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Gravity's Rainbow and Traversable Wormholes

Remo Garattini

Università degli Studi di Bergamo,
Dipartimento di Ingegneria e scienze applicate,
Viale Marconi,5 24044 Dalmine (Bergamo) ITALY
I.N.F.N. - sezione di Milano, Milan, Italy
*E-mail:remo.garattini@unibg.it

Francisco S. N. Lobo

Instituto de Astrofísica e Ciências do Espaço, Universidade de Lisboa,
Faculdade de Ciências, Campo Grande,
PT1749-016 Lisboa, Portugal.
E-mail: fslobo@fc.ul.pt

In the context of Gravity's Rainbow, we compute the graviton one-loop contribution to a classical energy in a traversable wormhole background, by considering the equation of state $p_r = \omega\rho$. The investigation is evaluated by means of a variational approach with Gaussian trial wave functionals. However, instead of using a regularization/renormalization process, we use the distortion induced by Gravity's Rainbow to handle the divergences.

1. Introduction

John A. Wheeler back in 1955 considered the possibility that spacetime could be undergoing topological fluctuations at the Planck scale[1]. This changing spacetime is best known as the *spacetime foam*, and can be a model for the quantum gravitational vacuum. Indeed, Wheeler also considered wormhole-type solutions, denoted as geons, obtained from the coupled equations of electromagnetism and general relativity, as objects of the spacetime quantum foam connecting different regions of spacetime at the Planck scale[1]. In this proceedings, we consider the possibility of quantum fluctuations in the context of Gravity's Rainbow[2–4]. The latter consists of a distortion of the spacetime metric at energies of the Planck energy, and for this purpose a general approach, denoted as deformed or doubly special relativity, was developed in order to preserve the relativity of inertial frames, maintain the Planck energy invariant and impose that in the limit $E/E_P \to 0$, the speed of a massless particle tends to a universal and invariant constant, c.

More specifically, we explore the possibility that wormhole geometries are sustained by their own quantum fluctuations, in the context of Gravity's Rainbow[5]. We consider a semi-classical approach, where the graviton one-loop contribution to a classical energy in a background spacetime is computed through a variational approach with Gaussian trial wave functionals. The energy density of the graviton one-loop contribution to a classical energy in a wormhole background is considered as a self-consistent source for wormholes[6–8]. In this semi-classical context, we consider specific choices for the Rainbow's functions and find a plethora of wormhole solutions, including non-asymptotically flat geometries and solutions where

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the quantum corrections are exponentially suppressed, which provide asymptotically flat wormhole geometries.

In fact, the possibility that quantum fluctuations induce a topology change, in Gravity's Rainbow, has also been explored[9]. The energy density of the graviton one-loop contribution, or equivalently the background spacetime, was let to evolve, and consequently the classical energy was determined. Note that the background metric was fixed to be Minkowskian in the equation governing the quantum fluctuations, which behaves essentially as a backreaction equation, and the quantum fluctuations were let to evolve. Then, the classical energy, which depends on the evolved metric functions, was evaluated. Analysing this procedure, a natural ultraviolet (UV) cutoff was obtained, which forbids the presence of an interior spacetime region, and may result in a multipy-connected spacetime. Thus, in the context of Gravity's Rainbow, this process may be interpreted as a change in topology, and in principle results in the presence of a Planckian wormhole.

2. Gravity's Rainbow and the Equation of State

In Schwarzschild coordinates, the traversable wormhole metric can be cast into the form[10]

$$ds^2 = -\exp\left(-2\phi\left(r\right)\right)dt^2 + \left[1 - \frac{b\left(r\right)}{r}\right]^{-1}dr^2 + r^2 d\Omega^2. \tag{1}$$

where $\phi\left(r\right)$ is called the redshift function, while $b\left(r\right)$ is called the shape function and where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element of the unit sphere. Using the Einstein field equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, we obtain the following set of equations

$$\rho\left(r\right) = \frac{1}{8\pi G}\frac{b'}{r^2}, \tag{2}$$

$$p_r\left(r\right) = \frac{1}{8\pi G}\left[\frac{2}{r}\left(1 - \frac{b\left(r\right)}{r}\right)\phi' - \frac{b}{r^3}\right], \tag{3}$$

