The Distorted Wheeler-DeWitt Equation

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The Wheeler-DeWitt Equation represents a tool to study Quantum Gravity and Quantum Cosmology. Its solution in a very general context is, of course, impossible. To this purpose we consider some distortions of General Relativity like Gravity's Rainbow, Varying Speed of Light Cosmology, Generalized Uncertainty Principle deformations and Hořava-Lifshitz gravity which could allow the calculation of some observables like the cosmological constant. For simplicity we consider only the Mini-Superspace approach related to a Friedmann-Lemaître-Robertson-Walker space-time.

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1. Introduction

The Wheeler-DeWitt (WDW) equation represents the quantum version of the classical constraint describing the invariance with respect to time reparametrization[1]. In the context of the Friedmann-Lemaître-Robertson-Walker space-time, it assumes the simple form

$$\begin{aligned} H\Psi(a) &= \left[-a^{-q}\left(\frac{\partial}{\partial a}a^{q}\frac{\partial}{\partial a}\right)+\frac{9\pi^2}{4G^2}\left(a^2-\frac{\Lambda}{3}a^4\right)\right]\Psi(a) \\ &= \left[-\frac{\partial^2}{\partial a^2}-\frac{q}{a}\frac{\partial}{\partial a}+\frac{9\pi^2}{4G^2}\left(a^2-\frac{\Lambda}{3}a^4\right)\right]\Psi(a)=0, \end{aligned} \tag{1}$$

where we have introduced the following line element

$$ds^2=-N^2dt^2+a^2(t)\,d\Omega_3^2 \tag{2}$$

and where we have denoted with $d\Omega_3^2=\gamma_{ij}dx^idx^j$ the metric on the three-sphere. N is the lapse function, $a(t)$ is the scale factor, G and Λ are the Newton's constant and the cosmological constant, respectively. We have also introduced a factor order ambiguity q. In Eq.(1) all the degrees of freedom except the scale factor $a(t)$ have been integrated and matter fields have been excluded. If the WDW equation is interpreted as an eigenvalue equation, one simply finds

$$H\Psi(a)=E\Psi(a)=0, \tag{3}$$

namely a zero energy eigenvalue. However, it appears that the WDW equation has also a hidden structure. Indeed Eq.(1) has the structure of a Sturm-Liouville eigenvalue problem with the cosmological constant interpreted as the associated eigenvalue. We recall to the reader that a Sturm-Liouville differential equation is defined by

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)+q(x)\,y(x)+\lambda w(x)\,y(x)=0 \tag{4}$$

and the normalization is defined by

$$\int_a^b dxw\left(x\right)y^*\left(x\right)y\left(x\right), \tag{5}$$

where the boundary conditions are momentarily suspended. It is a standard procedure, to convert the Sturm-Liouville problem (4) into a variational problem of the form

$$F\left[y\left(x\right)\right] = \frac{-\int_a^b dxy^*\left(x\right)\left[\frac{d}{dx}\left(p\left(x\right)\frac{d}{dx}\right)+q\left(x\right)\right]y\left(x\right)}{\int_a^b dxw\left(x\right)y^*\left(x\right)y\left(x\right)}, \tag{6}$$

with unspecified boundary condition. If $y\left(x\right)$ is an eigenfunction of (4), then

$$\lambda = \frac{-\int_a^b dxy^*\left(x\right)\left[\frac{d}{dx}\left(p\left(x\right)\frac{d}{dx}\right)+q\left(x\right)\right]y\left(x\right)}{\int_a^b dxw\left(x\right)y^*\left(x\right)y\left(x\right)}, \tag{7}$$

is the eigenvalue, otherwise

$$\lambda_1 = \min_{y(x)} \frac{-\int_a^b dxy^*\left(x\right)\left[\frac{d}{dx}\left(p\left(x\right)\frac{d}{dx}\right)+q\left(x\right)\right]y\left(x\right)}{\int_a^b dxw\left(x\right)y^*\left(x\right)y\left(x\right)}. \tag{8}$$

The minimum of the functional $F\left[y\left(x\right)\right]$ corresponds to a solution of the Sturm-Liouville problem (4) with the eigenvalue λ. In the case of the FLRW model we have the following correspondence

$$\begin{aligned} p\left(x\right) &\to a^q\left(t\right), \\ q\left(x\right) &\to \left(\frac{3\pi}{2G}\right)^2 a^{q+2}\left(t\right), \\ w\left(x\right) &\to a^{q+4}\left(t\right), \\ y\left(x\right) &\to \Psi\left(a\right), \\ \lambda &\to \frac{\Lambda}{3}\left(\frac{3\pi}{2G}\right)^2. \end{aligned} \tag{9}$$

