in terms of δ can be written as:

$$C_n = C_{n-}e^{r_n \Delta t} + \sum_{i \in \mathcal{I}} x_{in}^- v_{in}(1 - \chi^-) - \sum_{i \in \mathcal{I}} x_{in}^+ v_{in}(1 + \chi^+)$$
$$- \sum_{i, t(n) \le j \le t(n^+), k} [c_{in}^+(j,k) O_{in}^c(j,k) + p_{in}^+(j,k) O_{in}^p(j,k)](1 + \chi_o^+)$$
$$+ \sum_{i, t_{n-} < j \le t(n), k} [c_{in-}(j,k)\delta_{in}^c(j,k) + p_{in-}(j,k)\delta_{in}^p(j,k)] e^{r_n(t(n) - \tau_n(j))}(1 - \chi_o) \tag{2.49}$$

The third change from the previous model would be in the wealth equation. Again, the time interval of existing options would be $[t(n), t(n+)]$. Wealth then would Sum of cash held in the current node, investment in the assets and investment in the options.

$$W_n = C_n + \sum_{i \in \mathcal{I}} x_{in} v_{in} + \sum_{i, t(n) \le j \le t(n^+), k} [c_n(j,k) O_n^c(j,k) + p_n(j,k) O_n^p(j,k)], \ \forall\, n \in \mathcal{N} \tag{2.50}$$

This completes the multi-stage model where options are expiring on or before the next rebalancing stage.

Extension: Multi-stage model options expiring at any decision stage along the planning horizon)

Now, we consider options in the portfolio that expire at any decision stage along the planning horizon. Motivation behind having such a portfolio is that investor may be looking to hedge his portfolio over a longer period. This brings another complexity to the model, if an option contract has not expired then it must be reflected in the inventory equation in the children nodes. Comparing to the previous models, now options inventory are going to have different structure. We assume that purchasing of options contracts is allowed only at time 0.

We are now extending the model presented by Yin and Han, 2013b. This would require us to modify cash constraints at root node as well as for the nth node. At root node, time span for option purchase would now be between $t = 0$ and $t = T$.

$$C_0 = \bar{C} + \sum_{i \in \mathcal{I}} x_{i0}^- v_{i0}(1 - \chi^-) - \sum_{i \in \mathcal{I}} x_{i0}^+ v_{i0}(1 + \chi^+)$$
$$- \sum_{i, j \le T, k} [c_{i0}(j,k) O_{i0}^c(j,k) + p_{i0}(j,k) O_{i0}^p(j,k)](1 + \chi_o^+) \tag{2.51}$$

The option inventory at time 0 would remain the same as in the previous model.

$$c_{i0}(j,k) = c_{i0}^+(j,k), \forall (i,j,k) \tag{2.52}$$

$$p_{i0}(j,k) = p_{i0}^+(j,k), \forall (i,j,k) \tag{2.53}$$

However, at node n the inventory would be different. Since, no option purchase is allowed at the decision stages and only options that are bought in root node are allowed to carry to the next node, therefore, the position in the option in the current node would simply be the option position in the previous node, provided that option contract has not expired. If option contracts have expired then $c_{in}(j,k) = 0$ & $p_{in}(j,k) = 0$, otherwise

$$c_{in}(j,k) = c_{in-}(j,k), \forall (j,k) \tag{2.54}$$

$$p_{in}(j,k) = p_{in-}(j,k), \forall (j,k) \tag{2.55}$$

This change in inventory equations must be reflected in cash constraint at node n. So, the cash in node n would be the sum of cash carried from the previous node, cash inflow and outflow due to purchase in the underlying assets and cash coming in from the expiry of the options in that node.

$$\begin{aligned} C_n = C_{n-}e^{r_n \Delta t} &+ \sum_{i \in \mathcal{I}} x^-_{in} v_{in}(1-\chi^-) - \sum_{i \in \mathcal{I}} x^+_{in} v_{in}(1+\chi^+) \\ &+ \sum_{i,j=t(n),k} [c_{in-}(j,k)\delta^c_{in}(j,k) + p_{in-}(j,k)\delta^p_{in}(j,k)](1-\chi_o) \end{aligned} \tag{2.56}$$

Wealth in node n would be the sum of cash held and the investment in assets and options in that node.

$$W_n = C_n + \sum_{i \in \mathcal{I}} x_{in} v_{in} + \sum_{i,t(n)<j\leq T,k} [c_{in}(j,k)O^c_{in}(j,k) + p_{in}(j,k)O^p_{in}(j,k)], \ \forall\, n \in \mathcal{N} \tag{2.57}$$

This completes multistage model where we have options expiring at any decision stage along the planning horizon.

Extension: Multi-stage model for options expiring at any time point along the planning horizon)

Next, we add a little complexity to the model that options are expiring at any time point along the horizon (not necessarily on the decision stage). We would now extend the previous model with the equations developed in the beginning to consider options expiring before decision stages. The only equations that would change are cash balance equation and wealth equation in node n.

$$C_n = C_{n-}e^{r_n \Delta t} + \sum_{i \in \mathcal{I}} x_{in}^{-} v_{in}(1 - \chi^{-}) - \sum_{i \in \mathcal{I}} x_{in}^{+} v_{in}(1 + \chi^{+})$$
$$+ \sum_{i, t(n-) < j \leq t(n), k} [c_{in-}(j,k)\delta_{in}^{c}(j,k) + p_{in-}(j,k)\delta_{in}^{p}(j,k)] e^{r_n(t(n) - \tau_n(j))}(1 + \chi_o) \tag{2.58}$$

The difference here is the time interval on which the options are exercised, $(t(n-), t(n)]$ unlike at $t(n)$ in the previous model. This completes a multi-stage model where options are expiring at any time point along the horizon.

2.3.2 Multi-stage model: Buying, selling (no short selling) and exercising options at any time point along the horizon

Next, we introduce another complexity to the previous models. This time we allow selling of the options that have not expired. This allows us to transfer risk from the buyer of the option to writer of the option, for instance, we buy an option at time 0 to hedge our position before expiry at any decision stage the option was found to be in the money, so we can sell this option and can profit from its premium and buy another option to hedge our position. Option premiums are very lucrative, these are risky but at the same time can be rewarding as well. This is what we try to capture in the model discussed below, Davari-Ardakani, Aminnayeri, and Seifi, 2016 also attempted to develop a model like this, however, we do not agree with some technicalities of the model.

Asset Inventory Equation at root node

$$x_{i0} = \bar{x}_i + x_{i0}^{+} - x_{i0}^{-}, \; \forall\, i \in \mathcal{I} \tag{2.59}$$

Option Inventory Equation:

$$c_{i0}(j,k) = c_{i0}^{+}(j,k) - c_{i0}^{-}(j,k); \tag{2.60}$$

$$p_{i0}(j,k) = p_{i0}^{+}(j,k) - p_{i0}^{-}(j,k); \tag{2.61}$$

The options in the node n are equal to options bought (+ superscript) in that node minus options sold (- superscript) in that node. Since no short selling is allowed constraints are introduced in the later part of the model.

Cash Balance Constraint at root node:

$$C_0 = \bar{C} + \sum_{i \in \mathcal{I}} x_{i0}^{-} v_{i0}(1 - \chi^{-}) - \sum_{i \in \mathcal{I}} x_{i0}^{+} v_{i0}(1 + \chi^{+})$$
$$- \sum_{i, j \leq T, k} [c_{in}^{+}(j,k) O_{in}^{c}(j,k) + p_{i0}^{+}(j,k) O_{i0}^{p}(j,k)](1 + \chi_o^{+}) \tag{2.62}$$

While imposing cash balance constraint at root node, we make sure that no selling of options is allowed, as it is our assumption that there are no initial position in options, selling at time 0 would refer to short selling which is not in the scope of this model.

Asset Inventory Equation: At node $n \in \mathcal{N} - \{0\}]$

$$x_{in} = x_{in-} + x^+_{in} - x^-_{in},\ \forall\, i \in \mathcal{I}, \tag{2.63}$$

Asset inventory would have the same Markovian structure as in the previous multistage models.

Option Inventory Equations in node $n \in \mathcal{N} - \{0\}]$

$$c_{in}(j,k) = \begin{cases} c_{in-}(j,k) & \text{if } j \le t_n, \forall\, k, \\ c_{in-}(j,k) + c^+_{in}(j,k) - c^-_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

$$p_{in}(j,k) = \begin{cases} p_{in-}(j,k) & \text{if } j \le t_n, \forall\, k, \\ p_{in-}(j,k) + p^+_{in}(j,k) - p^-_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

We replicate the inventory equations to formulate the option inventory equations. The only difference comes from the maturity of the options. If an option is at maturity then the position in the option in the previous node is moved to the current node as it is and it settled in the cash equation. If not, then buying and selling in that option contract is allowed. The net position would be the sum of what is held previously and what is bought in the current node less what is sold in the current node.

Cash Balance Constraint in: $n \in \mathcal{N} - \{0\}]$

$$\begin{aligned} C_n = {} & C_{n-}e^{r_n\Delta t} + \sum_{i\in\mathcal{I}} x^-_{in}v_{in}(1-\chi^-) - \sum_{i\in\mathcal{I}} x^+_{in}v_{in}(1+\chi^+) \\ & - \sum_{i,t(n)\le j\le T,k} [c^+_{in}(j,k)O^c_{in}(j,k) + p^+_{in}(j,k)O^p_{in}(j,k)](1+\chi^+_o) \\ & + \sum_{i,t(n)\le j\le T,k} [c^-_{in}(j,k)O^c_{in}(j,k) + p^-_{in}(j,k)O^p_{in}(j,k)](1-\chi^-_o) \\ & + \sum_{i,t(n^-)\le j\le t(n),k} [c_{in}(j,k)\delta^c_{in}(j,k) + p_{in}(j,k)\delta^p_{in}(j,k)]e^{r_n(t(n)-\tau_n(j)})(1-\chi_o) \end{aligned} \tag{2.64}$$

Cash constraint clearly identifies cash outflow and inflow due to trade in underlying assets, any cash outflow due to options purchase, cash inflow due to options selling and expiries. Wealth in node n then would simply be the sum of cash available in that node, value of the underlying assets and options contracts.

$$W_n = C_n + \sum_{i\in\mathcal{I}} x_{in}v_{in} + \sum_{i,t(n)<j\le T,k} [c_{in}(j,k)O^c_{in}(j,k) + p_{in}(j,k)O^p_{in}(j,k)],\ \forall\, n \in \mathcal{N} \tag{2.65}$$

Policy constraints remain the same as in the previous model, non-negativity constraints are now extended to $c^+_{in}, p^+_{in}, c^-_{in}\&p^-_{in}$.

Policy Constraints:

$$\phi_L W_n \leq x_{in} v_{in} \leq \phi_U W_n,\ \phi_L = \{\phi_{iL}\}',\ \phi_{iL} \in [0,1], \tag{2.66}$$
$$\phi_U = \{\phi_{iU}\}',\ \phi_{iU} \in [0,1],\ \forall\, i \in \mathcal{I} \tag{2.67}$$

$$\phi_{icL} W_n \leq c_{in} O^c_{in}(j,k) \leq \phi_{icU} W_n, \quad \phi_{icL} \in [0,1],\ \phi_{icU} \in [0,1], \forall\, i, j, k \tag{2.68}$$

$$\phi_{ipL} W_n \leq p_{in} O^p_{in}(j,k) \leq \phi_{ipU} W_n, \quad \phi_{ipL} \in [0,1],\ \phi_{ipU} \in [0,1], \forall\, i, j, k \tag{2.69}$$

Non-negativity constraints:

$$x^+_{in} \geq 0,\ x^-_{in} \geq 0,\ x_{in} \geq 0,\ \forall\, i \in \mathcal{I},\ \forall\, n \in \mathcal{N}_{0,T-1} \tag{2.70}$$

$$\forall\, n \in \mathcal{N}_{0,T-1} \quad \begin{array}{c} c_{in}(j,k) \geq 0,\ p_{in}(j,k) \geq 0, \\ c^+_{in}(j,k) \geq 0, p^+_{in}(j,k) \geq 0, \\ c^-_{in}(j,k) \geq 0, p^-_{in}(j,k) \geq 0 \end{array} \tag{2.71}$$

$$x^-_{i0} \leq \bar{x}_i,\ \forall\, i \in \mathcal{I} \tag{2.72}$$

Cash Constraint:

$$0 \leq C_n \leq \gamma_C W_n,\ \forall n \in \mathcal{N},\ \gamma_C \in [0,1] \tag{2.73}$$

No decision is made on the leaf nodes.

$$x^+_{in} = 0;\ x^-_{in} = 0;\ c_{in} = 0,\ p_{in} = 0,\ \forall\, n \in \mathcal{N}_T,\ \forall\, i \in \mathcal{I} \tag{2.74}$$

This completes the model where buying and selling of option is allowed before expiry.

Model Validation

We now validate this model where buying and selling of options take place before their expiry. Table 2.5 shows how the price evolves in the scenario corresponding to 75th percentile of the price distribution at the horizon. We have price for equity, bond and commodity indexes and prices for ATM call and put options expiring at 1 month, 3 month and 6 month time.It can be seen clearly that after the maturity of options their price in the scenario is 0. Table 2.6, 2.7 and 2.8 show hold, buy and selling decisions made in the model. Option payoffs at expiry is either 0 or some positive value, depending on if it expires ITM or OTM. So, in the GAMS code option payoff can be treated like a payoff of an asset that goes to 0 at a certain point, forcing the algorithm to not buy when the price in the next node is 0 and sell all the quantity if price in the next node is 0.

Table 2.6, 2.7 and 2.8 show that options were bought and sold before the expiry. Call option on Equity with 6 month expiry was bought in the 3rd stage and then sold in the 4th stage, and was again bought in the 5th stage to drive the optimal investment strategy.

TABLE 2.5: Mode Validation: Buy/Sell Options *Price Evolution*

Month	0	1	2	3	4	5	6
Equity	1870.8	1904.4	1845.5	2061.6	2060.3	2055.3	2103
Bond	1931.2	1931	1950.6	1944.1	1925.7	1928.6	1921.7
Commodity	5018.1	4997.7	4916.2	4830.4	4708.4	4573.3	4409.2
Eq_Call_1M	16.552	33.594	0	0	0	0	0
Eq_Call_3M	24.198	39.398	2.6274	190.75	0	0	0
Eq_Call_6M	33.178	48.527	12.228	193.43	191.66	185.46	232.18
Eq_Put_1M	16.552	0	0	0	0	0	0
Eq_Put_3M	24.198	3.6751	27.118	0	0	0	0
Eq_Put_6M	33.178	9.6151	34.013	0.0004548	6.70E-06	1.17E-10	0

TABLE 2.6: Model Validation: Buy/Sell options before expiry (*holding decisions*)

Month	0	1	2	3	4	5	6
Equity	26.439	0	27.528	24.981	0	24.384	24.384
Bond	25.839	52.268	26.082	28.976	58.074	29.631	29.631
Commodity	0	0	0	0	0	0	0
Eq_Call_1M	26.439	0	0	0	0	0	0
Eq_Call_3M	0	0	27.528	0	0	0	0
Eq_Call_6M	0	0	0	24.981	0	24.384	24.384
Eq_Put_1M	0	0	0	0	0	0	0
Eq_Put_3M	0	0	0	0	0	0	0
Eq_Put_6M	0	0	0	0	0	0	0

The intrinsic value of the option is driven by the price of its underlying security and time to maturity. When the price was declining the option was sold and when it was increasing, it was held in the portfolio. This completes validation of the model developed for selling options before their expiry.

2.4 A Generic Multi-Stage Model: Long/Short Positions in Options

So, far we have talked about long position in options. Now, we are going to introduce a generic model that considers short selling of options. Short-selling of options would allow us to make profit the price movements in the underlying and it would help us in implementing hedged strategies like butterfly spread.

TABLE 2.7: Model Validation: Buy/Sell Options (*buying decisions*)

Month	0	1	2	3	4	5	6
Equity	26.439	0	27.528	0	0	24.384	0
Bond	25.839	26.429	0	2.8944	29.098	0	0
Commodity	0	0	0	0	0	0	0
Eq_Call_1M	26.439	0	0	0	0	0	0
Eq_Call_3M	0	0	27.528	0	0	0	0
Eq_Call_6M	0	0	0	24.981	0	24.384	0
Eq_Put_1M	0	0	0	0	0	0	0
Eq_Put_3M	0	0	0	0	0	0	0
Eq_Put_6M	0	0	0	0	0	0	0

TABLE 2.8: Model Validation: Buy/Sell Options (*selling decisions*)

Month	0	1	2	3	4	5	6
Equity	0	26.439	0	2.5467	24.981	0	0
Bond	0	0	26.186	0	0	28.443	0
Commodity	0	0	0	0	0	0	0
Eq_Call_1M	0	26.439	0	0	0	0	0
Eq_Call_3M	0	0	0	27.528	0	0	0
Eq_Call_6M	0	0	0	0	24.981	0	0
Eq_Put_1M	0	0	0	0	0	0	0
Eq_Put_3M	0	0	0	0	0	0	0
Eq_Put_6M	0	0	0	0	0	0	0

We are going to replicate the option inventory equations introduced in the above model. However, we would separate inventory for long and short positions. We introduce $c^l_{in}, c^{l+}_{in}, c^{l-}_{in}$ for *hold*, *buy* and *sell* decisions in long call options and $c^s_{in}, c^{s+}_{in}, c^{s-}_{in}$ for *hold*, *buy* and *sell* decisions in short call options. Similarly, we have variables for put options, $p^l_{in}, p^{l+}_{in}, p^{l-}_{in}, p^s_{in}, p^{s+}_{in}, p^{s-}_{in}$ for *hold*, *buy* and *sell* decisions in long and short put options. The asset inventory equations remain the same. This model in special cases should be equivalent to the models discussed by, Topaloglou, Vladimirou, and Zenios, 2011 and Yin and Han, 2013b. The model goes as follows:

Asset Inventory Equation: At root node

$$x_{i0} = \bar{x}_i + x^+_{i0} - x^-_{i0},\ \forall\, i \in \mathcal{I} \tag{2.75}$$

The options in the node n are equal to options bought (+ superscript) in that node minus options sold (- superscript) in that node.

Options Inventory Equation:

For long position:

$$c^l_{i0}(j,k) = c^{l+}_{i0}(j,k) - c^{l-}_{i0}(j,k); \tag{2.76}$$

$$p^l_{i0}(j,k) = p^{l+}_{i0}(j,k) - p^{l-}_{i0}(j,k); \tag{2.77}$$

For short position:

$$c^s_{i0}(j,k) = c^{s+}_{i0}(j,k) - c^{s-}_{i0}(j,k); \tag{2.78}$$

$$p^s_{i0}(j,k) = p^{s+}_{i0}(j,k) - p^{s-}_{i0}(j,k); \tag{2.79}$$

The cash constraint in node 0 tackles cash inflow and outflow due to sale and purchase in the underlying assets, any long position in call and put options is modeled as cash outflow and short position in option is modeled as cash inflow.

Cash Constraint at node 0:

$$\begin{aligned} C_0 = \bar{C} &+ \sum_{i\in\mathcal{I}} x^-_{i0} v_{i0}(1-\chi^-) - \sum_{i\in\mathcal{I}} x^+_{i0} v_{i0}(1+\chi^+) \\ &- \sum_{i,j\leq T,k} [c^{l+}_{in}(j,k) O^c_{in}(j,k) + p^{l+}_{i0}(j,k) O^p_{i0}(j,k)](1+\chi^+_o) \\ &+ \sum_{i,j\leq T,k} [c^{s+}_{in}(j,k) O^c_{in}(j,k) + p^{s+}_{i0}(j,k) O^p_{i0}(j,k)](1-\chi^-_o) \end{aligned} \tag{2.80}$$

Asset Inventory Equation: At node $n \in \mathcal{N} - \{0\}]$
It remains the same as there are no modifications done on this part of the model.

$$x_{in} = x_{in-} + x^+_{in} - x^-_{in},\ \forall\, i \in \mathcal{I}, \tag{2.81}$$

updating option inventory

For long Position:

$$c^l_{in}(j,k) = \begin{cases} c^l_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ c^l_{in-}(j,k) + c^{l+}_{in}(j,k) - c^{l-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

$$p^l_{in}(j,k) = \begin{cases} p^l_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ p^l_{in-}(j,k) + p^{l+}_{in}(j,k) - p^{l-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

For Short Position:

$$c^s_{in}(j,k) = \begin{cases} c^s_{in-}(j,k) & \text{if } j \le t_n, \forall\, k, \\ c^s_{in-}(j,k) + c^{s+}_{in}(j,k) - c^{s-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

$$p^s_{in}(j,k) = \begin{cases} p^s_{in-}(j,k) & \text{if } j \le t_n, \forall\, k, \\ p^s_{in-}(j,k) + p^{s+}_{in}(j,k) - p^{s-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

We replicate the inventory equations to formulate the long/short option inventory equations. The only difference comes from the maturity of the options. If an option is at maturity then the position in the option in the previous node is moved to the current node as it is and it settled in the cash equation. If not, then buying and selling in that option contract is allowed. The net position would be the sum of what is held previously and what is bought in the current node less what is sold in the current node. However, the short position would need to take into account for margin adjustments. This goes as follows.

$$m^c_{in}(j,k) = c^s_{in}(j,k)max(0, v_{in} - K^j_i)$$

$$m^p_{in}(j,k) = p^s_{in}(j,k)max(0, K^j_i - v_{in})$$

Cash Constraint in node n:

$$\begin{aligned} C_n = {} & C_{n-}e^{r_n\Delta t} + \sum_{i\in\mathcal{I}} x^-_{in}v_{in}(1-\chi^-) - \sum_{i\in\mathcal{I}} x^+_{in}v_{in}(1+\chi^+) \\ & - \sum_{t_n\le j\le T,k} [c^{l+}_{in}(j,k)O^c_{in}(j,k) + p^{l+}_{in}(j,k)O^p_{in}(j,k) - c^{s+}_{in}(j,k)O^c_{in}(j,k) - p^{s+}_{in}(j,k)O^p_{in}(j,k)] \\ & + \sum_{t_n\le j\le T,k} [c^{l-}_{in}(j,k)O^c_{in}(j,k) + p^{l-}_{in}(j,k)O^p_{in}(j,k) - c^{s-}_{in}(j,k)O^c_{in}(j,k) + p^{s-}_{in}(j,k)O^p_{in}(j,k)] \\ & + \sum_{t_{n^-}\le j\le t_n,k} [c^l_{in}(j,k)max(0, v_{in} - K^j_i) + p^l_{in}(j,k)max(0, K^j_i - v_{in}) \\ & - c^s_{in}(j,k)max(0, v_{in} - K^j_i) + p^s_{in}(j,k)max(0, K^j_i - v_{in})] \\ & - \sum_{k,j>t_n} (m^c_{in}(j,k) + m^p_{in}(j,k) \end{aligned} \tag{2.82}$$

The cash constraint here is the most complex of all the cash constraints discussed so far. It is a combination of cash outflows and inflows due to sale or purchase in underlying assets and long or short position in options, cash inflows from the options expiry and in addition, any margin outflow.

Wealth in node n would be the sum of cash available and value of investments in long assets and derivatives less the position in short derivatives contracts.

$$\begin{aligned} W_n = C_n + \sum_{i \in \mathcal{I}} x_{in} v_{in} + \sum_{t_n < j \leq T, k} [c^l_{in}(j,k) O^c_{in}(j,k) \\ + p^l_{in}(j,k) O^p_{in}(j,k) - c^s_{in}(j,k) O^c_{in}(j,k) - p^s_{in}(j,k) O^p_{in}(j,k)], \ \forall \, n \in \mathcal{N} \end{aligned} \tag{2.83}$$

Policy constraints remain the same as in the previous model, non-negativity constraints are extended to the new variables introduced.

