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A GRADIENT-BASED ECONOMIC MPC SUITABLE FOR INDUSTRIAL APPLICATIONS

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Resumen: Model predictive control (MPC) is typically implemented as a lower stage of a hierarchical control structure. The upper level stages - known as Real Time Optimizer (RTO) - are devoted to compute, by means of a stationary optimization, the targets that the dynamic control stage (MPC) should reach to economically optimize the operation of the process. A different alternative consist in incorporating the economic optimization performed by the RTO stage directly in the dynamic optimization that solves the MPC stage. In this way, a single stage control structure could be implemented, avoiding the frequent inconsistencies that shows the communication between the two stages. In this work a new MPC formulation that explicitly integrates the RTO structure into the dynamic control layer is presented. The main properties of the proposed strategy are the simplicity - provided that it uses a gradient-based approximation for the economic cost, the guarantee of stability and recursive feasibility and an extended domain of attraction. Several simulations of a subsystem of a fluid catalytic cracking (FCC) unit were performed to test the controller.

1. INTRODUCTION

Traditionally, process industries are controlled by a hierarchical control structure (Engell, 2007): at the top, an economic scheduler and planner determines the whole plant production (level, quality, etc.). The outputs of this layer are sent to a Real Time Optimizer (RTO), which is devoted to compute the stationary targets according to economic criteria. This optimizer is usually based on a complex nonlinear stationary model of the plant and so has a sampling time different from other layers. Then, the targets computed by the RTO are sent to the advanced control level (usually an MPC) which calculates the control actions necessary for the plant to reach the targets, taking into account a simplified dynamic model of the plant and the variable constraints. One well-known drawback of this hierarchical control structure is that the commu-

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nication between the economic/stationary and the dynamic layers may be inconsistent, producing in this way problems that go from unreachability of the targets to poor economic performances. As a result, a proper strategy to unify this (probably competing) objectives becomes highly desired from an operating point of view.

In (Zanin *et al.*, 2002) the authors present the formulation and industrial application of a combined RTO/MPC controller applied to a fluidized-bed catalytic cracker, FCC, in which the RTO economic cost function is part of the MPC cost function. In (De Souza *et al.*, 2010), the gradient of the economic objective function is included in the controller cost function, in order to obtain a computational low-cost strategy. This solution allows one to solve the resulting control/optimization problem as a single QP problem, and the results are promising from both, theoretic and practical points of view.

In this work, we resume the idea to use the gradients of the economic cost to simplify the one-layer MPC cost, in the context of an MPC formulation suitable to include additional objectives to the dynamic control one (Limon *et al.*, 2008; Ferramosca *et al.*, 2009; Gonzalez and Odloak, 2009). However, instead of applying to the system the optimal solution of an approximated problem, we apply a solution that is the convex combination of an arbitrary feasible solution and an approximated solution. In this way, a suboptimal MPC strategy is obtained, which ensures recursive feasibility and convergence to the optimal steady state in the economic sense, with a reduced computational cost. Furthermore, the MPC formulation is based on a mode-decoupled velocity system that derives in a controller with an extended domain of attraction (i.e., for open-loop stable systems, the domain of attraction is the largest the system allows for).

2. PROBLEM STATEMENT

Consider a system described by a linear time-invariant discrete time model

$$x^+ = Ax + B\Delta u \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $\Delta u \in \mathbb{R}^m$ is the current control increment and x^+ is the successor state. The solution of this system for a given sequence of control inputs $\mathbf{\Delta u} = \{\Delta u(0), \cdots, \Delta u(j-1)\}$ and an initial state x is denoted as $x(j) = \phi(j; x, \mathbf{\Delta u})$, $j \in \mathbb{I}_{\geq 1}$, where $x = \phi(0; x, \mathbf{\Delta u})$. The state of the system and the control input applied at sampling time k are denoted as $x(k)$ and $\Delta u(k)$ respectively. The system is subject to hard constraints on state and input:

$$x(k) \in X, \quad \Delta u(k) \in \Delta U \tag{2}$$

for all $k \geq 0$, where $X \subset \mathbb{R}^n$ and $\Delta U \subset \mathbb{R}^m$. It is assumed that the following assumption holds.

Assumption 1. The pair (A, B) is controllable and the state is measured at each sampling time. Furthermore, the set X is convex and closed, ΔU is convex and compact and both sets contain the origin in its interior.

