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Gravity's Rainbow and Black Hole Entropy

Remo Garattini

Abstract. We consider the effects of Gravity's Rainbow on the computation of black hole entropy using a dynamical brick wall model. An explicit dependence of the radial coordinate approaching the horizon is proposed to analyze the behavior of the divergence. We find that, due to the modification of the density of states, the brick wall can be eliminated. The calculation is extended to include rotations and in particular to a Kerr black hole in a comoving frame.

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1. Introduction

In recent years many attempts to modify gravity at the fundamental level have been proposed. Some of them, like $f(R)$ theories[1], have been considered to basically change the large scale structure of the space-time and some others have been conceived to modify the short scale behavior. Among them, Gravity's Rainbow (GRw) seems to be promising in dealing with Ultra-Violet divergences (UV). Indeed, in a series of papers GRw has been used to avoid any regularization/renormalization scheme which appear in conventional Quantum Field Theory calculations like one loop corrections to classical quantities[2]. This amazing property has been applied also to black holes and in particular to the computation of black hole entropy[3]. In this last case, the idea is to avoid to introduce a cut-off of Planckian size known as "*brick wall*"[4]. The "*brick wall*" appears when one uses a statistical mechanical approach to explain the famous Bekenstein-Hawking formula[5, 6]

$$S_{BH} = \frac{1}{4} A / l_P^2, \tag{1}$$

relating the entropy of a black hole and its area. Indeed, when one tries to adopt such an approach, one realizes that the density of energy levels of single-particle excitations is divergent near the horizon. Of course, several attempts have been done to avoid the introduction of the *brick wall*. For instance, without modifying gravity at any scale, it has been suggested that the *brick wall* could be absorbed in a renormalization of Newton's constant[7, 8, 9], while other authors approached the problem of the divergent brick wall using Pauli-Villars regularization[10, 11, 12]. Other than GRw other proposals have been made in the context of modified gravity. For instance, non-commutative geometry introduces a natural thickness of the horizon replacing the 't Hooft's brick wall[16] and *Generalized Uncertainty Principle* (GUP) modifies the Liouville measure[13, 14, 15]. To understand how GRw works we need to define two unknown functions $g_1(E/E_P)$ and $g_2(E/E_P)$ having the following property

$$\lim_{E/E_P \to 0} g_1(E/E_P) = 1 \qquad \text{and} \qquad \lim_{E/E_P \to 0} g_2(E/E_P) = 1. \tag{2}$$

In this formalism introduced by Magueijo and Smolin[18], the Einstein's field equations are replaced by a one parameter family of equations

$$G_{\mu\nu}\left(E\right) = 8\pi G\left(E\right) T_{\mu\nu}\left(E\right) + g_{\mu\nu}\Lambda\left(E\right), \tag{3}$$

where $G\left(E\right)$ is an energy dependent Newton's constant and $\Lambda\left(E\right)$ is an energy dependent cosmological constant, respectively. They are defined so that $G\left(0\right)$ is the physical Newton's constant and $\Lambda\left(0\right)$ is the usual cosmological constant. In this context, the Schwarzschild solution of (3) becomes

$$ds^2\left(E\right) = -\left(1-\frac{2MG\left(0\right)}{r}\right)\frac{dt^2}{g_1^2\left(E/E_P\right)}+\frac{dr^2}{\left(1-\frac{2MG(0)}{r}\right)g_2^2\left(E/E_P\right)}+\frac{r^2}{g_2^2\left(E/E_P\right)}\left(d\theta^2+\sin^2\theta d\phi^2\right) \tag{4}$$

and it can be easily generalized in the following way

$$ds^2\left(E\right) = -\exp\left(-2A\left(r\right)\right)\left(1-\frac{b\left(r\right)}{r}\right)\frac{dt^2}{g_1^2\left(E/E_P\right)}+\frac{dr^2}{\left(1-\frac{b(r)}{r}\right)g_2^2\left(E/E_P\right)}+\frac{r^2}{g_2^2\left(E/E_P\right)}d\Omega^2. \tag{5}$$