Policy Constraints:

$$\phi_L W_n \leq x_{in} v_{in} \leq \phi_U W_n, \ \phi_L = \{\phi_{iL}\}', \ \phi_{iL} \in [0,1], \tag{2.84}$$

$$\phi_U = \{\phi_{iU}\}', \ \phi_{iU} \in [0,1], \ \forall \, i \in \mathcal{I} \tag{2.85}$$

$$\phi_{icL} W_n \leq c_{in} O^c_{in}(j,k) \leq \phi_{icU} W_n, \quad \phi_{icL} \in [0,1], \ \phi_{icU} \in [0,1], \forall \, i,j,k \tag{2.86}$$

$$\phi_{ipL} W_n \leq p_{in} O^p_{in}(j,k) \leq \phi_{ipU} W_n, \quad \phi_{ipL} \in [0,1], \ \phi_{ipU} \in [0,1], \forall \, i,j,k \tag{2.87}$$

Non-negativity constraints:

$$x^+_{in} \geq 0, \ x^-_{in} \geq 0, \ x_{in} \geq 0, \ \forall \, i \in \mathcal{I}, \ \forall \, n \in \mathcal{N}_{0,T-1} \tag{2.88}$$

$$\begin{aligned} & c^l_{in}(j,k) \geq 0, \ p^l_{in}(j,k) \geq 0, \\ & c^s_{in}(j,k) \geq 0, \ p^s_{in}(j,k) \geq 0, \\ \forall \, n \in \mathcal{N}_{0,T-1} \quad & c^{l+}_{in}(j,k) \geq 0, p^{l+}_{in}(j,k) \geq 0, \\ & c^{l-}_{in}(j,k) \geq 0, p^{l-}_{in}(j,k) \geq 0 \\ \forall \, n \in \mathcal{N}_{0,T-1} \quad & c^{s+}_{in}(j,k) \geq 0, p^{s+}_{in}(j,k) \geq 0, \\ & c^{s-}_{in}(j,k) \geq 0, p^{s-}_{in}(j,k) \geq 0 \end{aligned} \tag{2.89}$$

$$x^-_{i0} \leq \bar{x}_i, \ \forall \, i \in \mathcal{I} \tag{2.90}$$

Cash Constraint:

$$0 \leq C_n \leq \gamma_C W_n, \ \forall n \in \mathcal{N}, \ \gamma_C \in [0,1] \tag{2.91}$$

No decision is made on the leaf nodes.

$$x^+_{in} = 0; \ x^-_{in} = 0; \ ncl^+_{in} = 0, \ npl^+_{in} = 0, \ \forall \, n \in \mathcal{N}_T, \ \forall \, i \in \mathcal{I} \tag{2.92}$$

This completes the model where buying and short selling of option is allowed before expiry.

2.4.1 Theoretical Validation of the Generic Model

The theoretical model developed above can be reduced to the single-stage model we started with, or to the subsequent models we developed. As the asset inventory equations remain the same throughout all these models and the only difference comes from the options intricacies. We see the option inventory equation of the generic model first and then start making reductions to it:

For long Position:

$$c^l_{in}(j,k) = \begin{cases} c^l_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ c^l_{in-}(j,k) + c^{l+}_{in}(j,k) - c^{l-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

$$p^l_{in}(j,k) = \begin{cases} p^l_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ p^l_{in-}(j,k) + p^{l+}_{in}(j,k) - p^{l-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

For Short Position:

$$c^s_{in}(j,k) = \begin{cases} c^s_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ c^s_{in-}(j,k) + c^{s+}_{in}(j,k) - c^{s-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

$$p^s_{in}(j,k) = \begin{cases} p^s_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ p^s_{in-}(j,k) + p^{s+}_{in}(j,k) - p^{s-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

We first remove the short-selling feature from this model, then we get rid of c^s, p^s variables. The model reduces to the buy/sell (long) model with the following inventory equations. (c^l, p^l are replace by c, p respectively)

$$c_{in}(j,k) = \begin{cases} c_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ c_{in-}(j,k) + c^{+}_{in}(j,k) - c^{-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

$$p_{in}(j,k) = \begin{cases} p_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ p_{in-}(j,k) + p^{+}_{in}(j,k) - p^{-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

Next we remove the selling feature from the model to reduce it to a multi-stage model where options are expiring at any stage (bought only at time 0)

$$c_{in}(j,k) = c_{in-}(j,k), \forall (j,k), j > t(n) \tag{2.93}$$

$$p_{in}(j,k) = p_{in-}(j,k), \forall (j,k), j > t(n) \tag{2.94}$$

Next we reduce it to a model where options are expiring at the subsequent stages, Yin and Han, 2013b. The option inventory is due to the option bought in that node.

$$c_{in}(j,k) = c^{+}_{in}(j,k), \forall (j,k), j = t(n+) \tag{2.95}$$

$$p_{in}(j,k) = p^{+}_{in}(j,k), \forall (j,k), j = t(n+) \tag{2.96}$$

Finally, to a single stage model, where in node n, nc,np are equal to 0, Topaloglou, Vladimirou, and Zenios, 2011.

$$c_{0n}(j,k) = c^{+}_{0i-}(j,k), \forall (j,k), j = T \tag{2.97}$$

$$p_{0n}(j,k) = p^{+}_{0i}(j,k), \forall (j,k), j = T \tag{2.98}$$

Model Validation: Generic Model

We are now going to validate the model where short positions on options are considered. To validate the model, we are going to put some constraints in the model that would restrict the naked short selling of options. We are interested in short selling but in a hedged environment, short position in options could expose the portfolio to unlimited losses. Therefore, we implement a vertical spread (bull call spread) strategy to check its implications in the optimization process.

A bull call spread is formed by buying a call option in-the-money and going short on a call option which is out-of-the-money, with the same expiry. The constraint for bull call spread would be:

$$\sum_{k \in K^*} c^l_n(j,k) + \sum_{k \in K^{**}} c^s_n(j,k) \le \phi_s W_n, \forall j = t(n), n \in \mathcal{N} \tag{2.99}$$

$$c^l_n(j,k) = c^s_n(j,k), \forall (j,k),\ n \in \mathcal{N}, \tag{2.100}$$

where K^* and K^{**} are sets of in-the-money (ITM) and out-of-the-money (OTM) options. We implemented this constraint in the model and validated it. In the figure 2.1, we show the payoff of a call bull spread formed using 5% ITM and OTM call options expiring in six months time. In the figure, x-axis and y-axis represent monthly time steps and strategy payoff respectively. Both profit and loss are capped.

TABLE 2.9: Model Validation: Short Position in options (Hold decisions)

Month	0	1	2	3	4	5	6
Equity	21.366	20.769	19.815	0	0	30.897	30.897
Bond	25.839	26.599	27.94	0	28.629	28.629	28.629
Commodity	0	0	0	0	0	0	0
call option (ITM)	106.83	103.85	99.076	0	0	0	0
CALL Option (OTM)	-106.83	-103.85	-99.076	0	0	0	0

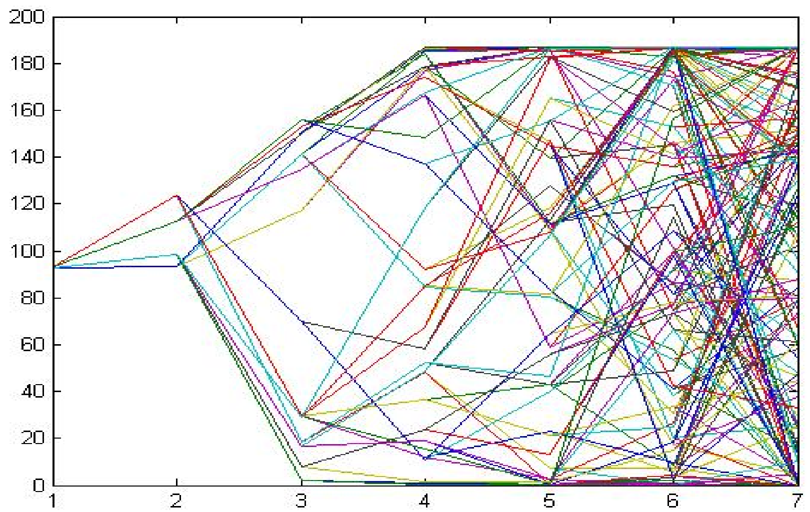

FIGURE 2.1: Call Bull Spread Payoff

TABLE 2.10: Model Validation: Short Position in Options (Price)

Month	0	1	2	3	4	5	6
Equity	1870.8	1893.1	1944.2	1979.5	1927.4	1760.3	1759.9
Bond	1931.2	1917.1	1921.5	1929.3	1940.1	1930.9	1927.5
Commodity	5018.1	4956	4925.3	4902.1	4791.3	4743.6	4611.5
Call Option 5% ITM	95.65	116.34	166.86	202.18	150.05	3.86	0
Call Option 5% OTM	2.59	3.9	13.82	28.12	3.75	0	0

It can be seen from the table 2.9 & 2.10 that short and long positions in options have changed as the price of the underlying was changing while the number of long and short options are exactly the same at each time step, the position always remained hedged. The buy/sell decisions in long/short positions in options drive the optimal investment policy.

So far, we have discussed cash settled European options in the optimization models. In the next Chapter (3), we are going to discuss an optimization model to update inventory using physical options.

Chapter 3

Model Extension Based on Derivatives' Inventory

So far, we have discussed models and approaches where options contracts are cash settled. In this chapter we explore physical settled options contracts in multi-stage programming framework, this could have a wide applications across various markets. For instance, a production that needs copper to manufacture its products can buy copper form a commodity exchange. Options contracts can be used to buy real copper commodity at the strike price of the option. If options purchased expire in the money at the maturity then the buyer would have the right to buy the underlying at the strike price which would be lower than the market price, this way a firm can accumulate more copper metal from the exchange at a better price than the price in the retail metal market. On the other hand, for a firm that has copper in its inventory fearing a decline in the prices of copper in future could use put options contracts to sell underlying at a price higher than market price.

This approach is not limited to a commodity trader, let us consider a broker who wants to purchase an amount of shares of a company XYZ over next few months for his client at a price lower than the market price. He can use options contracts to accumulate stocks, the objective function in this optimization program gives flexibility to the trader/broker to achieve an inventory with a certain buying cost, for instance, *VWAP* (Volume Weighted Average Price).

Barbaros and Bagajewicz (2004) introduce a model to use inventory and options to hedge financial risk, they analyze how the shape of the risk curves change when options are introduced in capacity expansion problem. The study showed that the usual assumption that with option contracts will by itself reduce the risk exposure at small profits is not always true, and that proper risk management tools are needed for this purpose as well. Some authors have also considered futures contracts to manage inventory. This has seen wide applications where energy derivatives are involved, (Bertochhi et al (2011)).

In this chapter we confine our research to the use options contracts. The rest of the chapter is organized as follows. Section 1 of the chapter introduces sets and parameters that are needed to formulate a stochastic programming model. Section 2 introduces a multistage model. In section 3 we discuss about possible extensions of the model. Section 4 is dedicated to the model validation.

3.1 Multi-stage model: Using options contracts to update the inventory

3.1.1 Sets, Parameters and Variables

We describe here parameters and sets needed to setup multistage model.

Sets

- $\mathcal{T}$, set of discrete time space indexed by t, $\mathcal{T} : t = \{0, 1, 2..., T\}$
- $\mathcal{N}$, Set of nodes in the scenario tree indexed by n, $\mathcal{N}_t$ is the set of nodes at time stage t
  (Every $n \in \mathcal{N}_t$ has a unique ancestor $n- \in \mathcal{N}_{t-1}$ and for $t \leq T - 1$ there exists a non-empty set of nodes $n+ \in \mathcal{N}_{t+1}$)
- $\mathcal{I}$, set of financial assets, indexed by i
- $\mathcal{O}$, set of vanilla options
  - $\mathcal{O}^c$ & $\mathcal{O}^p$ are set of call and put options respectively
  - $\mathcal{J}$, set of expiries of the options $\mathcal{O}$, indexed by j, $\mathcal{O}_j$ represents the set of options expiring at maturity j, $\mathcal{J} : j = \{J_1, J_2...\}$
  - $\mathcal{K}$, set of strikes of the options in $\mathcal{O}$, indexed by k, $\mathcal{O}_k$ represents the set of options with strike price K, $\mathcal{K} : k = \{K_1, K_2...\}$, K^j represents the vector of strikes at maturity j, $K^j = \{K^j_1, K^j_2, K^j_3....\}$ indexed by k^j

Input Parameters

- $\bar{x}_i$, initial position in asset $i \in \mathcal{I}$
- $\bar{c}$, is the initial available cash
- T, length of planning horizon
- χ^+ and chi^-, are the proportional transaction cost for purchase and sale in underlying
- χ^+_o, χ^-_o and χ_o, are proportional transaction costs on buying, selling and exercising option respectively.
- μ, user defined target
- v_{i0}, current price of the asset i per unit face value
- $O^c_{i0}(j, k)$, is the price of the European call option on asset i with expiry j and strike price k
- $O^p_{i0}(j, k)$, is the price of the European put option on the asset i with expiry j and strike price k

Now, we introduce parameters that would model the flow of information along the scenario tree.

Scenario Dependent Parameters

- $p(n)$, probability of node $n \in \mathcal{N}$ such that $\sum_{n \in \mathcal{N}_T} p(n) = 1$ and for every non-terminal node $p(n) = \sum_{m \in n+} p(m)$, $\forall n \in \mathcal{N}_t$, $t \leq T-1$
- v_{in}, price of asset i, in node n
- $O^c_{in}(j,k)$, is the price of the European call option on the asset i in node n, with strike price equal to $K_k, \forall K_k \in \mathcal{K}$ that expires at $t_j = J, j \in \mathcal{J}$
- $O^p_{in}(j,k)$, is the price of the European put option on the asset i in node n, with strike price equal to $K_k, \forall K_k \in \mathcal{K}$ that expires at $t_j = J, j \in \mathcal{J}$

Computed Parameters

Value of the initial portfolio

$$W\_0 = \bar{c} + \sum_i \bar{x_i} v_{i0}, \tag{3.1}$$

Decision Variables

- x^+_{in}, nominal amount of asset i purchased in node n
- x^-_{in}, nominal amount of asset i sold in node n
- x_{in}, nominal amount of asset i held in node n in the revised portfolio
- $c_{in}(j,k)$, units purchased of a European call option on asset i with expiry j and strike price k
- $p_{in}(j,k)$, units purchased of a European put option on asset i with expiry j and strike price k

We assume that there is no initial position in the options contracts.

$$\bar{c_{i0}} = 0, \forall i \in I, \bar{p_{i0}} = 0, \forall i \in I \tag{3.2}$$

Auxiliary Variables

- W_n, value of portfolio in node n

We consider a optimization problem to update inventory of the underlying assets over a six month horizon with monthly rebalancing stages. We assume that options that are available at a decision stage are expiring at the subsequent stage. This gives us flexibility to look into choices available to update inventory at the nearest maturity of the derivatives contracts. The optimal strategy is determined for the given parameter λ, by adopting risk-reward function mentioned in Chapter 2.

$$\max(1-\alpha)E[W^T] - \alpha R_\zeta \tag{3.3}$$

$$A_0 X_0 = D_0 \tag{3.4}$$

$$A_n X_{n-} + G_n X_n = D_n, \forall n \in \mathcal{N} \tag{3.5}$$

The first part is the expected wealth ($E[W^T]$) (*reward*) at the planning horizon T and the second part is a risk measure R_ζ (*risk*), λ is the risk aversion coefficient that defines how risk averse the investor is. Expected wealth is defined as $\sum p_n W_n, \forall n \in \mathcal{N}_T$. We discuss risk measure in later part in this section. $A, G\&D$ are the constraint matrices and define inventory balance equations, cash balance equations and other constraints model is subjected to. X_n are the control variables, this vector decides buy, sell and hold decision for each asset in each node of the tree. The risk measure R_ζ adopted is expected shortfall and is defined as:

$$R_\zeta = \sum [\bar{\mu} - W_n]^+ p_n, \forall n \in \mathcal{N}_T \tag{3.6}$$

Where, $\bar{\mu}$ is user defined target, any scenario that yields wealth (W_n) lower than this target would be reflected in the expected shortfall, shortfalls are then weighted by their probability (p_n) to calculate the expected shortfall. (10). Portfolio revisions imply, for $t = 0, 1, 2...T-1$ a transition along the tree from the portfolio allocation at the ancestor node to a new allocation through holding, buying and selling decisions on individual securities. The last possible revision is at stage $T-1$ with one period to go.

Since, the feasibility region and the optimal strategy are scenario dependent so their derivation requires the specification of the return coefficients and scenario probabilities along the tree. We present here next the set of decision variables and constraints actually implemented to characterize the random constraint matrices $A_n, G_n, D_n, \forall n \in \mathcal{N}$ and solve the problem.

We consider three types of constraints, to be satisfied: the *inventory balance equations* define the portfolio evolution over time; the *cash balance constraints* include in each node all cash inflows and outflows generated by the current strategy; the *upper and lower bounds* on the decision vector which define policy constraints on the adoptable strategy.

For each node n of the scenario tree and the asset/derivative i, the optimal strategy is defined through the following possible decisions, x_{in} is the nominal amount *held* in asset i in node n; $c_{in}(j,k)$ and $p_{in}(j,k)$ is the nominal amount *bought* in call and put options contract on asset i in node n with strike k and maturity j respectively; x_{in^+} refers to a *buying decision* in asset i in node n; while x_{in}^- refers to a *selling decision* in asset i in node n. All the decision variables are constrained to be non-negative.

Let $\bar{x_i}$ be the initial holding in asset $\mathcal{I}$ then the inventory balance equation at root node can be written as:

Inventory Balance Equation at root node

$$x_{i0} = \bar{x}_i + x_{i0}^+ - x_{i0}^-, \ \forall \ i \in \mathcal{I} \tag{3.7}$$

The inventory balance constraints reflect the decision problem Markovian structure: as time evolves, along each scenario, the portfolio evolution will be fully specified in nominal value through holding, buying and selling decisions. Each such decision generates, jointly with other commitments, cash flows in each node resulting in cash surpluses or deficits to be compounded to the following stage.

The cash balance constraint is imposed for the first stage in the following way; it takes into account the number of options contracts bought in the root node n.

$$\begin{aligned} C_0 =& \bar{c} + \sum_{i \in \mathcal{I}} x^-_{i0} v_{i0}(1-\chi^-) - \sum_{i \in \mathcal{I}} x^+_{i0} v_{i0}(1+\chi^+) \\ & - \sum_{j=t+1,k} [c_0(j,k) O^c_0(j,k) + p_0(j,k) O^p_0(j,k)](1+\chi^+_o) \end{aligned} \tag{3.8}$$

3.1.2 Options Payoff Modeling Approach

Option payoff is a non-linear function unlike other asset, we need to break this non-linearity somehow in order to make the problem simpler. We need to introduce some variables that can tell us whether the option at expiry is in the money or not. The advantage of such variables is two-fold, first, it would get rid of us of the max function in the equations, so that would translate the problem to a linear system making it easier for computations etc. Secondly, we are considering nominal amount model and these variables would help us in tracking number of options, that is going to be a great help in inventory update problem using physical settled options, it is shown in the later part of this chapter and then in Chapter 3 in more detail.

We define *moneyness* (δ) of the call option on asset i with maturity j and strike k in node n as:

$$\delta^c_{in}(j,k) = max(v_{in} - K^j_k, 0) \tag{3.9}$$

Similarly, *moneyness* for put option would be:

$$\delta^p_{in}(j,k) = max(K^j_k - v_{in}, 0) \tag{3.10}$$

We then define an indicator variable λ to check if the options are in-the-money or not, $\lambda = 1$, if $\delta \geq 0$, otherwise 0. $\lambda^c_{in}(j,k), \lambda^p_{in}(k,j)$ are the indicator variables for call and put option respectively having strike $k \in \mathcal{K}$ and expiring at $j \in \mathcal{J}$. The product of $lambda^c_{in}(j,k)$ & $\delta^c_{in}(j,k)$ is the option payoff at maturity.

We now write the equations and above constraints at node n (later than first stage). The options purchased in the previous stage are going to update the inventory (in case of in-the-money expiry of options). So, first we need to know if the options have expired in the money or not. This we can track by the indicator variables defined in chapter 2. If the option expires in the money then λ takes value equal to 1 otherwise 0. Buy decisions in options are define as c_n and p_n for call and put options. Since, we have adopted a nominal amount model, these c_n and p_n are actually number of options, therefore, if multiplied by λ we would know the quantities to added or subtracted from the inventory.

Assumption

To make the problem simple, we assume that there exists only stock index in the portfolio and options are available on stock index, $\mathcal{I} = \{1\}$, now, we can rewrite

$c_{in}(j,k)$ as $c_n(j,k)$.

Inventory Balance Equation at node n

$$x_{in} = x_{in-} + x_{in}^{+} - x_{in}^{-} + c_{n-}(j,k)\lambda_n^c(j,k) - p_{n-}(j,k)\lambda_n^p(j,k),\ \forall\, i \in \mathcal{I} = \{1\}, n \in \mathcal{N} - \{0\} \tag{3.11}$$

It is clear from the above equation that if a call option expires in-the-money then it would increase inventory, on the other hand, if a put option expires in-the-money then that would reduce the size of inventory. The variable λ defined in chapter 2 makes it very simple to model inventory equations.

Next, we write the cash balance constraint at node n.

Cash Balance Constraint at node n

$$\begin{aligned} C_n = {} & C_{n-}e^{r_n \Delta t} + \sum_{i\in\mathcal{I}} x_{in}^{-} v_{in}(1-\chi^{-}) - \sum_{i\in\mathcal{I}} x_{in}^{+} v_{in}(1+\chi^{+}) \\ & - \sum_{j=t(n^+),k} [c_n(j,k)O_n^c(j,k) + p_n(j,k)O_n^p(j,k)](1+\chi_o^{+}) \\ & + \sum_{j=t(n),k} k[p_{n-}(j,k)\lambda_n^p(j,k)](1-\chi_o) - \sum_{j=t(n),k} k[c_{n-}(j,k)\lambda_n^c(j,k)](1+chi_o) \end{aligned} \tag{3.12}$$

This equation reflects the cash inflow and outflow due to purchase and sell of underlying asset and options and options expiry which is leading to inventory modification. The $j = t(n^+)$ and $j = t(n)$ refer expiries at the next node and the current node respectively. We buy options that are expiring in the next node (as per our assumption) and we exercise options at the current node. If call option expires in-the-money, this would lead to cash outflow as inventory size is going to be increased. It is reflected in the equation as strike price times the number of options purchased. Similarly, put options expiring in-the-money reflect cash inflow, with reverse sign.

The wealth at node n is the sum of the market value times the units of all the assets and derivatives held.

$$W_n = C_n + \sum_{i\in\mathcal{I}} x_{in}v_{in} + \sum_{j=t(n^+),k} [c_n(j,k)O_n^c(j,k) + p_n(j,k)O_n^p(j,k)], \forall\, n \in \mathcal{N} \tag{3.13}$$

where $S_n = v_{1n}$ is the price of the stock index ($i = 1$)

The model includes constraints on the upper and lower bounds on investment in underlying security through equation 3.14. Let ϕ_L and ϕ_U be the set of lower and upper bounds on the underlying. Constraints on options in this model would be different from those discussed in Chapter 2. Now, the aim is to update inventory using call and put options. Buying put options give right to sell the underlying at strike price, therefore, we cannot have more physical put options in out portfolio than the number of owned underlying assets. This is reflected in the equation 3.16. It is important to note that we have only at-the-money strike options expiring at the next decision stage, this leaves us with exactly one type of option in the portfolio.

As for the call options, it gives us right to buy the underlying at strike price. Given the self-financing portfolio we are optimizing, we need to limit the number of call options we can buy to add underlying securities to our inventory at a decision stage, we don't want to run into a situation where we do not have enough cash to buy the underlying in case of in-the-money expiry of the call option. It is not the case where there is a choice of going for cash settlement or physical delivery. Only physical options are considered in this model. Equation 3.17 ensures this, product of strike price of the option and number of options is less then the cash available in that node (C_n). Equation 3.20 is the cash constraint, γ_C is the fractional wealth that is allowed to be kept in the cash account.