2.1 Dynamic system decoupling

It is assumed in this work that matrix A has $nss = m$ integrating eigenvalues, nun (pure) unstable eigenvalues and nst stable eigenvalues (González *et al.*, 2011). Therefore, matrix A in (1) can be decomposed into its stationary, unstable and stable modes using the Jordan decomposition:

$$\begin{aligned} A &= W\Lambda V \\ &= \begin{bmatrix} W_{ss} & W_{un} & W_{st} \end{bmatrix} \begin{bmatrix} \Lambda_{ss} & 0 & 0 \\ 0 & \Lambda_{un} & 0 \\ 0 & 0 & \Lambda_{st} \end{bmatrix} \begin{bmatrix} V_{ss}^T \\ V_{un}^T \\ V_{st}^T \end{bmatrix}, \end{aligned}$$

where $T \triangleq WV = I_n$, and Λ is a block diagonal matrix (Jordan canonical form), in which Λ_{ss}, Λ_{un} and Λ_{st} are upper triangular matrices. The columns of the linear maps $T_{ss} \triangleq W_{ss}V_{ss}^T$, $T_{un} \triangleq W_{un}V_{un}^T$ and $T_{st} \triangleq W_{st}V_{st}^T$ span the (complementary) stationary, unstable and stable subspaces or manifolds of the state space, $\mathcal{W}_{ss}$, $\mathcal{W}_{un}$ and $\mathcal{W}_{st}$, respectively; and they trivially satisfy $T_{ss} + T_{un} + T_{st} = T = I_n$. Since $\mathcal{W}_{ss} \oplus \mathcal{W}_{un} \oplus \mathcal{W}_{st} \equiv \mathbb{R}^n$, every state can be decomposed as $x = x^{ss} + x^{un} + x^{st}$, where $x^{ss} = T_{ss}x$ belongs to $\mathcal{W}_{ss}$, $x^{un} = T_{un}x$ belongs to $\mathcal{W}_{un}$ and $x^{st} = T_{st}x$ belongs to $\mathcal{W}_{st}$. Furthermore, given that $\mathcal{W}_{ss} \cap \mathcal{W}_{un} \cap \mathcal{W}_{st} \equiv \{0\}$, if $x \in \mathcal{W}_{ss}$ then $x^{st} = x^{un} = 0$, and so on for the others subspaces. As it is known, $\mathcal{W}_{ss} \subseteq \mathbb{R}^n$, $\mathcal{W}_{un} \subseteq \mathbb{R}^n$ and $\mathcal{W}_{st} \subseteq \mathbb{R}^n$ are invariant subspaces of the state space under the transformation A. Taking into account the model decomposition presented above, it is assumed that the original state constraint set is given by the decoupled set:

$$X \triangleq X_{ss} \oplus X_{un} \oplus X_{st}$$

where $X_{ss} \subseteq \mathcal{W}_{ss}$ (notice that this set includes the input constraints $u \in U$), $X_{un} \subseteq \mathcal{W}_{un}$ and $X_{st} \subseteq \mathcal{W}_{st}$ are closed convex sets. Furthermore, X_{st} is assumed to be a contractive set in $\mathcal{W}_{st}$.

2.2 Steady state characterization and economic optimum

If we consider the joint variable $(x, \Delta u)$, the state and input equilibrium subspace, associated to model (1), is given by

$$\mathcal{V}_{ss} = \mathcal{N}([(A-I) \quad B]) \subseteq \mathbb{R}^{n+m},$$

where $\mathcal{N}$ is the null space operator. Because of the velocity form of model (1) it can be shown that the steady state input set is given by the origin, $\Delta Uss = \{0\}$. So, the equilibrium subspace can be defined in $\mathbb{R}^n$, and is given by $\mathcal{W}_{ss} = \mathcal{N}(A-I)$

We define now the set of admissible stationary states as $X_{ss} = \{x \in X \mid x \in \mathcal{W}_{ss},\}$, which is a convex set in the equilibrium subspace. Now, taking into account the economic objectives, let us consider the following definition:

Definition 1. The optimal steady state, x_s, satisfy

$$\begin{aligned} x_s \;&=\; \arg\min_{x} f_{eco}(x,p) \\ & s.t.\; x \in X_{ss} \end{aligned}$$

where $f_{eco}(x,p)$ defines an economic cost function and p is a parameter that takes into account prices, costs or production goals. Notice that system input, u, which usually defines the optimal operating point, can always be described in terms of the states, according to model (1).

Assumption 2. The economic cost function $f_{eco}(x,p)$ is convex in x and twice differentiable.

In addition, according to most real cases, it is assumed that $f_{eco}(x,p)$ is nonlinear and its evaluation takes a significant computation time, provided that it is based on complex stationary models of the real plant.

3. THE ONE LAYER ECONOMIC MPC STRATEGY

In this section, the proposed controller is presented. The controller cost function is formulated following (Limon *et al.*, 2008) and is given by:

$$V_N(x,p;\mathbf{\Delta u}) = V_N^{dyn}(x,\mathbf{\Delta u}) + V_{ss}(x_{ss},p)$$

where $V_N^{dyn}(x,\mathbf{\Delta u}) = \sum_{j=0}^{N-1} \|x_j - x_{ss}\|_Q^2 + \|\Delta u_j\|_R^2 + \sum_{j=N}^{\infty} \|x_j - x_{ss}\|_Q^2$, $Q > 0$ and $R \geq 0$ are penalization matrices of appropriate dimension, and $V_{ss}(x_{ss},p) = f_{eco}(x_{ss},p)$. For any current state x, the optimization problem $P_N(x,p)$ to be solved is given by:

