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Essays on Stochastic Orderings in Portfolio Selection

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Abstract

Stochastic Orderings represent a relevant approach in portfolio selection for various reasons. Firstly, Stochastic Ordering are theoretically justified by Expected Utility theory. Typically, investors are classified according to their attitude toward risk. For each class of investors then, it is possible to define stochastic orderings coherent with investors' preference. Secondly, Stochastic Orderings are flexible enough to allow different definitions of efficiency suitable for each category of investors. This Thesis proposes several applications of Stochastic Orderings to portfolio selection problems. In the first chapter, an analysis of the relationship between Second order Stochastic Dominance efficient set and Mean Variance Efficient Frontier is proposed. Not only the two sets differ under many aspects, but the Global Minimum Variance portfolio and other Mean Variance Efficient portfolios are dominated in the sense of Second order Stochastic Dominance. Based on this fact, the chapter concludes proposing dominating strategies able to outperform the Global Minimum Variance portfolio. In the second chapter, starting from recent findings in the literature, that address the behavior of investors as non satiable, nor risk averting nor risk seeking, an extension of classic definition of Stochastic Dominance efficiency, linked to behavioral finance is given. In particular, investors' behavior changes according to market conditions. The last part of the chapter presents a methodology, based on estimation function theory, to test for portfolio efficiency with respect a general stochastic ordering. Both the analysis of efficiency for Second order of Stochastic Dominance and behavioral finance, questioned the validity of highly diversified choices. For this reason, this thesis concludes introducing Risk Diversification measures, a new class of functional quantifying the amount of idiosyncratic risk diversified among the assets in a portfolio. The Mean Risk Diversification Efficient Frontier is introduced, along with the concept of Mean Risk Diversification efficiency. The empirical analysis describes the relationship between risk aversion, Risk Diversification and classic diversification, and show how Risk Diversification based strategies perform under periods of financial distress.

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Chapter 1

Introduction

Portfolio selection is the process of finding optimal allocation of wealth across risky assets. Optimal allocation is better understood in terms of efficiency. The seminal work of Markowitz started it all (see Markowitz (1952a)). In the Mean Variance Efficient Frontier (MVEF), efficiency is given by the trade off between expected return and risk. Under mean variance efficiency, investors seek portfolio with the lowest possible variance for a given desired level of mean μ. Portfolios satisfying this condition are called Mean Variance Efficient (Markowitz (1952a)). The MVEF is then composed by portfolios with the minimum variance for all the admissible level of mean. It is possible to compare Mean Variance Efficient portfolios in terms of expected return for a unit of risk using the Sharpe Ratio, defined as the ratio between portfolio mean and standard deviation (see Sharpe (1964)). Then all investors, with mean variance type of preference would prefer, among all the Mean Variance Efficient portfolio, the one maximizing the Sharpe Ratio (see Black (1972) and Ingersoll (1987)). Such portfolio is called Markowitz market portfolio, or tangent portfolio. Following Markowitz analysis, Sharpe (1964), Lintner (1964), Mossin (1966) and Black (1972), developed one the most famous asset pricing model: the Capital Asset Pricing Model (CAPM). Under a series of strong assumption the CAPM established a relationship between any asset return and the market portfolio (see Harris (1972), Ingersoll (1987)). In particular, the expected excess return of any assets, over the risk free, is proportional to the expected excess return of the market portfolio, over the risk free rate (see Ingersoll (1987)). The terms expressing the proportionality relation is called the *beta* of the asset and is usually interpreted as the marginal contribution of the asset to the market portfolio risk.

Despite being fascinating in it's simplicity, Mean Variance efficiency suffers form limitations in its theoretical validity. First, Mean Variance preference implies that investors have increasing risk aversion, hypothesis which is usually rejected in many empirical studies (see Bawa (1975), Levy (1992), Levy and Levy (2002) and Ingersoll (1987)). Furthermore, it necessitates the underling asset return distribution to be elliptical (see for example Chamberlain (1983) and Bawa (1975)). Expected utility is an alternative approach in portfolio optimization overcoming such criticality. Expected Utility is based on an axiomatic description of investors preference (see Von Neumann and Morgenstern (2007)). Typically, investors preferences are represented via expected utility functions: an agent with a utility function u, prefers a portfolio P_1 over a portfolio P_2 if $\mathbb{E}[u(P_1)] \geq \mathbb{E}[u(P_2)]$. The optimal allocation for such investor would then be the solution of a expected utility maximization problem. Nevertheless, except in obvious circumstances, the true form of investors' utility function is not know. Typically, instead of looking for the optimal allocation for a given utility function, we classify investors according to their attitude toward risk, and then express an ordering coherent with investors' preference. Stochastic Dominance, and in general Stochastic Ordering, serves to this purpose (see for example Bawa (1975), Fishburn (1976),Mosler and Scarsini (1991) Levy (1992))). Stochastic Dominance is a partial ordering in the space of distribution functions. Let X and Y be two random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution functions $F_X(t) = \mathbb{P}[X \leq t]$ and $F_Y(t) = \mathbb{P}[Y \leq t]$ respectively. Classic definition of Stochastic Dominance are given in terms of iterative integral conditions on the distribution functions. X dominates Y is the First order of Stochastic Dominance (FSD) if $F_X(t) \leq F_Y(t)$ $\forall t \in \mathbb{R}$; X dominates Y in the Second order Stochastic Dominance if $\int_0^\infty F_X(u)du \leq \int_0^\infty F_Y(u)du$ $\forall t \in \mathbb{R}$. It can be shown that , FSD is coherent with the choice of non satiable investors, i.e. investors with non decreasing utility function ($u' > 0$) and SSD is coherent with the choice of non satiable and risk averse investors, i.e. investors with non decreasing and concave utility function ($u' > 0$ and $u'' \leq 0$).

Expected Utility and Stochastic Dominance have a different definition of efficiency than MVEF. A portfolio is said to be efficient, with respect to a given stochastic ordering, if it doesn't exist a dominating portfolio. Efficient allocations, with respect to a given stochastic ordering, are then optimal for the corresponding investors category. Stochastic orderings and MVEF theory are rival approaches in portfolio selection. In particular, there is a considerable stream of literature suggesting that MVEF is not consistent with SSD efficiency, implying that only under very rare circumstances a non satiable and risk averse investor would be a mean variance optimizer (see for example Borch (1969), Bawa (1975), Markowitz (2014) and Loistl (2015)). Based on this fact, Chapter 2 of this thesis serves as a justification for the usage of stochastic orderings in portfolio selection. It aims to compare MVEF and

Second order Stochastic Dominance efficient set (see Dybvig and Ross (1982) and Roman et al. (2006)). The two portfolio sets are indeed quite different, in terms of portfolio composition and moments. In fact, the only portfolio belonging to both sets is the one composed only by the asset with the maximum mean. An interesting result is linked with diversification. The diversification level in the two sets is quite different. Portfolio belonging to Second order Stochastic Dominance efficient set are in general less diversified than those in the MVEF. It is a common knowledge that high risk aversion should imply higher diversification. However, Second order of Stochastic Dominance is consistent with diversification only under strong assumption (see Wong (2007) and Egozcue and Wong (2010)). When applied to real market, Second order of stochastic dominance efficiency doesn't select highly diversified portfolio (see for example Mansini et al. (2007)). This last observation might serve as an alternative explanation of Statman Diversification puzzle, i.e. the diversification level observed in real market is lower than those predicted by MVEF (see Statman (2004)). Non satiable and risk averse investors prefer more concentrated portfolios with respect to Mean Variance efficient ones. The main results however, is linked with the different efficiency definitions in the two approaches. In fact, it turns out that portfolio belonging to the Mean Variance Frontier where the mean is "low"are second order stochastically dominated. This area comprehends the Global Minimum Variance Portfolio too. The second part of the chapter employs the non efficiency of the Global Minimum Variance Portfolio to construct dominating strategies that outperform it up to 150% in terms of wealth and other performance measures, such as Sharpe Ratio, Maximum DrawDown, and Rachev Ratio (see Biglova et al. (2004), Young (1998), Pedersen and Satchell (2002), Deng et al. (2005), Ruttiens (2013) and Ortobelli et al. (2013)).

According to recent finding in the literature, the risk averse assumption might be too strong to describe investors behavior. Typically, investors prefer more to less, i.e. are non satiable, and are neither risk averse nor risk seeker (see Markowitz (1952b), Kahneman and Tversky (1979), Tversky and Kahneman (1992) and Barberis and Thaler (2003)). Prospect Theory aims to describe this category of investors. It relies on four pillars: investors take decision according to relative change in wealth, rather than total or final wealth; investors are risk risk averse for gains and risk seeker for losses, i.e. have a S-shaped utility function; look at subjective probabilities rather than objective and suffers from framing effect (see Kahneman and Tversky (1979) and Tversky and Kahneman (1992)). On a similar approach, Markowitz in 1952 proposed that investors' utility functions have a fourfold behavior: convex-concave-convex-concave (see Markowitz (1952b)). Investors are then, risk averse for losses and risk seeker for gains, while in case of extreme events, are risk seeker for losses and risk averse for gains. Such typology of utility functions is addressed as inverse S-shaped utility.

Similarly to the case of Expected Utility theory, it is possible to define stochastic orderings even for the S-shaped and inverse s-Shaped utility functions. In particular, X dominates Y in the Markowitz Stochastic Dominance (MSD) if $\int_{-\infty}^{y} F_Y(u) - F_X(u)du \geq 0 \; \forall y \leq 0$ and $\int_{x}^{\infty} F_Y(u) - F_X(u)du \geq 0 \; \forall x \geq 0$. A MSD efficient allocation correspond to an optimal allocation for all the investors with inverse S-shaped utility functions (see Levy and Levy (2002) and Baucells and Heukamp (2006)). Similarly, X dominates Y in the sense of Prospect Theory Stochastic Dominance (PSD) if $\int_{y}^{o} F_Y(u) - F_X(u)du \geq 0 \; \forall y \leq 0$ and $\int_{0}^{x} F_Y(u) - F_X(u)du \geq 0 \; \forall x \geq 0$.

The main idea behind behavioral finance is that, according to market conditions, investors sometimes behave as risk averse, while sometimes as risk seeker. Classic definitions Stochastic dominance then, are not flexible enough to describe what efficiency is. To construct an ordering consistent with the preference of non satiable nor risk averse nor risk seeker investors, Chapter 3 considers a particular family of distribution (see Ortobelli (2001)). Distributions belonging to this family depend on reward and risk measures and, other distributional parameters. This family can be seen as an extension of the elliptical family, widely used in finance and portfolio theory (see for example Owen and Rabinovitch (1983) and Adcock (2010)), where reward and risk measures serves as location and scale parameters. Under minimal assumptions, stochastic dominance conditions are extended in the case of general risk and reward measures. Firstly, in the case where the mean is assumed to be the reward measure, to guarantee second order stochastic dominance efficiency, it is not necessary for the risk measure to be convex. Convexity of risk measures guarantee that diversification doesn't increase risk (see Artzner et al. (1999) and Rachev et al. (2008)). This in some sense confirms the empirical finding of Chapter 2. Diversification is not a necessary condition for Second order Stochastic Dominance efficiency and thus, optimality for non satiable and risk averse investors. Secondly, in the case where a reward measure different than the mean is considered, it turns out that the behavior of non satiable nor risk averse nor risk seeker investors, changes according to market conditions. In particular agents behave, as non satiable risk averse, when the reward measure is lower than the mean, and as non satiable and risk seeker when the reward is higher. This allows to state that efficiency for non satiable, nor risk averse nor risk seeker investors, corresponds to the one for non satiable risk averse investors, in a market where the reward is lower than the expected return, and to the one of non satiable risk seeker investors, when the reward is higher.

The last part of the chapter combines these stochastic dominance relations with estimation function theory, in order to develop hypothesis testing methodology for portfolio efficiency (see Godambe and Thompson (1989) and Crowder (1986)). As an extension of Chapter 3, Chapter 4 presents an application on different data and different functional defining the ordering. The risk functional is based on a linear combination of tail Gini measures, and it corresponds to a weighted difference between a given percentage of worst a given percentage of best outcomes. Results on this ordering suggest that market portfolio is almost never efficient, even if in some situation, depending on the configuration of the functional, it can be hard to find a dominating portfolio.

Both Chapter 2 and Chapter 3 show applications of stochastic orderings in portfolio theory. In particular, definitions of efficient choice, under various investors point of view, are discussed. Starting from slightly different assumptions on investors behavior, both the chapters, come to similar conclusions on diversification. Diversification is not necessary for an efficient portfolio choice.

In ordering theory, diversification is understood in terms of majorization ordering. A portfolio is said to be more diversified than another if the ordered weights of the second majorizes those of the first. In other words, diversification consider the number of assets in which a positive proportion of wealth is invested, rather than *how* it is invested (see Marshall et al. (1943), Wong (2007) and Egozcue and Wong (2010)). Nevertheless, diversification ordering and stochastic dominance are consistent only under strong assumption, that are almost never satisfied by real data (see Samuelson (1967) and Ortobelli et al. (2018)). These last observations motivate Chapter 5. Chapter 5 proposes a new approach, and introduces Risk Diversification. Risk Diversification can be defined as the way idiosyncratic risk is diversified among portfolio's components. Some diversification measures already present in the literature can be seen as special cases of Risk Diversification measures (see Choueifaty and Coignard (2008), Vermorken et al. (2012), Clarke et al. (2013) and Flores et al. (2017)).

The empirical analysis introduces firstly, the mean risk diversification frontier. Similarly to Mean Variance Efficient Frontier, Mean Risk Diversification Efficient Frontier establishes a definition of efficiency. A portfolio is said to be Mean Risk Diversification Efficient if it belongs to the Mean Risk Diversification Efficient Frontier. To different risk measures, correspond different Mean Risk Diversification efficiency. Secondly, it shows how portfolio Risk Diversification based strategies perform under period of financial distress. The results suggests that the higher the risk aversion, the higher the concentration of a portfolio controlled for risk diversification.

In this thesis several datasets are considered. Chapters 2 and 5 consider a market composed by assets belonging to the Dow Jones Industrial Average index (DJIA). The DJIA is composed by 30 stocks of large, publicly owned and United States based company. Assets belonging to the DJIA index are well traded so any non-synchronous trading problems are avoided and the index itself represent a well diversified market portfolio (see Silvapulle and Granger (2001) and Skintzi and Refenes (2005)). Moreover, showing increasing correlation under periods of financial distress and having a well diversified index, it represents a good candidate to test Risk Diversification Measures in Chapter 5 (see Silvapulle and Granger (2001), Skintzi and Refenes (2005) and Preis et al. (2012)).

Chapter 3 analyses monthly observations Fama and French market portfolio and the six Fama and French benchmark portfolios formed on size and book-to-market equity ratio from July 1963 to October 2001 as this particular market portfolio represents a often used benchmark in testing for portfolio efficiency (see Fama and French (1993); Post and Kopa (2013); Kopa and Post (2015); Arvanitis and Topaloglou (2017)). Also it considers daily observation of assets belonging to Nasdaq and New York Stock Exchange (NYSE) from December 1995 to May 2017. NYSE and Nasdaq are the largest exchanges in world by market capitalization. Both the NYSE and Nasaq markets are formed by a large number of stocks. Since Chapter 4 addresses efficiency testing from the different perspective of risk averse and risk seeker investors, market formed by many assets are good candidates to distinguish between the two approaches. For the same reasons, the empirical analysis in Chapter 4 is based daily observation of assets belonging to the Standard and Poor's 500 (SP500) from January 2000 to June 2017.

Chapter 2

Pareto Optimal Choices vs Mean Variance Optimal Choices: a Paradigm of Portfolio Theory

Summary

In this chapter, we compare two of the main paradigms of portfolio theory: Mean Variance Efficient Frontier and Expected Utility. In particular, we aim to prove empirically that Mean Variance Efficient portfolios are sub-optimal for non satiable and risk averse investors. Firstly, we show that the Second order of Stochastic Dominance efficient set is the solution of a multi-objective optimization problem. Secondly, we perform an ex-ante and ex-post empirical analysis. In the ex-ante analysis, we compare Mean Variance and Second order of Stochastic Dominance Efficient sets, looking at portfolio moments, level of diversification and set distances. We also show that, the Global Minimum Variance portfolio and the part of the Mean Variance Efficient Frontier composed by highly diversified portfolio is second order stochastically dominated, providing a possible alternative explanation for the diversification puzzle. In the ex-post analysis, we construct Second order Stochastic dominating strategies that outperform the Global Minimum Variance portfolio in term of wealth and other performance measures.

2.1 Introduction.

Portfolio theory concerns investors taking decisions under uncertainty. If an investor knew future prices, he would invest everything in the asset that has the highest future return. Also, if there were more assets with that same future return, he would be indifferent towards any of those assets (or a linear combination of them) (see Markowitz (1991)). The crucial questions, at this point, are: how do investors decide? How do they combine information and probabilities in order to select the best option? In other words, are there some parameters that investors base their decision on? The main approach established in the literature to construct efficient set of portfolio, considers reward and risk measures (see among the others Markowitz (1952a), Ogryczak and Ruszczyński (2002) and Roman et al. (2006), Stoyanov et al. (2007)). While it is straightforward to pick the expected revenue as a reward measure, the choice of risk measure is less trivial. Here is where two of the most important theories in portfolio selection split into different paths. Mean Variance Efficient Frontier (MVEF) considers variance as risk measure, in particular the variance of the weighted sum of the assets in the portfolio Markowitz (1952a). Many authors have developed very well-known studies based on the seminal work of Markowitz, for example the theoretical and empirical studies on the Capital Asset Pricing Model (see Sharpe (1964), Lintner (1964) and-Mossin (1966)). The advancements of MVEF are also tied to estimation. It is well known that mean-variance optimization is very sensitive to estimation errors, in the sense that there exists a trade off between low number of observation implies estimation errors, while an higher number of data implies an higher likelihood of stationary parameters (see Broadie (1993), Chopra and Ziemba (1993)). The impact of such estimation errors is, however, different between mean and variance and depend on the level of risk tolerance imposed in the optimization (see among the other Chopra and Ziemba (1993)). Since expected value estimation hardly gives a reliable estimate, the literature suggests to apply specific estimation technique, such as shrinkage and concentrate the analysis only on the variance-covariance matrix (see among the other Chopra and Ziemba (1993), Broadie (1993), Jagannathan and Ma (2003) and Ledoit and Wolf (2004)). For this reason, among all the Mean Variance Efficient portfolios, the Global Minimum Variance (GMV) one has become the superstar. This particular portfolio has proved to be not very risk sensitive and also able to outperform in the out of sample analysis most, if not all, of the Mean Variance Efficient portfolios (see Clarke et al. (2011), DeMiguel et al. (2009)).

The other most important framework in portfolio selection is related to Expected Utility and decision theory under uncertainty. In decision theory under uncertainty, Von-Neumann and Morgenstern established the general rule under which all rationale investors take decision: an agent *prefers* an investments w.r.t. an other if its expected utility function is higher. We can always distinguish the choices of all rational investors using Stochastic Dominance rules. Given this fact we know that the set of optimal choices for non-satiable and risk averse investors is defined by Second order of Stochastic Dominance (SSD), which provides a selection criteria valid for all increasing and concave utility functions (see Bawa (1975) and Müller and Stoyan (2002)). Thus, to construct the efficient set for all non satiable and risk averse investors one may chose a coherent risk measure consistent with the SSD (see among the others Ogryczak and Ruszczyński (2002), Roman et al. (2006) and Mansini et al. (2007)).

Although relying on similar mathematical structure, the approaches lack of mutual coherency and, are different in implications and drawbacks. First of all MVEF is consistent with Von-Neumann and Morgenstern Expected Utility paradigm for non satiable and risk averse investors, only under either the assumption of elliptical distributed returns, or when the investors optimize a quadratic utility function which imply increasing absolute risk aversion (see for example Borch (1969), Feldstein (1969), Hakansson (1971), Porter et al. (1973) Bawa (1975)). MVEF provides a justification for diversification and, when no short sales are allowed implies the convexity of optimal choices set (efficient set). While diversification is of most interest for practical purpose, convexity, from an economic point of view, is a desirable property for such sets, in fact it allows the construction of two funds separation theorem, in which each optimal portfolio is a linear combination of a risk free and a the market portfolio (see Sharpe (1964) and Lizyayev and Ruszczyński (2012)).

On the other hand, variance is not a coherent measure and many authors, even Markowitz himself, questioned the use of variance as a measure of risk, and proposed different risk measure as the semi-variance or semi-deviation (see for example Mao (1970) and Artzner et al. (1999)). Also, the estimation of the GMV would need an high number of observations in order to beat a naive diversification strategy such as the $\frac{1}{N}$ portfolio in the out-of-sample type of analysis (see among the others Choueifaty and Coignard (2008), DeMiguel et al. (2009) and Clarke et al. (2011)). Typically, MVEF works as a dimensionality reduction technique. After having derived the MVEF, then a careful choice of a portfolio on the frontier would approximate the maximum of a great number of concave utility functions (see Markowitz (2014) and Loistl (2015)). In virtue of this approximation, many authors believe that lack of consistency with Expected Utility is not of major concern, form the portfolio management point of view. This statement is often supported by the fact that Expected Utility based models, often behave better in the in sample type of analysis, while are outperformed by MVEF based model in the out of sample studies (see for example Simaan (2014)).

Expected Utility and Stochastic Dominance are theoretically based and well established in the literature, since they provide a general criteria valid for typologies of agents sharing the same characteristics (see Müller and Stoyan

(2002)). However, SSD coherent models generally do not show neither convex efficient set nor diversification. A possible explanation is that SSD takes into account all the levels of risk aversion, while a considerable number of which, might be unrealistic (see for example Lizyayev and Ruszczyński (2012), Dybvig and Ross (1982), Ogryczak and Ruszczyński (2002) and Mansini et al. (2007)).

The main area of development of SSD is related to use of different kind of coherent risk measure in the construction of efficient set, and development of efficient algorithms (see Eeckhoudt et al. (2009), Hodder et al. (2014), Fábián et al. (2011)). For example, Ogryczak and Ruszczyński (1999) proposed a mean variance approach using semi-deviation as risk measure and De Giorgi (2005) and De Giorgi and Post (2008) extend this approach by giving conditions that stochastic dominance consistent risk measures must satisfy. Others, have developed tests to verify if a portfolio is stochastically non dominated by using suitable linear program or mixed integer linear programs (see for example Kuosmanen (2004),Post (2003), Kopa and Chovanec (2008), Kopa and Post (2015)), and other have developed methods and algorithms for portfolio problems with stochastic dominance constraints, in which the dominance is w.r.t. a given benchmark, using methodologies from the stochastic programming framework as the sample average approximation (see among the others Dentcheva and Ruszczyński (2006), Homem-de Mello and Mehrotra (2009) and Armbruster and Luedtke (2015)). A similar approach to Kuosmanen (2004) is found in Bruni et al. (2017), where they propose an approximation of Stochastic Dominance rules using the so called Enhanced Indexation (see also Guastaroba and Speranza (2014)).

In this chapter, we exploit Stochastic Dominance and multi-objective optimization to challenge the (MVEF) paradigm of portfolio theory. Our aim is twofold. Firstly, we construct and compare the two efficient sets. The Mean Variance Efficient set is composed by those portfolio belonging to the MVEF. The SSD efficient set is composed by AVaR (Average Value at Risk) Pareto optimal portfolio (see among the others Pflug (2000), Rockafellar and Uryasev (2000) and Rockafellar and Uryasev (2002)and Roman et al. (2006))[1]. To find Pareto optimal portfolios we resort to multi-objective optimization (see among the others Miettinen (2012), Mansini et al. (2007), Roman et al. (2006)). Then, we compare statistics of portfolio belonging to both sets, and also compute distributional distances between the two sets. Secondly, we aim to show that portfolio belonging to the MVEF are sub-optimal for

[1]The AVaR is also known as Conditional Value at Risk (CVaR), or Expected Shortfall (ES). Under the assumption of continuity of the return distributions, AVaR is a coherent risk measure (see Artzner et al. (1999)).

non satiable and risk averse investors. To do so, we propose optimization procedure taking into account that no non satiable and risk averse and non hold portfolios for which, at any level of risk aversion there exists a less risky portfolio.

We consider monthly observations of stocks belonging to Dow Jones Industrial Average index (DJIA) from 18^{th} of March 1997 to 14^{th} October 2017 and perform two type of analysis: ex-ante, or static analysis, where we construct and compare the two efficient sets, and ex-post, or dynamic analysis, where we compare the ex-post performances of the GMV and some strategies based on AVaRs. In the static analysis, we attest the non efficiency of the minimum variance portfolio in a market composed by N assets, in both cases, with and without the possibility to invest in a risk free rate. In addition, we find that more than 11% of portfolios of the Mean Variance Efficient set are second order stochastically dominated. All these portfolios share a common feature: a low expected value. This represents an empirical evidence against preference for highly diversified portfolio, for non satiable and risk averse investors. Moreover, we propose an alternative efficient frontier based on three parameters, Expected return, AVaR and Confidence levels, where we are able to interpret the Confidence levels as the opposite of risk aversion parameters. Finally, in the out of sample analysis, we construct $AVaR$ based strategies able to dominates the GMV portfolio in the SSD, and compare their ex-post performances. Most of all the AVaR based strategies beat the GMV up to 150% in terms of wealth. Moreover, the GMV presents the lowest ex-post expected return, Sharpe Ratio and Rachev Ratios, among the analyzed strategies.

In Section 2.2 we present the construction of the efficient set of portfolios for non satiable and risk averse agents, the related mathematical problem and its property. In Section 2.3 discuss the empirical result on DJIA with an in and out of sample type of analysis.

2.2 Efficient Choices for non Satiable and Risk Averse Investors

In this section, we define and describe the portfolio efficient sets for non satiable investors and non-satiable risk averse investors. Consider a market with N assets and denote $R = [R_1, \dots, R_N]'$ the random vector of returns[2].

[2] We call the return of i-th asset at time t over the period $[t, t+1]$ the value $R_{t+1,i} = p_{t+1,i}/p_{t,i} - 1$ where $p_{t,i}$ is the stock value of the asset i at time t. We sometimes consider gross returns $r_{t+1,i} = p_{t+1,i}/p_{t,i} = R_{t+1,i} + 1$ which are positive random variables or log-

Let $x \in \mathbb{R}^N$ denotes the portfolio weights, and assume that all the portfolios $X = R'x$ are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume also that no short sales are allowed, i.e. $x \in \Delta = \{x \in \mathbb{R}^N : \sum_{j=1}^N x_j = 1, : x_j \geq 0 : j = 1, \ldots, N\}$.

Generally, to describe the investors' optimal choices we can look at the ordering consistent with their preferences. According to non-satiable investors' preferences and to non-satiable risk-averse investors' preferences, we recall the following classic definitions of stochastic dominance orderings (see, among the other, Bawa (1975) , Mosler and Scarsini (1991) and Müller and Stoyan (2002)).

Definition 1. Let X and Y two portfolio of gross returns with distribution functions $F_X(t) = \mathbb{P}[X \leq t]$ and $F_Y(t) = \mathbb{P}[Y \leq t]$, respectively.

- All non-satiable investors prefer X to Y (i.e. $E(u(X)) \geq E(u(Y))$ for all non-decreasing utility functions u) or, equivalently, X dominates Y with respect to the first order of stochastic dominance ($X \, FSD \, Y$), if and only if $F_X(t) \leq F_Y(t)$, $\forall t \in \mathbb{R}$.

- All non-satiable risk-averse investors prefer X to Y (i.e. $E(u(X)) \geq E(u(Y))$ for all non-decreasing and concave utility functions u) or, equivalently, X dominates Y with respect to the second order of stochastic dominance ($X \, SSD \, Y$), if and only if $\int_{-\infty}^t F_X(u)du \leq \int_{-\infty}^t F_Y(u)du$, $\forall t \in \mathbb{R}$.

where the above inequalities are strict for at least one $t \in \mathbb{R}$.

FSD and SSD relationships can be expressed in terms of well known risk measures i.e., $X \ FSD \ Y$ if and only if $VaR_\alpha(X) \leq VaR_\alpha(Y) \ \forall \alpha \in [0, 1]$ with at least one strict inequality and, $X \ SSD \ Y$ if and only if $AVaR_\alpha(X) \leq AVaR_\alpha(Y)$, $\forall \alpha \in [0, 1]$ with at least one strict inequality, where $VaR_\alpha(X) = -F_X^{-1}(\alpha) = -\inf \{u : F_X(u) \geq \alpha\}$ is the Value at Risk, $AVaR_\alpha(X) = \frac{-1}{\alpha} \int_0^\alpha F_X^{-1}(u)du$ is the Average Value at Risk (see for example Ogryczak and Ruszczyński (2002) and Kopa and Chovanec (2008)). Typically, stochastic dominance relationships are implemented to find efficient choices for whole categories of investors. Following Kuosmanen (2004), a portfolio said to be SSD (or FSD) efficient, if another admissible portfolio, ables to dominate it in SSD (or FSD) sense, doesn't exist. We recall these definitions in terms of VaRs and AVaRs (see Roman et al. (2006)).

returns $z_{t+1,i} = \ln(r_{t+1,i})$ which are unbounded random variables. However, results are presented in terms of returns, i.e. $R_{t,i}$, for uniformity with the relevant literature in the field.

Definition 2. A portfolio X^* is said to be FSD efficient if and only if it is not FSD dominated by other portfolios i.e., it does not exist another portfolio $X = R'x$ s.t. $VaR_\alpha(X) \leq VaR_\alpha(X^*)$, $\forall\alpha \in [0,1]$ and $VaR_\alpha(X) < VaR_\alpha(X^*)$ for at least one $\alpha \in [0,1]$. A portfolio X^* is said to be SSD efficient if and only if it is not SSD dominated by other portfolios i.e., it does not exist another portfolio $X = R'x$ s.t. $AVaR_\alpha(X) \leq AVaR_\alpha(X^*)$ $\forall\alpha \in [0,1]$ and $AVaR_\alpha(X) < AVaR_\alpha(X^*)$ for at least one $\alpha \in [0,1]$. We call efficient set for non-satiable investors, all portfolios FSD efficient and, similarly, we call efficient set for non-satiable risk-averse investors, all portfolios SSD efficient.