$$p_t\left(r\right) = \frac{1}{8\pi G}\left(1 - \frac{b\left(r\right)}{r}\right)\left[\phi'' + \phi'\left(\phi' + \frac{1}{r}\right)\right] - \frac{b'r - b}{2r^2}\left(\phi' + \frac{1}{r}\right), \tag{4}$$

where $\rho\left(r\right)$ is the energy density, $p_r\left(r\right)$ is the radial pressure, and $p_t\left(r\right)$ is the lateral pressure. The conservation of the stress-energy tensor yields the following relation

$$p_r' = \frac{2}{r}\left(p_t - p_r\right) - \left(\rho + p_r\right)\phi'. \tag{5}$$

When Gravity's Rainbow comes into play, the line element (1) becomes[4]

$$ds^2 = -\exp\left(-2\phi\left(r\right)\right)\frac{dt^2}{g_1^2\left(E/E_P\right)} + \frac{dr^2}{\left(1 - b(r)/r\right)g_2^2\left(E/E_P\right)} + \frac{r^2}{g_2^2\left(E/E_P\right)}d\Omega^2 \tag{6}$$

and Einstein's Field Equations (2)–(4) can be rearranged to give

$$b' = \frac{8\pi G \rho\left(r\right) r^2}{g_2^2\left(E/E_P\right)}, \tag{7}$$

$$\phi' = \frac{b + 8\pi G p_r r^3/g_2^2\left(E/E_P\right)}{2r^2\left(1 - b(r)/r\right)}. \tag{8}$$

Now, we introduce the equation of state $p_r = \omega\rho$[11], and using Eq. (7), then Eq. (8) becomes

$$\phi' = \frac{b + 8\pi G\left(\omega g_2^2\left(E/E_P\right) b'\left(r\right)/\left(8\pi G r^2\right)\right) r^3/g_2^2\left(E/E_P\right)}{2r^2\left(1 - b(r)/r\right)}$$

$$= \frac{b + \omega b' r}{2r^2\left(1 - b(r)/r\right)}. \tag{9}$$

Considering a constant redshift function yields the following condition

$$b + \omega b' r = 0. \tag{10}$$

which provides the solution

$$b\left(r\right) = r_0 \left(\frac{r_0}{r}\right)^{\frac{1}{\omega}}, \tag{11}$$

where we have used the condition $b\left(r_t\right) = r_t$. In this situation, the line element (6) becomes

$$ds^2 = -\frac{1}{g_1^2\left(E/E_P\right)} dt^2 + \frac{dr^2}{1 - \left(\frac{r_0}{r}\right)^{1+\frac{1}{\omega}} g_2^2\left(E/E_P\right)} + \frac{r^2}{g_2^2\left(E/E_P\right)} d\Omega^2. \tag{12}$$

Note that the flaring-out condition entails the violation of the null energy condition[10], i.e., $p_r + \rho < 0$, so that considering the equation of state $p_r = \omega\rho$, the parameter is restricted by $\omega < -1$.

It is also possible to compute the proper radial distance modified by Gravity's Rainbow

$$l\left(r\right) = \pm \int_{r_0}^{r} \frac{dr'}{\sqrt{1 - \frac{b_{\pm}(r')}{r'}}}$$

$$= \pm \frac{r_0}{g_2\left(E/E_P\right)} \frac{2\omega}{\omega + 1} \sqrt{\rho^{\left(1+\frac{1}{\omega}\right)} - 1} \, {}_2F_1\left(\frac{1}{2}, \frac{1-\omega}{2\omega + 2}; \frac{3}{2}; 1 - \rho^{\left(1+\frac{1}{\omega}\right)}\right). \tag{13}$$

It is interesting to note that Eq. (10) works also for an inhomogeneous EoS. Indeed, the presence of the rainbow's function does not affect the form of (10) except for an explicit dependence on r of the ω parameter, so that $b(r) + \omega(r)b'(r)r = 0$ leads to the following general form of the shape function

$$b(r) = r_0 \exp\left[-\int_{r_0}^{r} \frac{d\bar{r}}{\omega(\bar{r})\bar{r}}\right]. \tag{14}$$