Policy Constraints:

$$\phi_L W_n \leq x_{in} v_{in} \leq \phi_U W_n,\ \phi_L = \{\phi_{iL}\}',\ \phi_{iL} \in [0,1], \tag{3.14}$$

$$\phi_U = \{\phi_{iU}\}',\ \phi_{iU} \in [0,1],\ \forall\, i \in \mathcal{I} \tag{3.15}$$

$$p_n(j,k) \leq x_{in}, i = 1 \tag{3.16}$$

$$k.c_n(j,k) \leq C_n, i = 1, \tag{3.17}$$

Non-negativity constraints:

$$x_{in}^{+} \geq 0,\ x_{in}^{-} \geq 0,\ x_{in} \geq 0,\ \forall\, i \in \mathcal{I},\ \forall\, n \in \mathcal{N}_{0,T-1} \tag{3.18}$$

$$c_n(j,k) \geq 0,\ p_n(j,k) \geq 0,\ \forall\, n \in \mathcal{N}_{0,T-1} \tag{3.19}$$

Cash Constraint:

$$0 \leq C_n \leq \gamma_C W_n,\ \forall n \in \mathcal{N},\ \gamma_C \in [0,1] \tag{3.20}$$

The following constraint imposes no decision on leaf node:

$$x_{in}^{+} = 0;\ x_{in}^{-} = 0;\ c_n = 0,\ p_n = 0,\ \forall\, n \in \mathcal{N}_T,\ \forall\, i \in \mathcal{I} \tag{3.21}$$

3.2 Possible Extension

Let us consider a trader/broker who holds shares of a company and is willing to add more shares of the same firm to his portfolio at a price lower than market price while avoiding risk pf any possible downside movement in the underlying's price. A possible solution to this problem would be to use call options to increase the inventory and to use put options to hedge the underlying against downside movement. This is a case where we mix of cash and physical settled contracts. In this case the equations (9 and 10) would change, the new formulation would be:

$$x_{in} = x_{in-} + x_{in}^{+} - x_{in}^{-} + c_{n-}(j,k)\lambda_n^c(j,k), \ \forall\, i \in \mathcal{I} = \{1\}, n \in \mathcal{N} - \{0\} \tag{3.22}$$

Compared to equation (9), in equation (21) there is no reduction in inventory size from the position in the put options. Hence, only call options adding to the inventory.

$$\begin{aligned} C_n = C_{n-}e^{r_n \Delta t} &+ \sum_{i \in \mathcal{I}} x_{in}^{-} v_{in}(1-\chi^{-}) - \sum_{i \in \mathcal{I}} x_{in}^{+} v_{in}(1+\chi^{+}) \\ &- \sum_{j=t(n^+),k} [c_n(j,k)O_n^c(j,k) + p_n(j,k)O_n^p(j,k)](1+\chi_o^{-}) \\ &+ \sum_{j=t(n),k} [p_{n-}(j,k)max(0,k-v_{in})](1-\chi_o^{-}) - \sum_{j=t(n),k} k[c_{n-}(j,k)\lambda_n^c(j,k)] \end{aligned} \tag{3.23}$$

In equation (22) put options expiring in the money adds cash to the portfolio while the cash outflow due to in-the-money expiry of the call options remains same as in the equation (10).

In other scenario where a trader/broker wants to reduce his inventory can formulate the model other way. In equation (23) only put option appears in the inventory equation.

$$x_{in} = x_{in-} + x_{in}^{+} - x_{in}^{-} - p_{n-}(j,k)\lambda_n^c(j,k), \ \forall\, i \in \mathcal{I} = \{1\}, n \in \mathcal{N} - \{0\} \tag{3.24}$$

In the cash constraint, call options contract is now cash settled, while the put options are still physical settled.

$$\begin{aligned} C_n = C_{n-}e^{r_n \Delta t} &+ \sum_{i \in \mathcal{I}} x_{in}^{-} v_{in}(1-\chi^{-}) - \sum_{i \in \mathcal{I}} x_{in}^{+} v_{in}(1+\chi^{+}) \\ &- \sum_{j=t(n^+),k} [c_n(j,k)O_n^c(j,k) + p_n(j,k)O_n^p(j,k)](1+\chi_o^{-}) \\ &+ \sum_{j=t(n),k} [c_{n-}(j,k)max(0,v_{in}-k)](1+\chi_o^{-}) + \sum_{j=t(n),k} k[p_{n-}(j,k)\lambda_n^c(j,k)] \end{aligned} \tag{3.25}$$

Chapter 4

Scenario Generation

A stochastic programming (SP) problem is a math programming problem, with values of some parameters replaced by distributions. SP can handle only discrete samples of limited size, so we need to approximate the distribution. The approximation is called a scenario tree.

4.1 Scenario Generation: Underlying Assets

Scenario tree is an information flow along the planning horizon. Scenario generation is a part of the stochastic optimization process where all the underlying forecast processes for assets and risk factors are translated to form a scenario tree that imposes information constraint on the decisions. The information flow is modelled by a filtration of sigma fields $\mathcal{A}_t, t = 1.....T$, which is associated to a stochastic input process $\xi = (\xi_1^T)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Typically, it is required that the $\sigma-$ field is generated by the random vector $\xi_1, \xi_2, ...\xi_T$. Then, the information or the nonanticipativity constraint means measurability of the decisions x_t with respect to $\mathcal{A}_\sqcup$ for every $t = 1, 2, 3...T$. $t = 1$ refers to the present or the root node of the scenario tree, therefore, $\mathcal{A}_1 = \{\emptyset, \Omega\}$. Figure **??** presents a sample scenario tree.

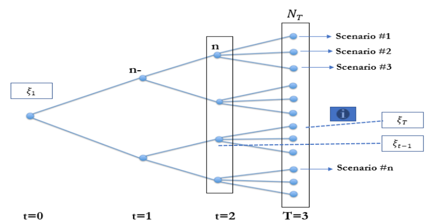

FIGURE 4.1: A Sample Scenario Tree

Any scenario-based approximation of the underlying probability distribution P of ξ has to reflect the growth of the $\sigma-$fields. Hence, the scenarios need to be tree-structured. In general, there are two ways to generate scenario trees, namely, (i) a tree-structure is prescribed and scenarios are generated via conditional distributions for increasing t starting with a root at t = 1, or (ii) in a first step a number of scenarios is generated for the whole horizon $t = 1, ...T$ based on the distribution P and according to some method, such as Monte Carlo simulation, Quasi-Monte Carlo, Quadrature rules using sparse grids and optimal quantization of probability measure. Secondly, a tree structure is generated successively by bundling scenarios. In our research, we rely on the first technique, we start from the root node, time $t = 1$

and produce information using the underlying pricing models to develop an information flow from root to leaf ($\mathcal{N}_T$) of the scenario tree. The sequences of realization are called *scenarios at stage t* $\xi_1, \xi_2, ...\xi_t$ given the conditional probability distribution of ξ_t conditioned by past realizations $\xi_1, \xi_2....\xi_{t-1}$

Having described what a scenario tree is and how the information flow is structured, we now move to the scenario generation of the underlying assets in our portfolio. As we are modelling an optimization problem from the perspective of an investor who wishes to maximize his wealth at horizon, which is six months, the assets in his portfolio are equity index, fixed income index and S&P GSCI index (formerly known as*Goldman Sachs Commodity Index*). Options contracts are available on Equity index. So, we need to introduce forecasting model for these assets.

We follow a two-layer approach to forecast these time series. The first layer is of the risk factors, we identify risk factors that describe the underlying assets. The second layer is an asset return formulation of the underlying time series derived from the risk factors and some exogenous variables. We identify short rate (r_n), long rate (l_n) and inflation rate (π_n) as risk factors, $j = 1, 2, 3$ respectively. The second layer of the process is to compute the values of the underlying processes B_n^k where $k = 1, 2, 3$ denote equity, fixed-income and commodity indexes respectively. We assume Cox-Ingersoll-Ross (CIR) dynamics for these risk factors. The coefficients α^j, $\omega^{j,*}$ and σ^j denote, respectively, the mean reversion coefficients, the long-term equilibrium and the standard deviation for each process, whereas, $t_n - t_{n-}$ is the differential time step between two consecutive nodes n and $n-$.

Correlation is introduced directly on the realizations e_n^r of three standard normal variables via the Cholesky elements c_j^r of the correlation matrix. Given the initial states $\omega^j(0)$ as ω_0^j for $t \in \mathcal{T}, n \in \mathcal{N}$, we have

$$\begin{aligned} \omega_n^j &= \omega_{n-}^j + \alpha^j(\omega_n^{j,*} - \omega_{n-}^j)(t_n - t_{n-}) \\ &\quad + \sigma^j \sqrt{(}\omega_{n-}^j)\sqrt{(}t_n - t_{n-})(\sum_{r=1,2,3} c_{j,r}e_n^r) \end{aligned} \tag{4.1}$$

The coefficients of the CIR process are estimated using *Maximum Likelihood Method*. In Table 4.1 we report the coefficients that are generated using this mechanism.

We adopt the models developed by Consigli et al., 2012 to model equity and bond index using these risk factors. Consider an equity benchmark that includes constant volatility σ_n, while the drift μ_n is random and depends on the market prices of risk λ_n and the interest rates r_n. The coefficient λ_n is market specific and reflect a varying risk aversion in the market. This is assumed to depend on the long interest rate l_n, inflation rate π_n and the recent market performance $(B_n/\bar{B})$.

$$\mu_n = r_n + \sigma\lambda_n, \tag{4.2}$$

$$\lambda_n = \beta_0 + \beta_1 l_n + \beta_2 \pi_n + \beta_3(B_n/\bar{B}) + e_n \tag{4.3}$$

$\bar{B}$ represents the constant average price over a given time while e_n is the realization of the standard normal variable. So, for given initial benchmark value B_0^1, the following price transition along the tree are derived:

$$B_n^k = B_0^k(1 + \mu_{n-}(t_n - t_{n-}) + \sigma\sqrt{t_n - t_{n-}}), \forall k = 1 \tag{4.4}$$

Using the risk factors we derive the fixed income benchmark tree model employing a duration-convexity approximation. Let $\bar{D}$ and $\bar{C}$ be the duration and convexity of the fixed income benchmark respectively, the evolution of the fixed income benchmark is determined through the following equation:

$$B_n^k = B_{n-}^k(1 - \bar{D}(l_n - l_{n-}) + 0.5\bar{C}(l_n - l_{n-})^2 + l_n(t_n - t_{n-})), \forall k = 2 \tag{4.5}$$

This completes our model for fixed income benchmark. Next, we introduce a model for pricing GSCI commodity index. Instead of using any established model for forecasting commodity, we adopt econometrics technique to develop a model for GSCI index. This requires to identify the key factors that best approximate the performance of GSCI index. We reviewed literature for this to know the drivers of commodity index.

Historically, commodities have shown positive correlation with inflation rate and change in inflation rate both in short-run and in long-run. Some studies have taken a longer-run perspective. Gorton and Rouwenhorst, 2006 find that correlations between commodity futures returns and inflation tend to rise and become statistically significant as the horizon lengthens. Adams et al., 2008 also conclude that correlations between commodities, measured using GSCI excess returns, and U.S. inflation rises with the investment horizon, although these positive correlations do not hold consistently for inflation in the euro area and Asia. Worthington and Pahlavani, 2007 presented evidence of the long-run hedging properties of gold based on a positive long-run relationship between gold and U.S. inflation in the post-war period.

Becker and Finnerty, 2000 attempted to incorporate futures leverage into the analysis by constructing levered indexes, which scale futures returns by a multiplier. They find that commodity futures serve as an inflation hedge, with the degree of protection increasing as the commodity futures are levered. This gives us the motivation to include inflation and change in inflation rate as the factors for developing a model for GSCI index.

Commodities have also shown negative correlation with stocks and bond market and therefore, been a good financial instrument for diversifying a portfolio. A high negative correlation has existed between stock and commodity prices over the past 140 years, Zapata, Detre, and Hanabuchi, 2012. Some early observers of commodity markets Bannister and Forward, 2002; Rogers, 2007 note that the history of U.S. stock and commodity prices has been characterized by recurring super cycles that last several decades. These observations make it evident to include the performance of the stock and bond market as drivers for commodity index. Since, commodities prices are heavily driven by demand and supply in the physical market, it is necessary to include lagged variables of dependent variable in order to derive the return dynamics.

We reviewed different approaches to estimate this model, Vector Autoregressive method (VAR), Vector Error Correction Mechanism (VECM), and the third method where we formulate an autoregressive distributed lagged (ARDL) model.

The classical VAR method and VECM method are used to generate scenarios. The key concern here is the number of parameters estimated in VAR and VECM methods that the model could actually be inefficient and in that case it owuld be vulnerable to type II error. Conventional regression estimators, including VARs, have good properties when applied to covariance-stationary time series, but encounter difficulties when applied to non-stationary or integrated processes which is the case here. These difficulties were illustrated by Granger and Newbold, 1974 when they introduced the concept of spurious regressions. If we have two independent random walk processes, a regression of one on the other will yield a significant coefficient, even though they are not related in any way. VECM method is the extension of VAR method where two or more time series are co-integrated. The model is fit to the first differences of the non-stationary variables, but a lagged error-correction term is added to the relationship. This addition of term leads to loss in degree of freedom.

Next, we consider ARDL *(autoregressive distributed lags)* method where we include one lagged variable of GSCI index with a constant term, we include equity market performance and fixed income market performance as exogenous variables and inflation rate as a risk factor. The equation for modelling GSCI index returns $(j = 4)$ is the following:

$$\begin{aligned}\omega_n^j = \beta_0^j + \beta_1^j (B_n^1 - B_{n-}^1)/B_{n-}^1 + \beta_2^j (B_n^2 - B_{n-}^2)/B_{n-}^2 + \beta_3^j \omega_n^{j=3} + \beta_4^j \omega_{n-}^j \\ + \beta_5 (\omega_n^{j=3} - \omega_{n-}^{j=3}) + \sigma^j \sqrt{t_n - t_{n-}} e_n\end{aligned} \tag{4.6}$$

$$B_n^k = B_{n-}^k (1 + \omega_n^{j=4}) \tag{4.7}$$

We estimated the parameters of the above model using OLS method. On the next page we present the model coefficients for the period May 2008 to May 2014, first we transform the monthly data to annual data points by taking annual returns at monthly frequency on rolling basis. It turns out that the performance of the equity index, change in inflation rate and lagged variable of first order of GSCI index returns are significant in approximating GSCI returns. The model has an impressive R-squared value and the error terms did not show autocorrelation upto 12th order that is significant at 99% confidence level.

GSCI Model: OLS, using observations 2008:05–2014:05 (T = 73)
Dependent variable: GSCI

	Coefficient	Std. Error	t-ratio	p-value
SP500	0.173856	0.0542959	3.2020	0.0021
dinfl	7.03558	1.87614	3.7500	0.0004
GSCI_1	0.813304	0.0394863	20.5971	0.0000

Mean dependent var	−0.010112	S.D. dependent var	0.269500
Sum squared resid	0.496064	S.E. of regression	0.084182
R^2	0.905274	Adjusted R^2	0.902568
$F(3, 70)$	222.9916	P-value(F)	9.65e–36
Log-likelihood	78.60761	Akaike criterion	−151.2152
Schwarz criterion	−144.3438	Hannan–Quinn	−148.4768
$\hat{\rho}$	−0.119793	Durbin's h	−1.087260

LM test for autocorrelation up to order 12 –
Null hypothesis: no autocorrelation
Test statistic: LMF = 1.62913
with p-value = $P(F(12, 58) > 1.62913)$ = 0.108659

```
Breusch-Godfrey test for autocorrelation up to order 12
OLS, using observations 2008:05-2014:05 (T = 73)
Dependent variable: uhat

             coefficient   std. error   t-ratio    p-value
  ---------------------------------------------------------
  SP500      -0.0618944    0.0710367    -0.8713    0.3872
  dinfl       1.65489      2.37431       0.6970    0.4886
  GSCI_1      0.0338920    0.0580509     0.5838    0.5616
  uhat_1     -0.124469     0.164143     -0.7583    0.4513
  uhat_2      0.0160500    0.152508      0.1052    0.9165
  uhat_3     -0.111580     0.140176     -0.7960    0.4293
  uhat_4     -0.108847     0.139960     -0.7777    0.4399
  uhat_5      0.173034     0.135852      1.274     0.2079
  uhat_6      0.149434     0.130343      1.146     0.2563
  uhat_7     -0.136355     0.128837     -1.058     0.2943
  uhat_8     -0.228446     0.136237     -1.677     0.0990   *
  uhat_9      0.00359541   0.134938      0.02664   0.9788
  uhat_10     0.154117     0.131051      1.176     0.2444
  uhat_11    -0.363626     0.138877     -2.618     0.0113   **
  uhat_12     0.0181454    0.152357      0.1191    0.9056

  Unadjusted R-squared = 0.252091

Test statistic: LMF = 1.629127,
with p-value = P(F(12,58) > 1.62913) = 0.109

Alternative statistic: TR^2 = 18.402627,
with p-value = P(Chi-square(12) > 18.4026) = 0.104

Ljung-Box Q' = 23.6291,
with p-value = P(Chi-square(12) > 23.6291) = 0.0228
```

We calibrate and run the above models for the period May 2008 to March May 2014. The figures 4.2- 4.7 below show the effectiveness of the models presented. Table 4.1 presents the estimates of the parameters obtained by fitting Cox-Ingersoll-Ross model. These parameters are used in set of equations 4.1. Starting from 15th

TABLE 4.1: Estimates of the CIR model fitted on the risk factors

	Cholesky Matrix			Speed	Level	Sigma
Short Rate	1.1417	0	0	0.8265	0.6885	0.5538
Long Rate	0.6558	0.4583	0	0.6019	3.3989	0.3954
Inflation Rate	0.4665	-0.1847	1.4898	0.01	0.2057	0.0476

May 2014, we simulate rates and prices for risk factors and indexes for the next six months at monthly time step by taking into account history from May 2008. We adopt [1 10 3 3 3 3] tree structure for the scenario generation process. The fan plot shows the distribution of the rates or index prices over the next six months. The red zone in the fan plot is the region where most of the scenarios are realized. The yellow ones have low regional density. The dark black line in the plots is the mean scenario which is defined as the 50th percentile of the distribution of the terminal stage values. The blue dotted line is the actual market dynamics, it extends from 12 data points in the past (one year history) to the next six data points in the future (monthly values).

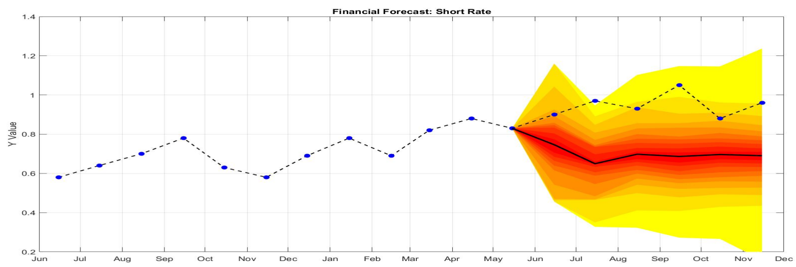

FIGURE 4.2: Short Rate

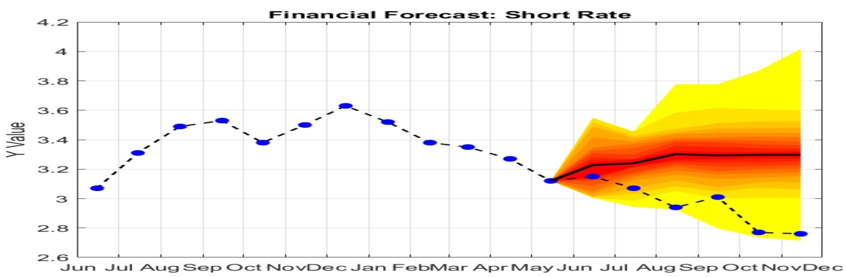

FIGURE 4.3: Long Rate

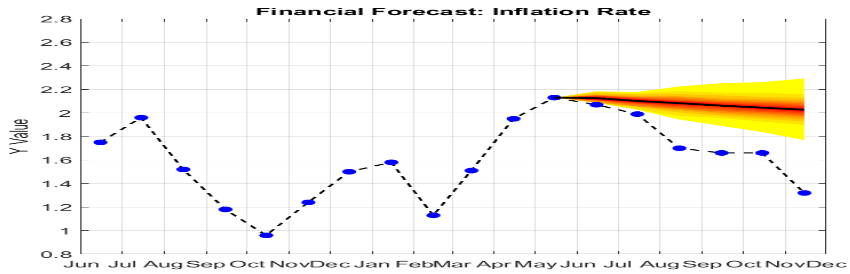

FIGURE 4.4: Inflation Rate

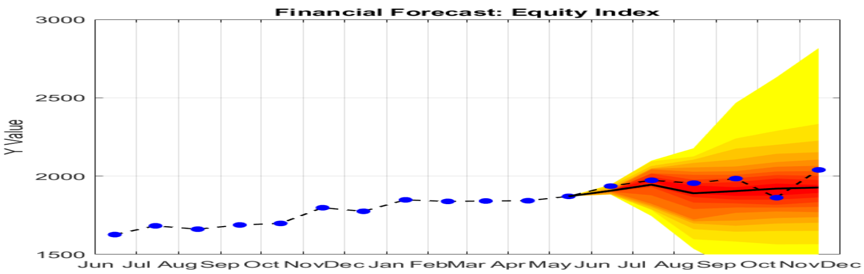

FIGURE 4.5: S&P500 Equity Index

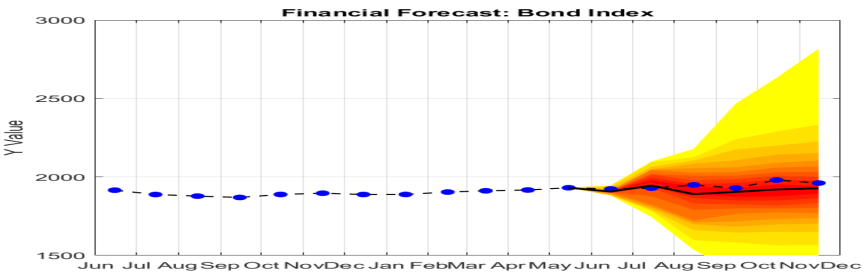

FIGURE 4.6: US-AGG Bond Index

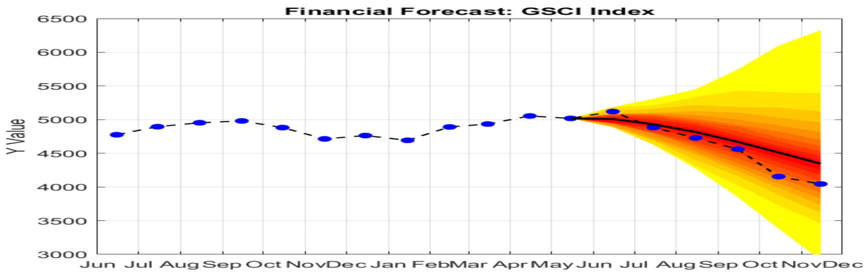

FIGURE 4.7: GSCI Index

What is more relevant from the above figures is that first month value was captured in all the simulations for all the risk factors and as well as for all the indexes.

4.2 Scenario Generation: Options

To price options on a scenario tree has been a challenging and debatable topic for researchers for many years. Starting from the Black-Scholes formulation to price options in a risk neutral environment the literature has expanded to many empirical studies. Several empirical studies show that Black-Scholes mis-price the deep OTM options, Rubinstein, 1985.

Dempster and his collaborators (Dempster, Hutton, and Richards, 1998; Dempster and Hutton, 1999; Dempster and Richards, 2000) developed and tested many techniques to price American options within Black-Scholes framework. So far the volatility had been treated as a constant parameter. The structure of asset prices is then calibrated using binomial and trinomial lattices. Rubinstein, 1994, Jackwerth and Rubinstein, 1996 and Derman, Kani, and Chriss, 1996 worked on these approaches. While in the late 90s when stochastic optimization was becoming popular a little work had been done price derivatives on a multinomial scenario tree.