Still this definition is based on infinitely many conditions. Stochastic dominance rules indeed provide selection criteria based on continuous constraints which, in practice, is most of the time not affordable since it translates into infinitely many constraints (see Bawa (1975)). However, when we assume a finite probability space Ω with T elements and with uniform probability, then, according to Kuosmanen (2004) and Kopa and Chovanec (2008), we can consider only T confidence levels $\alpha_i = \frac{i}{T}$ $(i = 1,\ldots,T\ )$ for ordering portfolios with rispect to FSD and SSD. Typically, we identify the (FSD, SSD) optimal choices considering T "equiprobable" historical observations and thus, SSD (FSD) efficient portfolios satisfy the following "empirical"condition: $X^* = R'x$ is said to be SSD (FSD) efficient if and only if $\nexists\ X = R'x$ s.t. $AVaR_{\alpha_i}(X) \leq AVaR_{\alpha_i}(X^*)$ with $\alpha_i = \frac{i}{T}$ for every $i = 1,\ldots,T$ and $AVaR_{\alpha_j}(X) < AVaR_{\alpha_j}(X^*)$ for at least one j ($\nexists\ X = R'x$ s.t. $VaR_{\alpha_i}(X) \leq VaR_{\alpha_i}(X^*)$ with $\alpha_i = \frac{i}{T}$ for every $i = 1,\ldots,T$ and $VaR_{\alpha_j}(X) < VaR_{\alpha_j}(X^*)$ for at least one j). Such typology of sets is called Pareto optimal and, typically are the solution sets of a multi-objective optimization problems (see among the others Roman et al. (2006), Miettinen (2012) and Luc (2016)). As a matter of fact, the solutions $x^* \in \Delta$ of multi-object problems $\min_{x\in\Delta} f(x)$ where the function: $f : \Delta \to \mathbb{R}^T$ s.t. $f(x) = (f_1(x), ..., f_T(x))$ are called Pareto optimal because $\nexists\ x \in \Delta$ s.t. $f_i(x) \leq f_i(x^*)$ $\forall i$ and $f_j(x) < f_j(x^*)$ for at least j. In our case, f_i is either VaR_{α_i} or $AVaR_{\alpha_i}$, for every $i = 1, ..., T$ and x is the vector of portfolio weights belonging to the convex set $\Delta \subset \mathbb{R}^N$. The literature on finding Pareto optimal of a multi-objective problem is wide and broad. In particular, when the multivalued function f is continuously differentiable we know that $x^* \in \Delta$ is Pareto optimal for f if and only if it solves the following optimization problem for any $i \in \{1,\ldots,T\}$:

$$\begin{aligned} &\min_{x\in\Delta}\ f_i(x) \\ &s.t.\quad f_j(x) - m_j \leq 0 \qquad j = 1, ..., T;\ j \neq i \end{aligned} \tag{2.1}$$

for some values m_j. This transformation of the multi-object problem in a minimization problem of a real function with constraints is also known as the ϵ-constrained method. Since the minimization of VaR could give more than one local optimum, determining the efficient set of all non-satiable investors is not a simple problem and will be object of future researches according to the seminal work of Bawa (1976) in a mean variance framework. For this reason in this chapter we essentially discuss and study the characteristics of the portfolio SSD efficient set. Thus, according to the ϵ-constrained method for SSD efficient set, a necessary and sufficient condition for X to be a Pareto optimal portfolio and, consequently, SSD efficient portfolio is that x solves the following optimization problem:

$$\begin{aligned} \min_{x \in \Delta} \quad & AVaR_{\alpha_i}(X) \\ s.t. \quad & AVaR_{\alpha_k}(X) \leq m_k \qquad \forall k \neq i \end{aligned} \tag{2.2}$$

for all $i = 1, ..., T$ and some values m_k.

Remark 1. *According to Roman et al. (2006) and Luc (2016), determinig the SSD efficient portfolios can be rearranged as a multi-objective linear program. Then, the efficient set of a multi-objective optimization consists in faces of the feasible set. Moreover if a relative interior point of a face is Pareto optimal, every point of the face is Pareto optimal. These properties should enlighten the discussion of the non-convexity of the Pareto set (see Dybvig and Ross (1982)). One of the reasons behind the success of the MVEF is the convexity of the optimal portfolio weights. In fact this property opened the door for the Two Funds Separation Theorem and the Capital Asset Pricing Model (Mossin (1966), Lintner (1964), Sharpe (1964)). However, as pointed out by Dybvig and Ross (1982) and Lizyayev and Ruszczyński (2012), the efficient set for non-satiable risk-averse investors is a finite union of convex sets, and therefore it's generally not convex. Thus, it could happen that the market portfolio is not SSD efficient. Recall that, it is possible to derive SSD efficiency tests applying different optimality conditions for multi-objective linear problem (see Kopa and Post (2015) and reference therein).*

The following Proposition identify a property of the SSD dominated portfolios (the inefficient ones).

Proposition 1. *Assume $X = R'x$ is second order stochastically dominated portfolio. Then for all $i = 1, ..., T$ there exists a portfolio, say $Y_{(i)} = R'y$, such that:*

$$AVaR_{\alpha_i}(Y_{(i)}) < AVaR_{\alpha_i}(X) \tag{2.3}$$

Proof. Assume that for a given i it doesn't exist a portfolio $Y_{(i)}$ such that $AVaR_{\alpha_i}(Y_{(i)}) < AVaR_{\alpha_i}(X)$. Then, X is a solution of the problem (2.2) and it is SSD efficient, that is in contradiction with our hypothesis. $\square$

An economic interpretation of this result is the following: agents prefer outcome in which the allocation of undertaken risk is optimal. In other words, if a non-satiable risk-averse investor can undertake less risk, he or she will go for it. The main impact of this result is, however, in the application. In the empirical analysis we will apply Proposition (1) to the portfolios belonging to the MVEF in order to verify their SSD efficiency. We know that MVEF portfolios are efficient for risk averse investors because if all risk averse investors prefer a portfolio X to a portfolio Y, then the portfolio X should have lower variance than portfolio Y. However, it could be that optimal MVEF portfolios are not efficient for non-satibale risk-averse investors. Then, call $X_{mv} = R'x_{mv}$ to verify if it is SSD efficient, we solve the following optimization problem:

$$\begin{aligned} \min_{x\in\Delta} \quad & AVaR_{\alpha_i}(X) \\ s.t. \quad & AVaR_{\alpha_k}(X) \leq AVaR_{\alpha_k}(X_{mv}) \qquad \forall k \neq i \end{aligned} \tag{2.4}$$

for all the i. Clearly, if we can find at least one portfolio solution of such problem, it implies that X_{mv} is SSD dominated. Moreover, if X_{mv} is SSD dominated, then by Proposition (1) the feasible set of the Problem (2.4) is non-empty for any $i = 1, ..., T$. Let us give a simple example with some given historical observations.

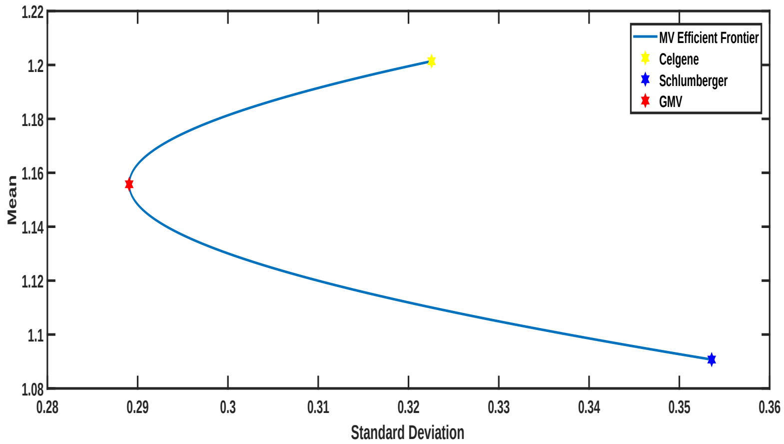
Example 1. *Consider the return series of two assets (Celegene and Schlumberger) with 10 years of yearly observations (from June 2006 to July 2016) and compute the MVEF between them*[3]*. Let X and Y be portfolios of returns and arrange them in ascending order, i.e. $X_{[1]} \leq ... \leq X_{[T]}$ and $Y_{[1]} \leq ... \leq Y_{[T]}$. Then, X second order stochastically dominates Y if and only if*

$$\frac{-1}{k}\sum_{i=1}^{k}\left(X_{[i]} - Y_{[i]}\right) \leq 0 \qquad \forall k = 1, ..., T. \tag{2.5}$$

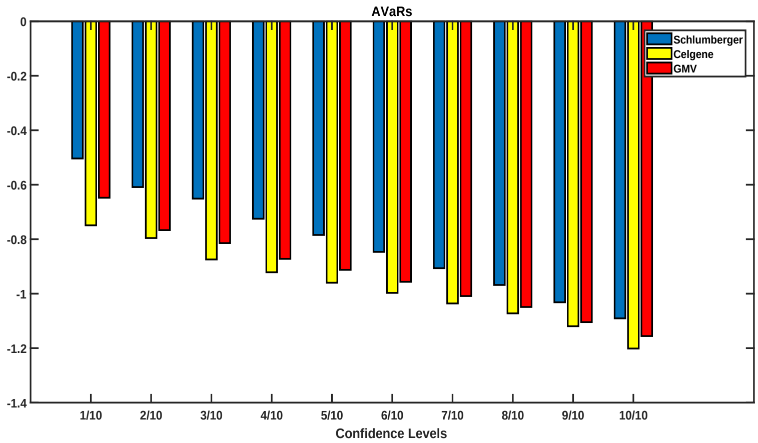
Observe that formula (2.5) represents the differences of all $AVaR_{\alpha_k}$ levels. Figure (2.1) shows the MVEF and the $AVaR_{\alpha_k}$ of the two assets and the

[3]Celgene adjusted gross return series, rounded at the third decimal is R_C = [0.268, 0.114, −0.251, 0.062, 0.184, 0.032, 0.938, 0.498, 0.327, −0.157]′. Schlumberger adjusted gross return series, rounded at the third decimal is R_S = [0.399, 0.265, −0.496, 0.023, 0.539, −0.264, 0.158, 0.621, −0.286, −0.053]′ The GMV portfolio gross return series, rounded at the third decimal is R_{GMV} = [0.322, 0.176, −0.352, 0.046, 0.330, −0.091, 0.616, 0.549, 0.074, −0.115]′.

global minimum variance (GMV) portfolio. As we see from panel (b), Celgene's $AVaRs$ smaller than the GMV one implying that it second order stochastically dominates the GMV portfolio. In addition the difference is strict for all $AVaR_{\alpha_i}$ levels according to Proposition (1).



(a) MVEF between Schlumberger and Celgene. The frontier is computed with yearly observations from June 2006 to July 2016. In this example, Celegene presents the highest mean, while Schlumberger presents the highest standard deviation. The GMV portfolio weights are 0.412 and 0.588



(b) Cumulative sums of Schlumberger, Celgene and GMV portfolio. As we see Celgene cumulative sums are higher than GMV one in each of the "state of the world"and hence satisfies condition (2.5).

Figure 2.1: Example with only two assets

In the next section we are going to construct the AVaR Pareto optimal set for portfolios composed by assets belonging to the DJIA.

2.3 Second Order Stochastic Dominance Efficient Set in Practice

We consider monthly observations of stocks belonging to DJIA index from 18^{th} of March 1997 to 14^{th} October 2017. The dataset is composed by those assets belonging to the DJIA at the day 14^{th} October 2017. For the ex-ante analysis, the total number of assets used is 28. The two missing stocks are VISA, which went public in 2008, and CISCO System. In the ex-post analysis, VISA is then included in the dataset after the 20^{th} March 2008.

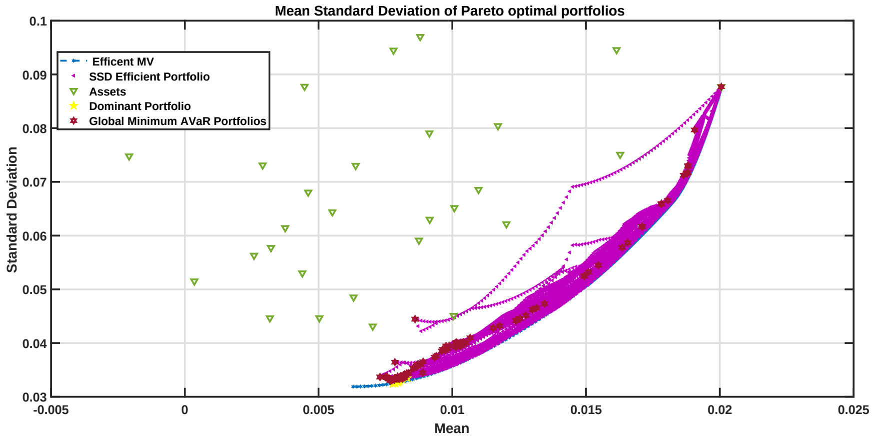
We proceed the analysis on gross returns. Let $p_{t,i}$ be the stock value of the asset i at time t, then its return over the period $[t, t+1]$ is defined as $y_{t,i} = p_{t+1,i}/p_{t,i}$. Establishing dominance among returns also implies the dominance among log-return, but the converse is not necessarily true (see for example Fishburn (1964)). Under this construction, we can consider gross returns as positive random variables implying that we cannot assume any kind of underling elliptical distribution. Results are presented in terms of returns, i.e. $y_{t,i}* = p_{t+1,i}/p_{t,i} - 1$, for uniformity with the relevant literature in the field.

2.3.1 Ex-Ante Analysis

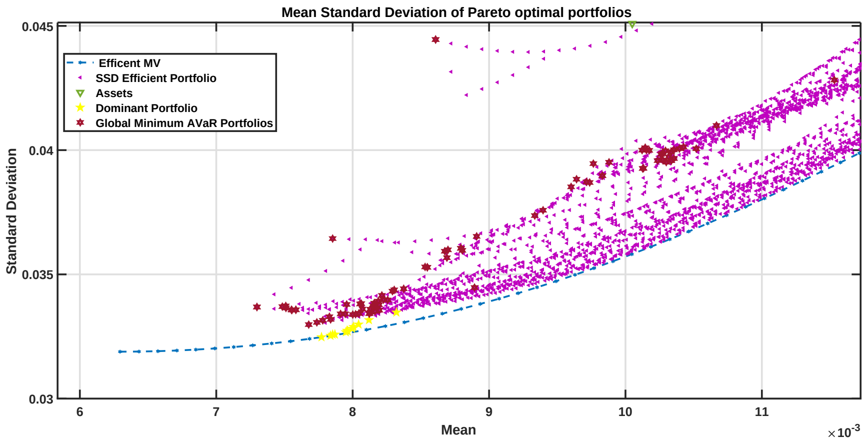
For ex-ante analysis we consider 10 of years of monthly observations of assets belonging to the dataset, i.e. from 13^{th} of October 2007 to 14^{th} of October 2017[4].

The aim of this part is to compare empirically, MVEF and SSD efficient sets. The crucial idea, is to show that the two sets differ. First, we solve the quadratic program associated to the Mean Variance Efficient set and solve Problem (2.2) to construct the SSD efficient set. Second, for each Mean Variance Efficient portfolio, according to Proposition (1), we solve Problem (2.4). If we cannot find, for each level of risk aversion α_i, the less risky portfolios $R'z$ it means that the *tested* Mean Variance Efficient portfolio is also efficient for non satiable and risk averse investors. The MVEF is composed by 100 portfolios, while the total number of SSD efficient portfolio is 12000. Then, we project onto the *Expected Value*, *Standard Deviation* plane. Figure 2.2 shows the result: (a) depicts the entire projection, while (b) is an enlargement of the first part. These graphs give interesting insights on the relationship between mean variance efficiency and AVaR Pareto optimal portfolios. Blue triangles with line is the classical MVEF, the purple triangles are the SSD ef-

[4]The number of total assets used in the ex-ante analysis is 28. The two missing stocks are VISA, which went public in 2008, and CISCO System.



(a) Comaprison between mean variance efficient frontier and AVaR Pareto optimal portfolios.



(b) Enlargement of the first part.

Figure 2.2: Comparison between mean variance efficient frontier and AVaR Pareto optimal portfolios. The asset universe is composed by monthly observations of assets belonging to the DJIA index, from the 18th of March 1997 to 13th of May 2007. The dashed blue line is the MVEF, purple triangles represent the mean and the standard deviation of the SSD efficient portfolios, green triangles represent the mean and the standard deviation of the stand alone assets. The yellow stars depict the mean and the standard deviation of portfolio dominating the GMV, i.e. portfolio solutions of problem 2.4, while red stars are the 120 portfolios of global minimum AVaR. Panel (a) shows the entire Mean-St.Deviation plan. As wee can see, there is an area nearby the Mean Variance Efficient Frontier formed by SSD efficient portfolios. Panel (b) shows an enlargement of the first part. The yellow stars are portfolios able to dominates the Mean Variance Efficient w.r.t. SSD.

ficient portfolios, the green triangles are the 28 assets, the yellow stars are the AVaR Pareto optimal portfolios that second orders stochastically dominate MVEF portfolios, the red stars are the 120 global minimum AVaR portfolios. As expected, the asset with maximum mean over the period belong to both efficient sets, while none of the others belong either to the mean variance nor the SSD efficient sets. From panel (a), which depicts the entire picture, we see that the majority of SSD efficient portfolios lies nearby the MVEF, but still there is a conspicuous number with higher standard deviation. Also their projection on the Mean-St.Deviation plane present two"kinks"around the points $(0.015, 0.07)$ and $(0.015, 0.055)$. The inner kink might suggests that the in the blank zone in the SSD efficient portfolio projection lie some SSD efficient portfolio that we are not able to construct due to the discretization in Problem (2.2). The other kink is probably due to the statistics of the asset belonging DJIA index. The only asset that belongs to the SSD efficient set is the one with the highest mean, and therefore the mean and standard deviation of the other assets affect the shape of the SSD efficient projection. From (b) we see that the first part of MVEF is second order stochastically dominated, approximately the first 12%, confirming the theoretical results in Ortobelli (2001) and also coherently with Roman et al. (2006) and Mansini et al. (2007).

MVEF selection criteria is based on the comparison between the first two central distributional moments, while stochastic dominance, in general, compares whole distributions. In the literature the first approach is also referred as in terms of primary probability functionals, while the second is in terms of simple probability functionals (see among the other Ortobelli et al. (2009) and Rachev et al. (2011)). Therefore, to better understand the differences between the two efficient sets, we first report some portfolio statistics, which concerns the first typology of orderings, and then study some distributional distances linked to FSD and SSD. Since we are dealing with a large number of portfolios, we divide the area of the mean-standard deviation plane into 10 parts, equally spaced apart on the mean axis.

In the first case, we compute the average of the following statistics of portfolio belonging to each areas: expected return, standard deviation, skewness, kurtosis, sum of squared portfolio's weights and number of assets which present a non-zero weight.

Table 2.1 reports the results: the first row of each areas refers to the Mean Variance Efficient set, while the second row refers to the AVaR Pareto optimal set. As expected, in general the mean is approximately the same across the efficient sets while the AVaR Pareto optimal set presents higher standard deviations. The most interesting area is the first, where the GMV lies, where portfolios belonging to the AVaR Pareto optimal set present a less negative

Table 2.1: Average of mean, standard deviation, skewness, kurtosis, sum of squared weights and number of invested assets, of portfolios belonging to each of the ten groups.

Range(10^{-3})		Mean	St. Dev.	Skewness	Kurtosis	$\sum_i x_i^2$	♯ Assets
$[0.0063, 0.0077)$	MV	0.0069	0.0320	-1.6861	10.869	0.128	11
	SSD	0.0075	0.0338	-1.3034	9.091	0.166	16.7
$[0.0077, 0.009)$	MV	0.0083	0.0330	-1.4574	9.867	0.171	9.3
	SSD	0.0085	0.0345	-1.5937	11.393	0.181	19.6
$[0.009, 0.0104)$	MV	0.0096	0.0351	-1.3924	9.882	0.198	8.2
	SSD	0.0097	0.0371	-1.6562	12.239	0.186	20.1
$[0.0104, 0.0118)$	MV	0.0110	0.0382	-1.4003	10.412	0.207	8.1
	SSD	0.0111	0.0406	-1.7076	13.213	0.189	20.2
$[0.0118, 0.0132)$	MV	0.0124	0.0420	-1.3616	10.490	0.214	7.1
	SSD	0.0124	0.0446	-1.6037	12.906	0.206	19.7
$[0.0132, 0.0145]$	MV	0.0138	0.0465	-1.2889	10.250	0.237	6.2
	SSD	0.0138	0.0490	-1.5095	12.451	0.220	18.7
$[0.145, 0.0159)$	MV	0.0152	0.0517	-1.3077	10.754	0.234	5
	SSD	0.015	0.0541	-1.4584	12.218	0.236	18.3
$[0.0159, 0.0173)$	MV	0.0166	0.0580	-1.2955	10.960	0.274	5
	SSD	0.0166	0.0600	-1.4283	12.092	0.282	17.6
$[0.0173, 0.0187)$	MV	0.0180	0.0653	-1.2302	10.563	0.393	4
	SSD	0.0180	0.0662	-1.3672	11.294	0.389	16.5
$[0.0187, 0.0201]$	MV	0.0193	0.0776	-0.8590	7.666	0.704	3
	SSD	0.0193	0.0786	-0.9546	7.828	0.726	14.1

skewness and a lower kurtosis, on average. Moreover, in all other areas SSD efficient portfolio present more negative skewness and higher kurtosis than Mean Variance Efficient portfolios. The relation with diversification can be seen in the last two columns; the portfolio diversification level is similar, as explained by the sum of squared weights, but the number of invested assets is higher in the AVaR Pareto optimal set. This implies that portfolios in the AVaR Pareto optimal set present an higher number of invested assets, but with only few of them with a considerable invested amount. In other words, these portfolio are more concentrated than those in the Mean Variance Efficient sets. Combining this result with the previous picture (panel (b)), we see that the part of the MVEF where diversification in higher is also second order stochastically dominated. In other words, a part of the Mean Variance Efficient portfolios are not optimal for non-satiable and risk averse investors. Since this part of the efficient frontier is composed by highly diversified portfolios this result might in some sense, give and explanation to the diversification puzzle in portfolio theory (see for examplesStatman (2004) and Egozcue et al. (2011)). Agents do not really seek for high diversification because, according to these results, highly diversified portfolios appear to be second order stochastically dominated and therefore sub-optimal for non-satiable and risk averse investors.

Having showed the differences in terms of mean statistics between the two sets, we proceed to consider differences in terms of distributions. We consider three different distributional distances. Let $X = R'x$ and $Y = R'y$ be two portfolio with weights x and y respectively. Call $d_1(X,Y)$ the following distance:

$$d_1(X,Y) = \sum_{i=1}^{N} |x_i - y_i|$$

$d_1(X,Y)$ define a distance between the weights of two chosen portfolio, it takes value in the interval $[0,2]$, where 0, by definition, imply that the two portfolio are equal, and 2 implies that the two portfolio are completely different in the sense that, the two portfolio do not invest in the same set of assets.

The second distance we consider is a particular type of Levy quasi-semidistance (see Rachev (1991) and Rachev et al. (2011)). It can be defined as:

$$d_2(X,Y) = \max_t |F_X^{-1}(t) - F_Y^{-1}(t)|$$

$d_2(X,Y)$ takes values in $[0,\infty]$, can be interpreted as a measure of closeness of distribution graphs and, it can be considered as a distance in terms of Value at risk. Moreover, d_2 has a nice relation with FSD, in the sense that *metrizes* the preference relation induced by FSD[5].

Similarly, a distance in terms of AVaR and linked with the SSD can be defined as follows:

$$d_3(X,Y) = \max_t \left| \int_0^t F_X^{-1}(u) - F_Y^{-1}(u) du \right|.$$

In each of the 10 areas, we compute the distances between the Mean Variance Efficient and the SSD efficient portfolios. Table (2.2) reports minimum, mean and maximum values.

[5]see Rachev et al. (2011) for a complete and detailed discussion on this.

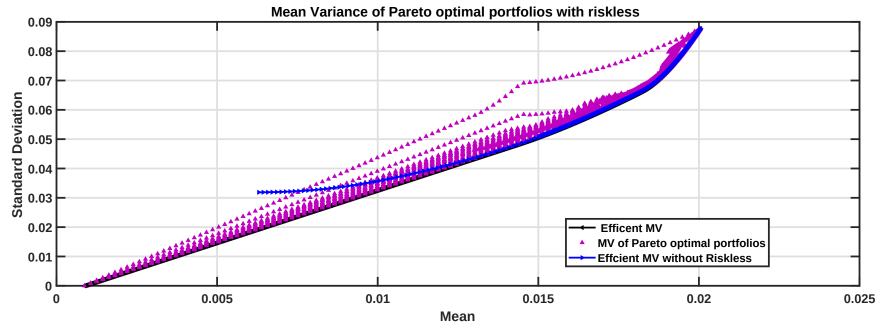
Table 2.2: Minimum, mean and maximum values for all the distances between portfolios belonging to SSD efficient set and MVEF.

Range	d_1 mean	d_1 min	d_1 max	d_2 mean	d_2 min	d_2 max	d_3 mean	d_3 min	d_3 max
$[0.0063, 0.0077)$	0.620	0.529	0.737	0.012	0.011	0.0141	0.099	0.076	0.135
$[0.0077, 0.009)$	0.275	0.191	0.361	0.008	0.006	0.0109	0.036	0.026	0.048
$[0.009, 0.0104)$	0.122	0.099	0.162	0.005	0.003	0.0070	0.014	0.010	0.022
$[0.0104, 0.0118)$	0.114	0.093	0.145	0.005	0.003	0.0085	0.016	0.011	0.021
$[0.0118, 0.0132)$	0.208	0.191	0.221	0.007	0.006	0.0084	0.020	0.016	0.028
$[0.0132, 0.0145]$	0.094	0.063	0.148	0.005	0.003	0.0086	0.015	0.009	0.022
$[0.145, 0.0159)$	0.042	0.022	0.062	0.003	0.001	0.0058	0.012	0.004	0.020
$[0.0159, 0.0173)$	0.093	0.032	0.137	0.007	0.003	0.0116	0.029	0.008	0.048
$[0.0173, 0.0187)$	0.069	0.006	0.101	0.006	0.0007	0.0088	0.025	0.002	0.040
$[0.0187, 0.0201]$	0.013	0	0.027	0.002	0	0.0041	0.005	0	0.013

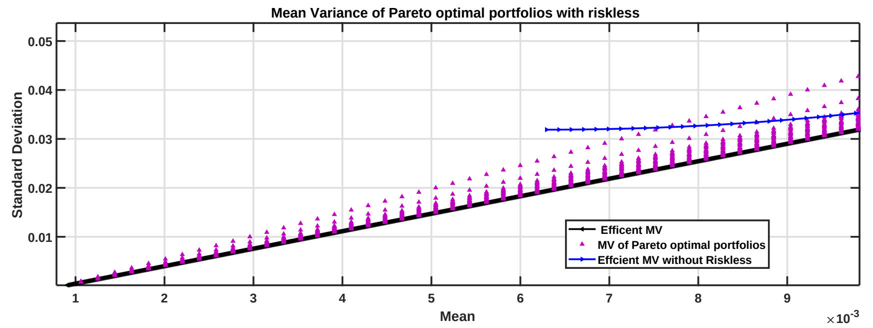
Distances d_2 and d_3 are set distances: when the minimum is equal to 0 implies that the intersection between the two sets is non-empty; when the maximum distance is 0 implies that the Mean Variance Efficient set is a subset of the SSD one. In other words, when the minimum of $d_2 = 0$ (or $d_3 = 0$) then, at least a Mean Variance Efficient portfolio belongs to the SSD efficient sets, while the maximum of $d_2 = 0$ implies that all Mean Variance Efficient portfolio in that area are SSD efficient. The economic interpretation of this, is that , when $d_2 = 0$ then, there exists at least one Mean Variance efficient portfolio which is at least as *preferable* as the SSD efficient one, according to all non satiable investors' preferences. When $d_3 = 0$, the same is valid but for all non satiable and risk averse investors (see Rachev et al. (2011)).

The first three columns of Table (2.2) show that portfolio dissimilarity is higher in the area near the GMV than in the other, more than 25% of the portfolio composition is different, and tends to decrease as the mean increase. Note that the minimum values for the area 10 is zero for all the distances because the asset with the maximum mean belongs to both efficient sets. Also set distances remain remarkable, even if they tend to decrease as the mean increases. As we see from the results for d_2 and d_3 none of the Mean Variance Efficient portfolios belong to the SSD efficient sets, rather than the maximum mean asset. This doesn't imply that all the MVEF is second order stochastically dominated but more reasonably, that the discretization, due to computational feasibility, in some sense affects the results.

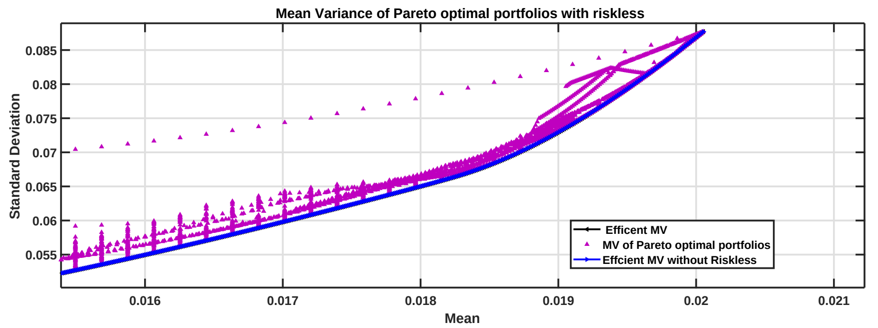
We perform the same empirical analysis including now also the risk free



(a) Comaprison between mean variance efficient frontier and AVaR Pareto optimal portfolios with risk free.

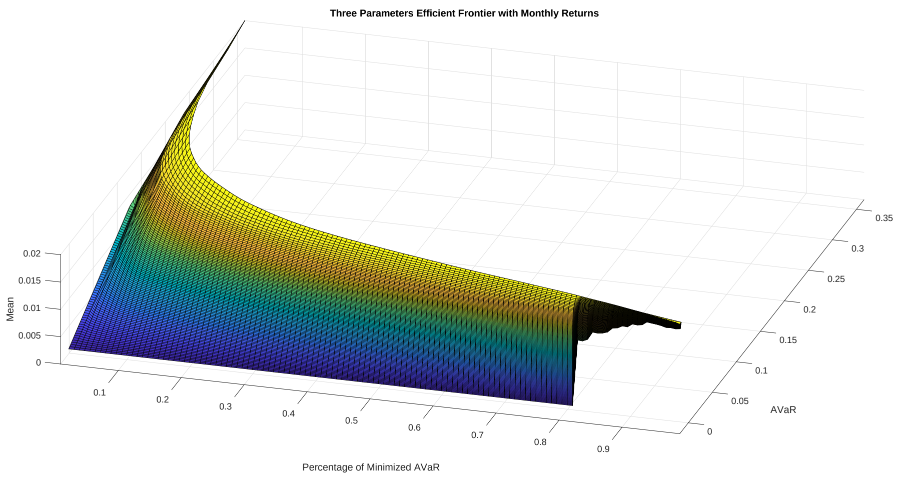


(b) Enlargement of the first part.