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The situation appears completely different when a polytropic with an inhomogeneous parameter ω is considered. Indeed, when the polytropic EoS, i.e., $p_r = \omega(r) \rho^\gamma$, is plugged into Eq. (8), one arrives

$$
\phi' = \frac{b + 8\pi G p_r r^3/g_2^2 (E/E_P)}{2r^2 \left(1 - \frac{b(r)}{r}\right)} = \frac{b + 8\pi G \left(\omega(r) \rho^\gamma\right) r^3/g_2^2 (E/E_P)}{2r^2 \left(1 - \frac{b(r)}{r}\right)}
$$

$$
= \frac{b + (8\pi G)^{1-\gamma} \omega(r) \left(b(r)'\right)^\gamma r^{3-2\gamma} g_2^{2(\gamma-1)} (E/E_P)}{2r^2 \left(1 - \frac{b(r)}{r}\right)}. \tag{15}
$$

We can always impose that $\phi(r) = C$, but this means that

$$
b + (8\pi G)^{1-\gamma} \omega(r) \left(b(r)'\right)^\gamma r^{3-2\gamma} g_2^{2(\gamma-1)} (E/E_P) = 0 \tag{16}
$$

and a dependence on $g_2(E/E_P)$ appears. For this reason, in this contribution, we will fix our attention on the case when ω is a constant.

3. Self-sustained Traversable Wormholes, Gravity's Rainbow and Phantom Energy

In this Section, we shall consider the formalism outlined in detail in Refs.[6,7], where the graviton one loop contribution to a classical energy in a wormhole background is used. A traversable wormhole is said to be "*self sustained*" if

$$
H_\Sigma^{(0)} = -E^{TT}, \tag{17}
$$

where E^{TT} is the total regularized graviton one loop energy and $H_\Sigma^{(0)}$ is the classical term. When we deal with a spherically symmetric line element, the classical Hamiltonian reduces to

$$
H_\Sigma^{(0)} = -\frac{1}{2G} \int_{r_0}^\infty \frac{dr\, r^2}{\sqrt{1 - b(r)/r}} \frac{b'(r)}{r^2 g_2(E/E_P)}
$$

$$
= \frac{1}{2G} \int_{r_0}^\infty \frac{dr\, r^2}{\sqrt{1 - b(r)/r}} \frac{b(r)}{r^3 g_2(E/E_P)\,\omega}, \tag{18}
$$

where we have used the explicit expression of the scalar curvature in three dimensions and the form of Eq. (10). Following Ref.[5], the self-sustained equation (17) becomes

$$
-\frac{b(r)}{2Gr^3 g_2(E/E_P)\,\omega} = \frac{2}{3\pi^2}(I_1 + I_2), \tag{19}
$$

where the r.h.s. of Eq. (19) is represented by

$$
I_1 = \int_{E^*}^\infty E \frac{g_1(E/E_P)}{g_2^2(E/E_P)} \frac{d}{dE} \left(\frac{E^2}{g_2^2(E/E_P)} - m_1^2(r)\right)^{\frac{3}{2}} dE \tag{20}
$$

and

$$
I_2 = \int_{E^*}^\infty E \frac{g_1(E/E_P)}{g_2^2(E/E_P)} \frac{d}{dE} \left(\frac{E^2}{g_2^2(E/E_P)} - m_2^2(r)\right)^{\frac{3}{2}} dE, \tag{21}
$$

respectively. E^* is the value which annihilates the argument of the root while $m_1^2(r)$ and $m_2^2(r)$ are two r-dependent effective masses. Of course, I_1 and I_2 are finite for appropriate choices of the Rainbow's functions $g_1(E/E_P)$ and $g_2(E/E_P)$. With the help of the EoS, one finds