Topaloglou, Vladimirou, and Zenios, 2008b talked about two techniques for pricing options on a scenario tree. Starting with multinomial tree where an optimization program needs to be run to calculate the risk neutral probabilities. Hence, it gets computationally inefficient as the size of the underlying tree expands. The second method however, is an empirical approach that extends the Black-Scholes framework to take into account higher moments (skewness and kurtosis). So, the calculation of the option prices becomes much simpler. As the price of an option can be expressed in a simple linear equation.

So far in the literature, where stochastic optimization techniques have been used on a portfolio that includes options, either multinomial tree approach has been used or Black-Scholes model or moment matching methods have been used. Most of the researchers, have considered options expiring at the next decision stage and therefore, it becomes less relevant to price option, as there is no worth of the options at the subsequent decision stage, either they expire in-the-money adding cash to the cash account or they expire worthless.

It becomes more relevant to price options in multi-stage stochastic programming framework when options are not expiring at the next stage. As they carry a value that needs to be correctly calculated to avoid any spurious profits or losses.

We use here a simple yet effective approach to price options on a scenario tree. We utilize the Greeks information of options to price them. The Greeks are the quantities representing the sensitivity of the price of derivatives such as options to a change in underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent. We use Delta-Gamma approximation, Estrella and Kambhu, 1997 and it works well for pricing options at one month time step which is consistent with the discrete time steps of planning horizon of our investment planning problem.

In general, in-the-money options will move more than out-of-the-money options, and short-term options will react more than longer-term options to the same price change in the stock. This fluctuation of prices with maturity and moneyness add more complexity to the option pricing. Since, OTM options have low premium, so even a small absolute change in the option price value would reflect a higher change in relative terms. An OTM option trading at \$2.5 if has a positive absolute change of \$0.5 that means it has sored up 20%, while the absolute change is small. Such options can proved to be very lucrative if the price of the underlying security moves in favorable direction. However, pricing of such options on scenario tree can generate spurious profits and loss.

The delta-gamma approximation is used to estimate option price movements if the underlying stock price changes. This approach is better than the delta approximation approach which is linear and since the option price is a non-linear function of the stock price we need to introduce another sensitivity parameter that can tackle this non-linearity. To take account of this we can use gamma to make our option price estimate more precise. Delta-gamma makes our approximation non-linear.

The delta-gamma approximation for call options is can be expressed with the following equation.

$$c(S_{T+1}) = c(S_T) + \Delta(S_T)(S_{T+1} - S_T) + 0.5\Gamma(S_T)(S_{T+1} - S_T)^2 \tag{4.8}$$

where, $\Delta(S_T)$ is the delta of the stock option on the underlying series S at time T

where, $\Gamma(S_T)$ is the gamma of the stock option on the underlying series S at time T

$\Delta(S_T)$ is approximated using Black-Scholes probabilities $d1$ and $d2$, where $N(d1)$ is the delta of the call option and $N(d2)$ is the probability that option would expire in the money.

Gamma of call option is expressed as:

$$\Gamma = K \exp^{-rt} \phi(d2)/(S^2\sigma\sqrt{t}) \tag{4.9}$$

The same formula can also be used for put options, delta of put option is negative, so if the price of the underlying would increase means the price of the put option would decrease. We tested the reliability of this approach on options market data and found it quiet consistent. We present below some results to make it clear for the readers that this approach can be used in scenario tree nodal framework. Then we move on to the nodal formulation of these equation to make it consistent with the optimization models presented earlier.

We test the accuracy of this approach on real market data. The method is applied to call and put options, both in-the-money and out-of-the-money of different strikes and maturities. The figure below shows market price of call and put options and the model price, the second subplot in the figures show tracking error with respect market price. Each bar in the figures is names as 'Call12142000', which corresponds to European type call option expiring in December 2014 that has strike price 2000. The first four characters correspond to type of European option, next two characters correspond month of expiry, then the following two characters correspond to year of expiry and the last four characters are the strike price of the option.

In the figure 4.8, call option prices are predicted on 15th July,2014 for 15th August, 2014. Since, we have planning horizon of six months with monthly rebalancing stages, so it becomes more relevant to check the accuracy of this method at monthly frequency. It can be seen from the figure that options that are deep out-of-the-money have the highest tracking error in price prediction. We define tracking error as current market price less the model price divided by the current market price. As explained earlier, when the price value is small then even a small absolute change would be reflected as a high relative change, this is clearly seen in the OTM options price prediction here. Options trading at USD 2 are predicted to have price around USD 3.5, which is actually 75% more than the market price. While in the case of at-the-money or in-the-money options the tracking error remains quite small. As the price of the option increases accuracy of the delta-gamma price approximation increases.

The same behaviour in price prediction (fig 4.9) is observed in call options price prediction on 25th August, 2014 at 24th September, 2014. Higher the option premium, lower the tracking error. The same is observed with put options prices (figures 4.10 and 4.11). It is interesting to note that in most of the cases tracking error remains lower than 5%, the delta-gamma approximation may be a rough approach to option pricing but still comparable to the approaches discussed by Topaloglou, Vladimirou, and Zenios, 2008b, we observe more error when we price out-of-the-money options.

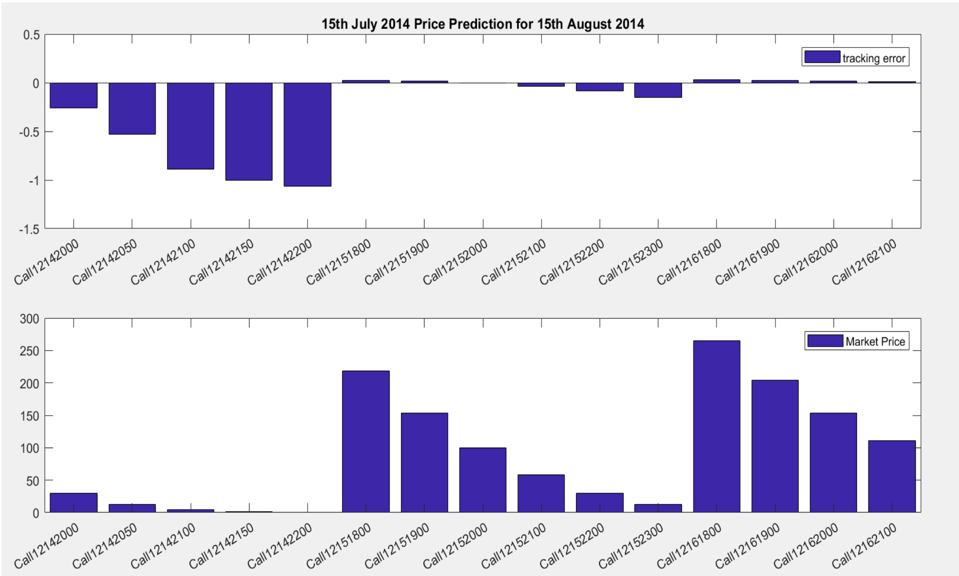

FIGURE 4.8: Call Option Price Prediction

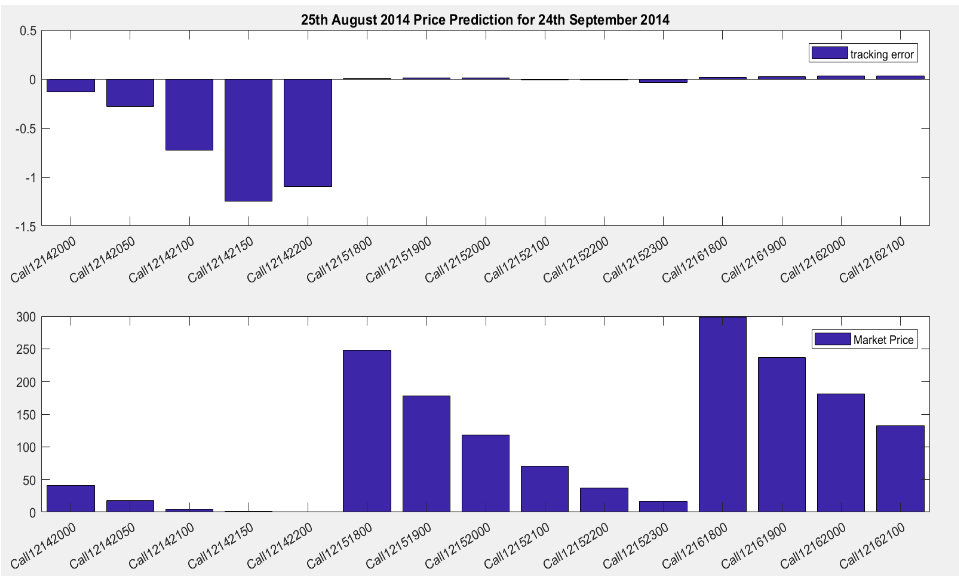

FIGURE 4.9: Call Option Price Prediction

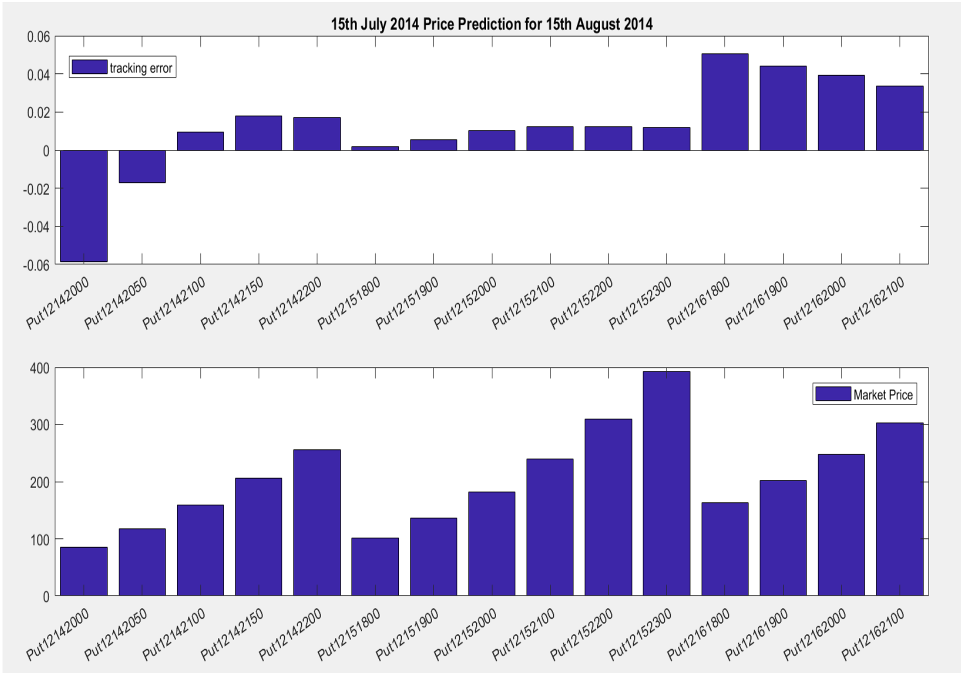

FIGURE 4.10: Put Option Price Prediction

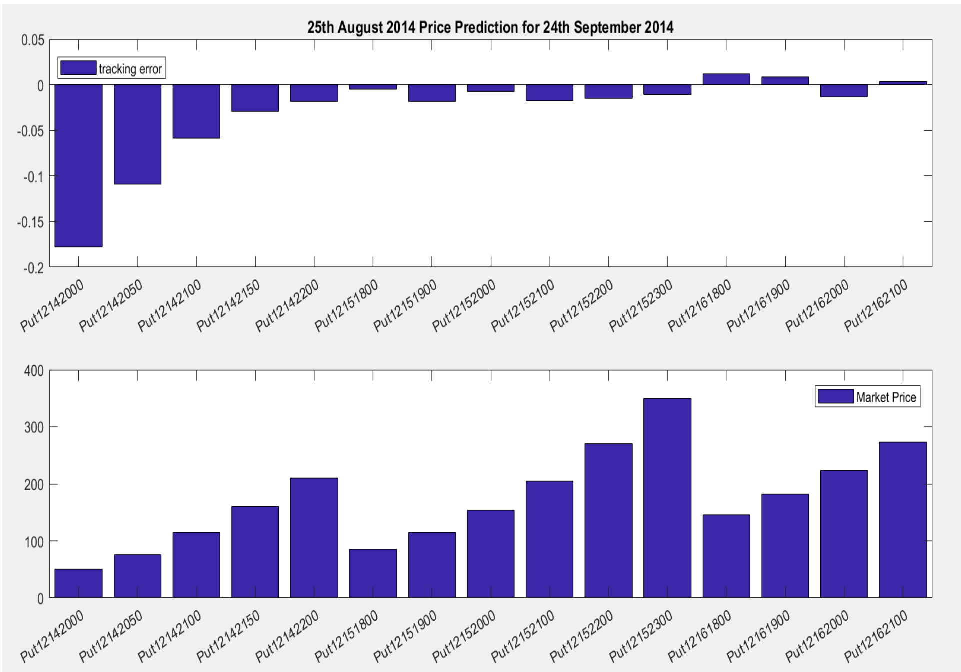

FIGURE 4.11: Put Option Price Prediction

A less relevant analysis in this context is the option price prediction on daily basis, this however, has shown impressive results. Fig 4.12 and Fig 4.13 shows market price and the model price in the upper subplot, on the right y-axis is the performance of the underlying S&P500 index. In the lower subplot, we plot the tracking error and it is found to be within 5% from the actual price. This motivates that shorter the price prediction time step, more accurate the price would be. This also motivates to follow this approach where we have nested simulation in the event tree or when optimization problem has frequent rebalancing stages.

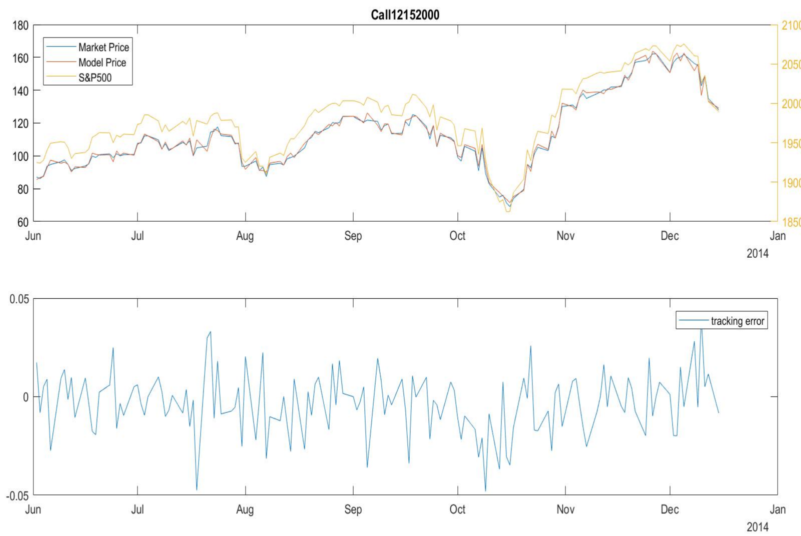

FIGURE 4.12: Daily price prediction using Delta-Gamma Approximation

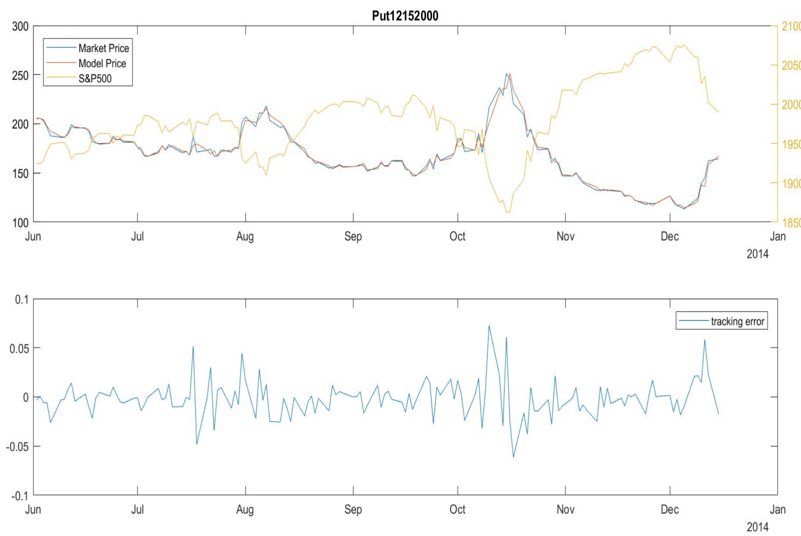

FIGURE 4.13: Daily price prediction using Delta-Gamma Approximation

4.2.1 Nodal Representation of Delta-Gamma Approximation for Option Pricing

It is important to map the delta-gamma approximation in a logical conformable way to the description of the underlying process and the discrete time dynamic stochastic optimization program in order to complete the definition of a stochastic program.

Let $c_{in}(j,k)$ be the price of call option on asset i in node n having maturity j and strike k. Let $\Delta^c_{in}(j,k)$ and $\Gamma^c_{in}(j,k)$ be the delta and gamma Greeks of that option in node n. The equation then can be rewritten as:

$$c_{in}(j,k) = c_{in-}(j,k) + \Delta^c_{in}(j,k)(v_{in} - v_{in-}) + 0.5\Gamma^c_{in}(j,k)(v_{in} - v_{in-})^2 \tag{4.10}$$

Similarly, the equation for pricing put option would be:

$$p_{in}(j,k) = p_{in-}(j,k) + \Delta^p_{in}(j,k)(v_{in} - v_{in-}) + 0.5\Gamma^p_{in}(j,k)(v_{in} - v_{in-})^2 \tag{4.11}$$

4.3 Arbitrage Free Pricing and In-sample Stability Analysis

It is important to check the scenarios for arbitrage opportunities. Any such opportunity would generate spurious profits in the wealth distribution obtained from the optimization program, hence, making it difficult to analyse actual realised returns, we may see unrealistic gains in the that make no sense in the real world. No-arbitrage scenario generation has been discussed by many researchers. Some famous studies were, Klaassen, 1997,Klaassen, 1998, Dupačová, Consigli, and Wallace, 2000,Høyland and Wallace, 2001 and Consiglio, Carollo, and Zenios, 2016.

We adopt Klaassen, 2002 approach for precluding arbitrage opportunities in multi-stage scenario tree. In Ingersoll, 1987, two types of arbitrage opportunities are discussed, *type 1*, where it is possible to construct a zero investment portfolio that has non-negative payoff in all states of the world and *type 2* arbitrage opportunities, where we construct a portfolio with negative wealth and end up with non-negative payoff in at least one state of the world. The inclusion or exclusion of type 1 arbitrage opportunity does not imply inclusion or exclusion of type 2 arbitrage opportunity. Hence, both opportunities should be checked while generating scenario trees. Klaassen, 2002 presented a single check by which it is possible to check both type of arbitrage opportunities at the same time. If the set of equations 4.12 has a strictly positive solution then arbitrage opportunities of type 1 and type 2 do not exist, where N is the number of children nodes at the subsequent decision stage, R_n is the return in children nodes. If there exists a solution $(X > 0)$ to this system of linear equation then scenario tree between between time t and $t+1$ is arbitrage free. It is important to note that the method assumes equally probable scenarios in the future states of the world and this may not be the case in real world. Consiglio, Carollo, and Zenios, 2016, discussed two types of system of equations where they considered scenarios with equal and different probabilities. We do not get into this approach and limit ourselves to equally probable scenarios and adopt Klaassen, 2002 approach.

The set of equations 4.12, guarantees arbitrage free scenario tree from time t to $t+1$. In multi-stage setting, this needs to be checked for all the stages before the horizon.

The methodology we follow is shown in the pseudo-code 1. We generate a scenario tree at first node by generating risk factors and the price of the underlying securities discussed in section 4.1, we check for arbitrage opportunities, if exist, we generate the scenario tree again until we get arbitrage free scenario tree. Once we obtain an arbitrage free scenario tree at the first node, we proceed to the next node and generate arbitrage free for its children nodes. We do this until we have populated all the nodes of the scenario tree for all the underlying assets. Once, we have generated underlying asset scenario tree, we can compute option prices in all the nodes using delta-gamma approximation discussed earlier. Another important aspect here is to discuss any arbitrage opportunities due to options in the investment universe. This can be checked while solving the optimization problem, we set no constraints on the investment in options and if this gives us an unbounded solution then it means arbitrage opportunities exist.

$$\sum_{n=1}^{N} v_n(1 + R^n_{i,t+1}) = 1, \ \forall\, i = 1, 2, 3.. \tag{4.12}$$

Algorithm 1 Arbitrage-free scenario tree generation (Underlying assets)

given $\alpha^j, \omega^{j,*}, \sigma^j$
given $\omega^j(0)$ and $B^k(0)$ $\forall j = 1, 2, 3$ and $k = 1, 2, 3$
$\mathcal{N}_{0,T-1}$: Set of nodes from stage 1 to stage T-1
procedure NOARBITRAGETREE($\mathcal{N}_{0,T}$)

    for each item n in $\mathcal{N}_{0,T-1}$ **do**
        $N \leftarrow$ children nodes of n
        while No arbitrage **do**
            given $\omega^j(n-)$ and $B^k(n-)$ $\forall j = 1, 2, 3$ and $k = 1, 2, 3$
            compute $\omega^j(n)$ and $B^k(n)$ $\forall j = 1, 2, 3$ and $k = 1, 2, 3$
            Solve equations 4.12
            if $X > 0$ **then**
                No arbitrage = True
            end if
        end while
    end for
end procedure

4.3.1 In-sample Stability

In optimization framework arbitrage free event trees are required to produce realistic results by avoiding any spurious profits arising from arbitrage opportunities. However, solution to these optimization problem depends on the distribution of the scenario tree. Different distribution can give different results, therefore, we must generate enough scenarios such that going beyond a certain number of scenarios the optimal solution does not change significantly. Hence, in-sample stability of these models is required.

We run the optimization program for different number of scenarios. Investment universe includes S&P 500 equity index, US Aggregate bond index, S&P GSCI commodity index and at-the-money call and put options on equity index, expiring in the

TABLE 4.2: Value of the objective function vs number of scenarios

Scenarios	Model Status	Value of the Objective Function	% Change in Obj Function Value
972	1	37318	
1215	1	37536	0.005824688
1458	1	38274	0.019470342
1701	1	37714	0.014739436
1944	1	37726	0.000318134
2187	1	37635	0.002415043

subsequent decision stages. We test the model discussed in section 2.3.1.

Table 4.2 shows how the value of the objective function changes with the number of scenarios. We adopt [1 4 3 3 3 3 3] branching structure. At each iteration, we increase the value of the first stage branching degree by 1. First column shows the number of scenarios, increasing from 972 to 2187 scenarios. Second column has got model status, its value equal to 1 means optimal solution has been achieved. Third column has the value of the objective function at optimality and the last column has the percentage change in the value of the objective function (wrt previous value) when we increase the number of scenarios. As we move from 972 number of scenarios to 2187, we see that value of the objective function does not change much when we reach 1701 number of scenarios. At 2187 scenarios, value of the objective function has changed 0.2% and this change is acceptable given a reasonably large number of scenarios. Next, we run the optimization program 50 times for different set of scenarios with the branching structure [1,9,3,3,3,3,3].