(c) Enlargement of the last part.

Figure 2.3: Comparison between mean variance efficient frontier and AVaR Pareto optimal portfolios with risk free . The asset universe is composed by monthly observation of the assets belonging to the DJIA index, from the 18th of March 1997 to 13th of May 2007 and a risk free rate equal to 0.0105. The dashed blue and black lines are the MVEF. The purple triangles are purple triangles represent the mean and the standard deviation of the SSD efficient portfolios. Panel (a) shows the entire Mean-St.Deviation plan. As wee can see, there is an area nearby the Mean Variance Efficient Frontier formed by SSD efficient portfolios. Panel (b) shows an enlargement of the first part.



(a) Three dimensional AVaR Efficient Frontier. This graph shows an Efficient Frontier atlerantive to the MVEF. The surface is composed by the Loretz curves of each SSD efficient portfolio.

Figure 2.4: Three dimensional AVaR efficient frontier

rate. The risk free rate is taken from the three moth treasury bill at the 17^{th} October 2017, and is equal to 0.0105. Graphical inspection of the result is reported in Figure 2.3. The blue line represent as before the mean variance frontier without the risk free rate, while the black line is the one with the risk free rate. As we see the behavior so far is exactly standard. The purple triangles are Pareto Optimal portfolios. The structure in panel (a) resemble Figure 2.2 panel (a), in the sense that all the Pareto optimal portfolios lie in an area near the MVEF. In panel (b) we focus on the first part. Enlargement (c) tells an even more interesting story. There is a point, after which the SSD efficient set converge to the MVEF. A possible explanation of this fact is the presence of the maximum mean portfolio on the MVEF. Since this portfolio is also non second order stochastically dominated, such convergence has to occur at a certain point.

Before moving to ex-post analysis we investigate one last features of the results. In Figure 2.4 is showed a particular three dimensional efficient frontier for monthly returns. We have on the two horizontal axis the $AVaR$s and the relative confidence level, the vertical axis considers the expected portfolio return (corresponding to -$AVaR_{\alpha_T}$). Thus for each confidence level $\frac{i}{120}$ (i=1,...,120) and for a fixed expected portfolio return we compute the minimum $AVaR$ portfolio.

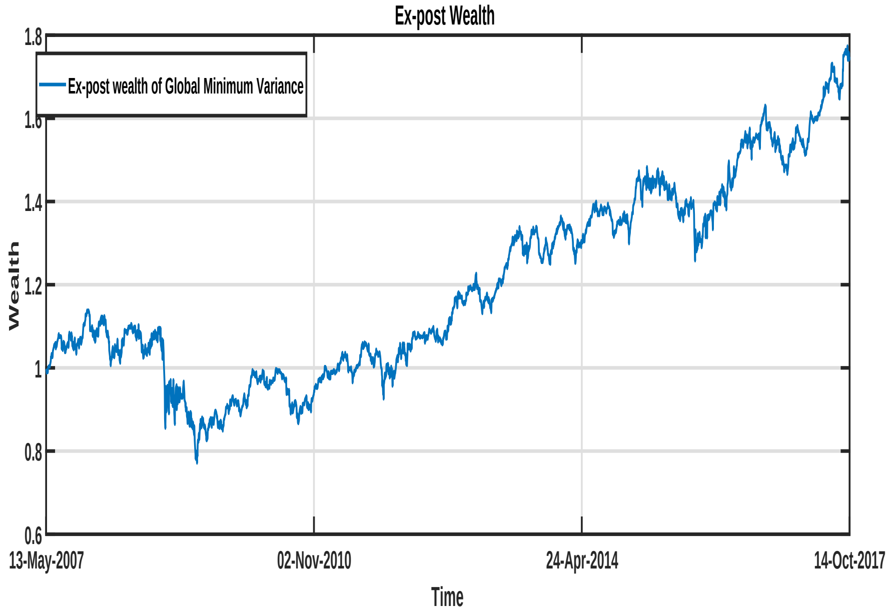
Figure 2.4 shows some kind of convexity with respect to confidence level $\frac{i}{120}$ which allows us to interpret it as a opposite of a risk aversion coefficient: a less risk averse investor prefers a AVaR at higher i. In other words, on this figure we have AVaR, a coherent risk measure, $\frac{i}{120}$ as risk aversion coefficient and expected return, as a reward measure, meaning that if an investor seeks a revenue level, given his or her risk aversion coefficient, must undertake at least the quantity of risk given by the global minimum AVaR. The surface in Figure 2.4 is composed by the Lorenz curves of all SSD efficient portfolios. For each portfolio is possible to determine aggressive and defensive securities among its components. Assets with Lorenz curve below the portfolio Lorenz curve helps to reduce the portfolio instability. Assets with Lorenz curve above the portfolio Lorenz curve tends to amplify the portfolio instability (see Shalit and Yitzhaki (1984)). Therefore, the assets with Lorenz curve below the surface are considered defensive by all the non satiable and risk averse investors, those above are considered aggressive.

2.3.2 Ex-Post Analysis

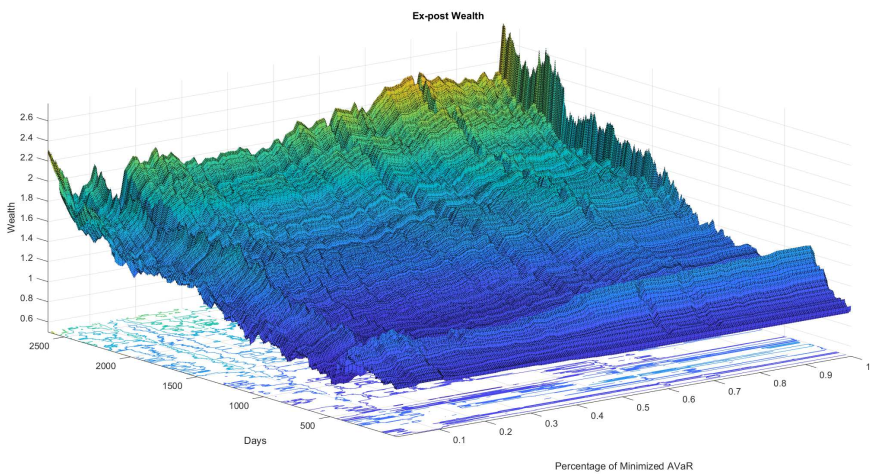
In this section we present an ex-post analysis. We divide the dataset into two parts: from 18^{th} March 1997 to 13^{th} May 2007, and from that date to end. In the first part of the dataset, we look for the 120 portfolio that dominates the GMV one (henceforth 120 AVaR strategies). To do so, we perform a rolling window analysis, with a time window of 10 years, solve Problem (2.4) where we recalibrate the portfolio each 21 trading days with transaction cost of 0.2%. Then on the second part we compare their ex-post performances in terms of wealth and other performance measures. Figure 2.5 shows the ex-post wealth of GMV and of 120 AVaR strategies. The GMV portfolio reach a level a level of wealth at the end of the period of 1.8. It shows a downwards during after the sub-prime crisis of 2008, and as expected it is not so much volatile. The AVaR strategies instead seem to reach an higher level of wealth but with higher volatility.

To see this, see consider Figure (2.6). In Figure(2.6) are reported the ex-post wealth of some selected AVaR strategies, the GMV, the maximum Sharpe Ratio and the equally wighted portfolio[6]. The red dense line is the ex-post behavior of the GMV, while the blue dense line is the behavior of the asset with maximum mean. All other dashed lines are represent the ex-post wealth of other AVaR strategies. The Black line is the maximum Sharpe Ratio and the gray line is the naive strategy. As we see, selected strategies outperform the GMV ex-post in terms of wealth up to 150%.

[6]The selected strategies are portfolios solution of Problem (2.4) with objective functions: $AVaR_{5\%}$, $AVaR_{10\%}$, $AVaR_{30\%}$, $AVaR_{50\%}$, $AVaR_{75\%}$, $AVaR_{90\%}$, $AVaR_{95\%}$, $AVaR_{100\%}$



(a) Ex-post wealth of the minimum variance portfolio.



(b) Ex-post wealth of the 120 AVaR Pareto optimal portfolios

Figure 2.5: Ex-post Strategies

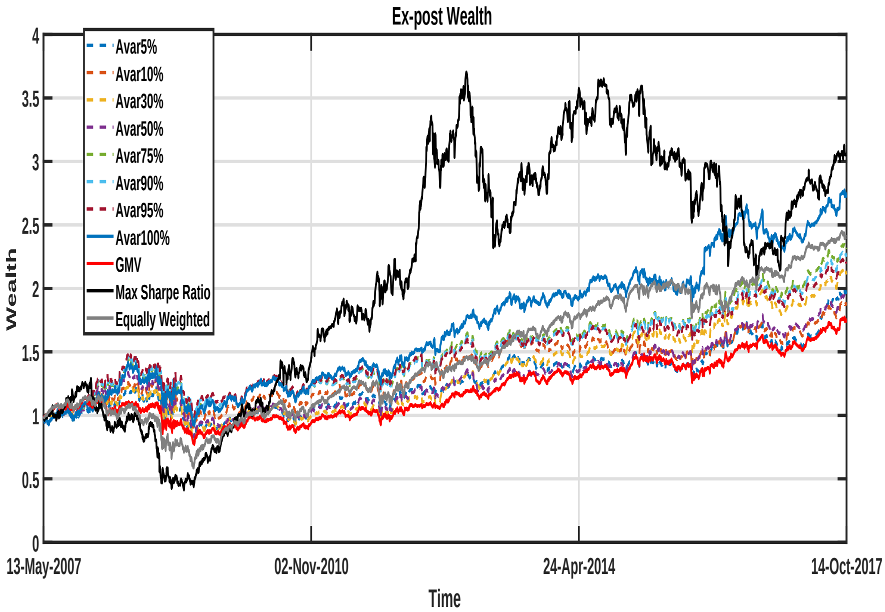


Figure 2.6: Ex-post wealth of selected strategies from May 2007 to October 2017 with . The selected strategies correspond to $AVaR_{5\%}$, $AVaR_{10\%}$, $AVaR_{30\%}$, $AVaR_{50\%}$, $AVaR_{75\%}$, $AVaR_{90\%}$, $AVaR_{95\%}$, $AVaR_{100\%}$ and the maximum Sharpe Ratio and a Equally weighted Portfolio.

We also consider five well known performance measures: Sharpe Ratio, MinMax Ratio, Maximum Drowdown, Ruttiens Ratio,Sortino and Satchell Ratio and Rachev Ratio. Let $X = R'x$ be a portfolio, then the Sharpe Ratio can be defined as

$$SR(X) = \frac{\mathbb{E}[X]}{\sqrt{\mathbb{V}(X)}}$$

The MinMax Ratio is defined as the ratio between expected value and portfolio value in the worst possible scenario. It is also consistent with the behavior of the most risk averse investor. It's defined as follows:

$$MM(X) = \frac{\mathbb{E}[X]}{-\min_s X_s}$$

where X_t is the portfolio value in the state of the world s. (see Young (1998) and Deng et al. (2005)). The Maximum DrawDown is the maximum of the drawdown function, defined in terms of the maximum of the portfolio up to time t and the portfolio at time t:

$$DD(X,t) = \frac{\max_{0 \leq \tau \leq t}(X_\tau - X_t)}{\max_{0 \leq \tau \leq t} X_\tau}$$

Then, the Maximum DrawDown of a portfolio X, over the period $[0,T]$ is:

$$MDD(X) = max_{0 \leq t \leq T} DD(X,t)$$

The Ruttiens Ratio is a dynamic performance ratio, in the sense that consider how returns change over time and, it's based on portfolio wealth. The risk measure is taken as the standard deviation of the quantity $C_t = W_t - (\frac{t}{T})(W_T - W_0)$ where W_t is the wealth of the portfolio at time t, i.e. (see Ruttiens (2013)):

$$Risk = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(C_t - \mathbb{E}[C])^2}$$

Then, the Ruttiens Ratio is defined as the excess wealth over the period over a discount rate proportional to the $Risk$ (see Ortobelli et al. (2013)):

$$Ruttiens(W_T) = \frac{W_T - 1}{1 + kRisk}$$

Finally, The Sortino and Satchell Ratio is the ratio of expected value and semi-standard deviation and suitable in case of asymmetric return distributions (see Pedersen and Satchell (2002)):

$$SS(X) = \frac{\mathbb{E}[X]}{\sigma(h)}$$

where $\sigma(h) = \frac{1}{T}\sum_{k=1}^{T}(h - X_k)_+$ and $(a)_+ = \max(a, 0)$. The Rachev Ratio is a performance measure which is specifically designed to consider the tail behavior of portfolio distribution and it is defined as a ratio of $AVaR$ at difference confidence levels (see Biglova et al. (2004)).

$$RR_{\alpha,\beta}(X) = \frac{AVaR_{\alpha}(-X)}{AVaR_{\beta}(X)}$$

Tables (2.3) and (2.4) report the ex-post performance measure values for the GMV, the selected AVaR strategies, a maximum Sharpe Ratio strategy (MSR) and the equally weighted portfolio (naive)[7]. The AVaR strategies reach an ex-post expected return, up to the 180% of the GMV. Even in the ex-post analysis the GMV presents the lowest standard deviation among all the selected strategies, and the MSR and naive strategies. Nevertheless, all other performance measures seem to advocate in favor of the other selected strategies. First of all, the GMV presents, ex-post, a lower value for all the performance ration than the AVaR strategies, Sharpe Ratio included. Moreover, it is possible also to infer some decision rules based on ex-post results. An agent with preferences consistent with Sharpe Ratio or with the Rachev Ratio, would prefer the portfolio solution of Problem (2.4) with $AVaR_{*}100\%$, over all the selected strategies. The MinMax Ratio suggests that the AVaR strategies seem to better respond to the worst possible scenario, specially those with higher confidence level. Ruttiens Ratio ex-post value advocates in favor of the $AVaR_{100\%}$, implying that the ex-post dynamic behavior of this strategies if favorable. As per the Sortino Satchel Ratio, portfolios solution of Problem (2.4) with AVaR as objective function at a confidence level at 3100% seem to be preferable for investors with semi-standard deviation/expected value type of preference. Moreover, all the AVaR strategies perform better even considering the Maximum DrawDown. Summarizing these results, all the selected AVaR strategies, are in general less risky and perform better ex-post than the GMV, even considering distribution tails, worst case scenario, dynamic point of view, asymmetry in returns distribution and drawdown.

Tables (2.3) and (2.4) also reports, as references, the MSR strategy and the naive portfolio. The MSR strategy reach the highest expected return among all analyzed strategies, but it's also riskier than the AVaR strategies, since it has an higher standard deviation, a lower Rachev Ratio for all the parameterization, the lowest MinMax Ratio and an higher Maximum DrawDown. The MSR seems to better performs in terms of expected returns and expost wealth, but it doesn't allocate the risk efficiently. The naive strategy

[7]We compute the Rachev Ratio with different parameters, $RR_{1\%,1\%}(X)$, $RR_{5\%,5\%}(X)$ and $RR_{50\%,10\%}(X)$ and the Ruttiens Ratio with $k = 1$

Table 2.3: Ex-post Performances of GMV and Selected Strategies. In this Table are reported the ex-post values of expected return, standard deviation, Sharpe Ratio, MinMax Ratio and Sortino Satchell Ratio . GMV still presents the lowest standard deviation, but performs poorly according to all the other chosen criteria.

Strategies	Mean	St.Dev.	$SR(X)$	MM(X)	$Ruttiens(W_T)$	$SS(X)$
GMV	0.0053	0.0327	0.1616	0.0584	0.6757	0.5031
$AVaR_{5\%}$	0.0072	0.0373	0.1928	0.0566	0.8682	0.6336
$AVaR_{10\%}$	0.0066	0.0364	0.1813	0.0523	0.7971	0.5749
$AVaR_{30\%}$	0.0069	0.0360	0.1935	0.0646	0.9336	0.6334
$AVaR_{50\%}$	0.0070	0.0369	0.1895	0.0644	0.8178	0.6174
$AVaR_{75\%}$	0.0081	0.0383	0.2119	0.0638	1.1667	0.7242
$AVaR_{90\%}$	0.0081	0.0382	0.2114	0.0633	1.1171	0.7274
$AVaR_{95\%}$	0.0079	0.0381	0.2066	0.0620	1.0677	0.7042
$AVaR_{100\%}$	0.0100	0.0392	0.2545	0.0784	1.524	0.9484
MSR	0.0130	0.0759	0.1682	0.7336	1.2980	0.6201
naive	0.0079	0.0446	0.1761	1.0120	1.2112	0.6293

is usually difficult to beat (see DeMiguel et al. (2009)). Even in our ex-post analysis, the naive strategy perform quite good in comparison with the AVaR strategies. It reaches a similar level of expected return and has a favorable dynamic of wealth. However it seems to be more exposed to drawdown and worst case scenario.

We also want to verify if portfolios still present some type of Stochastic Dominance even ex-post. In particular we want to test the presence of SSD, and Increasing and Convex order (ICX) which is equivalent to non-satiable risk seeker investor decisions. ICX can be defined coherently with the definitions of FSD and SSD but in terms of non decreasing and convex functions, i.e. $X\ ICX\ Y$ if and only if $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$, for all non decreasing and convex functions u. To determine the presence of SSD ex-post, we proceed again according to relation in 2.5. We sort portfolio returns in ascending(descending) and verify if cumulative sums at each time step of a AVaR strategy is lower(greater) than the minimum variance one. The results for the increasing and convex order are not surprising: all the AVaR strategies ICX dominates minimum variance portfolio. However, we have obtained that the AVaR strategies fail to dominate the GMV in the SSD sense[8].

[8]This result can be seen according to the results of Roll (1976) and Roll (1978) about the difference between the in-sample and out-of-sample type of analysis

Table 2.4: Ex-post Performances of GMV and Selected Strategies. In this Table are reported the ex-post values of Maximum DrawDown and Rachev Ratios. GMV has the second highest Maximum DrawDown and the lowest Rachev Ratio.

Strategies	$MDD(X)$		$RR_{\alpha,\beta}$	
		$RR_{1\%,1\%}(X)$	$RR_{5\%,5\%}(X)$	$RR_{50\%,10\%}(X)$
GMV	2.727	0.8708	0.9203	0.5609
$AVaR_{5\%}$	2.927	0.9280	1.2161	0.6683
$AVaR_{10\%}$	2.533	0.9618	1.1162	0.6438
$AVaR_{30\%}$	2.543	1.0369	1.0625	0.5987
$AVaR_{50\%}$	2.213	1.0508	1.1162	0.5929
$AVaR_{75\%}$	2.268	0.9532	1.1337	0.6205
$AVaR_{90\%}$	2.268	0.9532	1.1225	0.6261
$AVaR_{95\%}$	2.268	0.9532	1.1289	0.6308
$AVaR_{100\%}$	2.268	1.1866	1.3923	0.7244
MSR	3.759	0.9665	0.5153	0.0458
naive	4.044	0.8634	0.4710	0.0539

2.4 Conclusion.

Portfolio selection deals with investors taking decision under uncertainty. In this chapter we have analyzed two of the main approaches proposed in the literature: MVEF and expected utility.

Since the seminal work of Markovitz in 1952, many studies had been developed during the 60's (see Markowitz (1952a), Sharpe (1964), Lintner (1964) and Mossin (1966)). The properties of the Mean Variance Efficient set determined the main reason behind its success. Convexity of the efficient set, implies the efficiency of the market portfolios and, opened the door for the two fund separation and the Capital Asset Pricing Models. Diversification is a real issue for practitioner which has to fulfill regulatory constraints. Expected utility and stochastic dominance provide general criteria for entire classes of agents. In particular, SSD gives a decision rule valid for all non satiable and risk averse investors. Here the efficient set is not necessary convex and, non satiable and risk averse investors seem to prefer highly concentrated portfolios rather than diversified ones. Moreover, the two approaches are mutually consistent only if returns follow an elliptical distribution and if the agents has mean variance type of preferences.

We used multi-object optimization in order to construct set of efficient portfolios for all non satiable and risk averse investors, and then compared their performances with Mean Variance Efficient ones. In Section 2.2, we have discussed the definition of SSD rule and how a multi-objective minimization problem, having as objective vector the Average value at Risk, for all the admissible levels, can be used in the computing of the efficient set. Since the AVaR, under the hypothesis of equiporbable scenarios, can be easily linearize, we discussed the efficiency of the market portfolio. Such portfolio need not to belong to the efficient set for all non satiable and risk averse investors. We also showed that, as a consequence of the transferable assumption of the preference relation for non satiable and risk averse investors, agents will not hold a portfolio for which, at any level of risk aversion there exists a less risky portfolio.

In the empirical analysis of section 2.3 we find that approximately the 12% of portfolio belonging to MVEF are second order stochastically dominated. These portfolios belong to the part of the MVEF where the diversification is higher. This result might be a possible explanation for the Statman diversification puzzle since neither market portfolio nor diversification are sought by risk averse and non satiable investors (see Statman (2004)). Portfolios belonging to the two efficient sets, present, on average different distributional moments. In particular, for low level of mean, SSD efficient portfolios seem to have less negative skewness, lower kurtosis and tend to be more concentrated, than those belonging to the MVEF. Moreover, MVEF efficient portfolios and SSD efficient portfolios are remarkably different under distributional perspective.

Also, we have provided empirical evidence in considering the confidence levels of the Average value at risk as the opposite of risk averse coefficient and, depicted which is the minimum quantity of risk an agent is willing to undertake, given a risk aversion level. In the out of sample analysis we compared the performances of the GMV and 120 AVaR strategies able to dominate it. Some of these strategies are able to beat the GMV up to 150% and, also perform better in terms of Sharpe Ratio, Rachev Ratio, MinMax Ratio, Ruttiens Ratio and Sortino and Satchell Ratio. These results suggest that, in general, the AVaR strategies reach higher level of expected return without increasing too much the standard deviation and the riskiness of the portfolio. A Sharpe Ratio based strategy, in our case, will gain more in terms of expected return, but undertaking an "unjustified"amount of risk. In other words, accepting an higher level of standard deviation might lead to more profitable and less risky strategy, in terms of AVaR.

The future impact of this study is both positive and normative. Positive in the sense that, our study represent a valuable asset for the development of Capital Asset Pricing Models coherent with SSD, since we have given insight on the efficient choices for non satiable and risk averse investors. Normative, because we have shown some minimization problems, whose solutions are able to outperform the GMV portfolio in terms of wealth and with respect to different performance measures. Also using our three parameter efficient frontier, given a risk averse coefficient one can determine which is the minimum level of risk that must be undertaken in order to reach a determined reward.

Chapter 3

Testing for Parametric Orderings Efficiency

Summary

In this chapter, we develop and empirically compare semi-parametric tests to evaluate the efficiency of a benchmark portfolio with respect to different stochastic orderings. Firstly, we classify investors' choices when returns depend on a finite number of parameters: a reward measure, a risk measure and other parameters. We extend Stochastic Dominance theory under minimal assumptions on reward and risk measures. We prove that, when choices depend on a finite number of parameters and, reward measure is isotonic with investors' preference, agents behave as non satiable and risk averse when the reward measure is lower than the mean, and behave as non satiable and risk seeker when the reward measure is greater than the mean. Then, we introduce a new stochastic ordering, consistent with choices of a non satiable, nor risk averse nor risk seeker investors. Secondly, we propose a methodology to semi-parametric tests for the efficiency of a portfolio, when the return distribution is uniquely identified by four parameters, using estimation function theory. Finally, we empirically test whether the Fama and French market portfolio, as well as the NYSE and the Nasdaq indexes are efficients with respect to different stochastic orderings

3.1 Introduction

Several studies in decision theory under uncertainty, have categorized investors according to their utility functions. Typically, investors are considered to be, non satiable and risk averse, i.e. with non decreasing and concave utility function, or non satiable and risk seeker, i.e. with non decreasing and convex utility function. Nevertheless, it is very hard to know a priori the exact functional shape of agents' utility functions. For these reasons, many hypothesis tests assessing the efficiency of a portfolio are based on stochastic dominance (Gibbons et al. (1989)). Stochastic dominance ranks random variables comparing their distribution functions. To each order of stochastic dominance, it corresponds an investors category behavior. The most used stochastic dominance order in portfolio selection are, the First order of Stochastic Dominance (FSD), consistent with the preference of non satiable investors, Increasing and Concave order (ICV), consistent with the preference of non satiable and risk averse investors and, Increasing and Convex order (ICX), consistent with the preference of non satiable and risk seeker investors (see among the others Kroll and Levy (1980), Levy (1992) and Shaked et al. (1994)).

Stochastic dominance efficiency test related literature can be tracked down to the combination of majorization ordering with optimization problems (see for example Post (2003), Kuosmanen (2004) and Linton et al. (2005), Gibbons et al. (1989)). Moreover, Post and Kopa (2013) and Kopa and Post (2015) developed a linear problem formulation which can be implemented in testing for any nth order of stochastic dominance (see also, Kopa and Post (2015) and references therein). Other testing methodologies are based on Kolmogorov statistics, and take into account distances between benchmark distributions (see among the others Scaillet and Topaloglou (2010) and Barrett and Donald (2003)).

Several studies in behavioral finance suggests that investors prefer more to less and are neither risk averse nor risk seeker (see for example Markowitz (1952a), Kahneman and Tversky (1979), Tversky and Kahneman (1992), Levy (1992) and Barberis and Thaler (2003))[1]. It is not clear then, whether investors' preference vary with market conditions, and in particular under period of financial distress. Moreover, it is well known that asset returns exhibit skewness and excess kurtosis (see for example Rachev et al. (2011)). For these reasons, we propose a new approach able to test for efficiency of a portfolio from the perspective of non satiable, nor risk averse nor risk seeker investors and to consider also higher moments of return distributions. The aims of this paper is twofold. Firstly, we categorize non satiable investors preference, under the assumption that return distributions depend to on a finite number of parameters. We extend stochastic dominance conditions for FSD, ICV and ICX , when return distributions depend on a positive homo-

[1]Efficiency test for Prospect theory Stochastic Dominance and Markovitz Stochastic Dominance can be found in Arvanitis and Topaloglou (2017)

geneous and translation equivariant reward measure, a positive homogeneous and translation invariant risk measure and other distributional parameters. We show that, non satiable, nor risk averse nor risk seeker investors risk attitude changes according to market conditions. In particular, in a market where the reward measure is higher than the expected return, investors behave as non satiable and risk seeker, while when the reward measure is lower then the expected return, they behave as non satiable and risk averse. Then we define a new stochastic ordering, coherent with the preference of non satiable, nor risk averse nor risk seeker investors, that we call Rachev ordering, identified by a reward measure based on a linear combination of Conditional Value at Risk (see Biglova et al. (2004), Ortobelli et al. (2009), and Ortobelli et al. (2013), Pflug (2000), Rockafellar and Uryasev (2000)). Secondly, exploiting estimation function theory, we propose a methodology to test whether a given portfolio is efficient, with respect to ICV, ICX and Rachev ordering (see Godambe and Thompson (1989)). Finally, we propose an empirical analysis by testing whether the Fama and French market portfolio (see Fama and French (1993)) can be considered efficient according to the proposed semi-parametric tests. Starting with Banz (1981), small caps portfolio are of particular interest in behavioral finance, since they earn a return that defies rational expectation[2]. To apply our methodology to a large scale problem, we also test whether the NYSE and the Nasdaq market indexes are efficient during the period June 2006 to May 2017.

This paper is organized in four sections. In Section 2, we propose some ordering criteria when the portfolios distributions are uniquely determined by a reward measure, a deviation measure and a finite number of other parameters. In Section 3, we propose semi-parametric tests based on the estimating function theory. In Section 4, we perform the empirical analysis. Section 5 briefly summarizes the main results.

3.2 Optimal choices depending on a finite number of parameters

In this section we classify choices for different categories of investors when return distribution depends on a finite number of parameters. We focus on the ordering of parametric choices consistent with investors' preferences. Such typology of orderings can be addressed into the class of FORS orderings (see Ortobelli et al. (2009) and Ortobelli et al. (2013)). We consider First

[2]Similar conclusions on small-cap can be found in Post and Kopa (2013) and Arvanitis and Topaloglou (2017).

Order of Stochastic Dominance (FSD), Concave Order (also called Rothshild-Stiglitz (RS or CV) order, see Rothschild and Stiglitz (1971)), Increasing and Concave , also called Second Order of Stochastic Dominance (ICV or SSD) and Increasing and Convex Order (ICX), that are respectively consistent with the preference of non satiable investors, risk averse investors, non satiable and risk averse investors and non satiable and risk seeker investors[3]. Recall the classical definitions of different Orders of Stochastic Dominance.

Definition 3. Given a pair of random variables W and Y, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution functions F_W and F_Y respectively, we say:

- W dominates Y in the sense of First Order of Stochastic Dominance (i.e.W FSD Y) if and only if $F_W(\lambda) \leq F_Y(\lambda)\ \forall\, \lambda \in \mathbb{R}$, or equivalently W FSD Y if and only if $E(u(W)) \geq E(u(Y))$ for all non-decreasing functions u.

- W dominates Y in the sense of Second Order of Stochastic Dominance (i.e.W ICV Y) if and only if $\int_{-\infty}^{\lambda} F_W(t)dt \leq \int_{-\infty}^{\lambda} F_Y(t)dt\ \forall\, \lambda \in \mathbb{R}$, or equivalently W ICV Y if and only if $E(u(W)) \geq E(u(Y))$ for all non-decreasing and concave functions u.

- W dominates Y in the sense of Concave Order of Stochastic Dominance (i.e.W CV Y) if and only if W ICV Y and $E(W) = E(Y)$, or equivalently W CV Y if and only if $E(u(W)) \geq E(u(Y))$ for all concave functions u.

- W dominates Y in the sense of Increasing and Convex Order of Stochastic Dominance (i.e.W ICX Y) if and only if $\int_{\lambda}^{\infty} 1 - F_W(t)dt \geq \int_{\lambda}^{\infty} 1 - F_Y(t)dt\ \forall\, \lambda \in \mathbb{R}$, or equivalently W ICX Y if and only if $E(u(W)) \geq E(u(Y))$ for all non-decreasing and convex functions u.

- W dominates Y in the sense of Convex Order of Stochastic Dominance (i.e.W CX Y) if and only if W ICX Y and $E(W) = E(Y)$, or equivalently W CX Y if and only if $E(u(W)) \geq E(u(Y))$ for all convex functions u and if and only if Y CV W.

Where all the above inequalities are strict for at least a real λ and for at least one utility function u.

[3]Classic Stochastic dominance order can be addressed into the FORS orderings (see a series of paper by Ortobelli et al. (2009) and Ortobelli et al. (2013)). FORS ordering class contains also behavioral orderings often used in the recent financial literature (see among others Levy (1992), the Markowitz ordering (Markowitz (1952a)) and prospect behavioral type ordering (Tversky and Kahneman (1992)).