$$
\begin{cases}
m_1^2(r) = \frac{6}{r^2}\left(1 - \frac{b(r)}{r}\right) + \frac{3}{2r^3\omega}b(r)(\omega + 1) \\[2mm]
m_2^2(r) = \frac{6}{r^2}\left(1 - \frac{b(r)}{r}\right) + \frac{3}{2r^3\omega}b(r)\left(\frac{1}{3} - \omega\right)
\end{cases}
\tag{22}
$$

and on the throat, $r = r_0$, the effective masses reduce to

$$
m_1^2(r_0) = \frac{3}{2r_0^2\omega}(\omega + 1) \qquad
\begin{cases}
> 0 & \text{when} & \omega > 0 & \text{or} & \omega < -1 \\
< 0 & \text{when} & -1 < \omega < 0
\end{cases}
$$

$$
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad .
$$

$$
m_2^2(r_0) = \frac{3}{2r_0^2\omega}\left(\frac{1}{3} - \omega\right) \qquad
\begin{cases}
> 0 & \text{when} & 1/3 > \omega > 0 \\
< 0 & \text{when} & \omega > 1/3 & \text{or} & \omega < 0
\end{cases}
\tag{23}
$$

However, to have values of ω compatible with the flaring-out condition and the violation of the null energy condition, only the case $\omega < -1$ is allowed. It is easy to see that if we assume

$$
g_1(E/E_P) = 1 \qquad g_2(E/E_P) =
\begin{cases}
1 & \text{when} & E < E_P \\[2mm]
E/E_P & \text{when} & E > E_P
\end{cases}
, \tag{24}
$$

Eq. (19) becomes, close to the throat,

$$
-\frac{1}{2Gr_0^2\omega} = \frac{2}{\pi^2}\left(\int_{\sqrt{m_1^2(r)}}^{E_P} E^2\sqrt{E^2 - m_1^2(r_0)}dE + \int_{\sqrt{m_2^2(r)}}^{E_P} E^2\sqrt{E^2 - m_2^2(r_0)}dE\right),
\tag{25}
$$

where $m_1^2(r_0)$ and $m_2^2(r_0)$ have been defined in Eq. (23). Since the r.h.s. is certainly positive, in order to have real solutions compatible with asymptotic flatness, we need to impose $\omega < -1$, that it means that we are in the Phantom regime. With this choice, the effective masses (23) become, on the throat

$$
m_1^2(r_0) = \frac{3}{2r_0^2\omega}(\omega + 1), \tag{26}
$$

$$
m_2^2(r_0) = -\frac{3}{2r_0^2\omega}\left(\frac{1}{3} - \omega\right), \tag{27}
$$

and Eq. (25) simplifies to

$$
1 = -\frac{4r_0^2\omega}{\pi^2 E_P^2}\left[\int_{\sqrt{m_1^2(r_0)}}^{E_P} E^2\sqrt{E^2 - \frac{3}{2r_0^2\omega}(\omega + 1)}dE\right.
$$

$$
\left. + \int_0^{E_P} E^2\sqrt{E^2 + \frac{3}{2r_0^2}\left|\frac{1}{3\omega} - 1\right|}dE\right]
\tag{28}
$$

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The solution can be easily computed numerically and we find

$$-1 \geq \omega \geq -4.5,$$
$$2.038 \geq x \geq 1.083.$$

Therefore we conclude that a wormhole which is traversable in principle, but not in practice, can be produced joining Gravity's Rainbow and phantom energy. Of course, the result is strongly dependent on the rainbow's functions which, nonetheless must be chosen in such a way to give finite results for the one loop integrals (20) and (21).

4. Summary and further comments

In this work, we have considered the possibility that wormhole geometries are sustained by their own quantum fluctuations, but in the context of modified dispersion relations. We considered different models regulated by the Rainbow's functions to analyse the effect on the form of the shape function, and found specific solutions for wormhole geometries. However, it is important to point out that this approach presents a shortcoming mainly due to the technical difficulties encountered. The variational approach considered in this proceedings imposes a local analysis to the problem, namely, we have restricted our attention to the behaviour of the metric function $b(r)$ at the wormhole throat, r_t. Despite the fact that the behaviour is unknown far from the throat, due to the high curvature effects at or near r_t, the analysis carried out in this context should extend to the immediate neighbourhood of the wormhole throat.

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