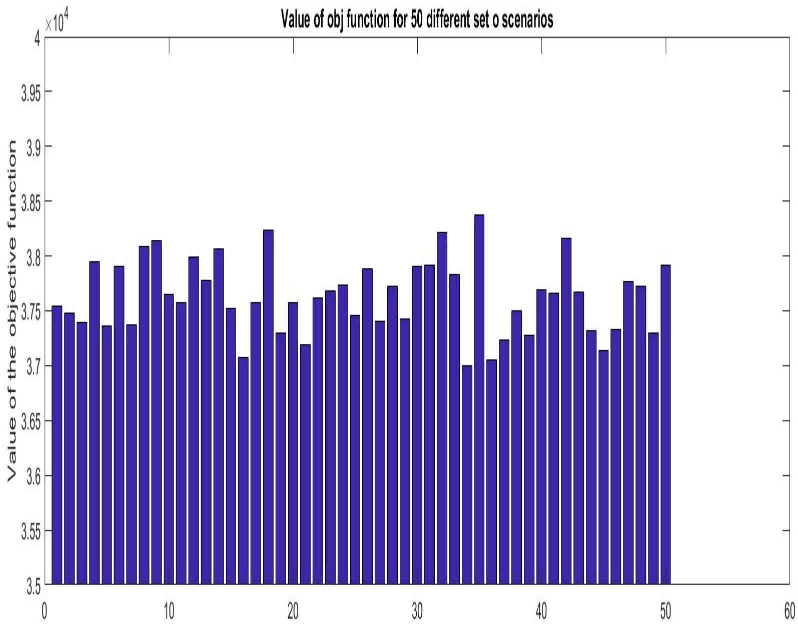

FIGURE 4.14: Value of the objective function

Figure 4.14 shows the value of the objective function for 50 different set of scenarios. It is found that the optimal value does not change significantly. Figure 4.15

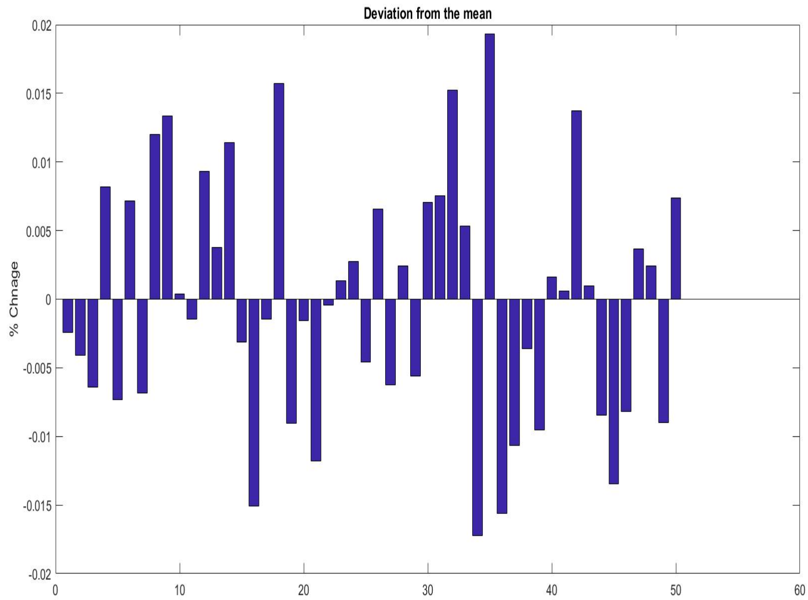

FIGURE 4.15: Value of the objective function

shows the deviation of each of the optimal objective function value from the mean of the 50 objective function value. It can be seen that the deviation is within the range of 2% and it confirms that in-sample results are pretty stable.

In the next chapter, we discuss results of the optimization models discussed in the previous chapter using the scenario generation techniques discussed here.

Chapter 5

Numerical Results

In this chapter we present results generated from the models discussed in the previous chapters, we collect some evidences from them. We start with the model where options are expiring in the next stage. We study the in-sample results of this model, then we check its performance on three year data (June 2014 - March 2017, in out-of-sample analysis case). We then present results for the model where options selling is allowed before their expiry and for the model where short position in options contracts are considered. Finally, we produce results for the inventory update model using derivatives and close the chapter with the summary of this research work and its possible extensions in future.

5.1 Multistage Model-Options Expiring at the next stage

We first present results of the multi-stage model where options are assumed to be expiring on the next decision stage. We model the problem from the perspective of an investor who wishes to maximize his wealth at the end of planning horizon which is six months. Investor is open to invest in equity, bond, commodity and options on equity index. We use the multi-stage model discussed in chapter 2. We go by doing the statistical analysis to generate scenario tree, branching structure considered for the scenario tree is [1 9 3 3 3 3 3], that generates 2187 scenarios for equity, bond, commodity and options. The model used for pricing these securities are discussed in chapter 4. Target set at the end of six months is 110000 USD. The objective function used is maximization of wealth penalized by a risk measure(expected average shortfall), risk aversion coefficient equal to 0.5. We use monthly data to develop statistical models. The available cash at the beginning is 100000$ and there is no position in any security. Data from May 2008-March 2017, is collected from Datastream5.1. We divide the data into two parts to carry out the in-sample and out-sample analysis. We truncate the data at May 2014 to see what adopted strategies have yielded in the future data. All the computation is done on Matlab 2013a and GAMS 2.7 platforms.

We consider a simple case where only equity and equity options are there in the portfolio. We see how the wealth distribution changes when we adopt different strategies. We test long straddle, strap, strip and protective put strategy. These strategies have been discussed by Topaloglou, Vladimirou, and Zenios, 2011 in stochastic programming framework. The model here we adopt is the one discussed in section 2.3.1, which is equivalent to the model presented by Yin and Han, 2013b. Table 5.1 shows the portfolio composition in the worst case scenario. We constraint the number of call and put options in our portfolio to not be more than the units of underlying equity index.

TABLE 5.1: Equity and long straddle on equity options

Month	1	2	3	4	5	6	7
Cash	0	4.59	9802.78	0	0	0	1847.2
Equity	99980.00131	99975.4133	90176.23932	99976.1059	99975.9037	99975.39972	98128.01214
Call Option	596.843014	601.8528273	542.1821531	613.1618504	607.1774831	609.0012182	0
Put Option	554.7901763	549.9272523	496.6752633	539.1974147	544.9805089	543.2367488	0
Wealth	101131.6345	101131.7834	101017.8767	101128.4652	101128.0617	101127.6377	99975.21214

Table 5.1 shows the portfolio composition in the worst scenario, we identify worst scenario corresponding to 0th percentile of wealth distribution at the leaf nodes (terminal stage). The first row in the table is the cash held at each decision stage, the second row gives the position in equity, 3rd and 4th row give position in call and put option, respectively, and the last row is the wealth of the portfolio over time. Month 0 correspond to decision taken at time 0, then subsequent decision stages are followed at one month frequency. We observe from the table that wealth decreases by 0.025% in the worst scenario over six months time, i.e. in the worst case if we are not able to make then at least we are not loosing a significant part of the wealth. Let us also have a look at the distribution of wealth at the final stage. We plot cumulative distribution function and histogram to see the frequency distribution. Fig 5.1 and 5.2 show CDF and histogram plot respectively.

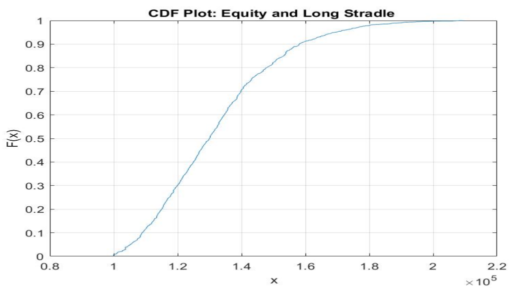

FIGURE 5.1: CDF plot: Equity Index and Long Straddle

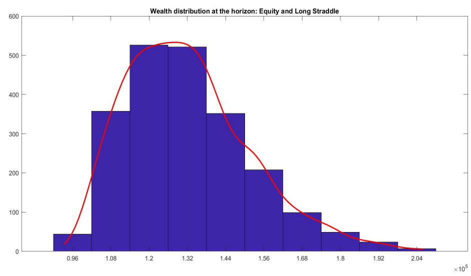

FIGURE 5.2: Wealth Distribution: Equity Index and Long Straddle

The distribution shows profit in most of the cases at horizon, maximum loss observed was 0.025%, however the probability of this event is very small. Most of the scenarios are centred around 25% gain. Distribution show more upside potential than downside risk. We present here next portfolio composition and wealth corresponding distribution and histograms in different cases, when strap, strip and protective put strategies are considered.

TABLE 5.2: Equity and strap strategy on equity options

Month	1	2	3	4	5	6	7
Cash	0	4.59	9802.78	0	0	0	1847.2
Equity	99980.00131	99975.4133	90176.23932	99976.1059	99975.9037	99975.39972	98128.01214
Call Option	1193.686028	1203.705655	1084.364306	1226.423516	1214.256813	1217.905601	0
Put Option	554.7901763	549.9272523	496.6752633	539.1974147	544.9805089	543.2367488	0
Wealth	101728.4775	101733.6362	101560.0589	101741.7268	101735.141	101736.5421	99975.21214

TABLE 5.3: Equity and strip strategy on equity options

Month	1	2	3	4	5	6	7
Cash	0	0	0	0	0	0	266.89
Equity	99980.00131	100118.1976	104033.0241	104901.5653	105248.5099	110088.7424	109955.2816
Call Option	1193.686028	1224.945622	1260.171555	1265.481166	1262.063972	1322.614478	0
Put Option	1109.580353	1083.059334	1137.222063	1151.8253	1162.857949	1213.891629	0
Wealth	102283.2677	102426.2025	106430.4177	107318.8718	107673.4318	112625.2485	110222.1716

TABLE 5.4: Equity and protective put strategy on equity index

Month	1	2	3	4	5	6	7
Cash	0	4.59	9802.78	0	0	0	1847.2
Equity	99980.00131	99975.4133	90176.23932	99976.1059	99975.9037	99975.39972	98128.01214
Call Option	0	0	0	0	0	0	0
Put Option	554.7901763	549.9272523	496.6752633	539.1974147	544.9805089	543.2367488	0
Wealth	100534.7915	100529.9306	100475.6946	100515.3033	100520.8842	100518.6365	99975.21214

It can be seen from the figures 5.1-5.8, that wealth distribution in each case is more concentrated in the upper region where wealth is higher than the starting wealth level which is 100000. The probability of facing losses is small. Almost, in all the scenarios wealth is seen to be increasing, may be because of arbitrage, we discuss later. We present worst case scenario for protective put case. We observe a decrease in wealth by 0.025%, it is however not less in magnitude as compared to other cases where loss was around 0.025% in the worst case. A protective put strategy is a sort of insurance against any down-trending market. Maximum loss in this strategy is the net premium paid to buy put option. Next, we plot distributions in various cases. Interestingly, when we double the number of put option as compared to call option (strip strategy) we see significant changes in the portfolio wealth in the worst case. In the worst case, when equity market is bearish, put options are proving profitable. We present a comparative analysis of all these strategies in fig 5.9 and 5.10.

Figure 5.9 and 5.10 summarize the four strategies, it is found that on the same set of scenarios protective put strategy has the least risk but at the same time it has the lowest profitability. Other strategies have outperformed protective put, strip gives the best returns in most of the scenarios, however, it is a scenario dependent phenomenon, had scenario of equity been increasing then strip was going to perform better. It is evident from the plots that protective put does give good protection but

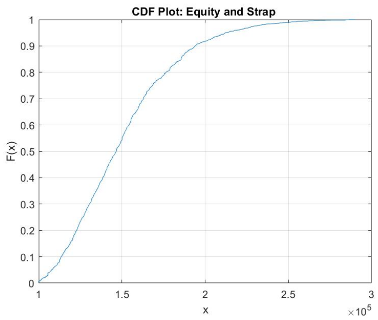

FIGURE 5.3: CDF plot: Equity and Strap Strategy

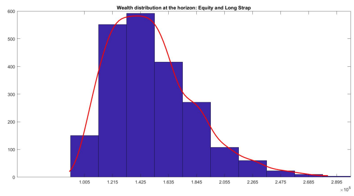

FIGURE 5.4: Histogram kernel fit:Equity and Strap Strategy

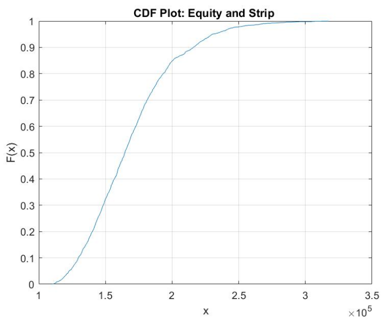

FIGURE 5.5: CDF plot: Equity and Strip Strategy

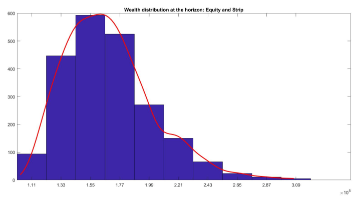

FIGURE 5.6: Histogram kernel fit:Equity and Strip Strategy

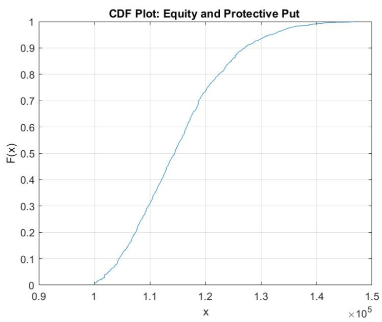

FIGURE 5.7: CDF plot: Equity and Protective Put

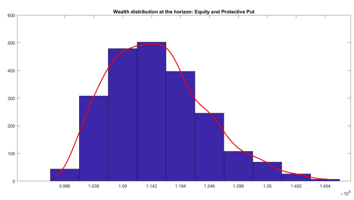

FIGURE 5.8: Histogram kernel fit:Equity and Protective Put

limits the upside potential, hence a mix of strategies could be more profitable here.

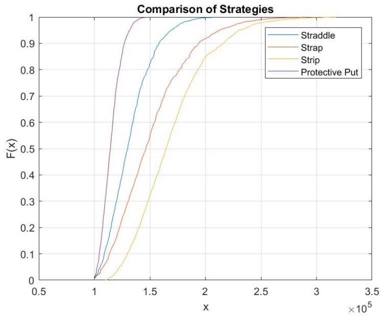

FIGURE 5.9: CDF plot for all the strategies on a portfolio of equity and equity options

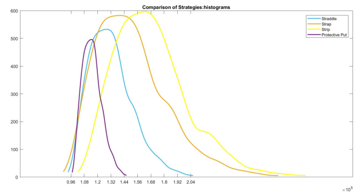

FIGURE 5.10: Histogram kernel fitting for all the strategies on a portfolio of equity and equity options

Now, we analyse a portfolio where we have equity, bond and commodity indexes and options available on equity index. We consider minimum 60% fixed income securities in our portfolio, commodities are capped at 15% and equity is free take any value between 0 to 40%. The cash account is also set free, there is no minimum cash requirement or liquidity issues. Table 5.5 shows the portfolio composition in the worst scenario where no options were included in the portfolio. Figure 5.11 and 5.12 plot the wealth distributions. We then consider cases where options are considered in the portfolio through straddle, strap, strip and protective put strategy using call and put options on equity index.

TABLE 5.5: Portfolio of equity, bond and commodity indexes (Worst scenario)

Month	1	2	3	4	5	6	7
Cash	0	24503.9	0	0	0	0	0
Equity	39980.00837	0	0	0	39953.15358	0	0
Bond	60000.00862	58809.34899	97674.37646	99230.25875	59929.73094	99025.86802	98921.67495
Commodity	0	14702.36145	0	0	0	0	0
Call Option	0	0	0	0	0	0	0
Put Option	0	0	0	0	0	0	0
Wealth	99980.017	98015.61044	97674.37646	99230.25875	99882.88452	99025.86802	98921.67495

TABLE 5.6: Equity, Bond, Commodity and equity options straddle strategy (worst scenario)

Month	1	2	3	4	5	6	7
Cash	0	0	0	0	0	103.54	0
Equity	39980.00837	40475.54377	0	0	0	37386.07574	35200.85535
Bond	60000.00862	60713.32762	98850.87892	99804.42491	59956.76313	61503.08741	60783.29137
Commodity	0	0	0	0	0	0	0
Call Option	238.6701948	239.8675324	0	0	0	227.1170084	0
Put Option	221.8537812	226.2348624	0	0	0	203.666453	0
Wealth	100440.541	101654.9738	98850.87892	99804.42491	59956.76313	99423.48661	95984.14672

TABLE 5.7: Equity, Bond, Commodity and Strap Strategy on equity options (worst scenario)

Month	1	2	3	4	5	6	7
Cash	0	0	0	0	0	103.73	0
Equity	39980.00837	40549.94618	0	0	0	37454.78275	35265.54643
Bond	60000.00862	60824.92141	99032.56676	99987.86537	60066.95793	61616.12253	60895.00359
Commodity	0	0	0	0	0	0	0
Call Option	477.3403896	480.7514527	0	0	0	455.1034566	0
Put Option	221.8537812	226.6609168	0	0	0	204.0996016	0
Wealth	100679.2112	102082.28	99032.56676	99987.86537	60066.95793	99833.83834	96160.55002

TABLE 5.8: Equity, bond, commodity and strip strategy on equity options (worst scenario)

Month	1	2	3	4	5	6	7
Cash	0	0	0	0	0	197.96	0
Equity	39980.00837	40475.54377	39754.18136	39498.36149	40332.52468	35740.20808	33651.18884
Bond	60000.00862	60713.32762	59631.2688	59247.53969	60498.79243	67109.34652	66323.93812
Commodity	0	0	0	0	0	0	0
Call Option	238.6701948	239.8675324	239.5366994	239.340033	242.44296	217.1667524	0
Put Option	443.7075624	452.5762384	436.8645321	431.5268514	444.3539371	389.4005556	0
Wealth	100662.3948	101881.3152	100061.8514	99416.76807	101518.114	103654.0819	99975.12696

TABLE 5.9: Equit, Bond, Commodity and Protective Put strategy using equity options (worst scenario)

Month	1	2	3	4	5	6	7
Cash	0	0	0	0	0	105.22	0
Equity	39980.00837	40401.1604	0	0	39897.7177	37992.44971	35771.78669
Bond	60000.00862	60601.73383	98669.19108	99620.98445	59846.56833	62500.63061	61769.15991
Commodity	0	0	0	0	0	0	0
Call Option	0	0	0	0	0	0	0
Put Option	221.8537812	225.8088081	0	0	219.7500106	206.9583821	0
Wealth	100201.8708	101228.703	98669.19108	99620.98445	99964.03604	100805.2587	97540.9466

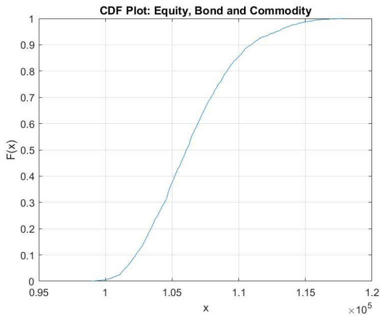

FIGURE 5.11: CDF: Equity, Bond & Commodity

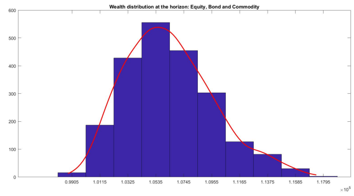

FIGURE 5.12: Hist: Equity, Bond & Commodity

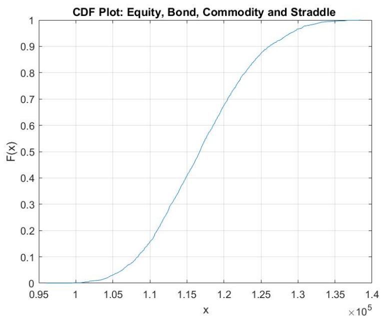

FIGURE 5.13: CDF: Equity, Bond, Commodity & Straddle Strategy on Equity Options

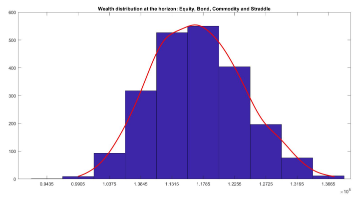

FIGURE 5.14: Hist: Equity, Bond, Commodity & Straddle Strategy on Equity Options

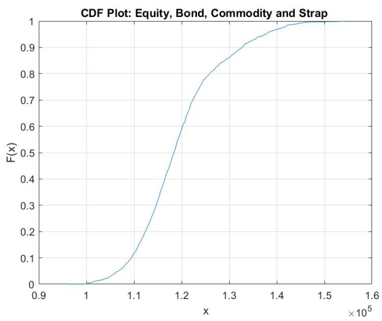

FIGURE 5.15: CDF: Equity, Bond, Commodity & Strap Strategy on Equity Options

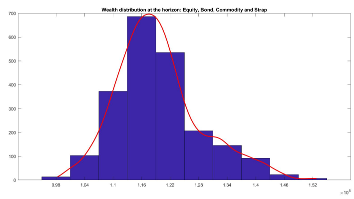

FIGURE 5.16: Hist: Equity, Bond, Commodity & Strap Strategy on Equity Options

Figures 5.11 -5.20 show optimal wealth distribution when options are used through straddle, strap, strip and protective put strategy in the portfolio. In all the distributions, wealth is increasing significantly at the planning horizon. However, this may not be the case in reality, we talk about this in the later part of this chapter. We also present a comparative analysis of all these strategies. Figures 5.21 and 5.22 show the wealth distribution in various strategies. It can be clearly seen that no option portfolio is trailing behind all other portfolios, while maintaining a close gap with protective put strategy. While the left tail of the no-option portfolio is longer than protective put portfolio. Other strategies however, have shown a different but expected behaviour. Straddle, Strip and Strap tend to outperform the other two portfolios. It is to be noted that these three strategies have longer left tails than the protective put strategy. It is because of the fact that these strategies work under certain market conditions. For instance, a straddle would have profit potential if the market experiences high volatility. If not, the investment in options to achieve a straddle strategy

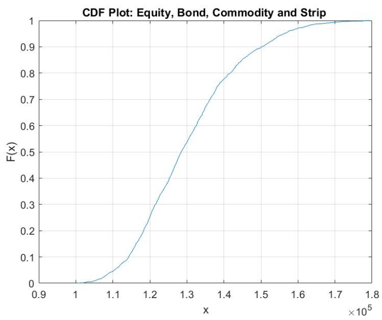

FIGURE 5.17: CDF: Equity, Bond, Commodity & Strip Strategy on Equity Options

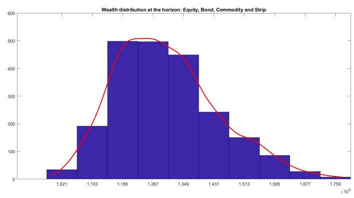

FIGURE 5.18: Hist: Equity, Bond, Commodity & Strip Strategy on Equity Options

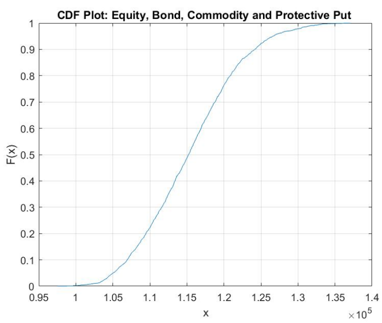

FIGURE 5.19: CDF: Equity, Bond, Commodity & Protective Put Strategy on Equity Options

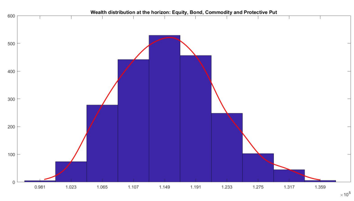

FIGURE 5.20: Hist: Equity, Bond, Commodity & Protective Put Strategy on Equity Options

would be worthless. The same applies to other strategies. It is to be noted that options are considered only on equity index, whilst we have other two securities in our portfolio as well. Considering options on individual security would definitely yield better results. Since, the equity fraction in this portfolio is capped at 40%, the effects of including options are minor but significant from risk management perspective.

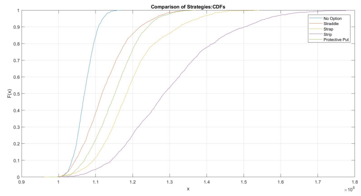

FIGURE 5.21: CDF plot: Equity, Bond, Commodity & Options Portfolio (options expiring at the next decision stage)

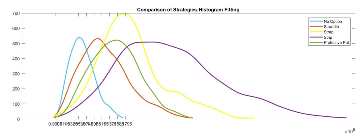

FIGURE 5.22: Histogram kernel fitting: Equity, Bond, Commodity & Options Portfolio (options expiring at the next decision stage)

Having seen the performance of this model on simulated data, its now time to check the out-of-sample performance of the multistage model where options are expiring at the next decision stage. Keeping the above constraints and initial parameters same, we run this algorithm for the period May 2014- March 2017. Figure 5.23 shows the portfolio performance using different strategies.