Let us now consider the optimal portfolio problem. Call $Z = [Z_1, \ldots, Z_N]'$ a vector of asset gross returns and, $w = [w_1, \ldots, w_N]$ a portfolio weights vector. Let any portfolio $W = w'Z$ be a random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Under institutional constraints, such as no short sales allowed (i.e. $w_i \geq 0$, $i = 1, \ldots, N$) and limited liabilities ($Z_i \geq 0$, $i = 1, \ldots, N$), we can assume that all portfolio gross returns belong to a scale invariant family, with positive translation property (see Ortobelli (2001)). We assume that all portfolio gross returns belong to a scale invariant family $\sigma\tau_q^+(\bar{a})$, with parameters $(\mu(W), \rho(W), a_1(W), \ldots, a_{q-2}(W))$, with the following properties:

1. For every distribution function $F \in \sigma_q^+(\bar{a})$, there exist a random variable having F as distribution.

2. Every distribution function $F \in \sigma_q^+(\bar{a})$ is weakly determined by the set of parameters $(\mu(X), \rho(W), a_1(W), \ldots, a_{q-2}(W))$, i.e. $F_W, F_Y \in \sigma_q^+(\bar{a})$, then

$$(\mu(W), \rho(W), a_1(W), \ldots, a_{q-2}(W)) = (\mu(Y), \rho(Y), a_1(Y) \ldots, a_{q-2}(Y))$$

implies $W \stackrel{d}{=} Y$, but the converse is not necessarily true.

3. $\mu(W)$ is a reward measure translation invariant, i.e for $t \geq 0$ $\mu(W + t) = \mu(W) + t$, and positive homogeneous, and $\rho(W)$ is a risk measure consistent with the additive shift, i.e. $\rho(X + t) \leq \rho(X)$ $\forall t \geq 0$, and positive homogeneous. Assumptions 1. and 2., are technical, guarantee that to every admissible set of parameters it corresponds a portfolio and implies that instead of looking at all the distribution it's sufficient to consider only certain quantities. Assumption 3. has nice economic interpretation. Translation invariance and consistency with additive shift implies that in a market where there exists a sure gain, i.e. a risk free rate, the reward measure shifts of the same sure amount, while riskiness doesn't increase. Positive homogeneity implies that position size affects, both riskiness and reward linearly. The first q moments of a distribution are admissible parameters for the $\sigma\tau_q^+(\bar{a})$ family, but the family also admits more general parametrization. In the following, we establish and extend dominance conditions when the $\sigma\tau_q^+(\bar{a})$ family depends on general reward and risk measures.

Theorem 1. *Assume all random admissible portfolios of gross returns belong to a $\sigma\tau_q^+(\bar{a})$ class. Let $W = w'Z$ and $Y = y'Z$ be random return of a couple of portfolios determined by the parameters:*

$$(\mu(W), \rho(W), a_1(W), ..., a_{q-2}(W)) \text{ and } (\mu(Y), \rho(Y), a_1(Y), ..., a_{q-2}(Y))$$

where, $a_i(W) = a_i(Y)$, $i = 1, \ldots, q - 2$, w and y are the portfolio weights vectors and Z the vector of gross returns. Then, the following implications

hold:
1. $\frac{\mu(W)}{\rho(W)} \geq \frac{\mu(Y)}{\rho(Y)}$ *and* $\rho(W) \geq \rho(Y)$ *(with at least one inequality strict) implies* W *FSD* Y*.*
2. Suppose $\dfrac{\mu(W)}{\rho(W)} = \dfrac{\mu(Y)}{\rho(Y)}$*. Then* $\rho(W) > \rho(Y)$ *if and only if* W *FSD* Y*.*
3. Suppose the risk measure ρ *is translation invariant (i.e.* $\rho(X + t) = \rho(X), \forall t \geq 0$*) and* $\rho(W) = \rho(Y)$ *. Then* W *FSD* Y*, if and only if* $\mu(W) > \mu(Y)$ *.*

Theorem (1) extends some results on FSD when the distributions depends on general reward and risk measures. The relationship between FSD, μ and ρ is similar to the case where the reward and risk measures are the first two moments (see for example Ortobelli (2001)).

In general, admissible reward measures for the $\sigma\tau_q^+(\bar{a})$ are isotonic with FSD, while risk measure are isotonic with FSD only when certain conditions on the risk reward ratio are met[4]. When conditions on risk reward ratio, in Theorem 1 are matched, all non satiable investors will prefer a portfolio with an higher risk measure over one with a lower. Moreover, when the risk measure is translation invariant, if two portfolios have the same risk measure, all non satiable investors will prefer the one with the higher reward measure.

Having extended FSD conditions in case of general risk and reward measure, we now consider the case of ICV and ICX. We first establish stochastic dominance relations when the $\sigma\tau_q^+(\bar{a})$ depends on the mean and on a positive homogeneous and translation invariant risk measure. Then, we extend those conditions also for the general reward measure case. We recall that, the mean (used as reward measure) is always isotonic either with a risk aversion or with a risk seeking behavior (because it can be seen as the expected utility of a linear positive function that is both convex and concave).

Remark 2. *Suppose every admissible portfolios of gross returns belong to a family* $\sigma\tau_q^+(\bar{a})$ *as uniquely characterized by the mean* μ*, a translation invariant risk measure* $\rho(X)$ *(i.e.* $\rho(X + t) = \rho(X), \forall t \geq 0$*) and other* $q - 2$ *scalar and translation invariant parameters* $a_1, ..., a_{q-2}$*. Let* $W = w'Z$ *and* $Y = y'Z$ *be a couple of portfolios determined by the parameters* $(E(W), \rho(W), a_1(W), ..., a_{q-2}(W))$ *and* $(E(Y), \rho(Y), a_1(Y), ..., a_{q-2}(Y))$*, where* $a_i(W) = a_i(Y)$*,* $i = 1, \ldots, q-2$*,* w *and* y *are the portfolio weights vectors and* Z *the vector of gross returns. Suppose* W *presents a different distribution respect to* Y *and* W *does not FSD* Y*. Then, according to Ortobelli (2001), assuming fixed* $a_1, ..., a_{q-2}$*, we*

[4]A reward measure μ is said to be isotonic with a preference order $\succ$ when, give two random variable, X and Y, if $X \succ Y$ then $\mu(X) \geq \mu(Y)$.

get

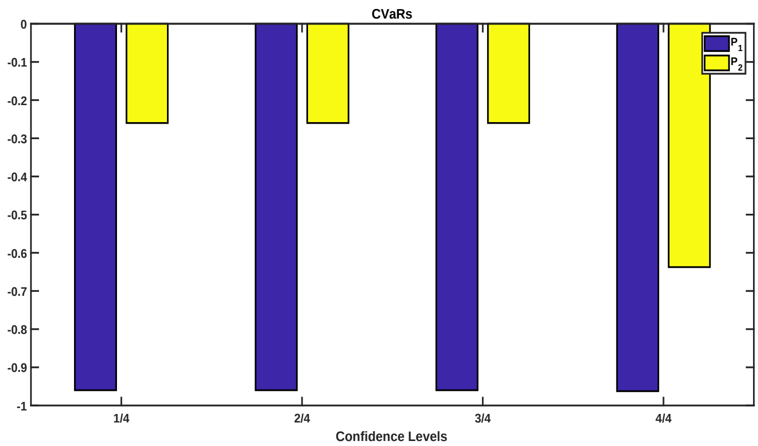
$$
\begin{aligned}
W\ ICV\ Y \ &\Leftrightarrow \mathbb{E}\left[W\right] \geq \mathbb{E}\left[Y\right]\ \textit{and}\ \rho(W) < \rho(Y) \\
W\ CV\ Y \ &\Leftrightarrow \mathbb{E}\left[W\right] = \mathbb{E}\left[Y\right]\ \textit{and}\ \rho(W) < \rho(Y)
\end{aligned} \tag{3.1}
$$

As an explanation of the previous results, consider the following example.

Example 2. *Consider a market with four states of the world and let P_1 and P_2 be two portfolios of gross returns whose distributions belong to the $\sigma\tau_4^+(\bar{a})$ family, weakly determined by the first four moments. Let gross return of P_1 and P_2 be summarized in the following table. Clearly P_1 does not FSD P_2,*

	P_1	P_2
1	0.96	1.77
2	0.96	0.26
3	0.96	0.26
4	0.97	0.26

and it's easy to see P_1 and P_2 have the same values of skewness and kurtosis, 2 and 4 respectively. The results of Remark 2 are then confirmed by, looking at Figure 3.1. In Figure 3.1 CVaRs at different confidence levels are depicted



(a) Conditional Value at Risk of portfolios P_1 and P_2 at confidence levels 0.25, 0.5, 0.75 and 1

Figure 3.1: Example with only two assets

and according to condition 2.5 in Chapter 2, P_1 ICV P_2. Finally condition 3.1 is satisfied by looking at the values of mean and standard deviation of P_1 and P_2, being respectively 0.9625 *and* 0.6375 *and,* 0.00002 *and* 0.57.

As a consequence of this remark we obtain the following corollary, presenting the ICX dominance conditions.

Corollary 1. *Suppose every admissible portfolios of gross returns belong to a family* $\sigma\tau_q^+(\bar{a})$ *as uniquely characterized by the mean* μ*, a translation invariant risk measure* $\rho(X)$ *(i.e.* $\rho(X + t) = \rho(X), \forall t \geq 0$*) and other* $q - 2$ *scalar and translation invariant parameters* $a_1, ..., a_{q-2}$*. Let* $W = w'Z$ *and* $Y = y'Z$ *be a couple of portfolios determined by the parameters* $(E(W), \rho(W), a_1(W), ..., a_{q-2}(W))$ *and* $(E(Y), \rho(Y), a_1(Y), ..., a_{q-2}(Y))$*, where* $a_i(W) = a_i(Y)$*,* $i = 1, \ldots, q-2$*,* w *and* y *are the portfolio weights vectors and* Z *the vector of gross returns. Suppose* W *presents a different distribution respect to* Y *and* W *does not FSD* Y*.Then, we get:*

$$\begin{aligned} W\ ICX\ Y\ &\Leftrightarrow \mathbb{E}[W] \geq \mathbb{E}[Y]\ \ and\, \rho(W) > \rho(Y) \\ W\ CX\ Y\ &\Leftrightarrow \mathbb{E}[W] = \mathbb{E}[Y]\ \ and\, \rho(W) > \rho(Y) \end{aligned} \tag{3.2}$$

Remark 2 and Corollary 1 establish necessary and sufficient conditions for ICV and ICX dominance when return distributions depend on the mean and a general risk measure ρ. When all gross returns belong to the $\sigma\tau_q^+(\bar{a})$ family, dominance conditions for ICX and ICV resemble those when the family is weakly determined by the first q moments. Notice that, the risk measure doesn't necessarily have to be convex. Furthermore, the risk measure discriminates between risk averse and risk seeker behavior.

A portfolio is said to be ICV (ICX) efficient, if a dominant portfolio in the sense of ICV (ICX) doesn't exist. As consequence of Remark 2 and Corollary 1, we state ICV and ICX efficiency conditions for portfolios whose random gross returns belong to the $\sigma\tau_q^+(\bar{a})$ family.

Efficiency Condition 1. *Suppose every admissible portfolios of gross returns belong to a family* $\sigma\tau_q^+(\bar{a})$ *uniquely characterized by the mean, a translation invariant risk measure* $\rho(X)$ *(i.e.* $\rho(X + t) = \rho(X), \forall t \geq 0$*) and other* $q - 2$ *scalar and translation invariant parameters* $a_1, ..., a_{q-2}$*. assumed to be fixed. Then, a portfolio* $W = w'Z$ *is said to be:*

1. *ICV efficient* $\Leftrightarrow\ \nexists Y = y'Z$ *such that* $\mathbb{E}[Y] \geq \mathbb{E}[W]$ *and* $\rho(Y) < \rho(W)$

2. *ICX efficient* $\Leftrightarrow\ \nexists Y = y'Z$ *such that* $\mathbb{E}[Y] \geq \mathbb{E}[W]$ *and* $\rho(Y) > \rho(W)$

ICV efficiency condition is based on (3.1) and ICV condition is based on (3.2). We will use the above efficiency conditions in the testing methodology in the next section. Nevertheless, Remark 2 and Corollary 1 do not describe the investors behavior when return distributions depend on a general reward measure. First, we need state the following proposition.

Proposition 2. *Suppose that a $\sigma\tau_2^+(\bar{a})$ distribution family admits two possible parametrization $(\mu_1(X), \rho(X))$ and $(\mu_2(X), \rho(X))$ with the same risk measure $\rho(X)$ that is translation invariant (i.e. $\rho(X+t) = \rho(X)$ for any real t). If there exists a random variable $X \in \sigma\tau_2^+(\bar{a})$ such that $\mu_1(X) > \mu_2(X)$, then $\mu_1(Y) > \mu_2(Y)$ $\forall Y \in \sigma\tau_2^+(\bar{a})$ and $(\mu_1(X) - \mu_2(X))/\rho(X)$ is a constant for all random variables $X \in \sigma\tau_2^+(\bar{a})$.*

Proposition 2 is mostly technical. It is based on the assumption that distributions belonging to the $\sigma\tau_q^+(\bar{a})$ admit two possible parameterization with different reward measures. In particular when the risk measure is translation invariant, the ratio between the difference of two admissible reward measures and risk measure is constant and, the order relation between reward measures is the same for all the random variables belonging to the family. The proposition holds also for scale and translation invariant families $\sigma\tau_2(\bar{a})$ of (non necessarily positive) random variables and it is theoretically indifferent using one or any other existing translational invariant scale parameter for a $\sigma\tau_2(\bar{a})$ (or $\sigma\tau_2^+(\bar{a})$) class. Moreover, the proposition suggests that the ranking among different portfolios given by the ratio between any reward measure and a risk measure is the same under the assumption that all portfolios belongs to $\sigma\tau_2(\bar{a})$ (or $\sigma\tau_2^+(\bar{a})$) family uniquely determined by two parameters. Observe that, when we fix all the parameters except the reward and the risk measures of a given $\sigma\tau_k^+(\bar{a})$ family, we obtain a particular $\sigma\tau_2^+(\bar{a})$ class. Therefore, every $\sigma\tau_k^+(\bar{a})$ class weakly determined by the parameters (μ, ρ, y) (where μ is a reward measure, ρ is the risk measure and $y \in B \subseteq R^{k-2}$ is the vector of the other parameters) can be seen as the union of two parametric families $V(\bar{y}) := \sigma\tau_2^+(\mu, \rho, \bar{y})$. For this reason, the above proposition can be applied to the respective components of the union. However, we cannot guarantee that the constant ratio between the difference of two admissible reward measures and the risk measure of the same component $V(\bar{y})$ remains equal for every other component of the union (i.e. it is not necessarily true that the ratio remains constant for all the distributions of the family $\sigma\tau_k^+(\bar{a})$). In fact, different ranking measures could penalize the asymmetry or the kurtosis parameters in a different way.

Thanks to Proposition 2, the following theorem allows to classify the choices of non satiable, nor risk averse nor risk seeker investors, when the $\sigma\tau_q^+(\bar{a})$ family depends on a positive homogeneous and translation equivariant reward measure.

Theorem 2. *Let every random admissible portfolios of gross returns be in a $\sigma\tau_q^+(\bar{a})$ family and this family admits two possible parametrization with different reward measures. Assuming that the risk measure is translation invariant* *(i.e. $\rho(X+t)=\rho(X)$) and the mean is one of the two possible reward measures, we define $W=w'Z$ and $Y=y'Z$ random return of a couple of portfolios determined by the parameters $(\mu(W),\rho(W),a_1(W),...,a_{q-2}(W))$ and $(\mu(Y),\rho(Y),a_1(Y),...,a_{q-2}(Y))$ (or $(E(W),\rho(W),a_1(W),...,a_{q-2}(W))$ and $(E(Y),\rho(Y),a_1(Y$ where $a_i(W)=a_i(Y)$, $i=1,\dots,q-2$. Then*

1. Whether $\mu(W)>E(W)$, the following implications hold:

- *1a) $\frac{\mu(W)}{\rho(W)}\geq\frac{\mu(Y)}{\rho(Y)}$ and $\mu(W)\geq\mu(Y)$ (with at least one inequality strict) implies W ICV Y.*
- *1b) $\mu(W)\geq\mu(Y)$ and $\rho(W)\leq\rho(Y)$ (with at least one inequality strict) implies W ICV Y.*
- *1c) W CX Y (i.e. Y dominates W in the Rothshild- Stiglitz sense, Y CV W) implies $\mu(W)>\mu(Y)$ and $\rho(W)>\rho(Y)$.*
- *1d) W ICX Y but W does not dominates at first order Y implies $\mu(W)\geq\mu(Y)$ and $\rho(W)>\rho(Y)$.*

2. Else, $\mu(W)<E(W)$ and we have that:

- *2a) W ICV Y implies $\mu(W)\geq\mu(Y)$.*
- *2b) W ICV Y but W does not dominates at first order Y (this assumption includes the case W CV Y) implies $\mu(W)\geq\mu(Y)$ and $\rho(W)<\rho(Y)$.*
- *2c) $\mu(W)\geq\mu(Y)$ and $\rho(W)\geq\rho(Y)$ (with at least one inequality strict) implies W ICX Y.*

Theorem 2 distinguishes two cases. When $\mu(W)>\mathbb{E}[W]$, points $1a$) and $1b$) describe ICV dominance conditions, while according to point $1c$) and $1d$), reward measure are isotonic with respect to ICX. When $\mu(W)<\mathbb{E}[W]$, according to points $2a$) and $2b$) reward measures are isotonic with ICV, while point $1c$) is a ICX dominance condition. Thanks to conditions $1d$) and $2b$) in Theorem 2 we can state ICV and ICX efficiency conditions when the $\sigma\tau_q^+(\bar{a})$ family depends on a general reward measure.

Efficiency Condition 2. *Assume every admissible portfolios of gross returns belong to a $\sigma\tau_q^+(\bar{a})$ family which admits two possible parameterization with different reward measures. Let a translation equivariant and positive homogeneous reward measure and the mean be the two possible reward measures for the $\sigma\tau_q^+(\bar{a})$ family. Let the risk measure be translation invariant and positive homogeneous and $a_1,\dots,a_{q-2}$ be other translation and scale invariant parameters, assumed to be fixed. Let a $W=w'Z$ be a portfolio. Then:*

1. *when $\mu(W) > \mathbb{E}[W]$, W is ICV efficient if $\nexists Y = y'Z$ such that $\mu(Y) \geq \mu(W)$ and $\rho(Y) < \rho(W)$;*

2. *when $\mu(W) < \mathbb{E}[W]$, W is ICX efficient if $\nexists Y = y'Z$ such that $\mu(Y) \geq \mu(W)$ and $\rho(Y) > \rho(W)$.*

The above efficiency conditions are weaker than the previous case. They depend on the relation between mean and reward measure, and are only necessary but not sufficient. They still resemble those where the mean is used as reward measure, but in this case are less of practical use. It is very often the case, in fact, that in practice we don't know whether conditions 1 or 2 in Theorem 2 are satisfied. Therefore, we suggest not to use the last efficiency condition in developing a test methodology and, instead, exploit Efficiency Condition 1.

Theorem 2 also implies that, admissible reward measures for the $\sigma\tau_q^+(\bar{a})$ family are isotonic with the investors' prevalent behavior (i.e. risk seeker or risk averse). In particular, the theorem classifies reward measures with respect to ICV and ICX orders:
1. those greater than the mean (for some fixed distributional parameters) are isotonic with risk seeking prevalent behavior;
2. those lower than the mean (for some fixed distributional parameters) are isotonic with risk averse prevalent behavior.
In particular, consider the Conditional Value at Risk (CVaR), defined as $CVaR_\alpha(W) = \int_0^\alpha F_W^{-1}(d)du$, where F_W^{-1} is the left inverse of the cumulative distribution function, i.e. $F_W^{-1}(u) = \inf\{x : \mathbb{P}[W \leq x] = F_W(x) \geq u\}$. When all portfolio gross returns belong to the $\sigma\tau_q^+(\bar{a})$ family, for any portfolio W, $\mu(W) = -CVaR_\alpha(W)$ is an admissible reward measure and always isotonic with ICV, since $CVaR_\alpha$ is a coherent risk measure and, in this case, μ is always lower than the mean. Similarly, $\mu(W) = -CVaR_\alpha(-W)$ is an admissible reward measure and always isotonic with ICX. Combining together $CVaR_\alpha(W)$ and $CVaR_\alpha(-W)$ it is possible to construct a functional, consistent with the behavior of non satiable, nor risk averse nor risk seeker investors (see for example Biglova et al. (2004), Ortobelli et al. (2009), and Ortobelli et al. (2013)). Therefore we propose the following functional. For any $\alpha, \beta, \lambda \in [0, 1]$:

$$\gamma_{\alpha,\beta,\lambda}(X) = \lambda CVaR_\alpha(X) - (1-\lambda)CVaR_\beta(-X) \tag{3.3}$$

We call $\gamma_{\alpha,\beta,\lambda}$ Rachev risk measure. The functional (3.3) (varying α, β, λ) identifies the distribution of X and it is consistent with FSD. By the properties of coherent risk measure, $\mu(X) = -\gamma_{\alpha,\beta,\lambda}(X)$ is an admissible reward measure for the $\sigma\tau_q^+(\bar{a})$ family, and when $\lambda = 0$ it is isotonic with

ICX while when $\lambda = 1$, it is isotonic with ICV (see Artzner et al. (1999), Rachev et al. (2008), Ortobelli et al. (2009) and Ortobelli et al. (2013)). For all the other values of λ, $\gamma_{\alpha,\beta,\lambda}$ matches the conditions in Theorem 2, and therefore, it also identifies the choices of non satiable, nor risk averse nor risk seeker investors .

Remark 3. *All the aggressive coherent utility functional, of the form*

$$\mu(X) = \lambda\nu_1(-X) - (1-\lambda)\nu_2(X)$$

where, ν_1 and ν_2 are coherent risk measures, are admissible reward measures for the $\sigma\tau_q^+(\bar{a})$ family, if they are positive for all admissible portfolios (see Artzner et al. (1999),Biglova et al. (2004) and Ortobelli et al. (2013)). Similarly, classical deviation measures, satisfy the properties of the $\sigma\tau_q^+(\bar{a})$ family, and typical scalar and translation invariant parameters are skewness and kurtosis .

We can now introduce a new Stochastic ordering, consistent with non satiable, nor risk averse nor risk seeker investors' preference, based on $\gamma_{\alpha,\beta,\lambda}$.

Definition 4. Given two real-valued random variables X, Y defined on $(\Omega, \mathcal{F}, \mathbb{P})$, we say that X dominates Y in the sense of Rachev ordering with parameters $\alpha, \beta, \lambda \in [0,1]$ (i.e. $X \geq^R_{\alpha,\beta,\lambda} Y$) if and only if $\gamma_{\alpha,\beta,\lambda}(X) \leq \gamma_{\alpha,\beta,\lambda}(Y)$, $\alpha, \beta, \lambda \in [0,1]$.

Thanks to Theorem 2, investors with preferences coherent with the Rachev ordering, behave as non satiable and risk averse, when $\gamma_{\alpha,\beta,\lambda}$ is lower than the mean, while behave as non satiable and risk seeker when $\gamma_{\alpha,\beta,\lambda}$ is higher than the mean. Therefore, it's possible to find portfolio that dominates a given benchmark in Rachev ordering sense exploiting ICV and ICX dominance conditions. The following example clarifies this point.

Example 3. *Let assume that all portfolios of gross returns distribution belong to the $\sigma\tau_4^+(\bar{a})$ family, weakly determined by parameters $(\mu(X), \rho(X), a_1(X), a_2(X))$. Suppose to have a portfolio with vector weights $x^{(P)} = [x_1^P, ..., x_n^P]'$ and parameters given by $\mu(P), \rho(P), s, k$, where the reward measure μ is translation equivariant and positive homogeneous, the risk measure ρ is translation invariant and positive homogeneous, and s and k are other scale and translation invariant parameters. If the portfolio P is not FSD dominated then, according to Corollary (1), Remark (2) and Theorem (2) all non-satiable risk seeker* *investors prefer portfolios solution of the following optimization problem:*

$$\begin{aligned}\max_{x_i, i=1,\ldots,N} \ & \mu(x'Z) \\ & x'\mathbb{E}[Z] \geq x^{(P)'}\mathbb{E}[Z] \\ & \rho(x'Z) \geq \rho(x^{(P)'}Z) \\ & a_1(x'Z) = s, \ a_2(x'Z) = k \\ & \sum_{i=1}^{N} x_i = 1, \ x_i \geq 0, \ i = 1, \ldots, N \end{aligned} \tag{3.4}$$

over portfolio P. Similarly, all non-satiable risk averse investors prefer the portfolios solution of the following optimization problems:

$$\begin{aligned}\max_{x_i, i=1,\ldots,N} \ & \mu(x'Z) \\ & x'\mathbb{E}[Z] \geq x^{(P)'}\mathbb{E}[Z] \\ & \rho(x'Z) \leq \rho(x^{(P)'}Z) \\ & a_1(x'Z) = s, \ a_2(x'Z) = k \\ & \sum_{i=1}^{N} x_i = 1, \ x_i \geq 0, \ i = 1, \ldots, N \end{aligned} \tag{3.5}$$

over portfolio P.

Feasible regions for Problems (3.4) and (3.5) coincide with the dominance conditions for ICX and ICV in Remark 2 and 1. With Problem 3.4, when the feasible set is non-empty, we select, among the portfolios that dominates P in the ICX sense, the one that has the maximum reward measure μ. Similarly, in case of Problem 3.5, we select the one with maximum reward measure among those dominating P in the ICV sense. In the next section we combine these results with estimation function theory to develop semi-parametric tests for ICV, ICX and Rachev ordering.

3.3 Testing choices depending on a finite number of parameters

In this section, we combine estimation function theory with the results of previous section to develop hypothesis testing procedure for Rahcev ordering, ICV and ICX efficiency. In the literature, several tests based on the

Kolmogorov-Smirnov statistic have been proposed to compare stochastic ordering preferences[5]. We discuss a semi parametric statistic obtained by estimating functions theory.

Let $R = (R_1, ..., R_T)$ be a random vector on a probability space and the distribution family of this vector is parametrized by $\xi = (\xi_1, ..., \xi_p)$. An estimating function (EF) $h(R_t, \xi)$ is called unbiased if $\mathbb{E}[h(R_t, \xi)] = 0$. Generally, the number of EFs is set equal to the number of parameters ξ_q ($q = 1, \ldots, \ p$) through the linear combinations of unbiased EFs $l_{\xi,q} = \sum_{t=1}^{T}\sum_{i=1}^{n} \delta_{q,i,t} h_i(R_t, \xi)$, with $q = 1, \ldots, p$ and $i = 1, \ldots, n$. These unbiased EFs $h_i(R_t, \xi)$ are also mutually orthogonal, i.e., for every $i \neq j$; $i,j = 1, \ldots,$ n, $\mathbb{E}[h_i(R_t, \xi) h_j(R_t, \xi)] = 0$. In particular, among every linear combination $l_{\xi,q} = \sum_{t=1}^{T}\sum_{i=1}^{n} \delta_{q,i,t} h_i(R_t, \xi)$ of unbiased mutually orthogonal EFs, we consider $l^*_{\xi,q} = \sum_{t=1}^{T}\sum_{i=1}^{n} \frac{\mathbb{E}\left[\frac{\partial h_i(R_t,\xi)}{\partial \xi_q}\right]}{\mathbb{E}\left[h_i^2(R_t, \xi)\right]} h_i(R_t, \xi)$ q=1,...,p as the optimal EFs. [6] Then, an estimate $\hat{\xi}$ of ξ is obtained by solving the system of equations $l^*_{\xi,q} = 0$, q=1,...,p. According to the estimating function theory the optimal EFs obtained as consistent[7] solution of equations $l^*_{\xi,q} = 0$, have the property

$$\sqrt{T}\left(\hat{\xi} - \xi\right) \to N(0, V_{EF}^{-1})$$

where N is normal distribution with zero mean vector and variance matrix V_{EF}^{-1}, with $V_{EF} = [v_{i,j}]_{i,j=1,\ldots,p}$ and $v_{i,j} = \mathbb{E}l\left[\frac{\partial l^*_{\xi,i}}{\partial \xi_j}\right]$ i,j=1,... p. Although in some cases we can easily obtain optimal solutions, it is common the introduction of convergent methods to compute the roots of equations starting by an approximate solution to get to the optimal estimates (Crowder (1986)). Typical examples of optimal estimating functions are those proposed by Godambe and Thompson (1989) based on the first four central moments of a given statistic. They propose a model with two unbiased and mutually orthogonal estimating functions:

$$h_1(R_t, \xi) = f(R_t) - \theta_1(\xi) \tag{3.6}$$
$$h_2(R_t, \xi) = (f(R_t) - \theta_1(\xi))^2 - \theta_2^2(\xi) - \theta_3(\xi)\theta_2(\xi)\,(f(R_t) - \theta_1(\xi)) \tag{3.7}$$

[5]See Beach and Davidson (1983); Anderson (1996); Davidson and Duclos (2000) and Scaillet and Topaloglou (2010).

[6]These estimation functions are called optimal since they present the lowest variance among all the $l_{\xi,q}$.

[7]The concept of consistent solution is given in Crowder (1986)

where f is a measurable real function $\mathbb{E}\left[f(R_t)\right] = \theta_1(\xi)$, $\theta_2^2(\xi) = \mathbb{E}\left[(f(R_t) - \theta_1(\xi))^2\right]$, and $\theta_3(\xi) = \dfrac{\mathbb{E}\left[(f(R_t) - \theta_1(\xi))^3\right]}{\theta_2^3(\xi)}$. Therefore the optimal estimating functions are given by

$$l^*_{\xi,q} = \sum_{t=1}^{T} \left(d_{q,t} h_{1,q}(R_t,\xi) + b_{q,t} h_{2,q}(R_t,\xi)\right)$$

where $d_{q,t} = \dfrac{\mathbb{E}\left[\frac{\partial h_1(R_t,\xi)}{\partial \xi_q}\right]}{\mathbb{E}\left[h_1^2(R_t,\xi)\right]}$ and $b_{q,t} = \dfrac{\mathbb{E}\left[\frac{\partial h_2(R_t,\xi)}{\partial \xi_q}\right]}{\mathbb{E}\left[h_2^2(R_t,\xi)\right]}$ q=1, ..., p. Under regularity assumptions the following proposition determines the class of consistent estimators for $\theta_1(\xi)$.