Since $a\left(t\right) \in [0, \infty)$, the normalization becomes

$$\int_0^\infty daa^{q+4}\Psi^*\left(a\right)\Psi\left(a\right), \tag{10}$$

where it is understood that $\Psi\left(\infty\right) = 0$. In the Mini-Superspace approach with a FLRW background, one finds

$$\frac{\int \mathcal{D}aa^q\Psi^*\left(a\right)\left[-\frac{\partial^2}{\partial a^2}-\frac{q}{a}\frac{\partial}{\partial a}+\frac{9\pi^2}{4G^2}a^2\right]\Psi\left(a\right)}{\int \mathcal{D}aa^{q+4}\Psi^*\left(a\right)\Psi\left(a\right)} = \frac{3\Lambda\pi^2}{4G^2}. \tag{11}$$

As a concrete case, fixing $q = 0$ and taking as a trial wave function $\Psi\left(a\right) = \exp\left(-\beta a^2\right)$, one finds $\Psi\left(a\right) \to 0$ when $a \to \infty$. Then the only solution allowed is complex and therefore it must be discarded[2]. Of course, the general q case is much more complicated[3]. Note that the global energy eigenvalue is still vanishing. What we can compute in the Sturm-Liouville formulation is the degree of degeneracy which is represented by the cosmological constant and the value of the cosmological constant itself. In the next section we give the general guidelines to build a Sturm-Liouville problem associated to the WDW equation when General Relativity (GR) is distorted. Units in which $\hbar = c = k = 1$ are used throughout the paper.

2. The Mini-Superspace Approach to the Cosmological Constant in Distorted Quantum Cosmology

In this section we will give some glimpses on how the WDW is modified when we distort GR. The reference metric will be the FLRW line element and we will consider modifications coming from Gravity's Rainbow, HL Gravity, VSL cosmology and GUP deformation. We begin to consider Gravity's Rainbow and HL Gravity.

2.1. *Gravity's Rainbow and HL Gravity*

Gravity's Rainbow is a distortion of the metric tensor around and beyond the Planck scale[4]. The basic ingredient is the definition of two arbitrary functions $g_1\left(E/E_{\rm Pl}\right)$ and $g_2\left(E/E_{\rm Pl}\right)$, which have the following property

$$\lim_{E/E_{\rm Pl}\to 0} g_1\left(E/E_{\rm Pl}\right) = 1 \qquad \text{and} \qquad \lim_{E/E_{\rm Pl}\to 0} g_2\left(E/E_{\rm Pl}\right) = 1. \tag{12}$$

For a FLRW metric, the deformation acts in the following way,

$$ds^2 = -\frac{N^2\left(t\right)}{g_1^2\left(E/E_{\rm Pl}\right)}dt^2 + \frac{a^2\left(t\right)}{g_2^2\left(E/E_{\rm Pl}\right)}d\Omega_3^2\ , \tag{13}$$

where

$$d\Omega_3^2 = \gamma_{ij}dx^i dx^j \tag{14}$$

is the line element on the three-sphere. Of course, ordinary gravity is recovered when $E/E_{\rm Pl} \to 0$. When we generalize Eq.(11) to include $g_1\left(E/E_{\rm Pl}\right)$ and $g_2\left(E/E_{\rm Pl}\right)$, the equation modifies in the following way

$$\left[-\frac{\partial^2}{\partial a^2} - \frac{q}{a}\frac{\partial}{\partial a} + U\left(a, E/E_{\rm Pl}\right)\right]\Psi\left(a\right) = 0, \tag{15}$$

where we have set $N = 1$ and defined the distorted potential as

$$U\left(a, E/E_{\rm Pl}\right) = \left[\frac{3\pi g_2\left(E/E_{\rm Pl}\right)}{2Gg_1\left(E/E_{\rm Pl}\right)}\right]^2 a^2\left[1 - \frac{a^2}{a_0^2 g_2^2\left(E/E_{\rm Pl}\right)}\right]\ , \tag{16}$$

with $a_0 = \sqrt{3/\Lambda}$. The potential $U\left(a, E/E_{\rm Pl}\right)$ has been used to discuss a possible alternative explanation of the inflation problem[5]. Moreover, always in the context of the FLRW space-time, it is possible to build a bridge between Gravity's Rainbow and HL theory[6]. Indeed, if one considers

$$g_1\left(E/E_P\right) \equiv g_1\left(E\left(a\left(t\right)\right)/E_P\right) \qquad g_2\left(E/E_P\right) \equiv g_2\left(E\left(a\left(t\right)\right)/E_P\right) \tag{17}$$

with the choice

$$g_1^2\left(E\left(a\left(t\right)\right)/E_P\right) f\left(A\left(t\right),a\right) = 1 \tag{18}$$

and

$$g_2^2\left(E\left(a\left(t\right)\right)/E_P\right) \frac{6}{a^2\left(t\right)} = \frac{6}{a^2\left(t\right)}\left[1 - \frac{2\kappa b}{a^2\left(t\right)} - \frac{4\kappa^2 c}{a^4\left(t\right)}\right], \tag{19}$$

one finds that the WDW equation becomes

$$-\frac{\partial^2}{\partial a^2} + \frac{(3\lambda-1)}{\kappa^2}24\pi^4 a^4\left(t\right)\left[\frac{6}{a^2\left(t\right)} - \frac{12\kappa b}{a^4\left(t\right)} - \frac{24\kappa^2 c}{a^6\left(t\right)} - 2\Lambda\right]\Psi\left(a\right) = 0. \tag{20}$$