We test straddle, strip, strap and protective put strategies, we report their performance along with the equity, bond and fixed-mix strategy (where equity constitutes 25%, bond, 60% and commodity 15% of the portfolio). We also add one more strategy where we allow speculation on call options on the basis of event trees generated

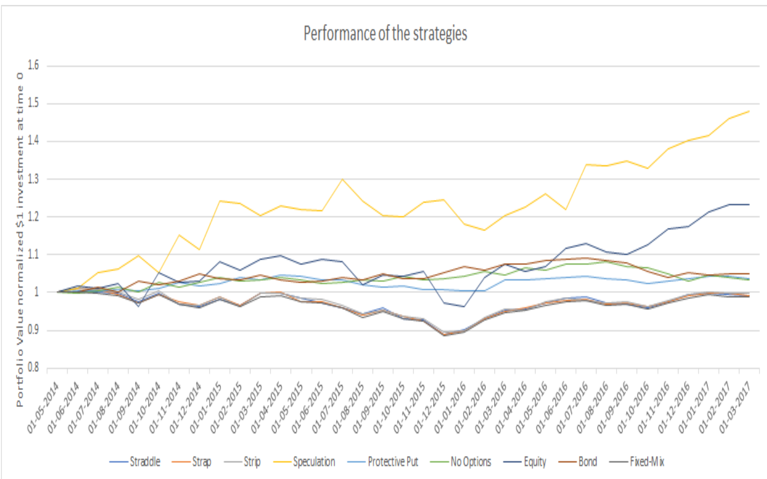

FIGURE 5.23: Out-of-sample: Portfolio performance of Equity, Bond, Commodity and Equity Options

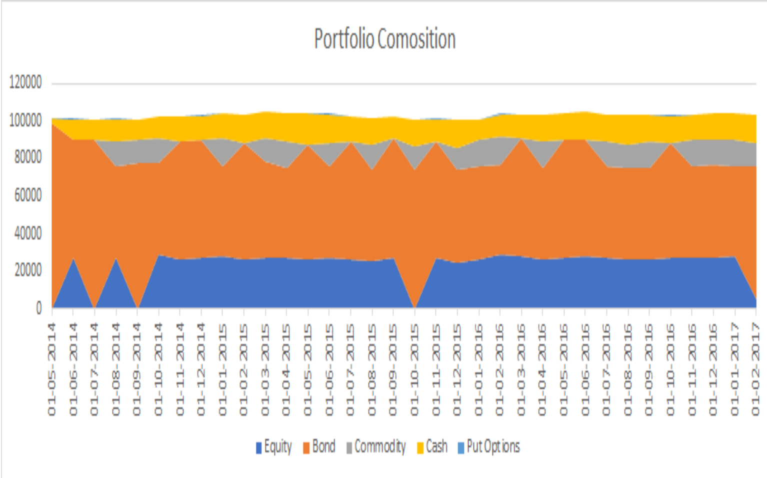

FIGURE 5.24: Out-of-sample: Portfolio composition of Equity, Bond, Commodity and Equity Options

at each rebalancing stage, we restrict investment in options at 1 basis point wealth of the portfolio.

Call options when used for speculation seem more profitable but also make portfolio more volatile. Portfolio where call options were bought for speculation has more volatility than the volatility of the equity index. However, the returns of this portfolio are correlated to the equity returns. Showing that, even a small amount of wealth used in call option made the portfolio more exposed to its underlying. While, bond index has minimum 60% weight in the portfolio, the returns are seen to be far from fixed income benchmark. Please see figure 5.24. On the other hand, protective put strategy tends to give good protection to the portfolio. The wealth level over three years remain more or less the same, it increased due to price increase of the underlying instruments. Speculation through options can have multi-fold advantages for traders depending on the risk-reward profile of the traders/investors. Figure 5.24 shows the portfolio composition in the protective put strategy. The option amount is not reflected in the chart, as it is really small compared to the investments in equity, bond and commodity indexes. It is important to note that straddle, strip and strap strategy have not given positive returns as found in the in-sample analyses. It is because of the fact that call and put options were not bought at every decision stage, a decision is made on the basis of forecasted data on the planning horizon and it doesn't seem to be very accurate forecast here. These strategies were also discussed by Topaloglou, Vladimirou, and Zenios, 2011 and they reported positive performance of the portfolio using these strategies.

5.2 Multistage model, Trading long options positions

So far, we have discussed models for optimization where options were expiring at the subsequent decision stage, in this section we are going to present results on trading long position on options, we have already discussed and validated a model for this in chapter 2. We assume the same initial conditions as in the cases discussed earlier in this chapter. The only dimension added is that the investor is now able sell options before their expiry. For this we consider that ATM equity options of 1 month, 3 month and 6 month expiry are available at time 0. We run the optimization model using the scenario generation models discussed in the previous chapters. We consider three different cases to understand the buy/sell effects on the portfolio. To study this it is important to allow portfolio to invest in options at different levels. Since, we have some position in the equity, we assume that only those many options contracts (both call and put) can be traded as the number of underlying equity units in the portfolio, that's the first case, in the second case, we increase this number to 5 times and then in the third case we increase this number to 10 times. Table 5.10 shows the portfolio composition in the mean scenario when as many options contracts are available for trading as the units of equity in the portfolio. Table 5.11 and 5.12 show the 2nd and 3rd case respectively. Figures 5.25 - 5.30 plot wealth distributions in these three cases.

TABLE 5.10: Options Buy/Sell (Mean Scenario)

Month	1	2	3	4	5	6	7
Cash	100000	40029.83	0	0	106293.49	0	0
Equity	0	0	39920.52577	0	0	41187.28114	42249.15858
Bond	0	59850.47151	59924.89573	105331.7368	0	64894.04202	65839.46927
Commodity	0	0	0	0	0	0	0
Call_Equity_1Month	0	0	0	0	0	0	0
Call_Equity_3Month	0	0	0	0	0	0	0
Call_Equity_6Month	0	0	29.40630226	0	0	0	0
Put_Equity_1Month	0	0	0	0	0	0	0
Put_Equity_3Month	0	0	0	0	0	0	0
Put_Equity_6Month	0	0	0	0	0	0	0
Wealth	100000	99880.30151	99874.8278	105331.7368	106293.49	106081.3232	108088.6279

TABLE 5.11: options Buy Sell (constrained to less than 5 times of underlying units)

Month	1	2	3	4	5	6	7
Cash	0	0	0	0	0	0	0
Equity	37332.21308	37664.17547	41838.16268	44136.91916	36361.9765	32336.69082	31057.21995
Bond	59880.2356	58978.59533	62922.42605	67679.59641	73901.04421	80073.52605	79334.22409
Commodity	0	0	0	0	0	0	0
Call_Equity_1Month	0	0	0	0	0	0	0
Call_Equity_3Month	2587.95272	0	0	0	0	0	0
Call_Equity_6Month	0	0	110.1298039	0	0	0	0
Put_Equity_1Month	0	0	0	0	0	0	0
Put_Equity_3Month	0	1654.898604	0	0	0	0	0
Put_Equity_6Month	0	0	0	982.8007893	12905.39231	12524.02817	18988.81613
Wealth	99800.4014	98297.6694	104870.7185	112799.3164	123168.413	124934.245	129380.2602

TABLE 5.12: Options Buy Sell (constrained to less than 10 times of underlying units)

Month	1	2	3	4	5	6	7
Cash	0	0	0	0	0	0	0
Equity	35059.37354	43164.46825	47218.32734	46833.79426	43651.20269	34762.22767	35058.12733
Bond	59880.2356	65048.86604	72149.96867	75857.7575	82027.63057	97525.48715	98014.29226
Commodity	0	0	0	0	0	0	0
Call_Equity_1Month	0	0	0	0	0	0	0
Call_Equity_3Month	4860.789229	0	0	0	0	0	0
Call_Equity_6Month	0	0	881.6431919	3738.045668	11033.88732	20246.50842	23000.1167
Put_Equity_1Month	0	0	0	0	0	0	0
Put_Equity_3Month	0	201.4413795	0	0	0	0	0
Put_Equity_6Month	0	0	0	0	0	0	0
Wealth	99800.39837	108414.7757	120249.9392	126429.5974	136712.7206	152534.2232	156072.5363

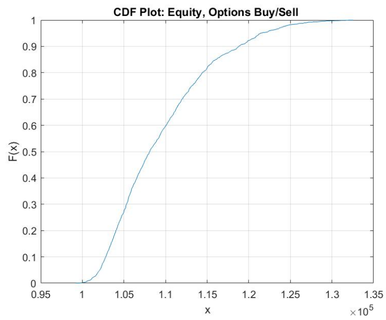

FIGURE 5.25: CDF plot: Equity, Bond, Commodity and Options of different expiries

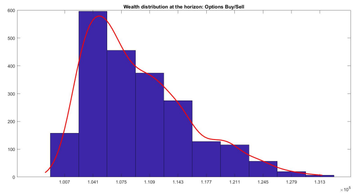

FIGURE 5.26: Hist: Equity, Bond, Commodity and options of different expiries

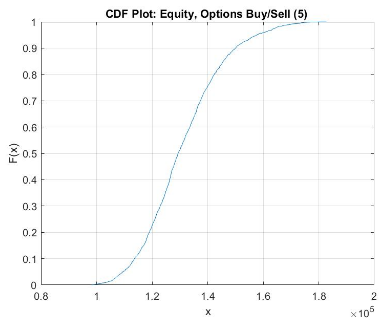

FIGURE 5.27: CDF plot: Equity, Bond, Commodity & trading long positions in options

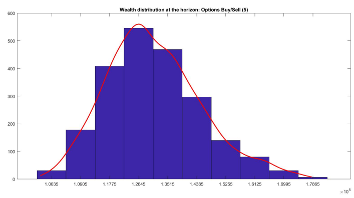

FIGURE 5.28: Histogram fitting: Equity, Bond, Commodity & long Options (5x underlying)

From the tables 5.10-5.12 we see that wealth increases sharply as we increase the amount invested in options. In the first case, a return of 9% was observed, in the second case when the amount invested in options increased to five times, the profit rises to 29% and finally in the third case it hits 56%. It should be noted that when we are increasing the investment in options by 5 or 10 times, the absolute change is investment is still relatively small compared to the portfolio value. In the three cases, higher volatility is observed with the increase in options investments. We now see the comparative results of these cases.

We plot CDFs for four different cases, when there are no options in the portfolio and other three cases are where option trading is allowed and we gradually increase the investment amount in that case. Wealth distribution plot of no-option portfolio is clearly trailing by other portfolios where options are allowed to buy/sell before expiry. As we increase the number of options available for trade the wealth distribution stretches towards right side, significantly. On the simulated data, it has outperformed the previous model where options were bought and exercised at the very next decision stage. Please see figure 5.32. Interestingly, the left tail of the distributions also move towards right. However, high investment in options is seen to make portfolios more volatile. This is evident from the wealth trajectory in the mean scenarios.

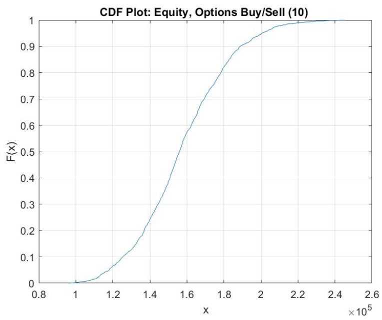

FIGURE 5.29: CDF: Equity, Bond, Commodity & Long Options (10x underlying)

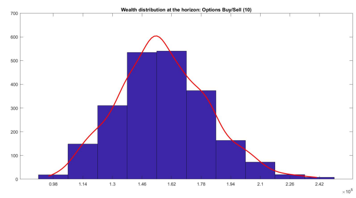

FIGURE 5.30: Hist: Equity, Bond, Commodity & Long Options

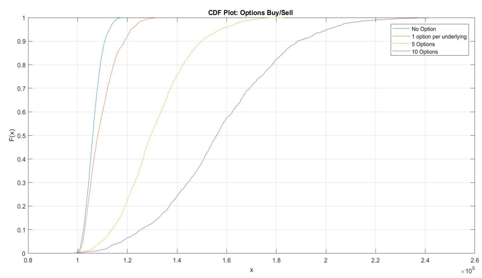

FIGURE 5.31: CDF Plots: Portfolio of Equity, Bond, Commodity and Long Options

5.3 Multistage model- Trading short options positions

In this section, we present results where we run a multistage model that takes into account short selling of options contracts. Since, short selling of options can lead to unlimited loses, so we restrict ourself from entering naked short positions on options contracts. Therefore, we implement strategies like bull call spread (as discussed in chapter 2) and put bear spread. We keep all the initial conditions same as in the cases discussed earlier in this chapter. Since, we test the model for different levels of moneyness to achieve bull call spreads. We take 5%, 10% and 15% ITM and OTM options to achieve bull call spread strategy.

Table 5.5 shows performance of the portfolio when no options are available for long/short selling. Table 5.13 shows portfolio composition and performance when call bull spread strategy is considered with options 15% moneyness levels, we build a call bull spread strategy using ITM and OTM options that are 15% in-the-money and 15% out-of-the-money. Figure 5.33 and 5.34 show wealth distributions in various cases. It is observed in all the cases that when options included in the portfolio its performance has improves significantly. Portfolio return in mean scenario is increasing as the moneyness level of ITM and OTM options increases. At the same time, volatility of the portfolio increases with increase in moneyness levels of options, it is evident from the fact that OTM options have high volatility in prices. We present a comparative analysis of all these strategies, figure 5.33 and 5.34 show wealth distribution of various strategies, portfolio without options, bull call spread of 5-10-15 % moneyness levels and portfolio that has access to both bull call spread and put bear spread.

TABLE 5.13: Portfolio of equity, bond, commodity and call bull spread on equity options, moneyness of the options 15% (mean scenario)

Month	1	2	3	4	5	6	7
Cash	499.42	0	0	0	0	0	0
Equity	34539.93204	0	0	35031.8528	35045.36272	33457.23263	35146.99602
Bond	59581.18928	99470.19898	99247.52086	59914.09494	60911.45718	66090.49009	66152.98496
Commodity	0	0	0	0	0	0	0
Call Bull Spread	5180.861736	0	0	4910.875948	5562.287721	7265.203518	8954.981897
Wealth	99801.40305	99470.19898	99247.52086	99856.82369	101519.1076	106812.9262	110254.9629

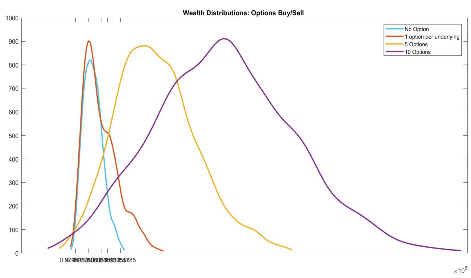

FIGURE 5.32: Wealth Distributions: Portfolio of Equity, Bond, Commodity and Long Options

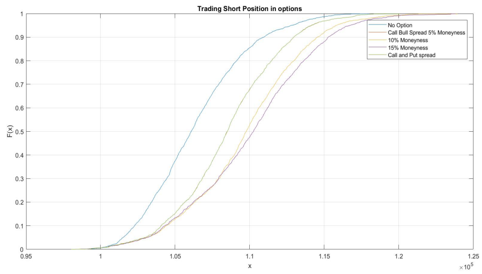

FIGURE 5.33: CDF Plots: Portfolio of Equity, Bond, Commodity and Long and Short position in Options through bull and bear spreads

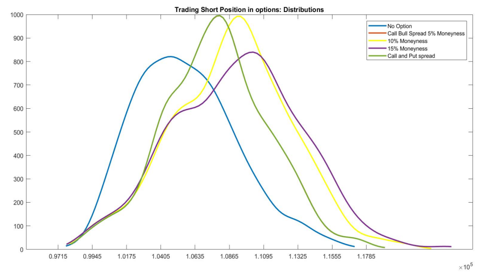

FIGURE 5.34: Wealth Distribution: Portfolio of Equity, Bond, Commodity and Options

All the portfolios with options outperform the portfolio that has no access to trade options. A clear pattern is observed in the CDF plot, when we increase the moneyness levels of the options, distribution curves shift towards right. In the strategy, where put bear spread and call bear spread both are considered have outperformed all the strategies. As it gives access to vertical spreads on both call and put options and hence able to make profit out of bullish and bearish market conditions.

5.4 Multi-stage Model to update inventory using options

In this section, we present results for the inventory update model. We consider a problem where investor is willing to increase his holdings in equity at a price lower than market price. His planning horizon is six months, monthly rebalancing is allowed on the portfolio. We apply the multistage model developed in chapter 4. We run the algorithm for the period May 2014- March 2017 and see how it has performed.

Figure 5.35 shows the months where algorithm achieved to update inventory using options. The y-axis is the buying cost with respect to the market price, WAP is the weighted average price of the equity index, as we have two possibilities to buy the equity index, one directly at the market price, second, by exercising call options. WAP is the weighted average price of the two. It can be seen from the figure that sometimes, algorithm bought equity at 6% lower than the market price. Maximum, saving observed was 10% in December'14. On average, 1.75% less price was paid compared to the market price. We have considered call option in this case study, profitability of this strategy relies on market momentum. If the market is bullish, it is going to be profitable.

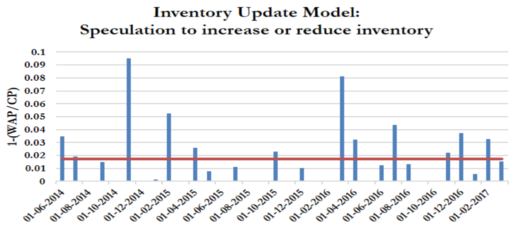

FIGURE 5.35: Inventory Update using Options

We include GAMS code for various cases in the appendix, readers may refer to it. Appendix A shows model that is used for optimizing portfolio of stocks, bonds, commodity with options available on equity that are expiring at the next decision stage, GAMS code for buying/selling of long options contracts and GAMS code for physical settled options contracts to update inventory.

Conclusion and Future Research

In this research, we discussed multi-stage stochastic programming techniques, cons and pros of including options in a portfolio and the combination of two. We reviewed the recent developments in the field of stochastic programming and options in general. Our motivation for the study was to explore both cash and physical settled options contracts within the framework of stochastic programming. We thoroughly reviewed work done Blomvall and Lindberg, 2003, Topaloglou, Vladimirou, and Zenios, 2011 and Yin and Han, 2013b and Davari-Ardakani, Aminnayeri, and Seifi, 2016 who developed single and multiple stage models to include options in a portfolio. We take some big steps in extending the work presented by these researchers. So, far in the literature only cash settled options contracts were considered, we extended it to physical settled contracts, where we actually show through out-of-sample results that inventory can actually be updated using call and options, depending on the objective of the investor.

We explored cash settled options contracts in detail. Other researchers had only considered buying and exercising features of the options, while we extend the literature to trade both long and short positions on call and put options. We have showed it through multiple models in this research report. We start with a single stage model where options expire at the horizon, then we extend the model to consider options expiring in the very next decision stage, a model similar to the model developed by Yin and Han, 2013b. Then we extend the model to consider options that are expiring at any decision stage along the planning horizon.

Difficulty to introduce such models lies in options pricing on scenario tree, many researchers have expressed their concern with this. We reviewed the work done by Topaloglou, Vladimirou, and Zenios, 2008b to price options on a scenario tree, where they talk about multinomial tree and numerical extension of Black-Scholes formula. To address this issue of option pricing on scenario tree, we relied on relatively simple technique. We introduce delta-gamma approximation for option pricing, as options payoff are non-linear, the non-linearity is captured by the gamma of the options in pricing formula, the technique sounds naive but has performed well in this context.

Once, we had an option pricing technique on scenario tree, we went on introducing more complexities to stochastic optimization models to play with options contracts. We then extended the model where we allow sell decision variables on options inventory, giving more degree of freedom to investors/traders. We then finally present a model we consider short selling of options, something that has not been touched by any researcher, to the best of our knowledge. We call it a generic model and we verify the model theoretically, how it reduced to multi-stage models or single stage model with options by setting a few variables equal to 0. We also present numerical validation for each model, where we actually verify that desired objective is achieved through equation implemented in the models.

Finally, we present some numerical results collected from these models. To run the models, we needed a statistical model for the underlying, we referred to Consigli et al., 2012 for equity and bond index model and we rely on econometrics technique to develop a first order autoregressive distributed lag model where we considered inflation, equity and performance of the bond index as exogenous variables. On the basis of these models we generate scenarios for the optimization program and collect some evidences.

Our research finds that options can increase profitability potential of a portfolio by many-fold, if included in a portfolio in the right amount. Options in all the portfolio optimization models have shown good performance, portfolio with options have outperformed portfolio without options. Protective put has provided protection to portfolio, while speculating through options have generated huge profits, at the same time they made the portfolio more volatile. Hence, it is a still a question if there can ba an optimal amount that can be assigned for speculation through options. We have seen in out out-of-sample results that minor speculation through options outperformed all other strategies significantly.

We also saw that options premiums are very lucrative and buying and selling them before expiry could be a profitable strategy. We saw that as we allocate more wealth for buying and selling in options our profit increases by up t0 50%, while the amount that is invested in the options is still relatively small compared to the portfolio value. We then finally introduce bull call spread and bear put spread strategies through general model that we developed. As we have mentioned before, we avoid taking naked short positions in options. We implemented vertical spread strategies and found that they have performed quite well compared to the portfolio where no options were considered. We considered both call bull spread and put bear spread and saw how well the portfolio performed, as it was protected from shocks in the market in either direction. In summary, our finding is that options should be included in a portfolio to improve profitability and to provide protection to portfolios.

We have taken some big steps forward from the existing literature, however, there are still many questions remain to be answered. Contribution of this PhD dissertation is in the modelling of dynamic stochastic programming models that can tackle options of different expiries and maturities. Most of the results that we presented here are in-sample results and we do not discuss any arbitrage opportunities arising from including options in the portfolio. In-sample results presented make money in almost all the scenarios, it is because of the strict constraints we are using on various asset classes and options, as a result, the in-sample results were not in sync with the out-sample results we observed. The other side of the problem is proper pricing of options which we have not touched in this thesis. The model we adopt should relate the volatility of the underlying process to the option pricing. This was not in the scope of this dissertation, so, we aim to accomplish that in our future research and we expect that with good option pricing method (such as the one recently discussed by Barkhagen and Blomvall, 2016) we should see more realistic results on the out-sample data.