Problem $P_N(x,p)$

$$\begin{aligned} &\min_{\mathbf{\Delta u}} V_N(x,p;\mathbf{\Delta u}) \\ &s.t. \\ &\quad x_0 = x, \\ &\quad x_{j+1} = Ax_j + B\Delta u_j, \qquad j \in \mathbb{I}_{0:N-1} \\ &\quad x_j \in X,\, \Delta u_j \in \Delta U, \qquad j \in \mathbb{I}_{0:N-1} \\ &\quad T_{ss}x_N = x_{ss}, \quad T_{un}x_N = 0, \end{aligned}$$

In this optimization problem, x and p are the parameters, while the sequence

$$\mathbf{\Delta u} = \{\Delta u(0), \cdots, u(N-1)\}$$

is the optimization variable. The control law is given by $\kappa_N(x,p) = \Delta u^0(0;x)$, where $\Delta u^0(0;x)$ is the first element of the solution sequence $\mathbf{\Delta u}^0(x)$.

Remark 2. Notice that $T_{ss}x_N$ is the stationary value that the system $x^+ = Ax$ will reach *asymptotically*, provided that no unstable modes are considered beyond N. As a result, the infinite summation term of the cost converges, and can be expressed as the sum of two single terms. A rank condition necessary to ensure that constraint $T_{un}x_N = 0$ can be fulfilled is shown in (Alamo *et al.*, 2012).

Remark 3. The domain of attraction of the controller derived from the iterative application of Problem $P_N(x,p)$ is given by the states that can be steered in N steps to the equilibrium-stable subspace of $\mathbb{R}^n$, $\mathcal{W}_{ss-st} = \mathcal{W}_{ss} \oplus \mathcal{W}_{st}$, fulfilling the constraints along the path. This set is the N-step controllable set from X to $X_{ss-st} = X_{ss} \oplus X_{st}$, and will be denoted as $\mathcal{X}_N$.

Given that the economic cost is generally based on a complex nonlinear model, problem $P_N(x,p)$ is not easy to solve, mainly when large dimension processes are considered. On the other hand, it is known that to ensure convergence and recursive feasibility a suboptimal solution of $P_N(x,p)$ could be used. In this context, instead of directly solve the complex one-layer problem, the convex combination of an easy-to-obtain feasible solution and an approximated optimal solution could be used to obtain a decreasing cost.

4. A FEASIBLE AND AN APPROXIMATED OPTIMAL SOLUTION TO THE ORIGINAL ECONOMIC OPTIMIZATION PROBLEM

Let us consider a feasible solution to problem $P_N(x,p)$,

$$\hat{\mathbf{\Delta u}} = \{\hat{\Delta u}_0, \hat{\Delta u}_1, \cdots, \hat{\Delta u}_{N-1}\}$$

This feasible solution is a sequence of control inputs which is associated to a corresponding (infinite) sequence of states:

$$\hat{\mathbf{x}} = \{\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_{ss}\}$$

Although the proper feasible solution to problem $P_N(x,p)$ is the input sequence $\hat{\mathbf{\Delta u}}$, we will name it as $(\hat{\mathbf{x}}, \hat{\mathbf{\Delta u}})$ in order to make explicit the associated state sequence. The usual way to obtain a feasible solution to problem $P_N(x,p)$, at a given sample time k, is by using the shifted solution of the same problem at time $k-1$. As will be shown later, this choice not only gives an easy-to-obtain feasible solution, but allows to prove the closed-loop convergence of the proposed strategy. For the initial sample time, $k=0$, when the $(k-1)$-solution is not available, a feasible solution can be obtained by solving an optimization problem with the cost $V_N^{dyn}(x;\mathbf{\Delta u})$, and the same constraints that problem $P_N(x,p)$. Let us consider now an a*pproximated optimal solution* to the original problem $P_N(x,p)$, which can be obtained by solving the following approximated problem:

Problem $P_N^{app}(x,p)$

$$\begin{aligned}
&\min_{\mathbf{\Delta u}} V_N^{app}(x,p;\mathbf{\Delta u})\\
&s.t.\\
&\quad x_0 = x,\\
&\quad x_{j+1} = Ax_j + Bu_j, \qquad j \in \mathbb{I}_{0:N-1}\\
&\quad x_j \in X,\ \Delta u_j \in \Delta U, \qquad j \in \mathbb{I}_{0:N-1}\\
&\quad T_{ss}x_N = x_{ss},\ \ T_{un}x_N = 0,
\end{aligned}$$

where the approximated cost is given by

$$V_N^{app}(x,p;\mathbf{\Delta u}) = V_N^{dyn}(x,\mathbf{\Delta u}) + \nabla V_{ss}(\hat{x}_{ss},p)\left[\, x_{ss} - \hat{x}_{ss}\,\right],$$

and $\nabla V_{ss}(\hat{x}_{ss},p)$ represents the gradient of V_{ss} with respect to x, evaluated at the point $\hat{x}_{ss}$.