The function $b\left(r\right)$ will be referred to as the "shape function" and it may be thought of as specifying the shape of the spatial slices. The location of the horizon is determined by the equation $b\left(r_H\right) = r_H$. On the other hand, $A\left(r\right)$ will be referred to as the "redshift function" and describes how far the total gravitational redshift deviates from that implied by the shape function. The line element (5) describes any spherically symmetric space-time. It is interesting to wonder what happens when one introduces rotations. Rotating black holes have a good description in terms of the Kerr metric which, in the context of GRw, becomes[19]

$$ds^2\left(E\right) = \frac{g_{tt}dt^2}{g_1^2\left(E/E_P\right)}+\frac{2g_{t\phi}dtd\phi}{g_1\left(E/E_P\right)g_2\left(E/E_P\right)}+\frac{g_{\phi\phi}d\phi^2}{g_2^2\left(E/E_P\right)}+\frac{g_{rr}dr^2}{g_2^2\left(E/E_P\right)}+\frac{g_{\theta\theta}d\theta^2}{g_2^2\left(E/E_P\right)}, \tag{6}$$

where

$$g_{tt} = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}, \qquad g_{t\phi} = -\frac{a\sin^2\theta\left(r^2+a^2-\Delta\right)}{\Sigma},$$
$$g_{\phi\phi} = \frac{\left(r^2+a^2\right)^2-\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta, \qquad g_{rr} = \frac{\Sigma}{\Delta}, \qquad g_{\theta\theta} = \Sigma, \tag{7}$$

and

$$\Delta = r^2 - 2MGr + a^2, \qquad \Sigma = r^2 + a^2\cos^2\theta. \tag{8}$$

Here M and a are mass and angular momentum per unit mass of the black hole, respectively. Δ vanishes when $r = r_\pm = MG \pm \sqrt{\left(MG\right)^2 - a^2}$, while g_{tt} vanishes when $r = r_{S\pm} = MG \pm \sqrt{\left(MG\right)^2 - a^2\cos^2\theta}$: they are not modified by GRw and the outer horizon or simply horizon is located at $r_+ = r_H$. Note that the Kerr metric modified by GRw (6) reduces to the standard rotating black hole background when $E/E_P \to 0$. In this contribution we will consider the effect of GRw on Black Hole entropy computation even for a rotating background. Units in which $\hbar = c = k = 1$ are used throughout the paper.

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2. GRw Entropy for a Schwarzschild Black Hole

To see what happens in practice for the Schwarzschild Black Hole, we define a real massless scalar field whose Euler-Lagrange equations are

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\phi=0. \tag{9}$$

The formalism has been outlined in detail in [3] and therefore we refer the reader to [3] for details. If ϕ has the separable form

$$\phi\left(t,r,\theta,\varphi\right)=\exp\left(-iEt\right)Y_{lm}(\theta,\varphi)f\left(r\right), \tag{10}$$

then the equation for $f\left(r\right)$ reads

$$\left[\frac{g_2^2\left(E/E_P\right)\exp\left(A\left(r\right)\right)}{r^2}\partial_r\left(r^2\exp\left(-A\left(r\right)\right)\left(1-\frac{b\left(r\right)}{r}\right)\partial_r\right)-V_l\left(r\right)\right]f_{nl}=0, \tag{11}$$

where

$$V_l\left(r\right)=\left(\frac{l(l+1)}{r^2}-\frac{E_{nl}^2g_1^2\left(E/E_P\right)\exp\left(2A\left(r\right)\right)}{1-\frac{b(r)}{r}}\right) \tag{12}$$

and where $Y_{lm}(\theta,\varphi)$ is the usual spherical harmonic function. In order to use the WKB approximation to compute the entropy, we define the following r-dependent radial wave number $k(r,l,E)$

$$k_r^2(r,l,E)\equiv\frac{1}{\left(1-\frac{b(r)}{r}\right)}\left[\exp\left(2A\left(r\right)\right)\frac{E^2h^2\left(E/E_P\right)}{\left(1-\frac{b(r)}{r}\right)}-\frac{l(l+1)}{r^2}\right], \tag{13}$$

where

$$h\left(E/E_P\right)=\frac{g_1\left(E/E_P\right)}{g_2\left(E/E_P\right)}. \tag{14}$$