To obtain Eq.(20), we have also defined

$$A\left(t\right) = \frac{1}{g_2\left(E\left(a\left(t\right)\right)/E_P\right)E_P}\frac{d}{dE}\left[g_2\left(E\left(a\left(t\right)\right)/E_P\right)\right]\frac{dE}{da} \tag{21}$$

and

$$f\left(A\left(t\right),a\right) = \left[1 - 2a\left(t\right)A\left(t\right) + A^2\left(t\right)a\left(t\right)^2\right]. \tag{22}$$

Moreover, by identifying

$$\begin{aligned} 3g_2 + g_3 &= b \\ 9g_4 + 3g_5 + g_6 &= c, \end{aligned} \tag{23}$$

it is immediate to recognize that Eq.(20) represents the WDW equation for the projectable version of a Hořava-Lifshitz gravity, without detailed balanced condition.

2.2. *VSL Cosmology*

A VSL cosmology model is described by the following line element

$$ds^2 = -N^2\left(t\right)c^2\left(t\right)dt^2 + a^2\left(t\right)d\Omega_3^2, \tag{24}$$

and where $c(t)$ is an arbitrary function of time with the dimensions of a $[length/time]$. Following[7–9], we assume that

$$c\left(t\right) = c_0\left(\frac{a\left(t\right)}{a_0}\right)^{\alpha} \tag{25}$$

where a_0 is a reference length scale. The form of the background is such that the shift function N^i vanishes. In this case, when π_a is promoted to an operator, we can write

$$\pi_a^2 \to -\left(\hbar c\left(t\right)\right)^2 a^{-q}\frac{\partial}{\partial a}a^q\frac{\partial}{\partial a}, \tag{26}$$

where we have introduced a factor ordering ambiguity. Thus the WDW equation $\mathcal{H}\Psi = 0$ simply becomes

$$\left(-\frac{\partial^2}{\partial a^2} - \frac{q}{a}\frac{\partial}{\partial a} + U_c(a)\right)\Psi(a) = 0, \tag{27}$$

where we have assumed that the factor ordering is not distorted by the presence of a VSL and we have set $N = 1$. The quantum potential is defined as

$$U_c(a) = \left(\frac{3\pi}{2G\hbar}\right)^2 a^2 c^6(t)\left(1 - \frac{\Lambda}{3}a^2\right) = \left(\frac{3\pi c_0^3}{2G\hbar a_0^{3\alpha}}\right)^2 a^{2+6\alpha}\left(1 - \frac{\Lambda}{3}a^2\right). \tag{28}$$

Note that the potential $U_c(a)$ vanishes in the same points where the original $U(a)$ has its roots.

2.3. *GUP Deformation*

The WDW equation deformed by a GUP can be derived modifying the original undeformed momentum conjugate to the scale factor $a(t)$

$$\tilde{\pi}_a = -i\frac{d}{da}, \tag{29}$$

into[10]

$$\pi_a = \tilde{\pi}_a(1 - \alpha||\tilde{\pi}_a|| + 2\alpha^2||\tilde{\pi}_a||^2), \tag{30}$$

where α is a coefficient with inverse momentum dimensions, in order to reestablish the correct powers. Then the WDW equation for a FLRW metric deformed by a GUP becomes

$$\left[\tilde{\pi}_a^2 - 2\alpha\pi_a^3 + 5\alpha^2\pi_a^4 + \left(\frac{3\pi}{2l_P^2}\right)^2 a^2\left(1 - \frac{\Lambda}{3}a^2\right)\right]\Psi(a) = 0. \tag{31}$$

This deformation of the WDW equation prevents the existence of singularities[10]. This is because this deformation modifies the uncertainty principle as

$$\Delta a\Delta\pi_a = 1 - 2\alpha < \pi_a > +4\alpha^2 < \pi_a^2 > . \tag{32}$$

Thus, we obtain a minimum value for the scale factor of the universe, $\Delta a \geq \Delta a_{min}$. Note that the action of the GUP is only on the kinetic term.

3. Summary

In this work, we have considered the WDW equation on a FLRW background and we have shown how such an equation is modified when some deviations from GR are considered. Even if we have only considered a Mini-Superspace approach with the scale factor $a(t)$ as unique degree of freedom, one has not to think that this procedure works only on a FLRW background. For instance, in a series of papers[12,13], it has been shown that Gravity's Rainbow can keep under control UV divergences, at least to one loop. This procedure has been widely tested on a spherically symmetric background. The same procedure can be applied also to Noncommutative geometries[14] and it has been extended to include also a $f(R)$ theory[15].

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