As it can be seen, this approximated optimal solution tries to optimize problem $P_N(x,p)$ by means of a simplified version of it. However, it should be noted that this solution is suboptimal (in the transient) with respect to the optimal solution to the original problem $P_N(x,p)$ and hence its direct application into the MPC scheme does not guarantee convergence of the closed-loop system to the optimal solution to the original problem $P_N(x,p)$.

5. IMPROVING THE FEASIBLE ECONOMIC MPC COST

Let us denote the optimal solution to problem $P_N^{app}(x,p)$ (which we named *approximated optimal solution*) as

$$\begin{aligned}
\mathbf{\Delta u}^* &= \{\Delta u_0^*, \Delta u_1^*, \cdots, \Delta u_{N-1}^*\}\\
\mathbf{x}^* &= \{x_1^*, x_2^*, \cdots, x_{ss}^*\}.
\end{aligned}$$

The *original cost* $V_N(x,\mathbf{\Delta u})$ corresponding to solutions $(\hat{\mathbf{x}}, \hat{\mathbf{\Delta u}})$ and $(\mathbf{x}^*, \mathbf{\Delta u}^*)$ are given, respectively, by:

$$\begin{aligned}
\hat{V} = V_N(x,p;\hat{\mathbf{\Delta u}}) &= \sum_{j=0}^{\infty} \|\hat{x}_j - \hat{x}_{ss}\|_Q^2 + \|\hat{\Delta u}_j\|_R^2\\
&\quad + V_{ss}(\hat{x}_{ss},p),\\
V^* = V_N(x,p;\mathbf{\Delta u}^*) &= \sum_{j=0}^{\infty} \|x_j^* - x_{ss}^*\|_Q^2 + \|\Delta u_j^*\|_R^2\\
&\quad + V_{ss}(x_{ss}^*,p)
\end{aligned}$$

Consider now a parameterized family of feasible solutions, given by the convex combination of the *feasible solution* and the *approximated optimal solution*:

$$\begin{aligned}
\mathbf{\Delta u}(\lambda) &= (1-\lambda)\hat{\mathbf{\Delta u}} + \lambda\mathbf{\Delta u}^*\\
\mathbf{x}(\lambda) &= (1-\lambda)\hat{\mathbf{x}} + \lambda\mathbf{x}^*.
\end{aligned}$$

with $\lambda \in [0,1]$. Define also the following performance indexes:

$$V(\lambda) = \sum_{j=0}^{\infty} \|x_j(\lambda) - x_{ss}(\lambda)\|_Q^2 + \|\Delta u_j(\lambda)\|_R^2 + V_{ss}(x_{ss}(\lambda),p),$$

which is the original cost $V_N(x,p;\mathbf{\Delta u})$ parameterized in λ, and

$$V_g(\lambda) = \sum_{j=0}^{\infty} \|x_j(\lambda) - x_{ss}(\lambda)\|_Q^2 + \|\Delta u_j(\lambda)\|_R^2 + \hat{V}_{ss}(\hat{x}_{ss},p) + \nabla V_{ss}(\hat{x}_{ss},p)\left[\, x_{ss}(\lambda) - \hat{x}_{ss}\,\right]$$

which is the original cost $V_N(x,p;\mathbf{\Delta u})$, with the economic cost $V_{ss}(x,p)$ replaced by its first order Taylor approximation.

Lemma 4. If $(\hat{\mathbf{x}}, \hat{\mathbf{\Delta u}}) \neq (\mathbf{x}^*, \mathbf{\Delta u}^*)$, then

$$V_g(1) < V_g(0). \tag{3}$$

PROOF. Notice that $V_g(0) = V_N^{app}(x,p;\hat{\mathbf{\Delta u}})$ and $V_g(1) = V_N^{app}(x,p;\mathbf{\Delta u}^*)$. Since $(\hat{\mathbf{x}}, \hat{\mathbf{\Delta u}}) \neq (\mathbf{x}^*, \mathbf{\Delta u}^*)$, $(\mathbf{x}^*, \mathbf{\Delta u}^*)$ is the optimal solution to problem $P_N^{app}(x,p)$, and $V_N^{app}(x,p;\mathbf{\Delta u})$ is convex in $(\mathbf{x}, \mathbf{\Delta u})$; then $(\mathbf{x}^*, \mathbf{\Delta u}^*)$ will produce a smaller cost than $(\hat{\mathbf{x}}, \hat{\mathbf{\Delta u}})$ - otherwise, the solution will be exactly $((\hat{\mathbf{x}}, \hat{\mathbf{\Delta u}})$ - i.e., $V_N^{app}(x,p;\mathbf{\Delta u}^*) < V_N^{app}(x,p;\hat{\mathbf{\Delta u}})$ and therefore $V_g(1) < V_g(0)$

Next, the main theorem of this work will be presented.