The number of modes with frequency less than E is given approximately by

$$n(E)=\frac{1}{\pi}\int_0^{l_{max}}(2l+1)\int_{r_H}^{R}\sqrt{k^2(r,l,E)}drdl, \tag{15}$$

Here it is understood that the integration with respect to r and l is taken over those values which satisfy $r_H\le r\le R$ and $k^2(r,l,E)\ge 0$. Thus, from Eq.(13) we get

$$\frac{dn(E)}{dE}=\frac{2}{\pi}\frac{d}{dE}\left(\frac{1}{3}E^3h^3\left(E/E_P\right)\right)\int_{r_H}^{R}dr\frac{\exp\left(3A\left(r\right)\right)}{\left(1-\frac{b(r)}{r}\right)^2}r^2. \tag{16}$$

Since the free energy can be written as

$$F=\frac{1}{\beta}\int_0^\infty\ln\left(1-e^{-\beta E}\right)\frac{dn(E)}{dE}dE, \tag{17}$$

where β is the inverse temperature measured at infinity. Plugging Eq.(16) into (17) we find

$$F_{r_H}=\frac{2}{\pi}\frac{1}{\beta}\int_0^\infty\ln\left(1-e^{-\beta E}\right)\frac{d}{dE}\left(\frac{1}{3}E^3h^3\left(E/E_P\right)\right)\left[\int_{r_H}^{r_1}drr^2\frac{\exp\left(3A\left(r\right)\right)}{\left(1-\frac{b(r)}{r}\right)^2}\right]dE \tag{18}$$

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and

$$F_R = \frac{2}{\pi}\frac{1}{\beta}\int_0^\infty \ln\left(1-e^{-\beta E}\right)\frac{d}{dE}\left(\frac{1}{3}E^3h^3\left(E/E_P\right)\right)\left[\int_{r_1}^{R} drr^2\frac{\exp\left(3A\left(r\right)\right)}{\left(1-\frac{b(r)}{r}\right)^2}\right]dE. \tag{19}$$

Assuming that $A(r) < \infty$, $\forall r \in [r_H, +\infty)$, F_R is dominated by large volume effects for large R and it will give the contribution to the entropy of a homogeneous quantum gas in flat space at a uniform temperature T when GRw is considered. If GRw does not come into play, then the radial part of F_{r_H} becomes divergent in proximity of r_H. On the other hand, if we allow that the "*brick wall* r_0" be affected by GRw, namely $r_0 \equiv r_0(E/E_P)$, then the radial integration F_{r_H} becomes

$$\int_{r_H+r_0(E/E_P)}^{r_1} drr^2\frac{\exp\left(3\Lambda\left(r\right)\right)}{\left(1-\frac{b(r)}{r}\right)^2} \simeq r_H^4\frac{\exp\left(3A\left(r_H\right)\right)}{\left(1-b'\left(r_H\right)\right)^2}\frac{1}{r_0\left(E/E_P\right)}$$

$$= r_H^3\frac{\exp\left(3A\left(r_H\right)\right)}{\left(1-b'\left(r_H\right)\right)^2}\frac{1}{\sigma\left(E/E_P\right)}, \tag{20}$$

where we have assumed that, in proximity of the throat the brick wall can be written as $r_0(E/E_P) = r_H\sigma(E/E_P)$ with