Proposition 3. *Suppose we have a sample $R = (R_1, ..., R_T)$ of i.i.d. observations. Consistent estimators of $\widehat{\theta}_1(\xi)$ are given by the solutions of equations $l^*_{\xi,q} = 0$ for q = 1, ..., p,:*

$$\widehat{\theta}_1(\xi) = \begin{cases} \frac{1}{T}\sum\limits_{t=1}^{T} f(R_t) + c_q - \left(c_q^2 - \frac{1}{T}\sum\limits_{t=1}^{T}\left(f(R_t) - \frac{1}{T}\sum\limits_{t=1}^{T} f(R_t)\right)^2 + \theta_2^2(\xi)\right)^{1/2} & \text{if } c_q > 0 \\ \frac{1}{T}\sum\limits_{t=1}^{T} f(R_t) + c_q + \left(c_q^2 - \frac{1}{T}\sum\limits_{t=1}^{T}\left(f(R_t) - \frac{1}{T}\sum\limits_{t=1}^{T} f(R_t)\right)^2 + \theta_2^2(\xi)\right)^{1/2} & \text{if } c_q \leq 0 \end{cases} \tag{3.8}$$

where $c_q = \frac{d_q - b_q a_1(\xi)\rho(\xi)}{2b_q}$, $d_q = \dfrac{E\left(\frac{\partial h_1(R_t,\xi)}{\partial \xi_q}\right)}{E(h_1^2(R_t,\xi))}$ and $b_q = \frac{E\left(\frac{\partial h_2(R_t,\xi)}{\partial \xi_q}\right)}{E(h_2^2(R_t,\xi))}$. Moreover, we get optimal estimators $\hat{\xi}_q$ when the regularity conditions of implicit function theorem are satisfied and we can determine the estimates in all its domain.

We combine optimal estimators and their limiting distributions, with the theoretical results of the previous section to derive a methodology to test for efficiency of a given portfolio, with respect to another, in the sense Rachev, ICV and ICX orderings.

3.3.1 Tests for ICV and ICX

Assume that all gross portfolio returns belong to the scale invariant family $\sigma\tau_q^+(\bar{a})$ weakly determined by the first four moments $(\mathbb{E}[X], \rho(X), a_1(X), a_2(X))$, where $\rho(X) = \mathbb{E}\left[(X - E(X))^2\right]^{1/2}$ $a_1 = \frac{\mathbb{E}\left[X - E(X))^3\right]}{\mathbb{E}[(X-E(X))^2]^{3/2}}$, $a_2 = \frac{\mathbb{E}\left[(X-E(X))^4\right]}{\mathbb{E}[(X-E(X))^2]^2}$.

Let P be the benchmark portfolio. Consider the ICX efficiency case first. We propose a two step procedure that involves solving an optimization problem, and then an hypothesis test. According to Efficiency Condition 1, portfolio P is ICX efficient if there doesn't exist another portfolio with greater or equal mean and greater $\rho(W)$. We consider the following optimization problem:

$$\begin{aligned} \max_{x_i, i=1,\dots,N} \; & x'\mathbb{E}[Z] \\ & \rho(x'Z) \geq \rho(x^{(P)'}Z) \\ & a_1(x'Z) = s, \; a_2(x'Z) = k \\ & \sum_{i=1}^{N} x_i = 1, \; x_i \geq 0, i = 1, \dots, N \end{aligned} \tag{3.9}$$

where s and k are the skewness and kurtosis of portfolio P. In Problem (3.9), among the portfolios with the same level of skewness and kurtosis, and at least the same level of standard deviation of portfolio P, we select the one with the maximum mean. Call x^{icx} the solution vector of Problem (3.9). Then, according to Corollary 1, perform the following hypothesis test:

$$H_0 : \mathbb{E}\left[x^{icx'}Z\right] - \mathbb{E}[P] \leq 0 \qquad H_1 : \mathbb{E}\left[x^{icx'}Z\right] - \mathbb{E}[P] > 0 \tag{3.10}$$

According to Corollary 1 and Efficiency Condition 1, under the null hypothesis P is ICX efficient. The alternative hypothesis implies that $x^{icx'}Z \; ICX \; P$. This is due to the fact that, with Problem (3.9), we seek for a portfolio able to dominate P in the ICX sense. If the mean of the candidate dominating portfolio is lower than the mean of P, it means that there doesn't exist any portfolio able to dominates it in the ICX sense[8]. The test statistic for the hypothesis test in (3.10) can be found using estimation function theory. In particular, let $R_t^{icx} = [R_1^{icx}, \dots, R_t^{icx}]$, $R_t^P = \left[R_1^P, \dots, R_t^P\right]$ be historical observations of portfolio gross returns of $x^{icx'}Z$ and P and call $\Delta R_t = R_t^{icx} - R_t^P$. Let us define the following:

$$\begin{aligned} \theta_1 &= \mathbb{E}[f(\Delta R_t)] \\ \theta_2^2 &= \mathbb{E}\left[(f(\Delta R_t) - \theta_1)^2\right] \\ \theta_3 &= \frac{\mathbb{E}\left[(f(\Delta R_t) - \theta_1)^3\right]}{\theta_2^2} \\ \theta_4 &= \frac{\mathbb{E}\left[(f(\Delta R_t) - \theta_1)^4\right]}{\theta_2^4} \end{aligned} \tag{3.11}$$

[8] In case the hypothesis test in (3.10) is performed using the historical observations of of gross returns of a portfolio not necessarily solution of Problem (3.4), the test simply verifies whether the chosen portfolio has higher mean than portfolio P.

where $f(\Delta R_t) = \Delta R_t$. The optimal estimators for θ_1, $\widehat{\theta}_1$, and θ_2, $\widehat{\theta}_2$ are the roots of the following optimal EFs:

$$
\begin{aligned}
l_{\theta_1} &= \sum_{t=1}^{T} \frac{-1}{\theta_2^2} \left(f(\Delta R_t) - \theta_1 \right) + \frac{\theta_3}{\theta_2^3 \left(\theta_4 - 1 - \theta_3^2 \right)} \left(\left(f(\Delta R_t) - \theta_1 \right)^2 - \theta_2^2 - \theta_3 \theta_2 \left(f(\Delta R_t) - \theta_1 \right) \right) \\
l_{\theta_2} &= \sum_{t=1}^{T} \frac{-2}{\theta_2^3 (\theta_4 - 1 - \theta_3^2)} \left(\left(f(\Delta R_t) - \theta_1 \right)^2 - \theta_2^2 - \theta_3 \theta_2 \left(f(\Delta R_t) - \theta_1 \right) \right)
\end{aligned}
\tag{3.12}
$$

The distribution of the optimal estimator then satisfies the following :

$$
\sqrt{(T)} \left(\begin{bmatrix} \widehat{\theta}_1 \\ \widehat{\theta}_1 \end{bmatrix} - \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \right) \rightarrow^d \; N \left(0, V^{-1} \right) \tag{3.13}
$$

where $V^{-1} = \dfrac{\theta_2^2}{4} \begin{bmatrix} 4 & 2\theta_3 \\ 2\theta_3 & \theta_4 - 1 \end{bmatrix}$. Then the rejection region for the hypothesis test in (3.10), is given by $C = \{\sqrt{T}\widehat{\theta}_1 > c\}$ where c is a non negative real number. This rejection region implies that we reject the null hypothesis when the difference between the parameters is positive and large *enough*. For a given test size $\alpha \leq 0.5$, c is chosen such that $\int_c^\infty \phi(x)dx = \alpha$, where ϕ is the density of the marginal distribution of θ_1 deriving from the limiting distribution in 3.13, so that:

$$
\lim_{T \to \infty} \mathbb{P}[\text{reject } H_0 | H_0 \text{ is true}] \leq \int_c^\infty \phi(x)dx = \alpha
$$

Similarly, we can define a testing methodology for ICV efficiency. In this case, we propose to solve first the following optimization problem:

$$
\begin{aligned}
\max_{x_i, i=1,\ldots,N} \; & x' \mathbb{E}\left[Z\right] \\
& \rho(x'Z) \leq \rho(x^{(P)'}Z) \\
& a_1(x'Z) = s, \; a_2(x'Z) = k \\
& \sum_{i=1}^{N} x_i = 1, \; x_i \geq 0, i = 1, \ldots, N
\end{aligned}
\tag{3.14}
$$

where in this case we select the portfolio with the maximum mean among those who have the same values of skewness and kurtosis and lower standard deviation than portfolio P. Then call x^{icv} the solution of Problem (3.14), and perform the following hypothesis test:

$$
H_0 : \mathbb{E}\left[x^{icv'} Z \right] - \mathbb{E}\left[P\right] \leq 0 \qquad H_1 : \mathbb{E}\left[x^{icv'} Z \right] - \mathbb{E}\left[P\right] > 0 \tag{3.15}
$$

Also in this case, under the null hypothesis P is ICV efficient, while under the alternative $x^{icv'}Z\ ICV\ P$. The test statistics is the root of the optimal EF in (3.12). Limiting distribution and rejection region are equivalent to the ICX case.

Hypothesis tests in (3.10) and (3.15), based on the rejection region of the form of C, are asymptotically efficient and unbiased, as shown in the following proposition.

Proposition 4. *When portfolio gross return distributions belong to a family $\sigma\tau_4^+(\bar{a})$ uniquely determined by $(\mathbb{E}[X], \rho(X), a_1(X), a_2(X))$ we can guarantee the existence of opportune value c, such that*

$$\lim_{T\to\infty} P(\textit{reject } H_0 \mid H_0 \textit{ is true}) \leq \alpha \textit{ and } \lim_{T\to\infty} P(\textit{reject } H_0 \mid H_0 \textit{ is } false) = 1$$

3.3.2 Test for Rachev Ordering

Assume that all portfolios gross returns belong to a scale invariant family $\sigma\tau_4^+(\bar{a})$ of positive random variables, with parameters $(\mu(X), \rho(X), a_1(X), a_2(X))$, where:

$$\begin{aligned}
\mu(X) &= E(f(X)) \\
\rho(X) &= E\left((f(X) - \mu(X))^2\right)^{0.5} \\
a_1(X) &= \frac{E\left((f(X) - \mu(X))^3\right)}{\rho^3(X)} \\
a_2(X) &= \frac{E\left((f(X) - \mu(X))^4\right)}{\rho^4(X)}
\end{aligned} \tag{3.16}$$

with $t_\alpha(X) = F_X^{-1}(\alpha)$, and $f(X) = 0.5\frac{I_{[X\geq t_\beta(X)]}X}{(1-\beta)} + 0.5\frac{I_{[X\leq t_\alpha(X)]}X}{\alpha}$. From Definition (4), we know that $X \geq_{\alpha,\beta,\lambda} P$ if and only if $\gamma_{\alpha,\beta,\lambda}(X) \leq \gamma_{\alpha,\beta,\lambda}(P)$ for all $\alpha, \beta, \lambda \in [0,1]$. Moreover, from Theorem 2 the reward measure based on the Rachev risk measure is isotonic with the the choice of non satiable risk averse investors, when is lower than the mean, and isotonic with the choice of non satiable risk seeker investors when it is greater than the mean. This allows to test for Rachev ordering from two different perspective. Under the perspective of risk seeker investors, we propose to solve first Problem 3.4, under risk averse perspective we propose to solve problem 3.5. Let x^R be de solution vector from either problem 3.4 or 3.5. Then, the test for Rachev ordering efficiency can be formulated as:

$$H_0 : \mu(x^{R'}Z) - \mu(P) \leq 0 \qquad H_1 : \mu(x^{R'}Z) - \mu(P) > 0 \tag{3.17}$$

Similarly to the previous cases, under the null hypothesis P is efficient in the Rachev ordering sense, while under the alternative $x^{R'}Z \geq_{\alpha,\beta,\lambda} P$. Test statistics, rejection region and asymptotic distribution are similar to previous section, where in 3.11

$$f(\Delta R_t) = 0.5\frac{I_{[\Delta R_t \geq t_\beta(\Delta R_t)]}\Delta R_t}{(1-\beta)} + 0.5\frac{I_{[\Delta R_t \leq t_\alpha(\Delta R_t)]}\Delta R_t}{\alpha}$$

Even in this last case, hypothesis test in 3.17 based on the rejection region of the form C is asymptotically efficient and unbiased.

Proposition 5. *When the portfolios belong to a family $\sigma\tau_4^+(\bar{a})$ uniquely determined by $(\mu(X), \rho(X), a_1(X), a_2(X))$ we can guarantee the existence of opportune values c, such that*

$$\lim_{T\to\infty} P\left(reject\ H_0 \,|\, H_0\ is\ true\right) \leq \alpha\ and\ \lim_{T\to\infty} P\left(reject\ H_0 \,|\, H_0\ is\ false\right) = 1$$

3.4 Empirical applications

In this section, we discuss the results of two empirical applications. Illustrating the potential of the proposed semi-parametric approach, we test whether different reward-risk measures rationalize the market portfolio. Firstly, we test the efficiency of the Fama and French benchmark portfolios to confirm some classical results in portfolio literature (see among the others Scaillet and Topaloglou (2010), Post and Kopa (2013) and Arvanitis and Topaloglou (2017)). Secondly, we examine the efficiency of some stock indexes in large scale markets (NYSE and Nasdaq). We deal with three different datasets. In the first case, we consider the Fama and French benchmark portfolios covering the period from July 1963 to October 2001 (460 monthly observations obtained from the homepage of Kenneth French (http://mba.tuck.dartmouth.edu/pages/faculty/ken.Fre
In the second case, we test the efficiency of Nasdaq composite index and NYSE composite index. We use 876 assets in the NYSE market and 386 in the Nasdaq from 28 December 1995 to 12 May 2017.

3.4.1 Case I. The Fama and French Market

This analysis considers the six Fama and French benchmark portfolios as a set of risky assets. They are constructed at the end of each June from the intersections of two size portfolios and three portfolios created according to the ratio between book equity (BE) and market equity (ME). The tested portfolio is the Fama and French market portfolio, which is the value-weighted average of all non-financial common stocks listed on NYSE, AMEX and Nasdaq, covered by CRSP and COMPUSTAT. A preliminary statistical analysis is reported in Table 3.1 showing the first four moments of the Fama and French market portfolio and the six benchmark portfolios.

Descriptive Statistics				
Benchmark portfolios	Mean	Std. Dev.	Skewness	Kurtosis
1	0.316	7.070	-0.337	-1.033
2	0.726	5.378	-0.512	0.570
3	0.885	5.385	-0.298	1.628
4	0.323	4.812	-0.291	-1.135
5	0.399	4.269	-0.247	-0.706
6	0.581	4.382	-0.069	-0.929
Fama and French Market portfolio	0.462	4.461	-0.498	2.176

Table 3.1: Descriptive statistics of monthly returns in percentage from July 1963 to October 2001 of the Fama and French market portfolio and the six Fama and French benchmark portfolios formed on size and book-to-market equity ratio. Portfolio 1 has low BE/ME and small size, portfolio 2 has medium BE/ME and small size, etc.

An interesting feature is the comparison between the behavior of the Fama and French market portfolio and the benchmarks. A mean-variance analysis illustrates the inefficiency of the market portfolio considering these two measures. In fact, it is clearly dominated by the benchmark portfolio 6 and the presence of other benchmark portfolios with higher mean or lower standard deviation suggests the possibility to create a Markowitz efficient frontier above the Fama and French portfolio. Thus, the market portfolio is mean-standard deviation inefficient. Analyzing the efficiency of the portfolio selection problem with semi-parametric test, we want to verify whether the market portfolio is stochastically dominated in the sense of ICX, ICV and Rachev ordering. We parametrically test (confidence level 95%) the efficiency of the Fama and French portfolio in the Rachev ordering, ICX and ICV. When we try to solve Problem 3.4, 3.5, 3.9 and 3.14, results show infeasible solutions for the all the orderings, confirming the results from Post and Kopa (2013) and Arvanitis and Topaloglou (2017). Moreover, all three tests concord on the results. In fact, the Rachev, ICX and ICV efficiency tests all agree in assessing the efficiency of the Fama and French Portfolio.

3.4.2 Case II. The NYSE and Nasdaq Markets

In the second case, we analyze the efficiency of the NYSE and Nasdaq composite indexes considering the four parameters. In particular, we test the possibility to build a portfolio from the set of the stock components that dominates the index in the ICX and ICV sense and, with respect to Rachev Ordering. The efficiency of the market portfolio changes according to size and time frames, as do the risk and the preferences of the investors. In order to control for those, we consider a rolling window type of analysis with a window of 10 years. When we test for ICX or ICV, we assume that all portfolio gross returns belong to the $\sigma\tau_4^+(\bar{a})$ family, weakly determined by, mean, standard deviation, skewness and kurtosis. We solve Problems (3.9) or (3.14), every 21 trading days. In total we solve 140 optimization problem per tests. Then we perform the hypothesis test in (3.10) and (3.15) at 95% of confidence level. When we test for Rachev ordering we assume that the portfolios are uniquely determined by the set of parameters in equations (3.16) and then test for Rachev ordering efficiency according to Theorem 2. Under the non satiable and risk seeker investors' perspective, we solve Problem 3.4, under non satiable and risk averse perspective, we solve Problem 3.5. Also in this case we solve 140 optimization problems per test. Then we perform the hypothesis test in (3.17) on each solution of each problem. Table 3.2 reports, for each test, the percentage of time we cannot reject the null hypothesis, i.e. when the benchmark portfolio is not efficient. We observe that most of the time, up to 89% of the cases, we cannot reject the null hypothesis for ICV and ICX, meaning that Nasaq and NYSE market portfolios are not efficient for non satiable risk averse and non satiable and risk seeker investors. Nevertheless, the ICX efficiency it's easier to obtain than ICV efficiency, because market portfolios are well diversified, and non satiable and risk seeker investors prefer more concentrated portfolios over well diversified one (see for example Egozcue and Wong (2010) and Ortobelli et al. (2018)). These facts confirm also results from Kopa and Post (2015) on the efficiency of the market portfolios. Similarly, in the case of Rachev ordering efficiency, we cannot reject the null hypothesis and high number of times, but still lower than ICV and ICX cases. Even for non satiable, nor risk averse nor risk seeker investors, Nasdaq and NYSE market portfolio are not efficient.

Rachev, ICX and ICV Order of Stochastic Dominance					
Portfolio Problem	Reward Measure	Ordering	Ordering Confidence level	% Dominance NYSE	% Dominance Nasdaq
3.9	mean	ICX	95%	89.286%	82.142%
3.14	mean	ICV	95%	87.857%	77.857%
3.4	Rachev(5%,5%)	Rachev	95%	70.714%	67.857%
3.4	Rachev(5%,10%)	Rachev	95%	85.714%	72.857%
3.4	Rachev(10%,5%)	Rachev	95%	69.285%	79.285%
3.4	Rachev(10%,10%)	Rachev	95%	62.143%	70.714%
3.5	Rachev(5%,5%)	Rachev	95%	83.571%	85.714%
3.5	Rachev(5%,10%)	Rachev	95%	73.571%	63.571%
3.5	Rachev(10%,5%)	Rachev	95%	66.428%	75%
3.5	Rachev(10%,10%)	Rachev	95%	84.286%	71.428%

Table 3.2: Percentage of times we get the dominance from June 2006 till May 2017 for a total of 140 optimizations. In the first part of the table are reported the results using the mean as reward measure. In this case the first row reports the percentage of time where the market portfolio is not ICX efficient. Similarly for the second raw, where the test is for ICV efficiency. The second part shows the results using the $-\gamma_{\alpha,\beta,\lambda}$ as reward measure. In this case it is possible to test for Rachev ordering from two different perspectives: non satiable and risk seeker investors and, non satiable and risk averse investors.

3.5 Conclusion

Stochastic dominance efficiency tests have been developed under the assumptions that investors are non satiable and risk averse (Kopa and Post (2015),Post (2003),Post and Kopa (2016) and Kuosmanen (2004)). Nevertheless, studies based on risk averse investors have proven to be too restrictive in many circumstances (see among the others Markowitz (1952a), Levy and Levy (2002) Kahneman and Tversky (1979), Barberis and Thaler (2003).). In this paper, we have developed a methodology to asses the efficiency of a given portfolio able to consider a wider class of investors. We have extended classic dominance conditions, when the return distributions depend on a reward measure, a risk measure and other distributional parameters, satisfying a minimal set of assumptions. In doing so, we also establish that non satiable, nor risk averse nor risk seeker investors, adjust their risk attitude according to market conditions. In a market where the reward measure is higher than the expected return, they behave as risk seeker, while when the reward measure is lower than the expected return, they behave as risk averse. Then, we have defined a new stochastic ordering coherent with the preference of non satiable, nor risk averse nor risk seeker investors. We have employed these results to develop efficiency tests for ICV, ICX and the newly introduced stochastic ordering. The test statistics are based on estimation function theory and its asymptotic properties. Finally, we illustrate the potential of the proposed statistic tests in two empirical applications. In the case of the Fama and French market portfolio, we reject the null hypothesis during all the period for all the 3 efficiency test. This results is coherent with related literature, assessing the efficiency of the Fama and French market portfolio (see for example Post and Kopa (2013), Arvanitis and Topaloglou (2017)) Empirical results indicate that we cannot reject the null hypothesis of the market portfolio in the Rachev, ICX and ICV orders. Then, we apply the test statistics to the NYSE and Nasdaq composite indexes. The results dictate that the two stock indexes are often dominated for all the three stochastic orderings.

The proposed methodology is general and can be applied to any stochastic ordering defined by a positive functional, satisfying positive homogeneity and translation equivariance.

Appendix A: Proofs

Proof of Theorem 1. *Implication 1:* As a consequence of the assumptions follows

$$h = \mu(W) - \mu(Y) \geq 0 \text{ and } \rho(W) \geq \rho(Y) \geq \rho(Y + t)$$

for every $t \geq 0$. Moreover, for every $t \geq 0$ the function $g(t) \equiv \dfrac{\mu(Y) + t}{\rho(Y + t)}$ is an increasing continuous positive function that tends to infinity for big values of t. As a consequence of definition of $\sigma\tau_q^+(\bar{a})$ family there exist $t \leq h$ such that the random variable $\frac{W}{\rho(W)}$ has the same parameters of $\frac{Y+t}{\rho(Y+t)}$ and hence $\frac{W}{\rho(W)} \overset{d}{=} \frac{Y+t}{\rho(Y+t)}$. Then, for every $\lambda \geq 0$:

$$\mathbb{P}\left[W \leq \lambda\right] \leq \mathbb{P}\left[\frac{Y + t}{\rho(Y + t)} \leq \frac{\lambda}{\rho(Y + t)}\right] \leq \mathbb{P}\left[Y \leq \lambda\right]$$

Observe that at least one of the two inequalities $\rho(W) \geq \rho(Y)$ and $h \geq 0$ is strict by hypothesis. Then, at least one of the previous inequalities is strict for some real $\lambda \geq 0$. Therefore, $W\ FSD\ Y$.

Implication 2: According to definition of $\sigma\tau_q^+(\bar{a})$ family, it follows

$$\frac{W}{\rho(W)} \stackrel{d}{=} \frac{Y}{\rho(Y)}$$

because the two random variables have the same parameters. If $\rho(W) > \rho(Y)$, then for every $t \geq 0$

$$\mathbb{P}\left[W \leq t\right] = \mathbb{P}\left[\frac{W}{\rho(W)} \leq \frac{t}{\rho(W)}\right] \leq \mathbb{P}\left[\frac{W}{\rho(W)} \leq \frac{t}{\rho(Y)}\right] = \mathbb{P}\left[Y \leq t\right]$$

and the above inequality is strict for some t. Conversely, if $W\ FSD\ Y$, then by the Skorokhod Representation Theorem, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two random variables X and Y defined on this space such that $X \geq Y$ and X , Y have the same distributions of W and Y. Since $\mu(X)$ is a simple reward measure $\mu(W) \geq \mu(Y)$ and must be $\rho(W) > \rho(Y)$ (Skorokhod (1977)).

Implication 3: If $W\ FSD\ Y$ then $\mu(W) \geq \mu(Y)$ because any reward measure is isotonic with FSD order. If $\mu(W) = \mu(Y)$, then $W \stackrel{d}{=} Y$ thus $\mu(W) > \mu(Y)$. Conversely if $\mu(W) > \mu(Y)$ then $W \stackrel{d}{=} Z = Y + \mu(W) - \mu(Y)$ and $W\ FSD\ Y$. □

Proof of Corollary 1. Recall that W CV Y if and only if $Y\ CX\ W$ (or equivalently $Y\ ICX\ W$ and $\mathbb{E}[W] = \mathbb{E}[Y]$) i.e. every risk seeker investor prefers Y to W. Thus, $W\ CX\ Y \Leftrightarrow \mathbb{E}[W] = \mathbb{E}[Y]$ and $\rho(W) > \rho(Y)$ as consequence of Remark **??**. When $\mathbb{E}[W] \geq \mathbb{E}[Y]$ and $\rho(W) > \rho(Y)$, then $Z = Y + \mathbb{E}[W] - \mathbb{E}[Y]$ belongs to $\sigma\tau_q^+(\bar{a})$ and from Remark **??**, $W\ CX\ Z$ and, $Z\ FSD\ Y$ if $\mathbb{E}[W] > \mathbb{E}[Y]$. Thus $W\ ICX\ Y$ for the transitive property of the ICX ordering (considering that FSD implies ICX).

Conversely, suppose $W\ ICX\ Y$, then $\mathbb{E}[W] \geq \mathbb{E}[Y]$. Since the case $W\ ICX\ Y$ and $\mathbb{E}[W] = \mathbb{E}[Y]$ has been previously studied, we assume $\mathbb{E}[W] > \mathbb{E}[Y]$. Suppose W and Y have the same risk (i.e., $\rho(W) = \rho(Y)$), then $Z = Y + \mathbb{E}[W] - \mathbb{E}[Y]$ has the same distribution of W (because Z has the same parameters of W) and $W\ FSD\ Y$ against the hypothesis. Suppose W presents lower risk than Y (i.e., $\rho(W) < \rho(Y)$), then $\frac{\mathbb{E}[W]}{\rho(W)} > \frac{\mathbb{E}[Y]}{\rho(Y)}$. Therefore, there exists a positive value $t > \mathbb{E}[W] - \mathbb{E}[Y] > 0$ such that $\frac{\mathbb{E}[W]}{\rho(W)} = \frac{\mathbb{E}[Y]+t}{\rho(Y)}$and, thus$\frac{W}{\rho(W)} \stackrel{d}{=} \frac{Y+t}{\rho(Y)}$ because they have the same parameters. Then, the distributions of W and Y intersect themselves in the point $M = \frac{t}{\frac{\rho(Y)}{\rho(W)}-1} > 0$ and for every $\lambda \leq M$, $F_W(\lambda) \leq F_Y(\lambda)$ and for every $\lambda > M$, $F_W(\lambda) \geq F_Y(\lambda)$. However, a value $\lambda > M$ such that $F_W(\lambda) > F_Y(\lambda)$

cannot exist, because distribution functions are right continuous and $\int_M^\infty (1 - F_W(u))\,du < \int_M^\infty (1 - F_Y(u))\,du$, against the assumption $w'Z\ ICX\ y'Z$. Thus, if $W\ ICX\ Y$ and $\rho(W) \le \rho(Y)$ implies $W\ FSD\ Y$ against the hypothesis, hence, it must be that $\rho(W) > \rho(Y)$. □

Proof of Proposition 1. Let $(\mu_1(X), \rho(X))$ and $(\mu_2(X), \rho(X))$ be two parametrization of the $\sigma\tau_2$ family . Observe that for every distribution functions $F_U, F_Y \in \sigma\tau_2^+(a)$, $F_{V_1} := F_{\frac{U-\mu_1(U)}{\rho(U)}} = F_{\frac{Y-\mu_1(Y)}{\rho(Y)}}$ and $F_{V_2} := F_{\frac{U-\mu_2(U)}{\rho(U)}} = F_{\frac{Y-\mu_2(Y)}{\rho(Y)}}$. Then for every $F_X \in \sigma\tau_2^+(\bar{a})$ identified by the parameters $(\mu_i(X), \rho(X))$, i=1,2 we get

$$F_X = F_{\rho(X)V_1+\mu_1(X)} = F_{\rho(X)V_2+\mu_2(X)}.$$

Thus, $V_1 + (\mu_1(X) - \mu_2(X))/\rho(X) \overset{d}{=} V_2$ and, $(\mu_1(X) - \mu_2(X))/\rho(X)$ is constant for every $F_X \in \sigma\tau_2^+(\bar{a})$.

□

Proof of Theorem 2. *Case 1:* Suppose $\frac{\mu(W)}{\rho(W)} \ge \frac{\mu(Y)}{\rho(Y)}$ and $\mu(W) \ge \mu(Y)$. From Theorem 1 if $\rho(W) \ge \rho(Y)$ implies $W\ FSD\ Y$ that implies $W\ SSD\ Y$. Thus assume $\rho(W) < \rho(Y)$. From Proposition (**??**), we know that $\dfrac{\mu(W)}{\rho(W)} - \dfrac{\mu(Y)}{\rho(Y)} = \dfrac{\mathbb{E}[W]}{\rho(W)} - \dfrac{\mathbb{E}[Y]}{\rho(Y)} \ge 0$. Since $\mu(W) > E(W)$ then $\dfrac{\mu(W) - \mathbb{E}[W]}{\rho(W)} = c > 0$ that implies $0 \le \mu(W) - \mu(Y) = \mathbb{E}[W] - \mathbb{E}[Y] + c\,(\rho(W) - \rho(Y))$, i.e., $0 < c\,(\rho(Y) - \rho(W)) \le \mathbb{E}[W] - \mathbb{E}[Y]$. Then by Ortobelli (2001) we know that $W\ SSD\ Y$. Similar considerations follow when we assume $\mu(W) \ge \mu(Y)$ and $\rho(W) \le \rho(Y)$. Moreover, if $W\ CX\ Y$ it is not possible that $\rho(W) \le \rho(Y)$ since $W\ ICX\ Y$ implies $W\ FSD\ Y$ against the hypothesis $E(W) = E(Y)$. Thus, it should be $\rho(W) > \rho(Y)$. From Proposition 1:

$$\mu(W) - \mu(Y) = \mathbb{E}[W] - \mathbb{E}[Y] + c\,(\rho(W) - \rho(Y)) = c\,(\rho(W) - \rho(Y)) > 0.$$

Similarly, for 1*d*) from Corollary 1 we deduce $\mathbb{E}[W] \ge \mathbb{E}[Y]$ and $\rho(W) > \rho(Y)$, then $c = \dfrac{\mu(W) - \mathbb{E}[W]}{\rho(W)} > 0$ that implies $\mu(W) - \mu(Y) = \mathbb{E}[W] - \mathbb{E}[Y] + c\,(\rho(W) - \rho(Y)) > 0$, i.e., $\mu(W) > \mu(Y)$.