Theorem 1. The following hold

(i) The pair $(\mathbf{x}(\lambda), \mathbf{\Delta u}(\lambda))$, for every $\lambda \in [0,1]$, provides a feasible solution to $P_N(x,p)$.
(ii) If $V_g(1) < V_g(0) = \hat{V}$, then there exists a $\tilde{\lambda} \in (0,1]$ such that $V(\tilde{\lambda}) < V(0) = \hat{V}$.

PROOF. Taking into account that the constraints of problem $P_N(x,p)$ are convex with respect to $(\mathbf{x}(\lambda), \mathbf{\Delta u}(\lambda))$ and that both the *feasible solution* $(\hat{\mathbf{x}}, \hat{\mathbf{\Delta u}})$ and the *approximated optimal* one $(\mathbf{x}^*, \mathbf{\Delta u}^*)$ are feasible for this problem, it results that any convex combination of them is a feasible solution. This proves the first claim of the theorem. We now proceed to prove the second claim of the theorem. The convexity of $V_g(\lambda)$ with respect to λ implies that $V_g(\lambda) \leq (1-\lambda)V_g(0) + \lambda V_g(1)$. Now, consider a point between $x_{ss}(\lambda)$ and $\hat{x}_{ss}$, which can be parameterized with a parameter θ as $x_{ss}(\theta) = (1-\theta)x_{ss}(\lambda)+\theta\hat{x}_{ss}$. Since $V_{ss}(\cdot,p)$ is twice differentiable, one can affirm that for every $\lambda \in [0,1]$ and $\theta \in [0,1]$, there exists the Hessian $H(\lambda,\theta) = H(x_{ss}(\theta))$; and, by the *mean value theorem*, it follows that

$$\begin{aligned} V_{ss}(x_{ss}(\lambda),p) &= V_{ss}(\hat{x}_{ss},p) \\ &+\nabla V_{ss}(\hat{x}_{ss},p)\left[\, x_{ss}(\lambda) - \hat{x}_{ss} \,\right] \\ &+\frac{1}{2}\left[\, x_{ss}(\lambda) - \hat{x}_{ss} \,\right]^T H(\lambda,\theta)\left[\, x_{ss}(\lambda) - \hat{x}_{ss} \,\right] \end{aligned}$$

for every $\lambda \in [0,1]$ and for some $\theta \in [0,1]$. With the last equality, we now have that for every $\lambda \in [0,1]$ and for some $\theta \in [0,1]$,

$$V(\lambda) = V_g(\lambda) + \lambda^2\left(\frac{1}{2}\right)\left[\, x^*_{ss} - \hat{x}_{ss} \,\right]' H(\lambda,\theta)\left[\, x^*_{ss} - \hat{x}_{ss} \,\right]. \quad (4)$$

Furthermore, since $x_{ss}(\theta)$ is a point between $x_{ss}(\lambda)$ and $\hat{x}_{ss}$, and $x_{ss}(\lambda)$, a point between x^*_{ss} and $\hat{x}_{ss}$, then an upper bound for the second term in (4) can be computed as:

$$\rho = \max_{\lambda\in[0,1]}\left(\frac{1}{2}\right)\left[\, x^*_{ss} - \hat{x}_{ss} \,\right]' H(\lambda)\left[\, x^*_{ss} - \hat{x}_{ss} \,\right],$$

where $H(\lambda) = H(x_{ss}(\lambda))$. Then

$$\begin{aligned} V(\lambda) &\leq V_g(\lambda) + \lambda^2\rho \\ &\leq (1-\lambda)V_g(0) + \lambda V_g(1) + \lambda^2\rho \\ &= V_g(0) - \lambda(V_g(0) - V_g(1) - \lambda\rho) \end{aligned}$$

Since $V_g(0) = \hat{V}$, hence

$$V(\lambda) \leq \hat{V} - \lambda(\hat{V} - V_g(1) - \lambda\rho)$$

Since it is assumed that $V_g(1) < V_g(0) = \hat{V}$, we obtain that $\hat{V} - \lambda(\hat{V} - V_g(1) - \lambda\rho)$ is positive for λ smaller than

$$\tilde{\lambda} = \min\left\{(\hat{V} - V_g(1))/\rho, 1\right\}, \quad (5)$$

which implies that

$$V(\lambda) < \hat{V}, \quad \forall\lambda \leq \tilde{\lambda}. \quad (6)$$

This means that for every λ smaller than $\tilde{\lambda}$, the pair $(\mathbf{x}(\lambda), \mathbf{\Delta u}(\lambda))$ provides not only a feasible solution to the original problem, but also an improved *original cost* when compared with the one corresponding to the feasible solution $(\hat{\mathbf{x}}, \hat{\mathbf{\Delta u}})$.

Remark 5. One can heuristically search for a value of λ that gives a cost $V(\lambda)$ smaller than $\hat{V}$. What theorem (1) ensures, is that this value of λ does exist.

Notice that θ can be obtained resorting to the Back Tracking technique (Boyd and Vandenberghe, 2006).