$$\sigma\left(E/E_P\right) \to 0, \qquad E/E_P \to 0. \tag{21}$$

Plugging Eq.(20) into F_{r_H} we obtain, after an integration by parts

$$F_{r_H} = -\frac{C_{r_H}}{3\beta r_H}\int_0^\infty \frac{E^3h^3\left(E/E_P\right)}{\sigma\left(E/E_P\right)}\left[\frac{\beta}{\left(\exp\left(\beta E\right)-1\right)} - \frac{\ln\left(1-e^{-\beta E}\right)}{E_P\sigma\left(E/E_P\right)}\sigma'\left(E/E_P\right)\right]. \tag{22}$$

$h(E/E_P)$ must be chosen in such a way to allow the convergence when $E/E_P \to \infty$, thus we assume that

$$h\left(E/E_P\right) = \exp\left(-\frac{E}{E_P}\right) \qquad \text{and} \qquad \sigma\left(E/E_P\right) = h^\delta\left(E/E_P\right)\left(\frac{E}{E_P}\right)^\alpha. \tag{23}$$

In particular, for $\delta = 0$; $\alpha = 2$, one finds that the entropy becomes

$$S = \beta^2\frac{\partial F_{r_H}}{\partial\beta} \underset{\beta E_P \gg 1}{\simeq} \frac{A_{r_H}E_P^2}{4}\frac{\exp\left(2A\left(r_H\right)\right)}{1-b'\left(r_H\right)}\frac{2}{9\pi}. \tag{24}$$

where we have used the expression for the surface gravity in the low energy limit. As we can see the "*brick wall*" does not appear.

3. GRw Entropy for the Kerr Black Hole

To discuss the entropy for a Kerr black hole we have two options: we can use a rest observer at infinity (ROI) or we can use a Zero Angular Momentum Observer (ZAMO)[20, 21]. The ROI frame is described by the line element (6) and the appropriate form of the free energy is the following

$$F = \frac{1}{\beta}\int_0^\infty dn\left(E\right)\ln\left(1-e^{-\beta\left(E-m\Omega\right)}\right). \tag{25}$$

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It is immediate to see that when we use a ROI, the problem of superradiance appears when the free energy (25) is computed in the range $0 < E < m\Omega$. On the other hand when a ZAMO is considered, the free energy (25) becomes similar to the one used for a Schwarzschild black hole (17). Basically this happens because near the horizon the metric becomes

$$ds^2 = -\frac{N^2 dt^2}{g_1^2(E/E_P)} + g_{\phi\phi}\frac{d\phi^2}{g_2^2(E/E_P)} + g_{rr}\frac{dr^2}{g_2^2(E/E_P)} + g_{\theta\theta}\frac{d\theta^2}{g_2^2(E/E_P)} \tag{26}$$

and the mixing between t and ϕ disappears. Moreover when we use a ZAMO frame, the superradiance does not come into play because there is no ergoregion. Indeed since we have defined

$$N^2 = g_{tt} - \frac{g_{t\phi}^2}{g_{\phi\phi}} = -\frac{1}{g^{tt}} = -\frac{\Delta \sin^2\theta}{g_{\phi\phi}}, \tag{27}$$

N^2 vanishes when $r \to r_H$. Therefore if we repeat the same steps which led us to the computation of (16), one finds

$$\frac{dn(E)}{dE} = \frac{1}{8\pi^2}\int d\theta d\bar{\phi}\int_{r_H}^{R} dr\left(-g^{tt}\right)^{\frac{3}{2}}\sqrt{g_{rr}g_{\theta\theta}g_{\phi\phi}}\frac{1}{3}\frac{d}{dE}\left(h^3(E/E_P)E^3\right), \tag{28}$$

where the solution of the massless Klein-Gordon equation (9) assumes the form

$$\phi(x) = \exp\left(-iEt + im + iK(r,\theta)\right) \tag{29}$$

with

$$k_r = \frac{\partial K(r,\theta)}{\partial r}, \qquad k_\theta = \frac{\partial K(r,\theta)}{\partial \theta} \tag{30}$$