Case 2: We know that $W\ SSD\ Y$ implies that $\mathbb{E}[W] - \mathbb{E}[Y] \ge 0$. If $W\ SSD\ Y$ and $\rho(W) \ge \rho(Y)$,by Ortobelli (2001), implies $W\ FSD\ Y$ that implies $\mu(W) \ge \mu(Y)$ for Theorem (**??**). If $W\ SSD\ Y$ and $\rho(W) < \rho(Y)$ then $\dfrac{\mu(W) - \mathbb{E}[W]}{\rho(W)} = c < 0$ because $\mu(W) < \mathbb{E}[W]$. Then using the same arguments of *Case 1* we find $0 \le \mu(W) - \mu(Y) = \mathbb{E}[W] - \mathbb{E}[Y] + c\,(\rho(W) - \rho(Y))$ that explains case 2a) that is $W\ SSD\ Y$ implies $\mu(W) \ge$

$\mu(Y)$ that $\mu(W) \geq \mu(Y)$. If W SSD Y and $\rho(W) \geq \rho(Y)$ we know by Ortobelli (2001) that implies W FSD Y. Thus, must be $\rho(W) < \rho(Y)$. To get 2c), assume $\mu(W) \geq \mu(Y)$ and $\rho(W) \geq \rho(Y)$ (with at least one inequality strict) implies $0 < \mu(W) - \mu(Y) - c\,(\rho(W) - \rho(Y)) = \mathbb{E}\,[W] - \mathbb{E}\,[Y]$ because $c = \dfrac{\mu(W) - \mathbb{E}\,[W]}{\rho(W)} < 0$. If $\rho(W) = \rho(Y)$ then W has the same distribution of $Z = Y + \mathbb{E}\,[W] - \mathbb{E}\,[Y]$ and thus W FSD Y that implies W ICX Y. If $\rho(W) > \rho(Y)$ then W CX $Z = Y + \mathbb{E}\,[W] - \mathbb{E}\,[Y]$ and Z FSD Y , thus W ICX Y. $\square$

Proof of Proposition 2. If we solve the estimating equations $l^*_{\xi,k} = 0$ in terms of $\theta_1(\xi)$ we obtain the two solutions:

$$\theta_1(\xi) = \frac{1}{T}\sum_{t=1}^{T} f\left(R_t\right) + c_q \pm \left(c_q^2 - \frac{1}{T}\sum_{t=1}^{T}\left(f\left(R_t\right) - \frac{1}{T}\sum_{t=1}^{T} f\left(R_t\right)\right)^2 + \theta_2^2(\xi)\right)^{1/2}.$$

However when $T \to +\infty$ the unique consistent admissible equations are:

$$\theta_1(\xi) = \frac{1}{T}\sum_{t=1}^{T} f\left(R_t\right) + c_q - \left(c_q^2 - \frac{1}{T}\sum_{t=1}^{T}\left(f\left(R_t\right) - \frac{1}{T}\sum_{t=1}^{T} f\left(R_t\right)\right)^2 + \theta_2^2(\xi)\right)^{1/2}$$

if $c_q > 0$ and

$$\theta_1(\xi) = \frac{1}{T}\sum_{t=1}^{T} f\left(R_t\right) + c_q + \left(c_q^2 - \frac{1}{T}\sum_{t=1}^{T}\left(f\left(R_t\right) - \frac{1}{T}\sum_{t=1}^{T} f\left(R_t\right)\right)^2 + \theta_2^2(\xi)\right)^{1/2}$$

if $c_q \leq 0$. $\square$

Proof of Proposition 3. When, H_0 is true, then $\theta_1 = \mathbb{E}\left[x^{icx'} Z\right] - \mathbb{E}\,[P] \leq 0$, which implies that:

$$\begin{aligned} \mathbb{P}\left[\sqrt{T}\widehat{\theta_1} > c\right] &\leq \mathbb{P}\left[\sqrt{T}\widehat{\theta_1} \geq c + \sqrt{T}\theta_1\right] \\ &= \mathbb{P}\left[\sqrt{T}(\widehat{\theta_1} - \theta_1) \geq c\right] \end{aligned}$$

Passing to the limit for $T \to \infty$ the last term converges to α.
When H_0 is false, we get that $\theta_1 = \mathbb{E}\left[x^{icx'} Z\right] - \mathbb{E}\,[P] > 0$, therefore:

$$\lim_{T\to\infty} \mathbb{P}\left[reject\ H_0 \,|\, H_0\ is\ false\right] = \lim_{T\to\infty} \mathbb{P}\left[\sqrt{T}(\widehat{\theta_1}) \geq c\right] = 1$$

$\square$

Chapter 4

On the Efficiency of Portfolio Choices

Summary

This chapter proposes semi-parametric tests to evaluate the efficiency of a given portfolio, with respect to a new stochastic ordering, called Gini order, coherent with the preference of non satiable, nor risk seeker nor risk averse investors. Firstly, the Gini order is introduced and its relation with risk aversion is discussed. Secondly, a testing methodology based on associated optimization problems is proposed. The solutions of such problems are designed specifically in order to dominates the tested portfolio in the Gini order. Finally, en empirical investigation of the efficiency of SP500 index is proposed.

4.1 Introduction

Expected Utility is one of the cornerstone of the modern portfolio theory. It relies on a finite number of axioms which characterize investors preferences. In particular, under the assumptions of rationality of preferences, an agent prefers an investment X over Y if and only if $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ where u is a given utility function (Von Neumann and Morgenstern (2007)). Typically, optimal choices for different categories of investors can be distinguished using Stochastic Dominance. Each stochastic ordering identifies a specific investor behavior. Efficient allocations with respect a given stochastic ordering are also optimal for the corresponding category of agents. First Order of Stochastic dominance is coherent with the behavior of non satiable investors, Second order of Stochastic Dominance is coherent with the preference of non satiable and risk averse investors, while efficient choices for non satiable and risk seeker investors are coherent with the Increasing and Convex order of Stochastic dominance (see for example Fishburn (1976) and Levy (1992)). The literature on testing stochastic dominance efficiency can be split into parametric and non parametric (for parametric tests see among the otherPost (2003) Kuosmanen (2004) and Kopa and Post (2015), and for non parametric see Gibbons et al. (1989), Linton et al. (2005) and Scaillet and Topaloglou (2010)). However, several studies have shown that investors prefer more to less and are nor risk seeker nor risk averse (see among the others Markowitz (1952b), Kahneman and Tversky (1979) and Levy and Levy (2002)). Therefore the classic definitions of Stochastic dominance appear to be too restrictive to fully characterize the investors preferences.

This chapter proposes semi-parametric tests for portfolio efficiency w.r.t. a new stochastic ordering coherent with non satiable and nor risk seeker nor risk averse investors. The new stochastic ordering, called Gini order, is defined in terms of *aggressive-choerent* functionals and specifically designed to take into account the tail behavior of random variable (Biglova et al. (2004) and Ortobelli et al. (2013)). It is strictly linked to Gini tail measure, which is an extension of the Gini index, widely used in economics and other social sciences, in particular in measuring disparities (see for example Gini (1921), Shalit and Yitzhaki (2005) and Ceriani and Verme (2012)). The functional defining the stochastic orderings depends on 3 parameters to which correspond different levels of risk aversion. Following Chapter 3, the testing methodology is a two step procedure.

First, assume that gross return distribution belong to a scale invariant family, weakly determined by four parameters, a reward measure, a risk measure and other distributional parameters linked to skewness and kurtosis. Then, check whether the chosen portfolio is dominated with respect to the First order of Stochastic dominance using the procedure in Scaillet and Topaloglou (2010). Second, solve optimization problems, whose solutions are specifically designed to dominate the tested portfolio with respect to the Gini order, and finally perform an hypothesis tests to verify whether the solutions dominated the portfolio or not.

The chapter is organized as follows: Section (4.2) introduces the Gini order, Section (4.3) describes the testing methodology. Finally, Section (4.4) presents an empirical application on the SP500.

4.2 Efficient Choices

Classic definitions of Stochastic Dominance rules are given in terms of iterative integral conditions. For instance, First order Stochastic Dominance is based on pairwise comparison of distribution functions, Second order Stochastic Dominance, is based on pairwise comparison of integral of distribution functions, and so on. Inverse stochastic dominance offers an alternative representation based on the left inverse of cumulative distribution functions (see among the others Dybvig (1988), Ogryczak and Ruszczyński (2002) and Kopa and Chovanec (2008)). Let X and Y be real random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution functions $F_X(t) = \mathbb{P}[X \leq t]$ and $F_Y(t) = \mathbb{P}[Y \leq t]$, $\forall\ t \in \mathbb{R}$, respectively. Call $F_X^{-1}(p) = \inf\{t : F_X(t) \geq p\}\ \forall p \in [0,1]$ the left inverse of F, then:

- X dominates Y in the sense of First Order of Stochastic Dominance (i.e.W FSD Y) if and only if $F_X^{-1}(t) \geq F_Y^{-1}(t)\ \forall t \in [0,1]$, or equivalently W FSD Y if and only if $E(u(X)) \geq E(u(Y))$ for all non-decreasing utility functions u.

- X dominates Y in the sense of Second Order of Stochastic Dominance (i.e.X SSD Y) if and only if $\int_0^t F_X^{-1}(\lambda)d\lambda \geq \int_0^t F_Y^{-1}(\lambda)d\lambda\ \forall\ t \in \mathbb{R}$, or equivalently X SSD Y if and only if $E(u(X)) \geq E(u(Y))$ for all non decreasing and concave utility functions u.

Both FSD and SSD, can be interpreted in terms of important financial quantities widely used in the recent literature developments: $X\ FSD\ Y$ if and only if $V@R_\alpha(X) \leq V@R_\alpha(Y)$, and $X\ SSD\ Y$ if and only if $CV@R_\alpha(X) \leq CV@R_\alpha(Y)$ for all confidence level α, where $V@R_\alpha(X) = -F_X^{-1}(\alpha)$ is the Value-at-Risk and $CV@R_\alpha = \frac{-1}{\alpha}\int_0^\alpha F_X^{-1}(p)dp$ is the Conditional-Value-at-Risk (see among the other Ogryczak and Ruszczyński (2002) and Kopa and Chovanec (2008)). Typically, stochastic dominance relations identifies optimal choices for the whole category of investors. A portfolio is said to be optimal, for a given stochastic ordering, if another portfolio, ables to dominates it, doesn't exists. Moreover, when a portfolio is non-dominated w.r.t. FSD is also optimal for all non satiable investors (i.e. with increasing utility functions), and when it is SSD non dominated, it is also optimal for all non satiable and risk averse investors (i.e. with increasing and concave utility functions).

Nevertheless, many studies have shown that investors prefer the "more"to "less"and are neither risk seeker nor risk averse (see for example Markowitz (1952b), Kahneman and Tversky (1979) and Levy and Levy (2002)). Therefore, the aforementioned stochastic orderings appear to be too restrictive.

In order to relax them and, construct stochastic ordering coherent with non satiable, nor risk seeker nor risk averse investors, consider the definition of coherent risk measures (see Artzner et al. (1999)). A coherent risk measure is a map $\nu : \Lambda \subset L^p(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, satisfying the following axioms:

- Monotonicity: $\forall X, Y \in \Lambda$ such that $X \leq Y$, $\nu(X) \geq \nu(Y)$
- Positive homogeneity: $\forall h \in \mathbb{R}$ and $X \in \Lambda$, $\nu(hX) = h\nu(X)$
- Translation invariance: $\forall \alpha \in R$ and $X \in \Lambda$, $\nu(X + \alpha) = \nu(X) - \alpha$
- Sub-additivity: $\forall X_1, X_2 \in \Lambda$, $\nu(X_1 + X_2) \leq \nu(X_1) + \nu(X_2)$

Sub-additivity and positive homogeneity imply also convexity. Convexity of coherent risk measures is of most importance from a practical point of view, because it means that diversification should not increase the amount of risk[1].

Combining together two coherent risk measure it is possible to construct functionals consistent with the choices of non satiable, nor risk averse nor risk seeker investors. Following Biglova et al. (2004) the resulting functionals are called *aggressive-coherent* and are defined as follows $\forall \lambda \in [0, 1]$:

$$\eta(X) = \lambda\nu_1(X) - (1 - \lambda)\nu_1(-X) \tag{4.1}$$

where ν_1 and ν_2 are coherent risk measures. The functional in (4.1) is consistent with non satiable, nor risk seeker nor risk averse investors' preferences (see Biglova et al. (2004) and Rachev et al. (2008)). The next proposition shows some properties of *aggressive-coherent* functional ν.

Proposition 6. *Let $\eta(X)$ be an* aggressive-coherent *functional defined as in (4.1). Then the following properties hold.*

- *Translation invariant: $\forall \alpha \in \mathbb{R}$ and $X \in \Lambda$, $\eta(X + \alpha) = \eta(X) - \alpha$*
- *Positive homogeneity: $\forall h \in \mathbb{R}$ and $X \in \Lambda$, $\eta(hX) = h\eta(X)$*
- *Monotonicity: $\forall X, Y \in \Lambda$,such that $X \geq Y$ $\eta(X) \leq \eta(Y)$*

Proof. Translation invariant:

$$\begin{aligned}
\eta(X + t) &= \lambda\nu_1(X + \alpha) - (1 - \lambda)\,\nu_2(-X - \alpha) \\
&= \lambda\,(\nu_1(X) - \alpha) - (1 - \lambda)\,(\nu_2(-X) + \alpha) \\
&= \lambda\nu_1(X) - (1 - \lambda)\nu_2(-X) - \alpha = \eta(X) - \alpha
\end{aligned}$$

[1] Note that some authors consider the axioms of positive homogeneity and sub-additivity too restrictive and, instead, assume directly convexity (see among the other Acerbi (2002))

Positive homogeneity:

$$\begin{aligned}\eta(hX) &= \lambda\nu_1(hX) - (1-\lambda)\nu_2(-hX) \\ &= h\left[\lambda\nu_1(X) - (1-\lambda)\nu_2(-X)\right] = h\eta(X)\end{aligned}$$

Monotonicity: note that by the monotonicity of coherent risk measures, $\nu(-X) \geq \nu(-Y)$, therefore:

$$\lambda\left(\nu_1(X) - \nu_2(Y)\right) - (1-\lambda)\left(\nu_2(-X) - \nu_2(-Y)\right) \leq 0$$

then, $\eta(X) \leq \eta(Y)$. □

The properties of ν are strictly linked to those of coherent risk measures, but sub-additivity, and hence convexity, for the aggressive coherent risk measure don't hold. By monotonicity, any *aggressive-coherent* functional is consistent with FSD, and as reported by Biglova et al. (2004) and Rachev et al. (2008), it is consistent with the preference of non satiable, nor risk averse nor risk seeker investors.

Given the general form of ν it is possible to derive different functionals, by specifying the coherent risk measures ν_1 and ν_2. In particular, consider the Gini type of risk measure. Based on extensions of Gini index, widely used in measuring social disparities, Gini type of measures have been proposed as an alternative way of measuring prospect variabilities (Konno and Yamazaki (1991), Shalit and Yitzhaki (2005)). The tail Gini measure (tGM), is a Gini type of measure which focus on the tail behavior of random variables and also allows specific consideration in terms of risk aversion. It is defined as the integral of the Lorentz Curve:

$$G_{\beta,\delta}(X) = -\frac{\delta(\delta-1)}{\beta^{\delta}}\int_0^{\beta}(\beta-u)^{\delta-2}L_X(u)du \tag{4.2}$$

where $L_X(p) = \int_0^p F_X^{-1}(p)$ is the Lorenz curve, δ ($\delta > 1$) governs the weight assigned to the lower β ($\beta \in [0,1]$) percentage of the portfolio distribution. There exists different formulation for the tGM, in particular for $\delta = 1$ it is possible to show that tGM is consistent with SSD (see Shalit and Yitzhaki (1994),Shalit and Yitzhaki (2005) and Ortobelli et al. (2013)). When, $\delta = 2$, $G_{\beta,2}(X) = G_{\beta}(X)$, in the discrete case correspond to the cumulative sum of the worst $\beta\%$ possible outcome. Furthermore, $G_{\beta}(X)$ is also consistent with dilation order (see Fagiuoli et al. (1999)).

Therefore functionals coherent with the preference of a non satiable, nor risk seeker nor risk averse investors behavior can be defined as difference of tGMs, i.e. $\forall, \alpha, \beta \in [0,1]$ and $\lambda \in [0,1]$:

$$\gamma_{\alpha,\beta,\lambda}(X) = \lambda G_{\alpha}(X) - (1-\lambda)G_{\beta}(-X) \tag{4.3}$$

The functional in (4.3), varying the parameters α, β and λ identifies the distribution of X and by Proposition (6) is consistent with the monotony order, and so, is a FORS risk measure (see Ortobelli et al. (2009)). Moreover, in the discrete settings, it corresponds to a weighted difference between the $\alpha\%$ worst and the $\beta\%$ best outcomes. It is possible to define the following new stochastic ordering consistent with the preference of a non satiable, nor risk seeker nor risk averse investor agent, called Gini order.

Definition 5. Given X and Y random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, X dominates Y in the Gini Order with parameter $\alpha, \beta \in [0,1]$ and $\lambda \in [0,1]$ (i.e. $X \geq^G Y$), if a and only if $\gamma_{\alpha,\beta,\lambda}(X) \leq \gamma_{\alpha,\beta,\lambda}(Y)$.

4.3 Optimal choices depending on finite number of parameters

Consider now a market with n assets. Denote with $Z = [Z_1, \ldots, Z_n]'$ and $x = [x_1, \ldots, x_n]'$ vectors of gross returns and portfolio weight, respectively. Under the assumptions of no short sales allowed ($x_i \geq 0$, $i = 0, \ldots, n$ and $\sum_{i=0}^{n} x_i = 1$) and limited liabilities ($Z_i \geq 0$, $i = 1, \ldots, n$), all portfolios $X = x'Z$ are positive random variables. Following Ortobelli (2001), assume that all portfolio gross return distributions belong to a scale invariant family $\sigma_4^+(\bar{a})$, with parameters $(\mu(X), \rho(X), a_1(X), a_2(X))$, with the following properties:
1. For every distribution function $F \in \sigma_4^+(\bar{a})$, there exist a random variable having F as distribution.
2. Every distribution function $F \in \sigma_4^+(\bar{a})$ is weakly determined by the set of parameters $(\mu(X), \rho(X), a_1(X), a_2(X))$, i.e. $F, G \in \sigma_4^+(\bar{a})$, then

$$(\mu(X), \rho(X), a_1(X), a_2(X)) = (\mu(Y), \rho(Y), a_1(Y), a_2(Y))$$

implies $X \stackrel{d}{=} Y$, but the converse is not necessarily true.
3. $\mu(X)$ is a reward measure translation invariant, i.e $\mu(X + t) = \mu(X) + t$ for all admissible t and positive homogeneous, and $\rho(X)$ is a risk measure consistent with the additive shift, i.e. $\rho(X + t) \leq \rho(X)$ $\forall t \leq 0$ and positive homogeneous.

Assumptions 1. and 2. are technical and guarantee that to every set of admissible parameters correspond a portfolio, and, concerning optimal choices is sufficient to look directly at the parameters. Assumption 3., instead, has nice economics interpretations. Translation invariance and consistency with additive shift implies that in a market where there exists a sure gain, i.e. a risk free rate, the reward measure shifts of the same sure amount, while riskiness doesn't increase. Positive homogeneity implies that position size affects, both riskiness and reward linearly. By Proposition (6), $\mu(X) = -\gamma_{\delta,\beta,\lambda}(X)$ is an admissible reward measure for the class $\sigma_4^+(\bar{a})$. Moreover, since $\gamma_{\delta,\beta,\lambda}(X)$ is consistent with the preference of non satiable, nor risk seeker nor risk averse investors, $\mu(X)$ is isotonic with the same preference order.

Let $R = [R_1, \ldots, R_T]'$ be a vector of historical observations of portfolio $X = x'Z$.

Then, a set of distributional parameters admissible for the $\sigma_q^+(\bar{a})$ is:

$$
\begin{aligned}
\mu(X) &= \mathbb{E}\left[f(R_t)\right] \\
\rho(X) &= \sqrt{\mathbb{E}\left[(f(R_t) - \mu(X))^2\right]} \\
a_1(X) &= \frac{\mathbb{E}\left[(f(R_t) - \mu(X))^3\right]}{\rho(X)^3} \\
a_2(X) &= \frac{\mathbb{E}\left[(f(R_t) - \mu(X))^4\right]}{\rho(X)^4}
\end{aligned}
\tag{4.4}
$$

with

$$
f(R_t) = \lambda\left(\frac{2}{\alpha^2}\sum_{k=1}^{t} R^a_{k:T}\mathbf{1}_{t<[T\alpha]}\right) - (1-\lambda)\left(\frac{2}{\beta^2}\sum_{k}^{t} R^d_{k:T}\mathbf{1}_{t<[T\beta]}\right)
$$

where $R^a_{k:T}$ and $R^d_{k:T}$ are the kth component of R sorted in ascending and descending order respectively and $\mathbf{1}_{t<[T\alpha]}$ is the indicator function equal to 1 when t is lower than $[T\alpha]$ which indicate the closest integer number to $T\alpha$. As established in Chapter 3, reward measures of the family $\sigma_4^+(\bar{a})$ are linked with SSD and ICX[2]. In particular, when a reward measure is greater than the mean of the distribution, it is also isotonic with the risk seeking prevalent behavior, while when it is lower than the mean is isotonic with the risk averse prevalent behavior. For example, consider Figure 4.1. In Figure 4.1 are depicted the values of mean and $\gamma_{0.05,0.1,0.5}$ and $\gamma_{0.1,0.1,0.5}$ of daily observations of SP500 with a rolling window of 1 year. As shown in the graph after 15 September 2008, which corresponds to Lehamnn Brother bankruptcy, $\gamma_{0.05,0.1,0.5}$ assumes lower values than the mean, meaning that, if before that date the functional was isotonic with risk seeking behavior, after that date is isotonic with risk averse behavior.

Let now, $P = p'Z$ be a portfolio, with parameters $(\mu(P), \rho(P), s, k)$, where s and k are skewness and kurtosis parameters respectively. According

[2]ICX stands for Increasing and Convex order, defined as X ICX Y if and only if $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all non satiable and risk seeker investors.

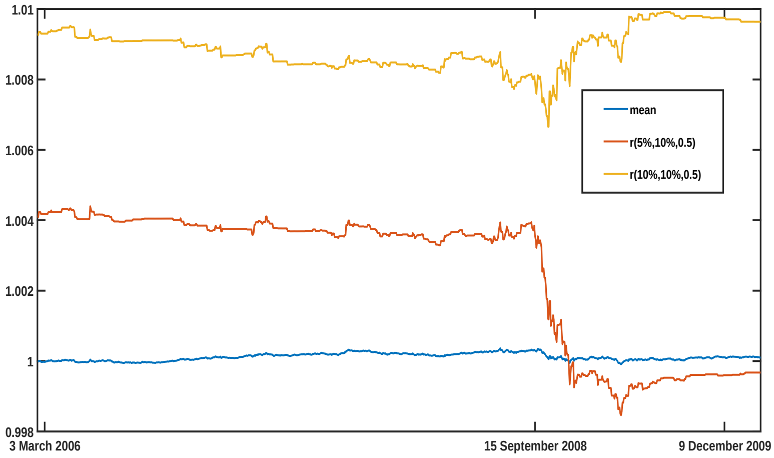


Figure 4.1: Comparison between mean and reward measures from 2006 to 2009

to Chapter 3, it is possible to semi-parametric test for efficiency of portfolio P, with respect to the Gini order, following a two steps procedure. Firstly, assume that $\sigma_4^+(\bar{a})$ is weakly determined by the set of parameters in (4.4) and then, solve optimization problems

$$\max_x \mu(X) = -\gamma_{\delta,\beta,\lambda}(X) \tag{4.5}$$

$$s.t. \quad \mathbb{E}[X] \geq \mathbb{E}[P]; \;\; \rho(X) \geq \rho(P) \tag{4.6}$$

$$a_1(X) = s; \;\; a_2(X) = k$$

$$\sum_{i=0}^{n} x_i = 1 \quad x_i \geq 0, i = 0, \ldots, n$$

or equivalently

$$\max_x \mu(X) = -\gamma_{\delta,\beta,\lambda}(X) \tag{4.7}$$

$$s.t. \quad \mathbb{E}[X] \geq \mathbb{E}[P]; \;\; \rho(X) \leq \rho(P) \tag{4.8}$$

$$a_1(X) = s; \;\; a_2(X) = k$$

$$\sum_{i=0}^{n} x_i = 1 \quad x_i \geq 0, i = 0, \ldots, n$$

The solution of problem (4.5) with constraints (4.6), is a portfolio that should dominates P, not only w.r.t. Gini order but also in the ICX sense. Similarly, portfolio solution of problem (4.7) under constraints (4.8) dominates P in the Gini order and also in the SSD sense. Call X^1 and X^2, the solutions of problems (4.5) and (4.7) respectively.

Secondly, perform the the following hypothesis test:

$$H_0 : \text{P is not dominated in the sense of Gini order} \qquad H_1 : \mu(X^i) - \mu(P) > 0$$

with rejection region defined as:

$$C = \{\sqrt{T}\widehat{\theta}(X^i, P) \geq c\}$$

for $i = 1, 2$, where c is a non negative real number and $\widehat{\theta}(X, P)$ is the optimal consistent estimator for the quantity $\theta(X, P) = \mu(X) - \mu(P)$. According to 3, such estimator is the root of a linear combination of unbiased and mutually orthogonal estimation functions (see Crowder (1986), Godambe and Thompson (1989)). Moreover, the limiting distribution of the test statistic is a gaussian distribution with mean $\sqrt{T}\theta$ and variance given by a function depending on the the second, third and fourth moment of θ (see Crowder (1986) and Godambe and Thompson (1989)).

4.4 Empirical study

The methodology presented in the previous sections can be used to test whether a given portfolio is efficient in the Gini order. As described before, the Gini order is coherent with the preference of non satiable, nor risk seeker nor risk averse investors. This section propose an empirical analysis of the efficiency of the SP500 index. The dataset is composed of daily observation from January 2000 to June 2017 of 386 stocks. Since investors' preference may change over time, the analysis is performed with a rolling window of approximately 4 year (1000 daily observation) where portfolio problems are solved every 21 trading days. According to the previous section, the efficiency can be tested as following a two step procedure:

1. Assume that the SP500 is weakly determined by the parameters in (4.4) and then test whether is not FSD dominated.
2. Solve optimizations problems (4.5) and (4.7) and then perform the hypothesis test H_0 vs H_1.

Table (4.1) reports a summary of the testing procedure. The first column indicates which optimization problem is solved and, the second points out with which parameters. Note that, the portfolio solutions of each problems in general differs, due to the fact that each problems characterize different investors preferences. The last column present the percentage of times the

Gini order Efficiency			
Portfolio Problem	Reward Parameters	Confidence Level	% Dominance NYSE
4.5	0.5, 5%, 5%	95%	95.38%
4.5	0.5, 5%, 10%	95%	99.42%
4.5	0.5, 10%, 15%	95%	97.11%
4.5	0.5, 5%, 15%	95%	5.73%
4.7	0.5, 5%, 5%	95%	100%
4.7	0.5, 5%, 10%	95%	99.42%
4.7	0.5, 10%, 15%	95%	100%
4.7	0.5, 5%, 15%	95%	100%

Table 4.1: Percentage of times we get the dominance from January 2000 till June 2017 for a total of 173 optimizations

null hypothesis is reject. In most of the cases the null and hence, the efficiency of the benchmark is rejected. Problem (4.5) with $G_{0.05,0.15,0.5}$ present a lower rate of rejection. Even if it admits feasible optimal solutions, it seems that the portfolio solution fails to dominates the benchmark and in this case is not possible to reject the null.

4.5 Conclusions

This chapter propose a methodology to test for portfolio efficiency w.r.t. a stochastic ordering coherent with the preference of non satiable, nor risk seeker nor risk averse investors. A new stochastic ordering has been defined in terms of an *aggressive-coherent* functional linked to the Gini tail measure. The proposed testing methodology is a two step procedure. Firstly, assume that gross return distributions belong to a scale invariant family weakly determined by four parameters. Distributional parameters are chosen coherently with the Gini order. Then, check whether the tested portfolio is not FSD dominated. Secondly, solve the proposed optimization problems whose solutions should dominate the tested portfolio w.r.t Gini order. Finally, perform the associated hypothesis tests to asses the efficiency of the tested portfolio. The last part of the chapter applies the proposed methodology to SP500. It turns out that during the analyzed period (2000-2017), the null is rejected almost all the time, implying that the SP500 index is dominated in the Gini order. This proposed procedure offers a way to test for efficiency w.r.t. Gini order and when the tested portfolio result dominated it also provide dominating portfolios.

Chapter 5

Risk Diversification

Summary

In this chapter, we propose a new approach to portfolio diversification based on risk measures. Risk Diversification Measures quantifies idiosyncratic risk diversified among the assets in a portfolio. Some of the diversification measures already established in the literature can be seen as special cases. We propose three empirical applications of four different Risk Diversification Measures taking into account various levels of risk aversion. Firstly, we discuss the mean-risk diversification efficient frontiers and secondly, we show how risk diversification based strategies perform under periods of financial distress. Finally, we attest the ability of risk diversification based portfolio strategies to outperform given market portfolios.

5.1 Introduction

Amongst the many aspects of portfolio selection, diversification is surely one of the most important. The seminal work of Markovitz provide the foundation of what diversification is in finance (see Markowitz (1952a)). In an efficient market and when all investors are mean variance optimizers, at equilibrium only the non-diversifiable risk is priced (see Sharpe (1964) and Mossin (1966)). Different approaches have, since then, been proposed in the literature. Starting from the empirical evidence that diversification benefits decrease when correlation increases, several studies have related diversification and correlation (see for example Levy and Sarnat (1970), Silvapulle and Granger (2001), Dopfel (2003) and Skintzi and Refenes (2005)). However, while correlation maybe an indicator of diversification benefit, it simply pairwise relates asset returns. Much effort has been done in better quantifying diversification. Statman (2004) propose to consider the Return Gap as a measure of benefit of diversification. The return gap is based on both correlation and standard deviation; when the benefit of diversification is low, return gap is low, while when standard deviation is high, return gap is high. The idea to consider not only correlation but also asset return standard deviations is also present in the Diversification Ratio (see Choueifaty and Coignard (2008) and Clarke et al. (2013)). The Diversification Ratio is based on the ratio between the weighted average of asset and portfolio standard deviations. Portfolio maximizing the Diversification Ratio it's called Most Diversified Portfolio, and it can be seen as the portfolio with the same level of correlation with each of its component (see Choueifaty and Coignard (2008)).