6. PROPOSED ALGORITHM

Based on the results presented in section 5, an iterative algorithm will be proposed now to obtain an MPC policy:

Algorithm 1. At each sample time k,

(1) compute the *feasible solution* $(\hat{\mathbf{x}}, \hat{\mathbf{\Delta u}})$ to problem $P_N(x,p)$, using the shifted solution applied to the system at the sample time $k-1$. If the current time is $k=0$, compute the *feasible solution* $(\hat{\mathbf{x}}, \hat{\mathbf{\Delta u}})$ by solving the reduced problem $P_N^{dyn}(x)$.
(2) compute the gradient of the economic cost function $V_{ss}(x,p)$ with respect to x, $\nabla V_{ss}(x,p)$.
(3) compute the *approximated optimal solution* to problem $P_N(x,p)$, $(\mathbf{x}^*, \mathbf{\Delta u}^*)$, by solving the approximated problem $P_N^{app}(x,p)$.
(4) compute the value of the parameter $\tilde{\lambda}$, as in (5).
(5) from the solution $(\mathbf{x}^0, \mathbf{\Delta u}^0) \triangleq (\mathbf{x}(\tilde{\lambda}), \mathbf{\Delta u}(\tilde{\lambda}))$, take the first input action of the sequence $\mathbf{\Delta u}^0$ to implement the implicit MPC control law, $\kappa_N(x,p) \triangleq \Delta u^0(0;x)$.

Remark 6. Notice that in the last step of the Algorithm (1), and provided that the sample time of the process is enough large, the solution $(\mathbf{x}(\tilde{\lambda}), \mathbf{\Delta u}(\tilde{\lambda}))$ can be iteratively improved, within the current sample time, to obtain a better approximation to the optimum.

7. CONVERGENCE OF THE PROPOSED CONTROLLER

To prove the convergence of the proposed MPC, we follows the usual steps found in the literature.

Theorem 2. Consider that assumption 1 holds, and consider a given parameter p for the economic cost $V_{ss}(x,p) = f_{eco}(x,p)$. Then, for any initial state $x \in \mathcal{X}_N$, the system controlled by the MPC control law derived from the application of Algorithm 1 at each time step k is stable and fulfills the constraints throughout the time. Furthermore, the closed-loop system converges asymptotically to a steady state x_s^0 that satisfy

$$x_s^* = \arg \min_{x \in X_{ss}} f_{eco}(x,p).$$

PROOF.

Feasibility: The feasibility follows directly from the fact that the set $\mathcal{X}_N$ is a control invariant set for system (A, B).

Convergence: Consider a state $x \in \mathcal{X}_N$, at a given time k. Consider also the solution defined in Algorithm (1), for this state,

$$\mathbf{\Delta u}^0(x) = \left\{\Delta u^0(0;x), \cdots, \Delta u^0(N-1;x)\right\},$$

and the corresponding state sequence

$$\mathbf{x}^0(x) = \left\{x^0(1;x), \cdots, x^0(\infty;x)\right\},$$

where $x^0(\infty;x) = x_{ss}^0(x)$ and $x_{ss}^0(x) \in X_{ss}$. Now, consider the state $x^+ = Ax + B\Delta u^0(0;x) = x^0(1;x)$, at time $k+1$, which is obtained by implementing the control law of step 5 of Algorithm 1, and define the following feasible solution for problem $P_N(x^+, p)$ at time $k+1$,

$$\tilde{\mathbf{\Delta u}} = \left\{\Delta u^0(1;x), \cdots, \Delta u^0(N-1;x), 0\right\},$$

which is a sequence made by shifting the sequence $\mathbf{\Delta u}^0(x)$ and adding a null control action. This solution has an associated state sequence, $\tilde{\mathbf{x}} = \left\{x^0(2;x), \cdots, x_{ss}^0(x), x_{ss}^0(x)\right\}$, where the additional state is given by $x_{ss}^0(x) = Ax_{ss}^0(x)$. Now, two consecutive cost functions will be compared. The cost function of Problem $P_N(x,p)$ corresponding to $\mathbf{\Delta u}^0(x)$ is given by

$$\begin{aligned} V_N^0(x) &= V_N(x, p; \mathbf{\Delta u}^0(x)) \\ &= V_N^{dyn}(x; \mathbf{\Delta u}^0(x)) + V_{ss}(x_{ss}^0(x), p). \end{aligned}$$

On the other hand, the cost function of Problem $P_N(x^+, p)$, at $k+1$, corresponding to $\tilde{\mathbf{\Delta u}}$, is given by

$$V_N(x^+, p; \tilde{\mathbf{\Delta u}}) = V_N^{dyn}(x^+; \tilde{\mathbf{\Delta u}}) + V_{ss}(x_{ss}^0(x), p).$$