Appendix A

GAMS Codes

GAMS code for options expiring at the next stage

```
$SET OnDebug N

$ONEMPTY

* OPTIONS
$EOLCOM //
$INLINECOM /* */

$INCLUDE "AllSets.inc";
$INCLUDE "Policy.inc";
$INCLUDE "Constants.inc";

alias (Tree,p,p2);
Node(Tree)=Yes;

alias (Node,n,n2,m);
set scenario(Tree);
alias (Scen, s);
alias (Stage, t);
alias (classSet,k);
alias (Security, i,j);

$INCLUDE "Price.inc";
$INCLUDE "StageofNode.inc";
$INCLUDE "NodeAncestor.inc";
$INCLUDE "NPM.inc";
$INCLUDE "Init_asset.inc";
$INCLUDE "class.inc";
$INCLUDE "Probability.inc";
$INCLUDE "Constants.inc";
$INCLUDE "delta_call.inc";
$INCLUDE "delta_put.inc";
Scalar Stage_Max;
    Stage_Max=smax(p,StageofNode(p));

* Define some sets
Set root(tree), leaf(tree);
root(tree)$(Ord(tree)=1) = Yes;
leaf(tree)$(StageOfNode(tree) eq Stage_Max) = Yes;

Set SpecGrade(i), InvGrade(i),equity(i),bond(i),gsci(i),callgrade(i),putgrade(i);
SpecGrade(i)$(class(i) ge 4)=Yes;
InvGrade(i)$(class(i) le 3)=Yes;
equity(i)$(class(i)eq 1)=Yes;
bond(i)$(class(i)eq 2)=Yes;
gsci(i)$(class(i)eq 3)=Yes;
callgrade(i)$(class(i)eq 4)=Yes;
putgrade(i)$(class(i)eq 5)=Yes;

scalar NFO;
NFO=2;

*scalar CapIniz;
```

```
*CapIniz = 100000;

* Initial Wealth definition
Parameter Wealth_0(tree);

        Wealth_0(root)= CapIniz + sum(i,Init_asset(i)*Price(i,root));

POSITIVE VARIABLES
 buy(p, i)
 sell(p, i)
 hold_asset(p, i)
 hold_debt(p)
 cash_hold(p)
 debt(p)
 debt_minus(p)
 debt_plus(p)
 shortfall
 nc(p)
 np(p)
 beta(p)

VARIABLES
   Expected_Wealth
   Wealth(p)
   reward
   risk
   xyz(p)

**** Objective function value ********************
   z          Objective function value type

EQUATIONS
   Inventory_Asset_Balance_Eq_0
   Inventory_Liab_Balance_Eq_0
   Cash_Flow_Balance_Eq_0
   Inventory_Asset_Balance_Eq
   Inventory_Liab_Balance_Eq
   Cash_Flow_Balance_Eq
   Def_Wealth
   Def_Expected_Wealth

*   Portfolio composition constraint
   Security_maximum_speculativ
   Security_minimum_speculativ
   Security_maximum_inv_grade
   Security_minimum_inv_grade
   expected_shortfall
   Cash_max
*    policy1
*    policy2
    policy3
    policy4
   equity_min
   equity_max
   bond_min
```

```
   bond_max
   gsci_min
   gsci_max


   Def_risk
   Def_reward
   ObjDef Objective function definition type;

* Inventory balance (nominal terms)
*** Asset Balance

Inventory_Asset_Balance_Eq_0(i,root)      ..
    hold_asset(root,i) =e= Init_asset(i)
    + buy(root,i) - sell(root,i);

Scalar chi_plus;
chi_plus=0.0002;
Scalar chi_minus;
chi_minus = 0.0002;

display i;

*nc.FX(n)=0;
*np.FX(n)=0;
buy.FX(n,callgrade(i))=0;
buy.FX(n,putgrade(i))=0;
sell.FX(n,callgrade(i))=0;
sell.FX(n,putgrade(i))=0;
*buy.FX(n,bond(i))=0;
*buy.FX(n,gsci(i))=0;
*sell.FX(n,bond(i))=0;
*sell.FX(n,gsci(i))=0;

* cash flow balance
Cash_Flow_Balance_Eq_0(root) ..
cash_hold(root)=e= CapIniz
- sum(i, buy(root,i)*Price(i,root)*(1+chi_plus))
+ sum(i, sell(root,i)*Price(i,root)*(1-chi_minus))
-sum(i$(class(i) eq 4), nc(root)*Price(i,root)*(1+0.0001))
-sum(i$(class(i) eq 5), np(root)*Price(i,root)*(1+0.0001));

********************************
*    next stage equations     *
********************************
Alias (parent, p, child);
Set anc(parent,child);
anc(parent,child) = Yes $(Ord(child) > 1 And Ord(parent) = NodeAncestor(child));

Inventory_Asset_Balance_Eq(i,anc(m,n))$(not root(n)) ..
hold_asset(n, i) =e= (hold_asset(m, i)+ buy(n, i) - sell(n, i) );

Scalar riskfree_rate;
riskfree_rate=0;
Scalar Sstep;
```

```
Sstep=1/12;



Cash_Flow_Balance_Eq(anc(m,n)) ..
cash_hold(n) =e= cash_hold(m)*(1+riskfree_rate*Sstep)
-sum(i, buy(n, i)*Price(i,n )*(1+chi_plus))
+sum(i, sell(n, i)*Price(i,n)*(1-chi_minus))
+sum(i$(class(i) eq 1), (nc(m))*delta_call(n)*(Price(i,n)-Price(i,m))*(1-0.0001))
+sum(i$(class(i) eq 1), (np(m))*delta_put(n)*(Price(i,m)-Price(i,n))*(1-0.0001))
-sum(i$(class(i) eq 4), nc(n)*Price(i,n)*(1+0.0001))
-sum(i$(class(i) eq 5), np(n)*Price(i,n)*(1+0.0001));


Wealth.FX(root) = Wealth_0(root);
*Def_Wealth(anc(m,n)) ..
Def_Wealth(anc(m,n)) ..
Wealth(n) =e= cash_hold(n) + sum(i, hold_asset(n,i)*Price(i,n))
+sum(i$(class(i) eq 4), nc(n)*Price(i,n))
+sum(i$(class(i) eq 5), np(n)*Price(i,n));

* Last Stage Constraints
buy.FX(leaf(n),i) = 0;
sell.FX(leaf(n),i) = 0;
nc.FX(leaf(n))=0;
np.FX(leaf(n))=0;
beta.FX(leaf(n))=0;

*Security_minimum_speculativ(n) ..  sum(SpecGrade(i), hold_asset(n,i)*Price(i,n)) =G= 0*sum(i, hold_asset(n,i)*Price(i,n));
*Security_maximum_speculativ(n) ..  sum(SpecGrade(i), hold_asset(n,i)*Price(i,n)) =L= 1*sum(i, hold_asset(n,i)*Price(i,n));

*Security_minimum_inv_grade(n) ..  sum(InvGrade(i), hold_asset(n,i)*Price(i,n)) =G= 0*sum(i, hold_asset(n,i)*Price(i,n));
*Security_maximum_inv_grade(n) ..  sum(InvGrade(i), hold_asset(n,i)*Price(i,n)) =L= 1*sum(i, hold_asset(n,i)*Price(i,n));

equity_min(n) ..  sum(equity(i), hold_asset(n,i)*Price(i,n)) =G= 0.0*Wealth(n);
equity_max(n) ..  sum(equity(i), hold_asset(n,i)*Price(i,n)) =L= 1*Wealth(n);

bond_min(n) ..  sum(bond(i), hold_asset(n,i)*Price(i,n)) =G= 0.6*Wealth(n);
bond_max(n) ..  sum(bond(i), hold_asset(n,i)*Price(i,n)) =L= 1*Wealth(n);

gsci_min(n) ..  sum(gsci(i), hold_asset(n,i)*Price(i,n)) =G= 0*Wealth(n);
gsci_max(n) ..  sum(gsci(i), hold_asset(n,i)*Price(i,n)) =L= 0.15*Wealth(n);

*policy1(n) .. sum(i$(class(i) eq 4), nc(n)*Price(i,n))=L= 0.025*Wealth(n);
*policy2(n) .. sum(i$(class(i) eq 5), np(n)*Price(i,n))=e= 0.025*Wealth(n);
*policy1(n) .. sum(i$(class(i) eq 4), nc(n)*Price(i,n))+sum(i$(class(i) eq 5), np(n)*Price(i,n))=g= 0.0*sum(equity(i),
hold_asset(n,i)*Price(i,n));
*policy2(n) .. sum(i$(class(i) eq 4), nc(n)*Price(i,n))+sum(i$(class(i) eq 5), np(n)*Price(i,n))=l= 1*sum(equity(i),
hold_asset(n,i)*Price(i,n));

policy4(n) .. np(n) =l= 1*sum(equity(i), hold_asset(n,i));
policy3(n) .. nc(n)=e=1*np(n);

Scalar Target;
Target = 110000;
```

```
expected_shortfall .. shortfall =g= sum(leaf(n), Probability(leaf)*(Target - Wealth(leaf)));


Cash_max(n) .. cash_hold(n) =l= 1*Wealth(n);


* Expected Final Wealth definition
Def_Expected_Wealth  .. Expected_Wealth =e=
sum(leaf(n), Wealth(leaf)*Probability (leaf));

Def_risk .. risk=e=shortfall;

Def_reward .. reward=e=(Expected_Wealth);

Scalar Lambda;
Lambda = 0.5;

ObjDef  .. z  =e= (1-Lambda)*(reward) - Lambda *risk;

MODEL Derivatives_basic /
   Inventory_Asset_Balance_Eq_0, Cash_Flow_Balance_Eq_0,
   Inventory_Asset_Balance_Eq, Cash_Flow_Balance_Eq,
   Def_Wealth,Def_Expected_Wealth,
*   Security_maximum_speculativ,
*   Security_minimum_speculativ,
*   Security_maximum_inv_grade,
*   Security_minimum_inv_grade,
   expected_shortfall,
   cash_max,
   Def_risk,Def_reward,
   equity_min,
   equity_max,
   bond_min,
   bond_max,
   gsci_min,
   gsci_max,

*   policy1,
*policy2,
policy4,
policy3,
   ObjDef
/;
```

GAMS code for options expiring at any stage

```
$SET OnDebug N

$ONEMPTY
$OFFDIGIT

* OPTIONS
$EOLCOM //
$INLINECOM /* */
```

```
$INCLUDE "AllSets.inc";
$INCLUDE "Policy.inc";
$INCLUDE "Constants.inc";

alias (Tree,p,p2);
Node(Tree)=Yes;

alias (Node,n,n2,m);
set scenario(Tree);
alias (Scen, s);
alias (Stage, t);
alias (classSet,k);
alias (Security, i,j);

$INCLUDE "Price.inc";
$INCLUDE "StageofNode.inc";
$INCLUDE "NodeAncestor.inc";
$INCLUDE "NPM.inc";
$INCLUDE "Init_asset.inc";
$INCLUDE "class.inc";
$INCLUDE "Probability.inc";
$INCLUDE "Constants.inc";
*$INCLUDE "delta_call_equity.inc";
*$INCLUDE "delta_put_equity.inc";
*$INCLUDE "delta_call_bond.inc";
*$INCLUDE "delta_put_bond.inc";
*$INCLUDE "delta_call_gsci.inc";
*$INCLUDE "delta_put_gsci.inc";

Scalar Stage_Max;
    Stage_Max=smax(p,StageofNode(p));

* Define some sets
Set root(tree), leaf(tree);
root(tree)$(Ord(tree)=1) = Yes;
leaf(tree)$(StageOfNode(tree) eq Stage_Max) = Yes;

Set SpecGrade(i), InvGrade(i), equity(i),bond(i),gsci(i),putgrade(i),callgrade(i),unput(i);
SpecGrade(i)$(class(i) ge 4)=Yes;
InvGrade(i)$(class(i) le 3)=Yes;
equity(i)$(class(i)eq 1)=Yes;
bond(i)$(class(i)eq 2)=Yes;
gsci(i)$(class(i)eq 3)=Yes;
putgrade(i)$(class(i) ge 7)=Yes;
callgrade(i) = SpecGrade(i)-putgrade(i);
unput(i) = (InvGrade(i) + putgrade(i));

display SpecGrade, InvGrade;

Scalar CapIniz;
CapIniz = 100000;

* Initial Wealth definition
Parameter Wealth_0(tree);
```

```
      Wealth_0(root)= CapIniz + sum(i,Init_asset(i)*Price(i,root));

POSITIVE VARIABLES
 buy(p, i)
 sell(p, i)
 hold_asset(p, i)
 hold_debt(p)
 cash_hold(p)
 debt(p)
 debt_minus(p)
 debt_plus(p)
 shortfall
 nc(p)
 np(p)
 beta(p)
 buys(p,i)
 sells(p,i)



VARIABLES
   Expected_Wealth
   Wealth(p)
   reward
   risk
   xyz(p)


**** Objective function value ********************
   z          Objective function value type

EQUATIONS
   Inventory_Asset_Balance_Eq_0
   Inventory_Short_0
   Inventory_Short_Eq
   Inventory_Liab_Balance_Eq_0
   Cash_Flow_Balance_Eq_0
   Inventory_Asset_Balance_Eq
   Inventory_Liab_Balance_Eq
   Cash_Flow_Balance_Eq
   Def_Wealth
   Def_Expected_Wealth

*  Portfolio composition constraint
   Security_maximum_speculativ
   Security_minimum_speculativ
*   Security_maximum_inv_grade
*   Security_minimum_inv_grade
   expected_shortfall
   Cash_max
   policy
   equity_min
   equity_max
   bond_min
```

```
  bond_max
  gsci_min
  gsci_max

  policy1
  policy2
  policy3
  policy4

  Def_risk
  Def_reward
  ObjDef Objective function definition type;

* Inventory balance (nominal terms)
*** Asset Balance

*if(Price(i,root)=0,
*           buy(root,i)=0;
*);

Inventory_Asset_Balance_Eq_0(i,root)      ..
    hold_asset(root,i) =e= Init_asset(i)
    + buy(root,i) - sell(root,i);
* + sells(root,SpecGrade(i));


*Inventory_Short_0(SpecGrade(i),root)              ..
*hold_asset(root,i) =e= Init_asset(i) + buys(root,i) - sells(root,i);


Scalar chi_plus;
chi_plus=0.002;
Scalar chi_minus;
chi_minus = 0.002;

nc.FX(n)=0;
np.FX(n)=0;
*buy.FX(n,SpecGrade(i))=0;
*sell.FX(n,SpecGrade(i))=0;

* cash flow balance
Cash_Flow_Balance_Eq_0(root) ..
cash_hold(root)=e= CapIniz
- sum(i, buy(root,i)*Price(i,root)*(1+chi_plus))
+ sum(i, sell(root,i)*Price(i,root)*(1-chi_minus));


Alias (parent, p, child);
Set anc(parent,child);
anc(parent,child) = Yes $(Ord(child) > 1 And Ord(parent) = NodeAncestor(child));

Loop( anc(m,n),
       hold_asset.FX(m,i)$(Price(i,n) eq 0)=0;
       buy.FX(m,i)$(Price(i,n) eq 0)=0;
);
```

```
Inventory_Asset_Balance_Eq(i,anc(m,n))$(not root(n)) ..
hold_asset(n, i) =e= (hold_asset(m, i)+ buy(n, i) - sell(n, i) );


Scalar riskfree_rate;
riskfree_rate=0;
Scalar Sstep;
Sstep=1/12;



Cash_Flow_Balance_Eq(anc(m,n)) ..
cash_hold(n) =e= cash_hold(m)*(1+riskfree_rate*Sstep)
-sum(i, buy(n,i)*Price(i,n )*(1+chi_plus))
+sum(i, sell(n,i)*Price(i,n)*(1-chi_minus));


Wealth.FX(root) = Wealth_0(root);
*Def_Wealth(anc(m,n)) ..
Def_Wealth(anc(m,n)) ..
Wealth(n) =e= cash_hold(n) + sum(i, (hold_asset(n,i))*Price(i,n));

* Last Stage Constraints
buy.FX(leaf(n),i) = 0;
sell.FX(leaf(n),i) = 0;
nc.FX(leaf(n))=0;
np.FX(leaf(n))=0;
beta.FX(leaf(n))=0;

*Security_minimum_speculativ(n) ..  sum(SpecGrade(i), hold_asset(n,i)) =G= 0*sum(i$(class(i) eq 1), hold_asset(n,i));
*Security_maximum_speculativ(n) ..  sum(SpecGrade(i), hold_asset(n,i)) =L= 0.01*sum(i$(class(i) eq 1), hold_asset(n,i));

*Security_minimum_inv_grade(n) ..  sum(InvGrade(i), hold_asset(n,i)*Price(i,n)) =G= 0*sum(i, hold_asset(n,i)*Price(i,n));
*Security_maximum_inv_grade(n) ..  sum(InvGrade(i), hold_asset(n,i)*Price(i,n)) =L= 1*sum(i, hold_asset(n,i)*Price(i,n));

equity_min(n) ..  sum(equity(i), hold_asset(n,i)*Price(i,n)) =G= 0*sum(i, hold_asset(n,i)*Price(i,n));
equity_max(n) ..  sum(equity(i), hold_asset(n,i)*Price(i,n)) =L= 1*sum(i, hold_asset(n,i)*Price(i,n));

bond_min(n) ..  sum(bond(i), hold_asset(n,i)*Price(i,n)) =G= 0.6*sum(i, hold_asset(n,i)*Price(i,n));
bond_max(n) ..  sum(bond(i), hold_asset(n,i)*Price(i,n)) =L= 1*sum(i, hold_asset(n,i)*Price(i,n));

gsci_min(n) ..  sum(gsci(i), hold_asset(n,i)*Price(i,n)) =G= 0*sum(i, hold_asset(n,i)*Price(i,n));
gsci_max(n) ..  sum(gsci(i), hold_asset(n,i)*Price(i,n)) =L= 0.15*sum(i, hold_asset(n,i)*Price(i,n));

*policy(n) .. sum(SpecGrade(i), hold_asset(n,i))=L= 1*sum(i$(class(i) eq 1), hold_asset(n,i));
policy(n) .. sum(i$(class(i) ge 4), hold_asset(n,i))=L= 1*sum(i$(class(i) eq 1), hold_asset(n,i));
*policy(n) .. sum(SpecGrade(i), hold_asset(n,i)*Price(i,n))=L= 0.0001*Wealth(n);


Scalar Target;
Target = 110000;

expected_shortfall .. shortfall =g= sum(leaf(n), Probability(leaf)*(Target - Wealth(leaf)));
Cash_max(n) .. cash_hold(n) =l= 1*Wealth(n);
```

```
* Expected Final Wealth definition
Def_Expected_Wealth  .. Expected_Wealth =e=
sum(leaf(n), Wealth(leaf)*Probability (leaf));

Def_risk .. risk=e=shortfall;

Def_reward .. reward=e=(Expected_Wealth);

Scalar Lambda;
Lambda = 0.5;

ObjDef  .. z  =e= (1-Lambda)*(reward) - Lambda *risk;

MODEL Derivatives_basic /
   Inventory_Asset_Balance_Eq_0, Cash_Flow_Balance_Eq_0,
   Inventory_Asset_Balance_Eq, Cash_Flow_Balance_Eq,
   Def_Wealth,Def_Expected_Wealth,
*Inventory_Short_0,
*Inventory_Short_Eq,
*   Security_maximum_speculativ,
*   Security_minimum_speculativ,
*   Security_maximum_inv_grade,
*   Security_minimum_inv_grade,
   expected_shortfall,
   cash_max,
   Def_risk,Def_reward,
   equity_min,
   equity_max,
bond_min,
bond_max,
gsci_min,
gsci_max,

policy,
   ObjDef
/;
```

GAMS code for inventory update model

```
$SET OnDebug N

$ONEMPTY

* OPTIONS
$EOLCOM //
$INLINECOM /* */

$INCLUDE "AllSets.inc";
$INCLUDE "Policy.inc";
$INCLUDE "Constants.inc";

alias (Tree,p,p2);
Node(Tree)=Yes;
```

```
alias (Node,n,n2,m);
set scenario(Tree);
alias (Scen, s);
alias (Stage, t);
alias (classSet,k);
alias (Security, i,j);

$INCLUDE "Price.inc";
$INCLUDE "StageofNode.inc";
$INCLUDE "NodeAncestor.inc";
$INCLUDE "NPM.inc";
$INCLUDE "Init_asset.inc";
$INCLUDE "class.inc";
$INCLUDE "Probability.inc";
$INCLUDE "Constants.inc";
$INCLUDE "delta_call.inc";
$INCLUDE "delta_put.inc";
Scalar Stage_Max;
     Stage_Max=smax(p,StageofNode(p));

* Define some sets
Set root(tree), leaf(tree);
root(tree)$(Ord(tree)=1) = Yes;
leaf(tree)$(StageOfNode(tree) eq Stage_Max) = Yes;

Set SpecGrade(i), InvGrade(i),equity(i),bond(i),gsci(i),callgrade(i),putgrade(i);
SpecGrade(i)$(class(i) ge 4)=Yes;
InvGrade(i)$(class(i) le 3)=Yes;
equity(i)$(class(i)eq 1)=Yes;
bond(i)$(class(i)eq 2)=Yes;
gsci(i)$(class(i)eq 3)=Yes;
callgrade(i)$(class(i)eq 4)=Yes;
putgrade(i)$(class(i)eq 5)=Yes;

Scalar CapIniz;
CapIniz = 100000;

* Initial Wealth definition
Parameter Wealth_0(tree);

       Wealth_0(root)= CapIniz + sum(i,Init_asset(i)*Price(i,root));

POSITIVE VARIABLES
 buy(p, i)
 sell(p, i)
 hold_asset(p, i)
 hold_debt(p)
 cash_hold(p)
 debt(p)
 debt_minus(p)
 debt_plus(p)
 shortfall
 nc(p)
 np(p)
 beta(p)
```

```
VARIABLES
   Expected_Wealth
   Wealth(p)
   reward
   risk
   xyz(p)

**** Objective function value *******************
   z          Objective function value type

EQUATIONS
   Inventory_Asset_Balance_Eq_0
   Inventory_Liab_Balance_Eq_0
   Cash_Flow_Balance_Eq_0
   Inventory_Asset_Balance_Eq
   Inventory_Liab_Balance_Eq
   Cash_Flow_Balance_Eq
   Def_Wealth
   Def_Expected_Wealth

*  Portfolio composition constraint
   Security_maximum_speculativ
   Security_minimum_speculativ
   Security_maximum_inv_grade
   Security_minimum_inv_grade
   expected_shortfall
   Cash_max
   policy1
   policy2
   equity_min
   equity_max
   bond_min
   bond_max
   gsci_min
   gsci_max


   Def_risk
   Def_reward
   ObjDef Objective function definition type;

* Inventory balance (nominal terms)
*** Asset Balance

Inventory_Asset_Balance_Eq_0(i,root)      ..
    hold_asset(root,i) =e= Init_asset(i)
    + buy(root,i) - sell(root,i);

Scalar chi_plus;
chi_plus=0.002;
Scalar chi_minus;
chi_minus = 0.002;

*nc.FX(n)=0;
```

```
*np.FX(n)=0;
buy.FX(n,callgrade(i))=0;
buy.FX(n,putgrade(i))=0;
*buy.FX(n,bond(i))=0;
*buy.FX(n,gsci(i))=0;

* cash flow balance
Cash_Flow_Balance_Eq_0(root) ..
cash_hold(root)=e= CapIniz
- sum(i$(class(i) le 3), buy(root,i)*Price(i,root)*(1+chi_plus))
+ sum(i$(class(i) le 3), sell(root,i)*Price(i,root)*(1-chi_minus))
-sum(i$(class(i) eq 4), nc(root)*Price(i,root)*(1+0.0001))
-sum(i$(class(i) eq 5), np(root)*Price(i,root)*(1+0.0001));

*******************************
*    next stage equations     *
*******************************
Alias (parent, p, child);
Set anc(parent,child);
anc(parent,child) = Yes $(Ord(child) > 1 And Ord(parent) = NodeAncestor(child));

Inventory_Asset_Balance_Eq(i,anc(m,n))$(not root(n)) ..
hold_asset(n,i) =e= hold_asset(m, i)+ buy(n, i) - sell(n, i) + (nc(m)*delta_call(n))$(class(i) eq 1);
Scalar riskfree_rate;
riskfree_rate=0;
Scalar Sstep;
Sstep=1/12;



Cash_Flow_Balance_Eq(anc(m,n)) ..
cash_hold(n) =e= cash_hold(m)*(1+riskfree_rate*Sstep)
-sum(i$(class(i) le 3), buy(n, i)*Price(i,n )*(1+chi_plus))
+sum(i$(class(i) le 3), sell(n, i)*Price(i,n)*(1-chi_minus))
+sum(i$(class(i) eq 1), (nc(m))*delta_call(n)*(Price(i,n)-Price(i,m))*(1-0.0001))
+sum(i$(class(i) eq 1), (np(m))*delta_put(n)*(Price(i,m)-Price(i,n))*(1-0.0001))
-sum(i$(class(i) eq 4), nc(n)*Price(i,n)*(1+0.0001))
-sum(i$(class(i) eq 5), np(n)*Price(i,n)*(1+0.0001));


Wealth.FX(root) = Wealth_0(root);
*Def_Wealth(anc(m,n)) ..
Def_Wealth(anc(m,n)) ..
Wealth(n) =e= cash_hold(n) + sum(i$(class(i) le 3), hold_asset(n,i)*Price(i,n))
+sum(i$(class(i) eq 4), nc(n)*Price(i,n))
+sum(i$(class(i) eq 5), np(n)*Price(i,n));

* Last Stage Constraints
buy.FX(leaf(n),i) = 0;
sell.FX(leaf(n),i) = 0;
nc.FX(leaf(n))=0;
np.FX(leaf(n))=0;
beta.FX(leaf(n))=0;

*Security_minimum_speculativ(n) ..  sum(SpecGrade(i), hold_asset(n,i)*Price(i,n)) =G= 0*sum(i, hold_asset(n,i)*Price(i,n));
```

```
*Security_maximum_speculativ(n) ..  sum(SpecGrade(i), hold_asset(n,i)*Price(i,n)) =L= 1*sum(i, hold_asset(n,i)*Price(i,n));

*Security_minimum_inv_grade(n) ..  sum(InvGrade(i), hold_asset(n,i)*Price(i,n)) =G= 0*sum(i, hold_asset(n,i)*Price(i,n));
*Security_maximum_inv_grade(n) ..  sum(InvGrade(i), hold_asset(n,i)*Price(i,n)) =L= 1*sum(i, hold_asset(n,i)*Price(i,n));

equity_min(n) ..  sum(equity(i), hold_asset(n,i)*Price(i,n)) =G= 0*sum(i, hold_asset(n,i)*Price(i,n));
equity_max(n) ..  sum(equity(i), hold_asset(n,i)*Price(i,n)) =L= 0.3*sum(i, hold_asset(n,i)*Price(i,n));

*bond_min(n) ..  sum(bond(i), hold_asset(n,i)*Price(i,n)) =G= 0.5*sum(i, hold_asset(n,i)*Price(i,n));
*bond_max(n) ..  sum(bond(i), hold_asset(n,i)*Price(i,n)) =L= 1*sum(i, hold_asset(n,i)*Price(i,n));

*gsci_min(n) ..  sum(gsci(i), hold_asset(n,i)*Price(i,n)) =G= 0*sum(i, hold_asset(n,i)*Price(i,n));
*gsci_max(n) ..  sum(gsci(i), hold_asset(n,i)*Price(i,n)) =L= 0.15*sum(i, hold_asset(n,i)*Price(i,n));

policy1(n) .. nc(n)=L= sum(equity(i),hold_asset(n,i));
policy2(n) .. np(n)=L= sum(equity(i),hold_asset(n,i));

Scalar Target;
Target = 120000;

expected_shortfall .. shortfall =g= sum(leaf(n), Probability(leaf)*(Target - Wealth(leaf)));


Cash_max(n) .. cash_hold(n) =l= 1*Wealth(n);



* Expected Final Wealth definition
Def_Expected_Wealth  .. Expected_Wealth =e=
sum(leaf(n), Wealth(leaf)*Probability (leaf));

Def_risk .. risk=e=shortfall;

Def_reward .. reward=e=(Expected_Wealth);

Scalar Lambda;
Lambda = 0.5;

ObjDef  .. z  =e= (1-Lambda)*(reward) - Lambda *risk;

MODEL Derivatives_basic /
   Inventory_Asset_Balance_Eq_0, Cash_Flow_Balance_Eq_0,
   Inventory_Asset_Balance_Eq, Cash_Flow_Balance_Eq,
   Def_Wealth,Def_Expected_Wealth,
*   Security_maximum_speculativ,
*   Security_minimum_speculativ,
*   Security_maximum_inv_grade,
*   Security_minimum_inv_grade,
   expected_shortfall,
   cash_max,
   Def_risk,Def_reward,
   equity_min,
   equity_max,
*bond_min,
*bond_max,
*gsci_min,
```

```
*gsci_max,

   policy1,
policy2,
   ObjDef
/;
```