Empirical evidence suggests that assets returns exhibit excess kurtosis and skewness (see for example Rachev et al. (2011)). Therefore quantifying diversification only by the first two moments might lead to incorrect decision. For this reason, Vermorken et al. (2012) developed a diversification measure based on the the Shannon entropy. Entropy measures the uncertainty of the entire portfolio distribution, rather than only the first two moments. The Diversification Delta is then based on the ratio between portfolio's entropy and weighted average of entropy of its component. Flores et al. (2017) proposed a modify version of Diversification Delta, called Revised Diversification Delta, which is left bounded, it measures the diversification of idiosyncratic risk in the portfolio and values the change in size in assets and portfolio in the same way. Exploiting Principal Component Analysis, Rudin and Morgan (2006) developed the Portfolio Diversification Index (PDI), based on the number of independent component of a portfolio. Meucci (2009) extended the PDI and introduced the diversification distribution.

In a different stream of literature, portfolio diversification is understood as the number of assets with non zero weights, rather than how the risk is allocated among the assets. Typically then, portfolio diversification is compared by using so called diversification orderings (see Marshall et al. (1943), Wong (2007) and Egozcue and Wong (2010)). Diversification ordering can be expressed as a comparison between the ordered statistics of two portfolios' weight vectors (seeWong (2007), Egozcue and Wong (2010) and Ortobelli et al. (2018)). Generally, the most used diversification measures are consistent with diversification ordering, such as the Herfindal index, and any Schur-convex function defined on portfolio weights.

In this chapter, we propose a new class of functionals, which we called Risk Diversification Measures (RDMs). RDMs are defined in terms of a given risk measure, and can be interpreted as the risk reduction benefit arising from risk diversification. With a positive and convex risk measure, the corresponding RDM can be seen as the percentage of idiosyncratic risk diversified among the portfolio components. When the risk measures satisfies all the axioms of coherency, then the corresponding RDM, values in the same way, changes in assets and portfolio size and it is left bounded. In this case we call it Coherent Risk Diversification Measures (CRDMs), and represents the percentage of capital requirement reduction arising from risk diversification. We prove that, under the assumption of elliptical distributed returns, any CRDM depends on assets return means and standard deviations and portfolio standard deviation. Moreover, any centered portfolio has CRDM equivalent to the Diversification Ratio. On the one hand, the class generalizes diversification measures such as Diversification Delta and Diversification Ratio (see Choueifaty and Coignard (2008) and Flores et al. (2017)). On the other hand, the functional belonging to the class are in general not consistent with diversification ordering, in the sense that there can exist two portfolios with the same ordered statistics, but with different diversification measures (see among the others Wong (2007) and Egozcue and Wong (2010)).

We also provide three empirical applications of RDM and CRDM. We consider a market composed by stock belonging to the Down Jones Industrial Average index (DJIA) from $3rd$ of January 2005 to $14th$ of October 2017 and perform a static and a dynamic analysis. DJIA represents a good candidate to test risk diversification measures. It shows increasing correlation under periods of financial distress, the index itself represent a well diversified portfolio, and stocks belonging to the index are highly traded (see Silvapulle and Granger (2001), Skintzi and Refenes (2005) and Preis et al. (2012)).

In the static analysis, we introduce the mean-risk diversification frontier. Similarly to the mean-variance efficient frontiers, it's formed by portfolios with the highest level of mean for a desired level of risk diversification. Since risk diversification depends on the risk measure, we construct and compare four different mean-risk diversification frontiers, corresponding to the standard deviation and Conditional Value at Risk with level of confidence, 90%, 95% and 99% (see Pflug (2000) and Rockafellar and Uryasev (2000)). We observe that all the mean-risk diversification frontiers are somewhat concave in the Risk Diversification Measure, and mean decreases as risk diversification increases. Comparing the average statistics of portfolio belonging to the efficient frontiers we observe that risk diversification measure based on Conditional Value at Risk with an high confidence parameter seems to better allocate risk diversification (see Rockafellar and Uryasev (2000) and Pflug (2000)). Moreover, since the confidence level can be seen as the risk aversion parameter, to an higher risk aversion it corresponds, on average, more concentrated portfolios.

In the dynamic analysis we test the adaptability of risk diversification measures based on the standard deviation and Conditional Value at Risk, at level 90%, 95% and 99%, to face periods of financial distress and to outperform a given market portfolios. During the year after Lehman Brother bankruptcy, all the risk diversification measure based strategies produce positive returns. In the second dynamic analysis we show that using risk diversification measure, portfolio can outperform market portfolios in terms of expected return, risk allocation and wealth up to 150%.

The rest of this chapter is organized as follows: Section 2 introduce the new class of risk diversification measure and Section 3 shows the results of the static and the dynamic analysis.

5.2 Coherent Diversification Measures

This section describes the new class of diversification measures and their properties. In particular, the new class of diversification measures, quantify the amount of risk diversified among the assets in the portfolio. Let $R = [R_1, \dots, R_N]'$ be a random vector of returns and $w \in \mathbb{R}^N$ be a vector of portfolio weights. Let $P = w'R$ be a portfolio defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and assume no short sales are allowed, i.e. $w \in \Delta$ where :

$$\Delta = \left\{ w \in \mathbb{R}^N : \sum_{i=1}^{N} w_i = 1 \text{ and } w_i \in [0,1],\ i = 1, \dots, N \right\}.$$

Let us recall the definition of coherent risk measure. A coherent risk measure is a map ν that associates to each "risky"asset a real value, i.e. $\nu : \Lambda \subset L^p(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ satisfying the following properties (see Artzner et al. (1999)):

- Monotonicity: $\forall X, Y \in \Lambda$ such that $X \leq Y$, $\nu(X) \geq \nu(Y)$
- Positive homogeneity: for $h \geq 0$ and $X \in \Lambda$, $\nu(hX) = h\nu(X)$
- Translation invariance: $\forall \alpha \in \mathbb{R}$ and $X \in \Lambda$, $\nu(X + \alpha) = \nu(X) - \alpha$
- Sub-additivity: $\forall X_1, X_2 \in \Lambda$, $\nu(X_1 + X_2) \leq \nu(X_1) + \nu(X_2)$

A positive homogeneous risk measure detects change in asset and portfolio size in the same way. Translation invariance implies that the presence of a sure gain in the market, e.g. a risk free rate, simply shifts the portfolio riskiness of the same sure amount. Positive homogeneity and sub-additivity imply convexity. Convexity implies that diversification should not increase risk. Coherent risk measures play an important role in the following definition.

Definition 6. Let $X = [X_1, \ldots, X_n]'$ be a vector of return, $w = [w_1, \ldots, w_n]'$ be a vector of portfolio weights and $\nu : \Lambda \in L(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a risk measure. A Risk Diversification Measure for a portfolio $P = w'X$ is a functional of the from:

$$D_\nu(P) = 1 - \frac{\nu(P)}{\sum_{i=1}^n w_i \nu(X_i)}$$

A portfolio P_1 presents higher Risk Diversification, with respect to the risk measure ν than a portfolio P_2, if $D_\nu(P_1) \geq D_\nu(P_2)$. When ν is a coherent risk measure, then D_ν is called Coherent Risk Diversification Measure.

The definition is fairly general. RDMS are defined as the ratio between the portfolio risk and the average risk of its components and given their structure, can be interpreted as risk reduction benefit arising from risk diversification. To any risk measure corresponds a different definition of risk and therefore a different definition of risk diversification. When the risk measure ν satisfies some of the axioms of coherency, the corresponding RDM exhibits the following properties.

Proposition 7. *Let $X = [X_1, \ldots, X_n]$ be a return vector, $w = [w_1, \ldots, w_n]$ be a weights vector of portfolio $P = w'X$ and let ν be a risk measure. Then the following statements hold.*

1. *If ν is a convex risk measure (and generally consistent with the preference of risk averse investors) and it assumes the same sign for all the admissible portfolios P, then $D_\nu \in [0, 1]$.*

2. *if ν is positive homogeneous and translation invariance and when $X_i = a_1 Z + b_i$ then $D(P) = 0$*

Proof. Point 1. is a direct consequence of the definition of convex function,since when ν is convex, for all the admissible portfolio we have:

$$\sum_{i=1}^n w_i \nu(X_i) - \nu\left(\sum_{i=1}^n w_i X_i\right) \geq 0.$$

To establish point 2. take $X = [X_1, \ldots, X_n]$ with $X_i = a_i X + b_i$. Then,

$$\begin{aligned}
D(P) &= 1 - \frac{\nu\left(\sum_{i=1}^n w_i X_i\right)}{\sum_{i=1}^n w_i \nu(X_i)} \\
&= 1 - \frac{\nu\left(\sum_{i=1}^n w_i (a_i X + b_i)\right)}{\sum_{i=1}^n w_i \nu(a_i X + b_i)} \\
1 &- \frac{\sum_{i=1}^n a_i w_i \nu(X) - \sum_{i=1}^n w_i b_i}{\sum_{i=1}^n a_i w_i \nu(X) - \sum_{i=1}^n w_i b_i} = 0
\end{aligned}$$

□

When the risk measure is convex and it assumes the same sign for all the assets, D_ν takes value between 0 and 1. In this case, RDM can be interpreted as the percentage of idiosyncratic risk diversified in the portfolio. A portfolio P having $D_\nu(P) = 0$ has the same level of risk diversification with respect to ν of a stand alone asset. When $D_\nu(P) = 1$ it means that all the risk has been diversified. Nevertheless, under institutional constraints, such as no short sales allowed, this level of risk diversification cannot be reached and, as long as the weight vector $w \in \Delta$ then any RDM is bounded away from 1. In particular, the maximum of any RDM is attained when the ratio between portfolio risk and average risk of its component reach its minimum. Portfolio reaching that level, then have then maximum risk diversification with respect to the risk measure ν. When the risk measure is positive homogeneous and translation invariant, then no diversification benefit can be earned from investments in linearly dependent assets. This is an important property for any RDM, because it assures that the risk diversification is detected correctly. Moreover, any RDM based on a positive homogeneous risk measure, are not affected by changes in assets and portfolio size.

Coherent risk measures are convex, positive homogeneous and translation invariant and describe the capital requirement to regulate the risk assumed by market participant (see Artzner et al. (1999) and Tasche (2006)). So, any Coherent Risk Diversification Measure (CRDM) can be interpreted as the percentage of capital requirement reduction, arising from risk diversification. Consider the Conditional Value at Risk of portfolio P, defined as (see Pflug (2000) and Rockafellar and Uryasev (2000)):

$$CV@R_\alpha(P) = \inf\left\{a + \frac{1}{1-\alpha}\mathbb{E}\left[(-P-a)^+\right]\right\}$$

Under this formulation the $CV@R_\alpha$ corresponds to the mean in the $(1-\alpha)\%$ worst scenarios. Then, we call the Diversification Conditional Value at Risk, i.e. $D_{CV@R_\alpha}$ the following CRDM:

$$D_{CV@R_\alpha}(P) = 1 - \frac{CV@R_\alpha(P)}{\sum_{i=1}^N w_i CV@R_\alpha(X_i)}$$

The $D_{CV@R_\alpha}$ is defined as the ratio between the $CV@R_\alpha$ of a portfolio P and the average $CV@R_\alpha$ of its components. To different levels of α it corresponds a different $CV@R_\alpha$ and, hence a different risk diversification.

Similar approaches to risk diversification have been proposed in the literature by two series of papers which considered similar functional to D_ν: the Diversification Ratio (DR), and Revised Diversification Delta (DD^*). DR is a functional based on the ratio of the weighted average of assets standard deviation and portfolio standard deviation (Choueifaty and Coignard (2008) and Clarke et al. (2013)). DD^* is based on the ratio between the exponential entropy of the portfolio and the weighted average of the exponential entropy of each assets (see Vermorken et al. (2012) and Flores et al. (2017)). Under the assumption of elliptical distributed returns, all the CRDMs, are linked to both DD^* and DR.

Remark 4. *Let the returns vector be distributed as an elliptical distribution with mean vector $\mu_X = [\mu_1, \ldots, \mu_n]'$ and covariance matrix Σ_X, $w = [w_1, \ldots, w_n]'$ be a vector of weights and ν be a coherent risk measure satisfying the following identity property: if $X =^d Y$ then $\nu(X) = \nu(Y)$. Then the D_ν for portfolio $P = w'X$ is given by:*

$$D_\nu(P) = 1 - \frac{\sqrt{w'\Sigma_X w}\nu(Z) - w'\mu_X}{w'\sigma_i \nu(Z) - w'\mu_X} \tag{5.1}$$

where Z is a random variable elliptical distributed with zero and mean and variance equals to 1, and $sigma_i$ is the standard deviation of asset i.

Proof. Since $X \sim Ell(\mu_x, \Sigma_X)$ then $P \sim Ell(w'\mu_X, w'\Sigma_X w)$. So by the properties of elliptical distributed random variables $P =^d w'\mu_X + \sqrt{w'\Sigma_X w}Z$, where $Z \sim Ell(0, 1)$.:

$$\begin{aligned} D_\nu(P) &= 1 - \frac{\nu(P)}{\sum_{i=1}^n w_i \nu(X_i)} \\ &= 1 - \frac{\nu(w'\mu_X + \sqrt{w'\Sigma_X} wZ)}{\sum_{i=1}^n w_i \nu(\mu_i + \sigma_i Z)} \\ &= 1 - \frac{\sqrt{w'\Sigma_X w}\nu(Z) - w'\mu_X}{w'\sigma_i \nu(Z) - w'\mu_X} \end{aligned}$$

□

This result can be extended to any family of distribution, weakly determined by a finite number of parameters (see Ortobelli (2001)). Under the assumptions of elliptical distribution, any CRDM depends on standard deviation of each assets, portfolio mean and standard deviation and, the risk measure evaluated at the standardized random variable. In the case of centered portfolio, D_ν is equivalent to DD^* and to DR,up to a one-to-one transformation (see Choueifaty and Coignard (2008) and Flores et al. (2017)).

CRDMs and, RDMS in general, are not proper diversification measures. Typically, diversification measures are consistent with majorization ordering.

We recall the definition of diversification ordering in terms of majorization ordering (see among the others Wong (2007), Egozcue and Wong (2010) and Ortobelli et al. (2018)).

Definition 7. Let $X = [X_1, \ldots, X_n]'$ be a vector of returns, $\alpha, \beta \in \Delta$ where $\Delta = \left\{w \in \mathbb{R}^n, w_i \in [0,1], \sum_{i=1}^n w_i = 1\right\}$, be portfolio weight vectors. Then $\alpha' X$ is more diversified in the sense of first order of majorization than $y'X$ if y dominates in the sense of the first order of majorization w, i.e. $y \succ_M w$ if $F_y(k) = \sum_{i=1}^k y_i^o \geq F_w(k)$ for $k = 1, \ldots, n$, where y_i^o is the i-th element of the vector β sorted in ascending order.

According to the definition, a portfolio is more diversified than another if, the former majorizes the latter, which is, if the ordered weights of the first portfolio are greater or equal that the ordered weights of the second portfolio. Example of functions consistent with diversification ordering are HH index, weights' vector moments and in general all Schur-convex functions applied on the space of weights (see Fastrich et al. (2014) and Ortobelli et al. (2018)). Diversification, *per-se*, is related to the number of assets in which the initial wealth is invested, rather than, how the initial wealth is invested. Therefore, there can exist portfolios with the same ordered weights, that is, portfolio with the same diversification according to the literature, but with different risk diversification, with respect to a risk measure ν. The following example clarifies this point.

Example 4. *Consider a market with 3 assets and 3 possible state of the world, described by the following matrix:*

$$X = \begin{bmatrix} 0.35 & 0.2 & 0.9 \\ 0.1 & -0.1 & 0.2 \\ 0.1 & 0.5 & -0.05 \end{bmatrix}$$

Take two portfolio weights vectors $w = [0.6, 0.25, 0.15]'$ *and* $y = [0.15, 0.25, 0.6]'$. *Consider two Risk Diversification measures:* $D_{CV@R_{67\%}}$ *and* D_σ*, where* σ *is the standard deviation. While having the same ordered weights, portfolios* Xw *and* Xy *have* $D_{CV@R_{67\%}}(Xw) = 0.16$ *and* $D_{CV@R_{67\%}}(Xy) = 0.1$ *and* $D_\sigma(Xw) = 0.29$ *and* $D_\sigma(Xy) = 0.22$.

Having discussed the basic properties of RDMs and their relationship with diversification in terms of majorization ordering, it is fundamental investigate for which investor category RDMs are designed. Several studies have shown that investors can be classified according to their attitude toward risk (see among the other Fishburn (1980), Rothschild and Stiglitz (1971), Levy (1992) and Levy and Levy (2002)). Investors prefer "the more to less"and can be risk averse, risk seeker and nor risk averse nor risk seeker . Typically, optimal choice for different category of investors can be distinguished using Stochastic Dominance. In particular, the choice of non satiable investors, i.e. with non decreasing utility function, are implied by the First Order of Stochastic Dominance.

Definition 8. Let X and Y be random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution functions $F_X(t) = \mathbb{P}[X \leq t]$ and $F_Y(t) = \mathbb{P}[Y \leq t]$ respectively. Then, X dominates Y in the sense of the First order of Stochastic Dominance, i.e. $X\, FSD\, Y$, if $F_X(t) \leq F_Y(t)$ $\forall t \in \mathbb{R}$, with at least one strict inequality.

Optimizing a risk measure consistent with FSD gives optimal choices for some non satiable investors[1]. Therefore, FSD consistency is a desirable property for any risk measure. The next proposition shows, under which conditions it is possible guarantee the FSD consistency of RDMs.

Proposition 8. *Let $X = [X_1, \dots, X_n]'$ be a vector of gross returns, $w = [w_1, \dots, w_n]'$ and $y = [y_1, \dots, y_n]'$ be portfolio weights and ν be a risk measure consistent with FSD. Assume that all the asset returns have the same risk measure ν. Then, if $w'X\, FSD\, y'X$ then $D_\nu(w'X) \geq D_\nu(y'X)$.*

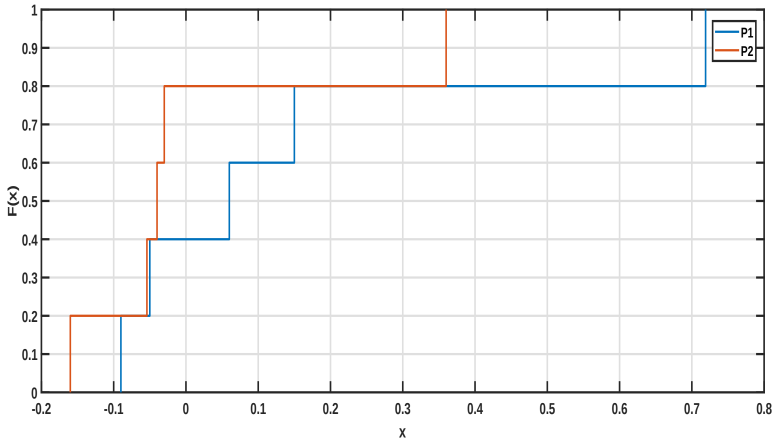
The condition in Proposition 8 is quite strong. In real applications, a market where all the assets returns share the same risk measure, hardly exists, but it is, in some sense, more general than the assumptions of Egozcue and Wong (2010). Nevertheless, the following counterexample shows that when such assumption is not satisfied, D_ν is not consistent with FSD.

Example 5. *Consider a market with 3 assets and 5 states of the worlds described by the following matrix*

$$X = \begin{bmatrix} 0.2 & -0.1 & 0 \\ 0.1 & -0.45 & 0.25 \\ 0.91 & 0 & -0.09 \\ -0.05 & 0.5 & -0.6 \\ -0.2 & 0.3 & 0.4 \end{bmatrix}$$

Take two portfolio weights vectors $w = [0.8, 0.1, 0.1]'$ and $y = [0, 0.4, 0.6]'$. Call $P_1 = Xw$ and $P_2 = Xy$ with cumulative density functions F_1 and F_2 respectively. As shown in Figure 5.1, P_1 FSD P_2, but $D_\sigma(P_1) = 0.21$ and $D_\sigma(P_2) = 0.47$ and, $D_{CVAR_{60\%}}(P_1) = 0.57$ and $D_{CVAR_{60\%}}(P_1) = 0.66$.

[1] A risk measure ν is consistent with FSD, if given two random variables X and Y, whenever $X\, FSD\, Y$ then $\nu(X) \leq \nu(Y)$.



(a) Cumulative distribution functions of portfolios P_1 and P_2. The condition for FSD in Definition ?? can be applied by a graphical inspection of the cdfs. Blue and red lines are cdfs of P_1 and P_2 respectively. Since F_1 lies beneath F_2, then P_1 FSD P_2

Figure 5.1: CDF of portfolios P_1 and P_2

The last example shows that in general, RDMs are not consistent with FSD. Therefore, maximizing a RDM would give a solution non consistent with FSD and sub-optimal for all non satiable risk investors[2]. We propose instead, to maximize the mean of a portfolio, for given values of a RDM. In particular, we consider the following optimization problem:

$$\max_{w} \mathbb{E}\left[w'X\right] \tag{5.2}$$

$$s.t.\ D_{\nu}(w'X) \geq \bar{d} \tag{5.3}$$

$$\sum_{i=1}^{n} w_i = 1 \ , \ 0 \leq w_i \leq 1, \ i = 1, \ldots, n$$

where $\bar{d}$ is a desired level of risk diversification. The solution will then depends on the values $\bar{d}$ and on the choice of the risk measure ν. For a given risk measure, similarly to the classic Mean Variance Efficient Frontier, solution of problem 5.2 can be expressed as a function of $\bar{d}$, i.e. $w_o = w_o(\bar{d})$. Then, the set of Mean Risk Diversification optimal portfolios with respect to ν, can be obtained by solving problem 5.2 for all the admissible level of $\bar{d}$. Finally, the curve $(w_o'\mathbb{E}[X], D_{\nu}(w_o'X))$ is called Mean Risk Diversification Efficient Frontier.

[2]Note that many authors have ignored this aspect, and proposed portfolio strategies that optimize a RDM (see Choueifaty and Coignard (2008), Clarke et al. (2013) and Vermorken et al. (2012)). Nevertheless, the lack of consistency of RDM implies that no non satiable investors would optimize any RDM.

In next section, we use Problem (5.2) in three different applications. Firstly, we compute and compare mean risk diversification efficient frontiers using $CV@R_\alpha$, with $\alpha = 90\%$, 95% and 99%, and standard deviation $\sigma(X) = \left(\mathbb{E}\left[(X - \mathbb{E}[X])^2\right]\right)^{1/2}$ as risk measures. Secondly, since the aim of risk diversification is to reduce idiosyncratic risk, we test the ability of RDMs to face periods of financial distress. Finally, we exploit constraint (5.3) to outperform given market portfolios.

5.3 Empirical Application

In this section we propose an empirical analysis of the newly introduced RDMs, in a static and a in a dynamic framework. We consider a market composed by assets belonging to the Dow Jones Industrial Average index (DJIA) from the $3rd$ of January 2005 to $13th$ October 2017. We consider only the assets present in the index for all the period[3]. In particular, assets belonging to the DJIA index are well traded so we avoid any non-synchronous trading problem and the index itself represent a well diversified market portfolio (see Silvapulle and Granger (2001) and Skintzi and Refenes (2005)). Moreover, it exhibits increasing correlation under periods of financial distress, implying that diversification benefit decreases when needed the most (see among the others Silvapulle and Granger (2001), Skintzi and Refenes (2005) and Preis et al. (2012)).

The objective of this section is threefold. In the static analysis, we investigate the mean-risk diversification efficient frontiers composition, comparing different RDMs. In the dynamic analysis we study the ability of RDMs to face periods of financial distress and finally, their ability to outperform given market portfolios.

5.3.1 RDMs Efficient Frontiers

In this subsection we introduce and describe efficient frontiers for $D_{CV@R_\alpha}$ and D_σ[4]. We consider 1 year of daily observation from $13th$ of October 2016 to $13th$ of October 2017. Similarly to the classic mean variance efficient frontier, for a given risk measure ν, the solution of Problem 5.2 depends on the value $\bar{d}$. Solving problem 5.2 for all the admissible level of d gives the mean-risk diversification efficient frontier. We consider $CV@R_\alpha$, with

[3] The excluded assets are VISA, which went public in 2008, and CISCO System.

[4] D_σ is in fact a one to one transformation of the DR in Choueifaty and Coignard (2008) and Clarke et al. (2013). We consider the RDM version of it to assure internal coherency in our empirical analysis.

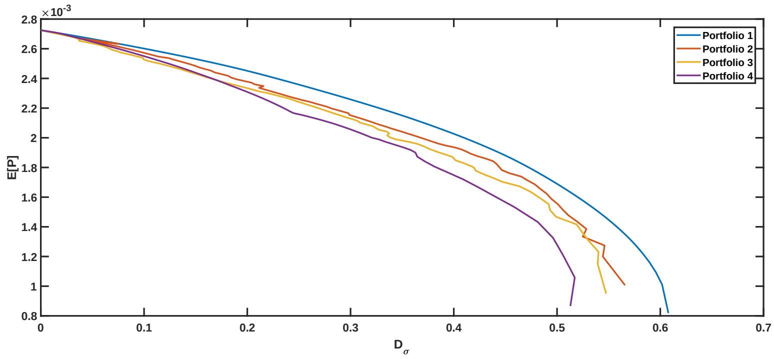
$\alpha = 90\%, 95\%$ and 99% and the standard deviation σ as risk measures and adapt constraint 5.3 accordingly. In order to estimate the four mean-risk diversification efficient frontiers, we need to identify the "admissible levels"for $\bar{d}$. Since each of the RMD measures risk diversification differently, we can expects to have different admissible levels $\bar{d}$. For each problems, the admissible levels $\bar{d}$ belong to the admissible interval $[0, \max_{w \in \Delta} D_\nu(P)]$. We divide the admissible interval into 100 parts and then and then solve problem 5.2 for each of the RDM 100 times.

Figure 5.2 shows the four mean-risk diversification efficient frontiers. Each panel depicts mean and RDM of all portfolios solution of Problem 5.2 with different constraint 5.3. *Portfolio 1, Portfolio 2, Portfolio 3, Portfolio 4* are portfolios corresponding to Mean-D_σ, Mean-$D_{CV@R_{90\%}}$, Mean-$D_{CV@R_{95\%}}$ and Mean-$D_{CV@R_{99\%}}$ respectively. Panel (a) depicts the Mean-D_σ (Portfolio 1), and the values of mean and D_σ for the portfolios belonging to the others Mean-Risk Diversification Frontiers. Panels (b), (c) and (d) are constructed similarly. Since $D_\nu(P) = 0$ when the portfolio is formed by a stand alone asset, the asset with maximum mean belongs to all the efficient frontiers. From the graphs we also note, that as risk diversification increases, mean decreases. Moreover, reduction magnitude varies across the RDMs. The shape of all efficient frontiers is somewhat concave, while the projection of each efficient frontier on different plans, is mostly irregular. Looking at the maximum level of risk diversification is, in general, higher in the Mean-$D_{CV@R}$s than in the Mean-D_σ. Moreover, such level increases with the risk aversion parameter α.

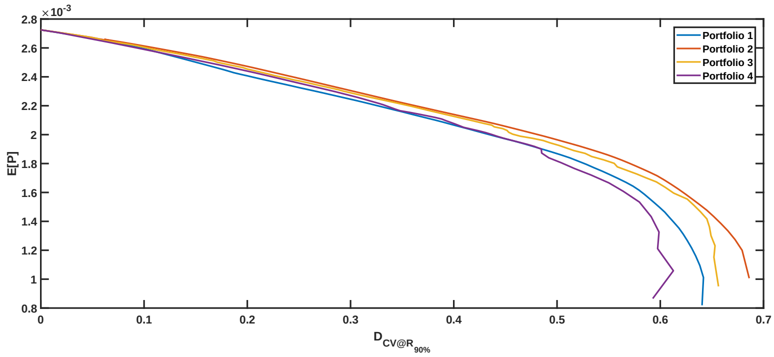
To better describe the features of Mean-Risk Diversification Efficient Frontiers, Table 5.1 provides some of their descriptive statistics. Since we are dealing with 400 portfolios, (1 optimal portfolio, for each level of $\bar{d}$ and each D_ν), we divide the efficient frontiers in 10 areas, equally spaced along the Risk-Diversification axis. Then we group portfolios belonging to each areas together. Since each RMD has a different range, we divide portfolios according to the range of $D_{CV@R_{99\%}}$, because it reaches the highest values. In each of the ten groups, we then compute the average of portfolios' mean, standard deviation, skewness and kurtosis, the average of the sum of squared portfolio's weights (i.e. the inverse of HH index) and the average number of assets with positive weight.

From Table 5.1, we see that on average expected return, standard deviation and kurtosis decrease as risk diversification increases. The evidence on diversification is though, mixed and can be seen from the last two columns.

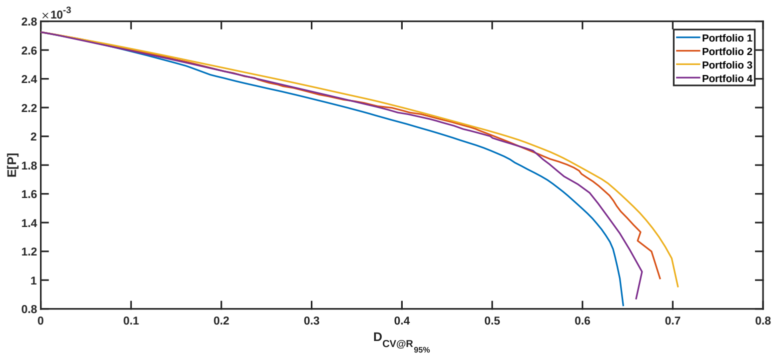
For "low"values of risk diversification, diversification increases with risk diversification. Nevertheless, this is not true for "high"values of risk diversification. As enlighten in Example 4, typically RDM are not consistent with



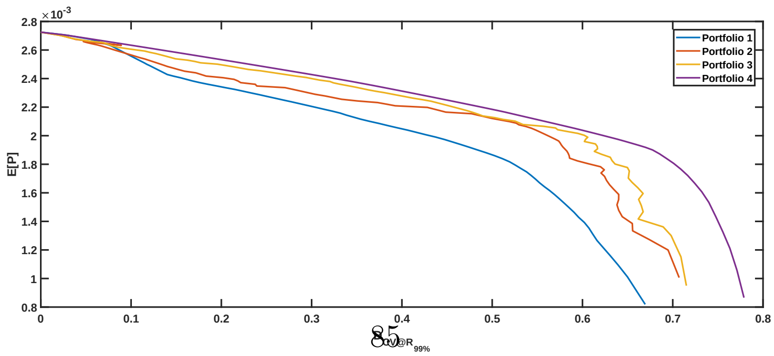
(a) Mean-D_σ Efficient Frontier.