If we compare now the consecutive costs, we have:

$$\begin{aligned} & V_N\,(x^+, p; \tilde{\mathbf{\Delta u}}) - V_N^0(x) = V_N^{dyn}(x^+; \tilde{\mathbf{\Delta u}}) \\ & -\ V_N^{dyn}(x; \mathbf{\Delta u}^0) \\ & =\ -\|x - x_{ss}^0(x)\|_Q^2 - \|\Delta u^0(0;x)\|_R^2 \\ & +\ \|x_{ss}^0(x) - x_{ss}^0(x)\|_Q^2 \\ & =\ -\|x - x_{ss}^0(x)\|_Q^2 - \|\Delta u^0(0;x)\|_R^2. \end{aligned} \quad (7)$$

Now, by Theorem (1), we have that the cost corresponding to the solution $\mathbf{\Delta u}^0(x^+)$, $V_N^0(x^+) = V_N(x, p; \mathbf{\Delta u}^0(x^+))$, is such that $V_N^0(x^+) < V_N$ $(x^+, p; \tilde{\mathbf{\Delta u}})$, because $\tilde{\mathbf{\Delta u}}$ is a feasible solution to problem $P_N(x^+, p)$, at time $k+1$. So, from (7), it follows that

$$\begin{aligned} V_N^0(x^+)\ -V_N^0(x) \leq& \\ -\ \|x - x_{ss}^0(x)\|_Q^2 &- \|\Delta u^0(0;x)\|_R^2. \end{aligned} \quad (8)$$

Since Q and R are definite positive, (8) implies that both, $\left|x - x_{ss}^0(x)\right|$ and $\left|\Delta u^0(0;x)\right|$ tends to 0 as $k \to \infty$, and so, the system converges to the steady state given by $x_{ss}^0(x)$.

Economic optimality: We have shown that the system converges to a steady state, that we will call, for the sake of simplicity, x_s. Now, we will show that the this steady state necessarily minimizes V_{ss}. Consider that $x_s \neq x_s^*$, where

$$x_s^* = \arg \min_{x \in X_{ss}} f_{eco}(x,p).$$

Let us define $x_s(\theta) = (1-\theta)x_s + \theta x_s^*$, with $\theta \in [0,1]$. Since both x_s^* and x_s are in X_{ss}, and this set is convex, then a convex combination of these points, $x_s(\theta)$, is also in X_{ss}. Furthermore, since by Assumption (2) V_{ss} is convex in x, we have that for a given value of p, $V_{ss}(x_s(\theta), p) \leq (1-\theta)V_{ss}(x_s, p) + \theta V_{ss}(x_s^*, p)$, and by optimality of x_s^*, we have $V_{ss}(x_s) > V_{ss}(x_s^*)$, and so $V_{ss}(x_s(\theta), p) \leq (1-\theta)V_{ss}(x_s, p) + \theta V_{ss}(x_s, p) = V_{ss}(x_s, p)$, for every $\theta \in [0,1]$. Assuming that the system is already stabilized at x_s, and defining $\mathbf{\Delta u}_{null} = \{0, \cdots, 0\}$, we have

$$\begin{aligned} V_N(x_s, p; \mathbf{\Delta u}_{null}) &= V_N^{dyn}(x_s, \mathbf{\Delta u}_{null}) + V_{ss}(x_s, p) \\ &= V_{ss}(x_s, p). \end{aligned}$$

Now, we will show that if we apply to the system a control sequence different from the null sequence, we can obtain a better cost. Let us consider the following control sequence:

$$\begin{aligned} \mathbf{\Delta u}(\theta) &= \begin{bmatrix} T_{un}Co_N \\ T_{ss}Co_N \end{bmatrix}^\dagger \begin{bmatrix} T_{un} \\ T_{ss} \end{bmatrix} (x_s^* - x_s)\theta \\ &= M_1\theta, \quad \text{for some } \theta \in (0,1) \end{aligned}$$

where $Co_j = [A^{j-1}B \quad A^{j-2}B \quad \cdots \quad B]$ is the generalized controllability matrix, † is the pseudo-inverse operator and, for simplicity, we assume that $rank(Co_N) = nun + m$. For an arbitrary small value of θ, this sequence is a feasible sequence that produces the following state sequence: $x_j(\theta) = x_s + [Co_1 \;\; 0_{n,(N-j)\cdot m}]\mathbf{\Delta u}(\theta) = x_s + [Co_j \;\; 0_{n,(N-j)\cdot m}]M_1\theta$, for $j \in \mathbb{I}_{0:N}$. This state sequence fulfills the constraints of problem $P_N(x_s, p)$ and tends asymptotically to the stationary value $x_s(\theta)$. The cost $V_N^{dyn}(x_s, \mathbf{\Delta u}(\theta))$ corresponding to this control and state sequences is given by

$$\begin{aligned} V_N(x_s, p; \mathbf{\Delta u}(\theta)) =& V_N^{dyn}(x_s, \mathbf{\Delta u}(\theta)) + V_{ss}(x_s(\theta), p) \\ =& \|[M_2M_1 - (x_s^* - x_s)]\theta\|_Q^2 + \|M_1\theta\|_R^2 \\ &+ \|(Co_NM_1 - (x_s^* - x_s))\theta\|_P^2 + V_{ss}(x_s(\theta), p) \end{aligned}$$

where

$$M_2 = \begin{bmatrix} Co_0 \ \ 0_{n,N\cdot m} \\ \vdots \\ Co_{N-1} \ \ 0_{n,m} \end{bmatrix}$$