Bibliography

Adams, Zeno et al. (2008). "Macroeconomic determinants of commodity futures returns". In: *The Handbook of Commodity Investing*, pp. 87–112.

Ahn, Dong-Hyun et al. (1999). "Optimal risk management using options". In: *The Journal of Finance* 54.1, pp. 359–375.

Aliprantis, Charalambos D, Donald J Brown, and Jan Werner (2000). "Minimum-cost portfolio insurance". In: *Journal of Economic Dynamics and Control* 24.11, pp. 1703–1719.

Aliprantis, Charalambos D, Paulo K Monteiro, and Rabee Tourky (2004). "Non-marketed options, non-existence of equilibria, and non-linear prices". In: *Journal of Economic Theory* 114.2, pp. 345–357.

Annaert, Jan et al. (2007). "Risk management of a bond portfolio using options". In: *Insurance: Mathematics and Economics* 41.3, pp. 299–316.

Bannister, B and P Forward (2002). "The Inflation Cycle of 2002 to 2015". In: *Legg Mason Equity Research, April* 19.

Barbaro, Andres and Miguel J Bagajewicz (2004). "Use of inventory and option contracts to hedge financial risk in planning under uncertainty". In: *AIChE journal* 50.5, pp. 990–998.

Barkhagen, Mathias and Jörgen Blomvall (2016). "Modeling and evaluation of the option book hedging problem using stochastic programming". In: *Quantitative Finance* 16.2, pp. 259–273.

Barkhagen, Mathias, Jörgen Blomvall, and Eckhard Platen (2016). "Recovering the real-world density and liquidity premia from option data". In: *Quantitative Finance* 16.7, pp. 1147–1164.

Barro, Diana and Elio Canestrelli (2009). "Tracking error: a multistage portfolio model". In: *Annals of Operations Research* 165.1, pp. 47–66.

— (2014). "Downside risk in multiperiod tracking error models". In: *Central European Journal of Operations Research* 22.2, pp. 263–283.

Becker, Kent and Joseph Finnerty (2000). "Indexed commodity futures and the risk and return of institutional portfolios". In:

Bertocchi, Marida, Giorgio Consigli, and Michael AH Dempster (2011). *Stochastic optimization methods in finance and energy: new financial products and energy market strategies*. Vol. 163. Springer.

Bertocchi, Marida, Vittorio Moriggia, and Jitka Dupačová (2000). "Sensitivity of bond portfolio's behavior with respect to random movements in yield curve: A simulation study". In: *Annals of Operations Research* 99.1, pp. 267–286.

Birge, JR and F Louveaux (1997). *Introduction to Stochastic Programming. Series in Operations Research and Financial Engineering*.

Blomvall, Jörgen and Per Olov Lindberg (2003). "Back-testing the performance of an actively managed option portfolio at the Swedish stock market, 1990–1999". In: *Journal of Economic Dynamics and Control* 27.6, pp. 1099–1112.

Bradley, Stephen P and Dwight B Crane (1975). "A dynamic model for bond portfolio management". In: *Stochastic Optimization Models in Finance*. Elsevier, pp. 487–499.

Brennan, Michael J and H Henry Cao (1996). "Information, trade, and derivative securities". In: *The Review of Financial Studies* 9.1, pp. 163–208.

Ç. Pınar, Mustafa (2009). "Measures of model uncertainty and calibrated option bounds". In: *Optimization* 58.3, pp. 335–350.

Carino, David R, David H Myers, and William T Ziemba (1998). "Concepts, technical issues, and uses of the Russell-Yasuda Kasai financial planning model". In: *Operations research* 46.4, pp. 450–462.

Carino, David R and William T Ziemba (1998). "Formulation of the Russell-Yasuda Kasai financial planning model". In: *Operations Research* 46.4, pp. 433–449.

Carino, David R et al. (1994). "The Russell-Yasuda Kasai model: An asset/liability model for a Japanese insurance company using multistage stochastic programming". In: *Interfaces* 24.1, pp. 29–49.

Carr, Peter, Xing Jin, and Dilip B Madan (2001). "Optimal investment in derivative securities". In: *Finance and Stochastics* 5.1, pp. 33–59.

Casey, Michael S and Suvrajeet Sen (2005). "The scenario generation algorithm for multistage stochastic linear programming". In: *Mathematics of Operations Research* 30.3, pp. 615–631.

Chen, Hsuan-Chi, San-Lin Chung, and Keng-Yu Ho (2011). "The diversification effects of volatility-related assets". In: *Journal of Banking & Finance* 35.5, pp. 1179–1189.

Consigli, Giorgio, Paolo Brandimarte, and Daniel Kuhn (2015). "Financial Optimization: optimization paradigms and financial planning under uncertainty". In: *OR Spectrum* 37.3, pp. 553–557.

Consigli, Giorgio and MAH Dempster (1998). "Dynamic stochastic programmingfor asset-liability management". In: *Annals of Operations Research* 81, pp. 131–162.

Consigli, Giorgio, Gaetano Iaquinta, and Vittorio Moriggia (2012). "Path-dependent scenario trees for multistage stochastic programmes in finance". In: *Quantitative Finance* 12.8, pp. 1265–1281.

Consigli, Giorgio et al. (2009). "The bond-stock yield differential as a risk indicator in financial markets". In: *The Journal of Risk* 11.3, p. 3.

Consigli, Giorgio et al. (2012). "Retirement planning in individual asset–liability management". In: *IMA Journal of Management Mathematics* 23.4, pp. 365–396.

Consiglio, Andrea, Angelo Carollo, and Stavros A Zenios (2016). "A parsimonious model for generating arbitrage-free scenario trees". In: *Quantitative Finance* 16.2, pp. 201–212.

Cont, Rama (2006). "Model uncertainty and its impact on the pricing of derivative instruments". In: *Mathematical finance* 16.3, pp. 519–547.

Cremers, KJ Martijn, Joost Driessen, and Pascal Maenhout (2008). "Explaining the level of credit spreads: Option-implied jump risk premia in a firm value model". In: *The Review of Financial Studies* 21.5, pp. 2209–2242.

Davari-Ardakani, Hamed, Majid Aminnayeri, and Abbas Seifi (2016). "Multistage portfolio optimization with stocks and options". In: *International Transactions in Operational Research* 23.3, pp. 593–622.

Dempster, MAH, Igor V Evstigneev, and Klaus Reiner Schenk-Hoppe (2008). "Financial markets. The joy of volatility". In: *Quantitative Finance* 8.1, pp. 1–3.

Dempster, MAH and JP Hutton (1999). "Pricing American stock options by linear programming". In: *Mathematical Finance* 9.3, pp. 229–254.

Dempster, MAH and Darren G Richards (2000). "Pricing American options fitting the smile". In: *Mathematical Finance* 10.2, pp. 157–177.

Dempster, MAH et al. (2007). "Designing minimum guaranteed return funds". In: *Quantitative Finance* 7.2, pp. 245–256.

Dempster, Michael AH, James P Hutton, and Darren G Richards (1998). "LP valuation of exotic American options exploiting structure". In: *RESEARCH PAPERS IN MANAGEMENT STUDIES-UNIVERSITY OF CAMBRIDGE JUDGE INSTITUTE OF MANAGEMENT STUDIES*.

Dempster, Michael AH and AM Ireland (1988). "A financial expert decision support system". In: *Mathematical models for decision support* 48, pp. 415–440.

Dentcheva, Darinka and Andrzej Ruszczyński (2006). "Portfolio optimization with stochastic dominance constraints". In: *Journal of Banking & Finance* 30.2, pp. 433–451.

Derman, Emanuel, Iraj Kani, and Neil Chriss (1996). "Implied trinomial tress of the volatility smile". In: *The Journal of Derivatives* 3.4, pp. 7–22.

Dert, Cees (1995). "Asset liability management for pension funds: a multistage chance constrained programming approach". In:

Dimson, Elroy and Massoud Mussavian (1999). "Three centuries of asset pricing". In: *Journal of Banking & Finance* 23.12, pp. 1745–1769.

Driessen, Joost and Pascal Maenhout (2007). "An empirical portfolio perspective on option pricing anomalies". In: *Review of Finance* 11.4, pp. 561–603.

Dupačová, Jitka (2002). "Applications of stochastic programming: achievements and questions". In: *European Journal of Operational Research* 140.2, pp. 281–290.

Dupačová, Jitka and Marida Bertocchi (2001). "From data to model and back to data: A bond portfolio management problem". In: *European Journal of Operational Research* 134.2, pp. 261–278.

Dupačová, Jitka, Giorgio Consigli, and Stein W Wallace (2000). "Scenarios for multistage stochastic programs". In: *Annals of operations research* 100.1-4, pp. 25–53.

Dupačová, Jitka, Nicole Gröwe-Kuska, and Werner Römisch (2003). "Scenario reduction in stochastic programming". In: *Mathematical programming* 95.3, pp. 493–511.

Estrella, Arturo and John Kambhu (1997). "Approximation of changes in option values and hedge ratios: how large are the errors?" In: *The Measurement of Aggregate Market Risk, Bank for International Settlements, November, available at http://www. bis. org/publ/ecsc07e. pdf*.

Eun, Cheol S and Bruce G Resnick (1988). "Exchange rate uncertainty, forward contracts, and international portfolio selection". In: *The Journal of Finance* 43.1, pp. 197–215.

— (1997). "International equity investment with selective hedging strategies". In: *Journal of International Financial Markets, Institutions and Money* 7.1, pp. 21–42.

Föllmer, Hans, Alexander Schied, and Terry J Lyons (2004). "Stochastic finance. an introduction in discrete time". In: *The Mathematical Intelligencer* 26.4, pp. 67–68.

Follmer, Hans and Dieter Sondermann (1986). "Contributions to Mathematical Economics". In: *North Holland*.

Fonseca, Raquel J and Berç Rustem (2012). "Robust hedging strategies". In: *Computers & Operations Research* 39.11, pp. 2528–2536.

Fonseca, Raquel J, Wolfram Wiesemann, and Berç Rustem (2012). "Robust international portfolio management". In: *Computational Management Science* 9.1, pp. 31–62.

Gaivoronski, Alexei A and Petter E De Lange (2000). "An asset liability management model for casualty insurers: complexity reduction vs. parameterized decision rules". In: *Annals of Operations Research* 99.1, pp. 227–250.

Gao, Pei-wang (2009). "Options strategies with the risk adjustment". In: *European Journal of Operational Research* 192.3, pp. 975–980.

Gondzio, Jacek, Roy Kouwenberg, and Ton Vorst (2003). "Hedging options under transaction costs and stochastic volatility". In: *Journal of Economic Dynamics and Control* 27.6, pp. 1045–1068.

Gorton, Gary and K Geert Rouwenhorst (2006). "Facts and fantasies about commodity futures". In: *Financial Analysts Journal* 62.2, pp. 47–68.

Granger, Clive WJ and Paul Newbold (1974). "Spurious regressions in econometrics". In: *Journal of econometrics* 2.2, pp. 111–120.

Harrison, J Michael and David M Kreps (1979). "Martingales and arbitrage in multiperiod securities markets". In: *Journal of Economic theory* 20.3, pp. 381–408.

Harrison, J Michael and Stanley R Pliska (1981). "Martingales and stochastic integrals in the theory of continuous trading". In: *Stochastic processes and their applications* 11.3, pp. 215–260.

Haugh, Martin B and Andrew W Lo (2001). "Asset allocation and derivatives". In:

Heitsch, Holger and Werner Römisch (2007). "A note on scenario reduction for two-stage stochastic programs". In: *Operations Research Letters* 35.6, pp. 731–738.

— (2009). "Scenario tree reduction for multistage stochastic programs". In: *Computational Management Science* 6.2, p. 117.

Ho, James K and Alan S Manne (1974). "Nested decomposition for dynamic models". In: *Mathematical Programming* 6.1, pp. 121–140.

Horasanlı, Mehmet (2008). "Hedging strategy for a portfolio of options and stocks with linear programming". In: *Applied mathematics and computation* 199.2, pp. 804–810.

Høyland, K (1998). "Asset liability management for a life insurance company: A stochastic programming approach". In: *Norwegian University of Science and Technology, Trondheim, Norway*.

Høyland, Kjetil and Stein W Wallace (2001). "Generating scenario trees for multistage decision problems". In: *Management science* 47.2, pp. 295–307.

Ingersoll, Jonathan E (1987). *Theory of financial decision making*. Vol. 3. Rowman & Littlefield.

Jackwerth, Jens Carsten and Mark Rubinstein (1996). "Recovering probability distributions from option prices". In: *The Journal of Finance* 51.5, pp. 1611–1631.

Kaut, Michal and Stein W Wallace (2003). "Evaluation of scenario-generation methods for stochastic programming". In:

King, Alan J (2002). "Duality and martingales: a stochastic programming perspective on contingent claims". In: *Mathematical Programming* 91.3, pp. 543–562.

Klaassen, Pieter (1997). "Discretized reality and spurious profits in stochastic programming models for asset/liability management". In: *European Journal of Operational Research* 101.2, pp. 374–392.

— (1998). "Financial asset-pricing theory and stochastic programming models for asset/liability management: A synthesis". In: *Management Science* 44.1, pp. 31–48.

— (2002). "Comment on "Generating scenario trees for multistage decision problems"". In: *Management Science* 48.11, pp. 1512–1516.

Kofman, Paul and Martin Martens (1997). "Interaction between stock markets: an analysis of the common trading hours at the London and New York stock exchange". In: *Journal of International Money and Finance* 16.3, pp. 387–414.

Kouwenberg, Roy (2001). "Scenario generation and stochastic programming models for asset liability management". In: *European Journal of Operational Research* 134.2, pp. 279–292.

Kuhn, Daniel (2008). "Aggregation and discretization in multistage stochastic programming". In: *Mathematical Programming* 113.1, pp. 61–94.

Leland, Hayne E (1980). “Who should buy portfolio insurance?” In: *The Journal of Finance* 35.2, pp. 581–594.
Liang, Jianfeng, Shuzhong Zhang, and Duan Li (2008). “Optioned portfolio selection: models and analysis”. In: *Mathematical Finance* 18.4, pp. 569–593.
Liu, Jun and Jun Pan (2003). “Dynamic derivative strategies”. In: *Journal of Financial Economics* 69.3, pp. 401–430.
Mausser, Helmut and Dan Rosen (1999). “Beyond VaR: From measuring risk to managing risk”. In: *Computational Intelligence for Financial Engineering, 1999.(CIFEr) Proceedings of the IEEE/IAFE 1999 Conference on*. IEEE, pp. 163–178.
Merton, Robert C, Myron S Scholes, and Mathew L Gladstein (1978). “The returns and risk of alternative call option portfolio investment strategies”. In: *Journal of Business*, pp. 183–242.
Mitra, Gautam and Katharina Schwaiger (2011). *Asset and liability management handbook*. Springer.
Muck, Matthias (2010). “Trading strategies with partial access to the derivatives market”. In: *Journal of Banking & Finance* 34.6, pp. 1288–1298.
Mulvey, John M (1996). “Generating scenarios for the Towers Perrin investment system”. In: *Interfaces* 26.2, pp. 1–15.
Mulvey, John M, Woo Chang Kim, and Changle Lin (2017). “Optimizing a portfolio of liquid and illiquid assets”. In: *Optimal Financial Decision Making under Uncertainty*. Springer, pp. 151–175.
Mulvey, John M, William R Pauling, and Ronald E Madey (2003). “Advantages of multiperiod portfolio models”. In: *The Journal of Portfolio Management* 29.2, pp. 35–45.
Neuberger, Anthony and Stewart Hodges (2002). “How large are the benefits from using options?” In: *Journal of Financial and Quantitative Analysis* 37.2, pp. 201–220.
Nielsen, Soren S and Stavros A Zenios (1993). “A massively parallel algorithm for nonlinear stochastic network problems”. In: *Operations Research* 41.2, pp. 319–337.
— (1996). “A stochastic programming model for funding single premium deferred annuities”. In: *Mathematical Programming* 75.2, p. 177.
Palma, André de and Jean-Luc Prigent (2008). “Hedging global environment risks: An option based portfolio insurance”. In: *Automatica* 44.6, pp. 1519–1531.
Papahristodoulou, Christos (2004). “Option strategies with linear programming”. In: *European Journal of Operational Research* 157.1, pp. 246–256.
Pflug, G Ch (2001). “Scenario tree generation for multiperiod financial optimization by optimal discretization”. In: *Mathematical programming* 89.2, pp. 251–271.
Pınar, Mustafa Ç (2007). “Robust scenario optimization based on downside-risk measure for multi-period portfolio selection”. In: *OR Spectrum* 29.2, pp. 295–309.
Rockafellar, R Tyrrell and Stanislav Uryasev (2002). “Conditional value-at-risk for general loss distributions”. In: *Journal of banking & finance* 26.7, pp. 1443–1471.
Rogers, Jim (2007). *Hot commodities: how anyone can invest profitably in the world's best market*. Random House Incorporated.
Römisch, Werner (2009). “Scenario generation in stochastic programming”. In:
Rubinstein, Mark (1985). “Nonparametric tests of alternative option pricing models using all reported trades and quotes on the 30 most active CBOE option classes from August 23, 1976 through August 31, 1978”. In: *The Journal of Finance* 40.2, pp. 455–480.
— (1994). “Implied binomial trees”. In: *The Journal of Finance* 49.3, pp. 771–818.
Scheuenstuhl, Gerhard and Rudi Zagst (2008). “Integrated portfolio management with options”. In: *European Journal of Operational Research* 185.3, pp. 1477–1500.

Sinha, Pankaj and Archit Johar (2010). "Hedging Greeks for a portfolio of options using linear and quadratic programming". In:

Sortino, Frank A and Robert Van Der Meer (1991). "Downside risk". In: *The Journal of Portfolio Management* 17.4, pp. 27–31.

Steil, Benn (1993). "Currency options and the optimal hedging of contingent foreign exchange exposure". In: *Economica*, pp. 413–431.

Topaloglou, Nikolas, Hercules Vladimirou, and Stavros A Zenios (2002). "CVaR models with selective hedging for international asset allocation". In: *Journal of Banking & Finance* 26.7, pp. 1535–1561.

— (2008a). "A dynamic stochastic programming model for international portfolio management". In: *European Journal of Operational Research* 185.3, pp. 1501–1524.

— (2008b). "Pricing options on scenario trees". In: *Journal of Banking & Finance* 32.2, pp. 283–298.

— (2011). "Optimizing international portfolios with options and forwards". In: *Journal of Banking & Finance* 35.12, pp. 3188–3201.

Villaverde, Michael (2004). "Hedging European and barrier options using stochastic optimization". In: *Quantitative Finance* 4.5, pp. 549–557.

Worthington, Andrew C and Mosayeb Pahlavani (2007). "Gold investment as an inflationary hedge: Cointegration evidence with allowance for endogenous structural breaks". In: *Applied Financial Economics Letters* 3.4, pp. 259–262.

Wu, Jason and Suvrajeet Sen (2000). "A stochastic programming model for currency option hedging". In: *Annals of Operations Research* 100.1-4, pp. 227–249.

Yin, Libo and Liyan Han (2013a). "International assets allocation with risk management via multi-stage stochastic programming". In: *Computational Economics*, pp. 1–23.

— (2013b). "Options strategies for international portfolios with overall risk management via multi-stage stochastic programming". In: *Annals of Operations Research* 206.1, pp. 557–576.

Zapata, Hector O, Joshua D Detre, and Tatsuya Hanabuchi (2012). "Historical performance of commodity and stock markets". In: *Journal of agricultural and applied Economics* 44.3, pp. 339–357.

Zenios, SA and WT Ziemba (2004). *Handbook of Asset–Liability Management. Handbooks in Finance Series*.

Zenios, Stavros A and William T Ziemba (2007). *Handbook of Asset and Liability Management: Applications and case studies*. Vol. 2. Elsevier.

Zhao, Yonggan and William T Ziemba (2008). "Calculating risk neutral probabilities and optimal portfolio policies in a dynamic investment model with downside risk control". In: *European Journal of Operational Research* 185.3, pp. 1525–1540.

Ziemba, William T and John M Mulvey (1998). *Worldwide asset and liability modeling*. Vol. 10. Cambridge University Press.

Zymler, Steve, Berç Rustem, and Daniel Kuhn (2011). "Robust portfolio optimization with derivative insurance guarantees". In: *European Journal of Operational Research* 210.2, pp. 410–424.