(b) Mean-$D_{CV@R_{90\%}}$ Efficient Frontier.



(c) Mean-$D_{CV@R_{95\%}}$ Efficient Frontier



(d) Mean-$D_{CV@R_{99\%}}$ Efficient Frontier.

Figure 5.2: Mean-Risk Diversification Efficient Frontiers

Table 5.1: Average of mean, standard deviation, skewness, kurtosis, sum of squared weights and number of invested assets, of portfolios belonging to each of the ten groups.

Range	RDM	Mean	St. Dev.	Skewness	Kurtosis	$\sum_i x_i^2$	$\sharp$ Assets
$[0, 0.078)$	D_σ	0.00268	0.0108	2.651	22.543	0.9186	2.166
	$D_{CV@R_{90\%}}$	0.00267	0.0109	2.667	22.782	0.9264	2
	$D_{CV@R_{95\%}}$	0.00268	0.0109	2.720	23.302	0.9345	2.636
	$D_{CV@R_{99\%}}$	0.00269	0.0109	2.726	23.307	0.9454	2.666
$[0.078, 0.156)$	D_σ	0.00257	0.0099	2.352	19.701	0.7382	3
	$D_{CV@R_{90\%}}$	0.00259	0.0104	2.462	20.852	0.8011	4
	$D_{CV@R_{95\%}}$	0.00258	0.0104	2.659	22.657	0.8262	3.727
	$D_{CV@R_{99\%}}$	0.00262	0.0104	2.640	22.321	0.8409	3
$[0.156, 0.234)$	D_σ	0.00246	0.0091	1.900	15.375	0.5760	3.25
	$D_{CV@R_{90\%}}$	0.00247	0.0097	2.252	19.014	0.6686	4.833
	$D_{CV@R_{95\%}}$	0.00248	0.0097	2.629	22.245	0.7165	4.272
	$D_{CV@R_{99\%}}$	0.00254	0.0099	2.530	20.953	0.7453	3
$[0.234, 0.311)$	D_σ	0.00231	0.0080	1.513	11.689	0.4233	5.5
	$D_{CV@R_{90\%}}$	0.00234	0.0088	2.180	18.198	0.5463	7.454
	$D_{CV@R_{95\%}}$	0.00238	0.0089	2.479	20.209	0.6045	4.909
	$D_{CV@R_{99\%}}$	0.00246	0.0094	2.388	19.093	0.6649	2.9
$[0.311, 0.389)$	D_σ	0.00214	0.0069	1.182	8.7405	0.3031	6.33
	$D_{CV@R_{90\%}}$	0.00222	0.0078	2.078	16.404	0.4401	7
	$D_{CV@R_{95\%}}$	0.00227	0.0082	2.217	17.109	0.4952	4.818
	$D_{CV@R_{99\%}}$	0.00237	0.0089	2.207	16.699	0.5972	2.3
$[0.389, 0.467)$	D_σ	0.00194	0.0059	0.897	6.2706	0.2125	7.8
	$D_{CV@R_{90\%}}$	0.00209	0.0070	1.769	12.213	0.3550	6.666
	$D_{CV@R_{95\%}}$	0.00214	0.0075	1.909	13.408	0.4051	5.363
	$D_{CV@R_{99\%}}$	0.00228	0.0085	2.016	14.149	0.5352	3
$[0.467, 0.545)$	D_σ	0.00165	0.0049	0.666	4.8942	0.1381	13.19
	$D_{CV@R_{90\%}}$	0.00195	0.0062	1.449	9.0999	0.2709	6.545
	$D_{CV@R_{95\%}}$	0.00201	0.0068	1.517	8.8859	0.3373	5.363
	$D_{CV@R_{99\%}}$	0.00205	0.0072	1.505	8.4170	0.3986	3.6
$[0.545, 0.623)$	D_σ	0.00124	0.0040	0.469	4.8144	0.0842	18.59
	$D_{CV@R_{90\%}}$	0.00175	0.0054	1.097	5.8251	0.1917	9.363
	$D_{CV@R_{95\%}}$	0.00182	0.0060	1.272	6.3980	0.2482	7.000
	$D_{CV@R_{99\%}}$	0.00217	0.0079	1.800	11.408	0.4768	3.363
$[0.623, 0.701)$	D_σ						
	$D_{CV@R_{90\%}}$	0.00137	0.0046	0.917	4.7324	0.1253	14
	$D_{CV@R_{95\%}}$	0.00144	0.0048	1.032	5.0948	0.1571	9.454
	$D_{CV@R_{99\%}}$	0.00191	0.0065	1.370	7.0917	0.3249	5.2
$[0.701, 0.778]$	D_σ						
	$D_{CV@R_{90\%}}$						
	$D_{CV@R_{95\%}}$	0.00095	0.0043	0.959	5.0430	0.1193	13
	$D_{CV@R_{99\%}}$	0.00148	0.0053	1.146	5.5559	0.2005	9

majorization ordering and therefore, the relationship with diversification remains unclear.

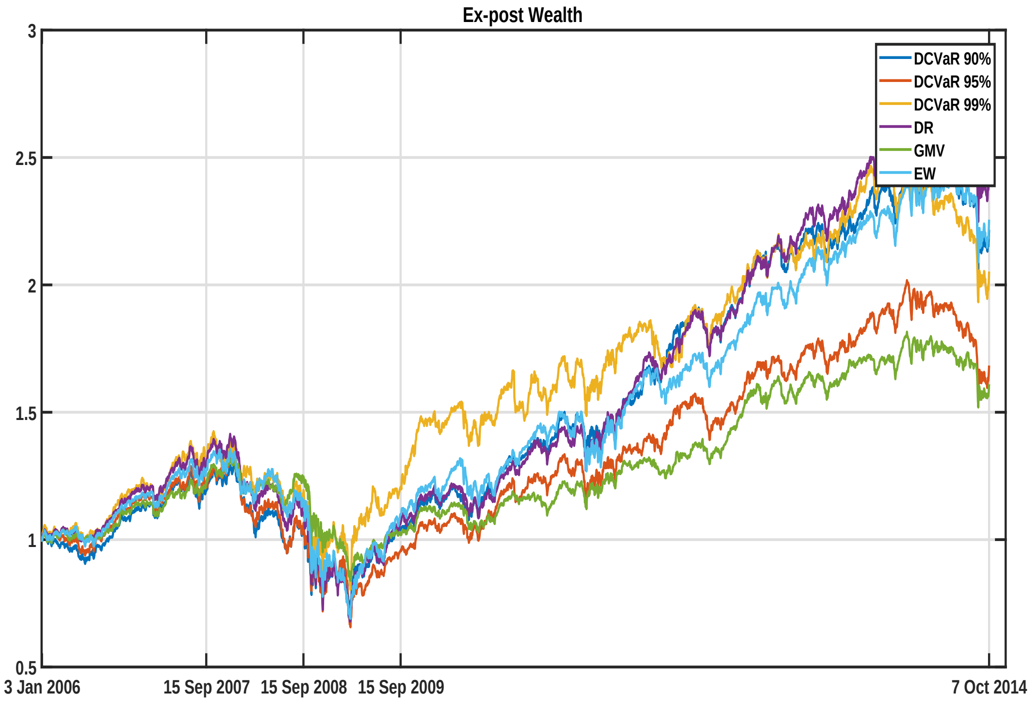
Except for the first area, where risk diversification is the lowest, portfolios belonging to Mean-D_σ have on average the lowest expected return, standard deviation, skewness and kurtosis. $D_{CV@R_{99\%}}$ has on average the higher expected return and standard deviation in all the areas. More interesting is the relationship with diversification and different RDM. Diversification decreases as the risk aversion parameter, i.e. the confidence level α of the $D_{CV@R_\alpha}$ increases. This can be seen by the last two column of Table 5.1, where on average sum of squared weights and average number of assets with positive portion of wealth invested are the lowest for $D_{CV@R_{99\%}}$. This seems to be a counter intuitive fact, but note that as enlighten by example 4, diversification and risk diversification differs. As a general comment, $D_{CV@R_{99\%}}$ seems to better allocate risk diversification, in terms of the first four moments of portfolio distribution than the other considered RDMs.

5.3.2 Dynamic Analysis

In this section, we investigate the ability of RDM to face periods of financial distress. In particular, we focus on the financial crisis of 2008. We perform a rolling window type of analysis with a 1 year window starting from $3rd$ January 2005, to $7th$ October 2014. We recalibrate every 21 trading days with transaction cost equal to 2% of the traded volume. In each period, we solve Problem 5.2, where $\bar{d} = \max D_\nu(P)$ for the four different risk measures. As reference we also compute the Global Minimum Variance portfolio (GMV) and the equally weighted portfolio (naive strategy).

Figure 5.3 shows the ex post wealth evolution of the four portfolio solution of Problem 5.2, the GMV portfolio and the naive strategy. As we see, during the financial crisis of 2008 all the strategies show a drop in the ex post wealth. Looking at the graph, we also observe that the portfolio controlled for $D_{CV@R_{99\%}}$ responds better with respect to the other in terms of wealth, being able to reach a comparable level of wealth with the *pre-crisis* period sooner. Out of the crisis period, the ex post wealth of $D_{CV@R_{99\%}}$ based portfolio is higher than the other for almost two years, but at the end of the period, portfolio based on D_σ reach the highest level. Surprisingly though, the equally weighted portfolio performs quite well outside the crisis period. GMV and $D_{CVAR_{95\%}}$ are less volatile and their ex post wealth remain on inferior levels.

We also evaluate the ex post performances in terms of expected return, standard deviation, Sharpe Ratio, $CV@R_\alpha$, with $\alpha = 90\%, 95\%$ and 99%. To better understand the relationship between diversification and risk diversifi-



(a) Ex post wealth of selected strategies.

Figure 5.3: Ex post wealth evolution

cation, we also compute the number of assets with positive weights for each strategy and finally, average transaction cost. Results are shown in Table 5.2.

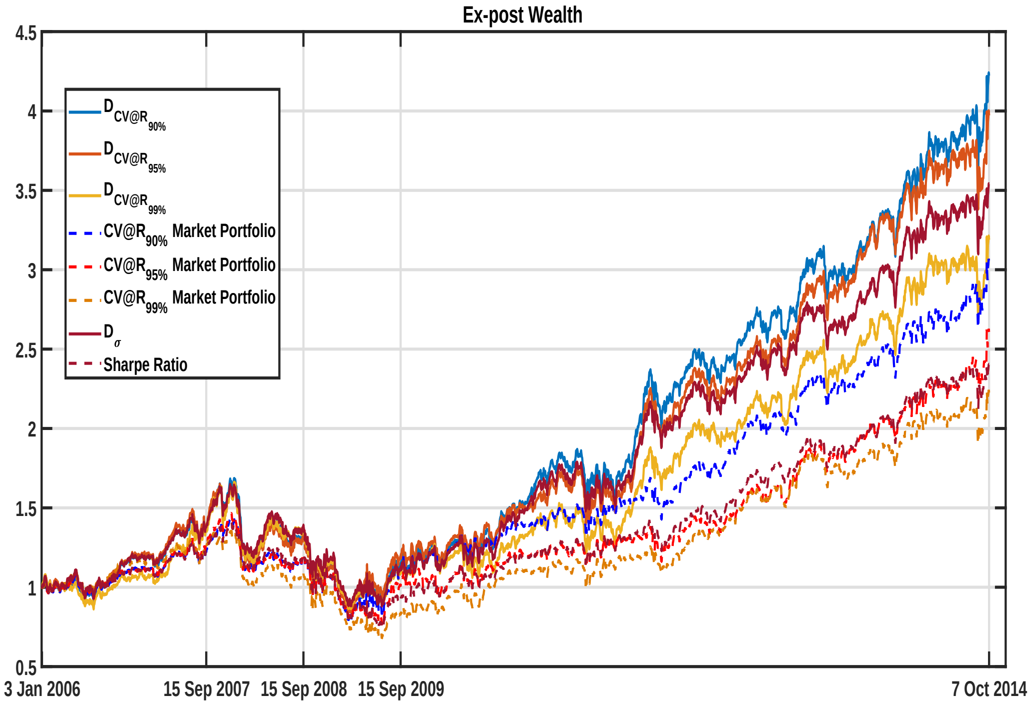
Table 5.2: Performances of $DCV@R_{90\%}$, $DCV@R_{95\%}$, $DCV@R_{99\%}$ D_σ, Global Minimum Variance and Equally Weighted portfolio. Crisis period begins the 15th of September 2008 and finishes the 15th September 2009.

Strategies	Mean	St.Dev.	$SR(X)$	$CV@R_\alpha$			$\sharp$ Assets	Transaction Cost
				$CV@R_{90\%}$	$CV@R_{95\%}$	$CV@R_{99\%}$		
Crisis Period								
$D_{CV@R_{90\%}}$	0.00038	0.0247	0.0154	0.0160	0.0202	0.0270	9.416	0,0009
$D_{CV@R_{95\%}}$	0.00008	0.0263	0.0031	0.0155	0.0199	0.0259	10	0,0010
$D_{CV@R_{99\%}}$	0.00057	0.0257	0.0220	0.0199	0.0239	0.0277	7.416	0,0013
D_σ	0.00028	0.0276	0.0101	0.0188	0.0238	0.0343	15.08	0,0005
GMV	-0.00056	0.0191	-0.0299	0.0124	0.0154	0.0198	28	0,0003
EW	0.00026	0.0271	0.0099	0.0215	0.0250	0.0346	28	0
After Crisis								
$D_{CV@R_{90\%}}$	0.00058	0.0083	0.0695	0.0151	0.0192	0.0293	10.930	0,0009
$D_{CV@R_{95\%}}$	0.00046	0.0086	0.0530	0.0156	0.0198	0.0313	8.662	0,0010
$D_{CV@R_{99\%}}$	0.00046	0.0093	0.0491	0.0168	0.0212	0.0325	5.825	0,0011
D_σ	0.00061	0.0083	0.0727	0.0152	0.0194	0.0295	13.80	0,0004
GMV	0.00034	0.0070	0.0486	0.0128	0.0162	0.0251	28	0,0004
EW	0.00051	0.0092	0.0558	0.0170	0.0219	0.0344	28	0

The first part of Table 5.2 shows the ex post performances from 15 September 2008, to 15 September 2009. All the strategies based on the RDMs show positive expected return during the year. $D_{CV@R_{99\%}}$ reaches the highest ex post mean and Sharpe ratio. Moreover, $D_{CV@R_{90\%}}$, $D_{CV@R_{95\%}}$ and $D_{CV@R_{99\%}}$ present ex post standard deviation lower than D_σ. The number of assets with positive weights confirms the preliminary results from the static analysis: diversification decreases as α increases. In other word, as the risk aversion parameter increases, the diversification decreases. This result might serve as an explanation of Statman Diversification puzzle (Statman (2004)). Investors controlling for risk diversification tend to hold a much more concentrated portfolio than mean variance optimizer. Moreover, holding more concentrated portfolio than GMV, during periods of financial distress, could lead to better ex post performance as long as the risk is efficiently diversified. Nevertheless, risk diversification comes with higher transaction cost. In particular, $D_{CV@R_\alpha}$ based portfolio have much higher transaction cost, than GMV or D_σ. Moreover, transaction cost increases with α, implying that an higher risk aversion demands, on one side, a more concentrated portfolio and on the other side, more active investment strategies. Finally, during the great financial crisis of 2008, risk diversification based on $CV@R_{90\%}$ and $D_CV@R_{99\%}$ better perform in terms of ex post expected return and risk, being, nevertheless more costly than the other considered strategies.

The second part of Table 5.2 shows ex post performances after the great financial crisis. All the strategies have positive ex post expected return. Here, D_σ seems to outperform the other strategies in terms of Sharpe Ratio and risk, being at the same time relatively a less expensive strategy than those based on $CV@R_\alpha$. Among the $D_{CV@R}$s strategies, $D_{CV@R_{90\%}}$ performs better than the others. Remarkably, also the equally weighted portfolio performs quite well. Moreover, the naive strategy has expected return, standard deviation, Sharpe Ratio, $CVA@R_{95\%}$ and $CVA@R_{99\%}$ not too dissimilar from the D_σ, while the transaction cost, obviously is null. This is in agreement with the literature since, the equally weighted portfolio turns out to be hard to beat using strategies based on the mean variance approach (see DeMiguel et al. (2009)).

Risk diversification aims to reduce the idiosyncratic risk of a position. For this reason we use risk diversification measure to construct portfolio with the same risk diversification of chosen market portfolio. Since we are considering, as risk measures, standard deviation and $CV@R_\alpha$, with $\alpha = 90\%, 95\%, 99\%$, we compute first the relative market portfolios (i.e. Sharpe Ratio and Mean-$CV@R$ ratios) solving the following optimization problem



(a) Ex post wealth of selected strategies and market portfolios.

Figure 5.4: Ex post wealth evolution

for each risk measure[5]:

$$\max_{w} \frac{\mathbb{E}\left[w'X\right]}{\nu(w'X)}$$
$$\sum_{i=1}^{n} w_i = 1$$
$$0 \leq w_i \leq 1,\ i = 1, \ldots, n$$

Call the solution vector w_m. Then, we solve Problem 5.2 with $\bar{d} = D_\nu(w_m'X)$. In other words, we consider the portfolio that maximize the mean, with the same level of risk diversification of the market portfolio for each risk measure.

Figure 5.4 shows the ex post wealth evolution of portfolio with the same level of risk diversification of the respective market portfolios. In comparison

[5]It is well known that portfolio solution of a maximization of the ratio of expected return and a risk measure, is the market portfolio with respect to that risk measure (see for example Stoyanov et al. (2007)).

with the evolution in Figure 5.3, ex post wealths appear to be less volatile during the crisis and reach an higher level of wealth at the end of the period. All the RDM based strategies beat the respective market portfolios in terms of final wealth, up to 150%. Similarly to the previous analysis, we compare the ex post performances during and after the first year of the great financial crisis. Table 5.3 reports the results.

Table 5.3: Performances of $DCV@R_{90\%}$, $DCV@R_{95\%}$, $DCV@R_{99\%}$, D_σ and the corresponding Market portfolios. Crisis period begins the $15th$ of September 2008 and finishes the $15th$ September 2009.

Strategies	Mean	St.Dev.	$SR(X)$	$CV@R_\alpha$			$\sharp$ Assets	Transaction Cost
				$CV@R_{90\%}$	$CV@R_{95\%}$	$CV@R_{99\%}$		
Crisis Period								
$D_{CV@R_{90\%}}$	-0.00014	0.0283	-0.0048	0.0436	0.0557	0.0763	5.83	0,0014
$D_{CV@R_{95\%}}$	0.00008	0.0289	0.0028	0.0431	0.0542	0.0745	5.92	0,0014
$D_{CV@R_{99\%}}$	-0.00041	0.0279	-0.0146	0.0424	0.0544	0.0754	5	0,0011
$\frac{\mathbb{E}[P]}{CV@R_{90\%}}$	0.00014	0.0279	0.0053	0.0427	0.0506	0.0673	4.25	0,0015
$\frac{\mathbb{E}[P]}{CV@R_{95\%}}$	-0.00032	0.0261	-0.0097	0.0402	0.0505	0.0738	4.25	0,0017
$\frac{\mathbb{E}[P]}{CV@R_{99\%}}$	-0.00061	0.0263	-0.0228	0.0386	0.0484	0.0668	4.25	0,0016
D_σ	-0.00036	0.0278	-0.0135	0.0421	0.0537	0.0734	15.08	0,0009
SR	-0.00058	0.0256	-0.0229	0.0352	0.0441	0.0634	5.08	0,0003
After Crisis								
$D_{CV@R_{90\%}}$	0.00100	0.0126	0.0796	0.0224	0.0285	0.0426	4.453	0,0014
$D_{CV@R_{95\%}}$	0.00093	0.0129	0.0726	0.0227	0.0289	0.0446	4.244	0,0014
$D_{CV@R_{99\%}}$	0.00085	0.0132	0.0645	0.0232	0.0294	0.0459	3.569	0,0014
$\frac{\mathbb{E}[P]}{CV@R_{90\%}}$	0.00079	0.0115	0.0694	0.0205	0.0262	0.0397	4.197	0,0013
$\frac{\mathbb{E}[P]}{CV@R_{95\%}}$	0.00075	0.0121	0.0628	0.0215	0.0275	0.0437	3.767	0,0013
$\frac{\mathbb{E}[P]}{CV@R_{99\%}}$	0.00078	0.0121	0.0651	0.0213	0.0275	0.0427	3.605	0,0015
D_σ	0.00088	0.0123	0.0723	0.0221	0.0279	0.0425	13.80	0,0012
SR	0.00074	0.0109	0.0683	0.0196	0.0248	0.0363	4.848	0,0004

The first part shows the performances during the year after the $15th$ of September 2008. As we see, the strategies producing positive expected returns are the $D_{CV@R_{95\%}}$ and the $CV@R_{90\%}$ market portfolio. In fact, the $CV@R_{90\%}$ market portfolio is the best performing strategy in this case, in terms of risk and expected return. Nevertheless, all the others RDM based strategies outperform their equivalent market portfolio in terms of expected return and Sharpe Ratio. All of the strategies seem to be quite concentrated during the 2008 except the D_σ based, that shows higher diversification. The $D_{CV@R}$ based strategies are more concentrated with respect to Table 5.2, while D_σ shows the same level of concentration. Transaction cost have increased for all the strategies, but still D_σ based one is the less expensive among the RDMs. The second part of the table shows the performances after the crisis. RDM based strategies, have higher expected return and Sharpe ratio, are more concentrated and have higher average transaction cost w.r.t Table 5.2.

5.4 Conclusion

Diversification has been of most importance in portfolio theory since the seminal work of Markovitz (Markowitz (1952a)). In this chapter, we have proposed a new class of diversification measures, called Risk Diversification Measure. The class extends some diversification measure already established in the literature (see Choueifaty and Coignard (2008), Clarke et al. (2013) and Flores et al. (2017)). The new measures depend on a given risk measure, and are defined as the ratio between portfolio risk and the average risk of its component. Under appropriate assumptions, RDMs can be interpreted as the percentage of idiosyncratic risk diversified in the portfolio.

We considered four RMDs: D_σ, and to take into account different levels of risk aversion, $D_{CV@R_\alpha}$ at various level α. For each of the RDM we propose three empirical application, on a market composed by stocks belonging to DJIA. In the first case, we introduce the mean-risk diversification efficient frontier. Similarly to Mean-Variance efficient frontiers, it's an empirical tool that analyze portfolios' efficiency in terms of expected return and risk diversification. Each of the efficient frontiers exhibit concavity w.r.t. risk diversification and the fact that as risk diversification increases, expected return decreases. Moreover, the level of risk diversification increases with risk aversion, while concentration decreases.

We also verify the ability of D_σ and $D_{CV@R_\alpha}$ with $\alpha = 90\%$, $\alpha = 95\%$ and 99% based portfolio strategies to face period of financial distress. We focus our analysis during the year after Lehman Brother bankruptcy. We observe that all the risk diversification based strategy produce positive expected return and have better performance in terms of risk reward ratio. The price to pay for the superior performance is an higher transaction cost, which implies more active investment strategies. Finally, we show how RDMs and CRDMs can be used to outperform given market portfolios.

Chapter 6

Conclusion

Portfolio Selection deals with taking decision under uncertainty. It aims to find the *best* wealth allocation among risky assets. To define what best allocation means, it is fundamental to a have definition of efficiency. The Markowitz's Mean Variance Efficient Frontier represent a milestone in Modern Portfolio Theory, and established the first definition of efficiency (Markowitz (1952a)). Even if, thanks to Mean Variance efficiency, many important results in Portfolio Selection theory have been established, it appears to be not suitable enough to describe investors behavior (see Mossin (1966), Sharpe (1964), Lintner (1964), Bawa (1975) and Levy (1992)). Here is where Expected Utility and Stochastic Dominance come to help. In Expected Utility it is possible to classify investors according to their attitude towards risk. Then with Stochastic Dominance and, generally, with stochastic orderings, it is possible to define efficiency for an entire category of investors. In particular, an efficient allocation with respect to a stochastic ordering, is also optimal for all the investors with preference to which the stochastic ordering is coherent with.

This thesis discussed several applications of stochastic orderings to portfolio selection problems. In particular, it studied efficiency from the point of view of investors with different attitude to risk. Chapter 2 compares Mean Variance efficiency, with Second order of Stochastic Dominance efficiency. In particular, it shows that the Second order of Stochastic Dominance efficient set is composed by a huge number of portfolios and, it is not necessarily convex. Portfolios belonging to it, are on average more concentrated than, those belonging to the Mean Variance Efficient Frontier. Moreover, the Global minimum Variance portfolio and other portfolio belonging to the Mean Variance Efficient Frontier with low expected value are Second order Stochastically dominated. These results question the validity of Mean Variance Efficiency for non satiable and risk averse investors. The last part of the chapter exploits the non efficiency of the Global Minimum Variance to construct dominating strategies, able to out perform it in terms of wealth and other performance measures. This results confirm other previous studies on the relationship between Mean Variance and Expected Utility (see Bawa (1975), Ingersoll (1987) and Levy (1992)).

Starting from recent findings in the literature, Chapter 2 considers efficiency for non satiable nor risk averse nor risk seeker investors. This particular category of investors is usually the subject of behavioral finance studies (Kahneman and Tversky (1979), Tversky and Kahneman (1992) and Barberis and Thaler (2003)). Chapter 3 assumes that asset return distributions belong to a family uniquely determined by a finite of parameter. This family of distribution can be seen as an extension of the elliptical family, which is a class widely used in portfolio selection and generally in finance. To set a link between distribution and investors' preference, the location parameter is understood as a positive homogeneous and translation invariant reward measure, the scale parameters is a positive homogeneous and translation invariant risk measure, while the other parameters can be chosen to be other distributional parameters, such as moments (see Ortobelli (2001)). Under this assumptions, the first part of the chapter extends classic Stochastic Dominance conditions. In particular, these conditions, do not necessitate for a convex risk measure. Moreover, non satiable nor risk averse nor risk seeker investors' behavior change according to market conditions. When the expected value is higher than the reward measure, investors behave as non satiable and risk averse, when the expected value is lower than the reward measure they behave as non satiable risk seeker. Thanks to this results, it is possible to define several orderings consistent with the preference of non satiable, nor risk averse nor risk seeker investors, of which the introduced Rachev Ordering is an example (see also Biglova et al. (2004)). The last part of the Chapter provide a methodology to test weather a given portfolio is efficient with respect to classic Stochastic dominance orderings and the Rachev Ordering, using estimation function theory (see Godambe and Thompson (1989) and Crowder (1986)). Finally, this methodology is applied to test the efficiency of the Fama and French, NYSE and Nasdaq market portfolios. The results are in linear with other tests presents in the literature, assessing the efficiency of the Fama and French market portfolios, while NYSE and Nasdaq seem are not (see among the other Scaillet and Topaloglou (2010), Kopa and Post (2015) and Arvanitis and Topaloglou (2017)). The newly introduced stochastic ordering describe the behavior of non satiable nor risk averse investors differently from Prospect Theory or Markowitz utility of wealth theory (see Markowitz (1952b) Kahneman and Tversky (1979), Tversky and Kahneman (1992)). Both in Prospect Theory and Markowitz utility, investors' behavior change according to change of wealth. The results on Rachev ordering suggest that the change in behavior due to relative wealth is in reality endogenous to the market. What causes the change in the investors' behavior is relationship between the reward, that embed investors preference, and expected return. Chapter 4 applies theoretical results from Chapter 3, in defining a new ordering called Gini ordering. The functional defining the ordering can be interpreted as a weighted difference between a given percentage of worst a given percentage of best outcomes. Results on this ordering suggest that market portfolio is almost never efficient for non satiable, nor risk averse nor risk seeker investors, even if in some situation, depending on the configuration of the functional, it can be hard to find a dominating portfolio.

Chapter 2 and 3 apply definitions of efficiency valid for different category of investors. Assuming different investors behavior nevertheless, reach the same conclusion on diversification. Given the results in Chapter 2, optimal portfolios for non satiable and risk averse investors have a lower level of diversification than Mean Variance Efficient ones. Chapter 3 attest that convexity of risk measures is not a necessary condition for optimality, not even in the case of non satiable, nor risk averse nor risk seeker investors. In ordering theory, diversification is defined in terms of majorization ordering. In particular, diversification consider the number of assets in which a positive proportion o wealth is invested, rather than how it is invested (see Marshall et al. (1943), Wong (2007) and Egozcue and Wong (2010)). These observations motivates Chapter 4. As alternative to classic diversification, Chapter 5 introduces the definition of Risk Diversification Measures, as measures quantifying the idiosyncratic risk diversified among the portfolio components. When the risk measure satisfies the axiom of coherency, the resulting risk diversification measure can be seen as the percentage reduction in capital requirement due to the portfolio composition (see Artzner et al. (1999) and Tasche (2006)). The second part of the chapter establish the Mean Risk Diversification Efficient Frontier. Similar to Mean Variance Efficient Frontier, Mean Risk Diversification Frontier carries a new definition of portfolio efficiency. In particular, to any risk measure it correspond a different portfolio efficiency. Observing the results of different frontiers, to an higher level of risk aversion it corresponds an higher level of Risk Diversification and a lower lever of Diversification. Jointly with the results of Chapter 1 and 2 this might serve as an explanation of the Statman's Diversification puzzle (i.e. the level of the observed diversification in real market is lower than the one predicted by Mean Variance Theory) (see Statman (2004)). No non satiable and risk averse investors, and investors controlling for risk diversification, even if the risk measure is convex, aim for highly diversified portfolio. The last part of the chapter presents an application of risk diversification under period of financial distress. During the Great Financial Crisis, risk diversification based strategies perform better in terms of risk/reward ratio than other competitors strategies, having at the same time higher concentration and higher transaction cost.

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Summary of Publications

In the following is reported a list of my publications during the PhD.

Journal Paper

- Malavasi M., Previtali R., Ortobelli S., Tichy T., Backtesting AVaR and VaR with Simulated Copula, 2016, Ekonomicka Revue – Central European Review of Economic Issues. doi: 10.7327/cerei.2016.03.02 .

Conference Proceedings

- Malavasi M., Previtali R., Ortobelli S., Nardella C., "Managing Risk with Simulated Copula", Managing and Modelling of Financial Risk 2016, VŠB-TU Ostrava, Czech Republic (2016).

- Malavasi M., Ortobelli S., "Semiparametric Tests for Behavioral Finance Efficiency", 11^{th} International Scientific Conference Financial Management of Firms and Financial Institutions, VŠB-TU Ostrava, Czech Republic (2017).