defined in such a way to use the WKB approximation. In proximity of the horizon, the free energy can be approximated by

$$F_{r_H} = \frac{1}{8\pi^2\beta}\int d\theta d\bar{\phi}\int_0^\infty \ln\left(1 - e^{-\beta E}\right)\frac{d}{dE}\left(\frac{1}{3}h^3(E/E_P)E^3\right)dE\int_{r_H}^{r_1} dr\left(-g^{tt}\right)^{\frac{3}{2}}\sqrt{g_{rr}g_{\theta\theta}g_{\phi\phi}} \tag{31}$$

which can be further reduced to

$$F_{r_H} \simeq \frac{C(r_H,\theta)}{8\pi^2\beta}\int_0^\infty \frac{\ln\left(1 - e^{-\beta E}\right)}{\sigma(E/E_P)}\frac{d}{dE}\left(\frac{1}{3}h^3(E/E_P)E^3\right)dE, \tag{32}$$

where

$$C(r_H,\theta) = \int d\theta d\bar{\phi}\left[\frac{\left(r_H^2 + a^2\right)^4 \sin\theta}{r_H\left(r_H - r_-\right)^2\Sigma_H}\right]. \tag{33}$$

With an integration by parts one finds

$$F_{r_H} = -\frac{C(r_H,\theta)}{24\pi^2\beta}\int_0^\infty \frac{E^3 h^3(E/E_P)}{\sigma(E/E_P)}\left[\frac{\beta}{(\exp(\beta E) - 1)} - \frac{\ln\left(1 - e^{-\beta E}\right)}{E_P\sigma(E/E_P)}\sigma'(E/E_P)\right]dE. \tag{34}$$

If we adopt the same choice of the previous section described by (23) and we fix our attention on the particular values $\delta = 0$ and $\alpha = 2$, one finds

$$\begin{aligned} F_{r_H} &= -\frac{C(r_H,\theta)}{24\pi^2\beta}\int_0^\infty\left[\frac{\beta E e^{-3E/E_P}}{(\exp(\beta E) - 1)} - 2e^{-3E/E_P}\ln\left(1 - e^{-\beta E}\right)\right]dE \\ &= -\frac{C(r_H,\theta)}{24\pi^2\beta}\left[\zeta\left(2, 1 + \frac{3}{\beta E_P}\right) + \frac{\beta E_P}{3}\left(\gamma + \Psi\left(1 + \frac{3}{\beta E_P}\right)\right)\right], \end{aligned} \tag{35}$$

where $\zeta(s,\nu)$ is the Hurwitz zeta function, $\Gamma(x)$ is the gamma function and $\Psi(x)$ is the digamma function. In the limit where $\beta E_P \gg 1$, at the leading order, one finds that the entropy can be approximated by

$$S = \beta^2 \frac{\partial F_{r_w}}{\partial \beta} = \frac{E_P^2}{36\beta} \int d\theta d\bar{\phi} \left[\frac{\left(r_H^2 + a^2\right)^4 \sin\theta}{r_H \left(r_H - r_-\right)^2 \Sigma_H} \right] \tag{36}$$

and even in this case the "*brick wall*" does not appear. Of course the entropy (36) can always be cast in the familiar form

$$S = \frac{A_H}{4G}, \tag{37}$$

where A_H is the horizon area. To summarize, we have shown that the ability of Gravity's Rainbow to keep under control the UV divergences applies also to rotations. However the connection between a ROI and a ZAMO has to be investigated with care[22]. Indeed in the ROI frame, the superradiance phenomenon appears, while in the ZAMO frame does not. Once the connection is established nothing forbids to extend this result to other rotating configuration like, for example, Kerr-Newman or Kerr-Newman-De Sitter (Anti-De Sitter).

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