The last cost can be re-written as:

$$V_N(\theta)=\theta^2(\|[M_2M_1-(x_s^*-x_s)]\|_Q^2+\|M_1\|_R^2 \\ +\|(Co_N M_1-(x_s^*-x_s))\|_P^2)+V_{ss}(x_s(\theta),p)$$

This cost trivially satisfies $V_N(0) = V_{ss}(x_s,p)$. Now, let us consider the derivative of $V_N(\theta)$ with respect to θ,

$$\frac{\partial V_N(\theta)}{\partial\theta} = 2\theta(\|[M_2M_1-(x_s^*-x_s)]\|_Q^2+\|M_1\|_R^2 \\ +\|(Co_N M_1-(x_s^*-x_s))\|_P^2)+\frac{\partial V_{ss}(x_s(\theta),p)}{\partial\theta}$$

If we now evaluate this derivative at $\theta = 0$, we have

$$\left.\frac{\partial V_N(\theta)}{\partial\theta}\right|_{\theta=0} = \left.\frac{\partial V_{ss}(x_s(\theta),p)}{\partial\theta}\right|_{\theta=0} < 0, \text{ if } x_s \neq x_s^*$$

The last inequality follows from the fact that $V_{ss}(x_s(\theta),p) < V_{ss}(x_s,p)$. This means that a $\hat{\theta}$ does exist, such the cost corresponding to move the system from x_s to $x_s(\hat{\theta})$ is smaller that the one corresponding to remain in the stationary state x_s. So, the closed-loop system converges to the economic optimal steady state x_s^*.

8. EXAMPLE

The properties of the proposed controller have been tested in a simulation example, on the fluid catalytic cracking unit (FCC) studied in (Zanin *et al.*, 2002) and (De Souza *et al.*, 2010). In this simplified version of the FCC system, the output to be controlled are the temperature in the dilute phase of the regenerator y_1 (C), the temperature in the dense phase of the regenerator y_2 (C), the conversion of the cracking reaction y_3 (%), the riser temperature y_4 (C). The inputs (that in the mode-decoupled model are implicitly included into the state vector) are the total air flow-rate of the two stage catalyst regenerator u_1 (ton/h), the valve opening of the regenerated catalyst u_2 (%), the gasoil feed flow-rate u_3 (m^3/h), the temperature of the feed u_4 (C). As for the controller setup, the weighting matrices of the MPC cost function have been takens as $Q = C'Q_yC$, where $Q_y = diag(0.2, 0.1, 0.1, 1)$, and $R = diag(5, 5, 5, 5)$. Matrix P is taken as the solution of the Lyapunov equation $P = A_{st}'PA_{st}+Q$. An horizon $N = 3$ has been considered.
The economic objective is to maximize the production of liquified petroleum gas (LPG). This function is a nonlinear function of the feed properties and the process operating condition (Zanin *et al.*, 2002) and is given by $V_O = -u_3 \times LPGV$, where $LPGV$ is the volumetric yield of LPG. The system has the following constraints on the inputs: $u_{max} = (228, 98, 406, 235)'$ and $u_{min} = (200, 50, 400, 234.9)'$. A zone control strategy has been adopted, in such a way that the outputs are required to lie into the zone given by $y_{max} = (705, 725, 95, 547)'$ and $y_{min} = (695, 695, 60, 540)'$. The sampling time is $T_s = 1$ min. The initial steady state is given by $y_{ss} = (697.3, 699.4, 78.2, 544.4)'$ and $u_{ss} = (220.7, 85, 404, 234.9)'$. The results of the simulation are presented in Figures 1, 2 and 3. Figure 1 shows the production of LPG, while

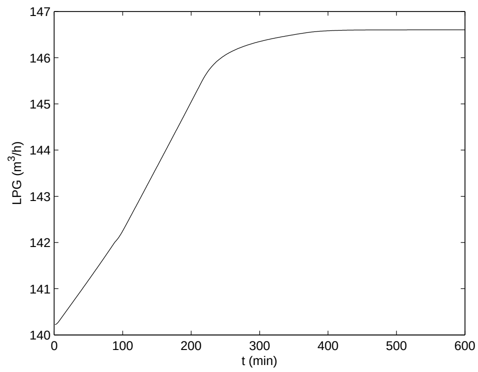

Fig. 1. Time evolution of the LPG production.

Figure 2 and Figure 3 show the time evolution of outputs and inputs, respectively. The evolution of the LPG production shows how the controller proposed in this work satisfies the economic objective in the same way as (De Souza *et al.*, 2010).
Notice also that, in order to maximize the production of LPG, input u_3 and u_4 are pushed by the controller to their maximum and minimum bounds, respectively. This indicates that all four degrees of freedom are used in order to maximize the LPG production, while constraints are always fulfilled. Figure 4 shows a comparison between

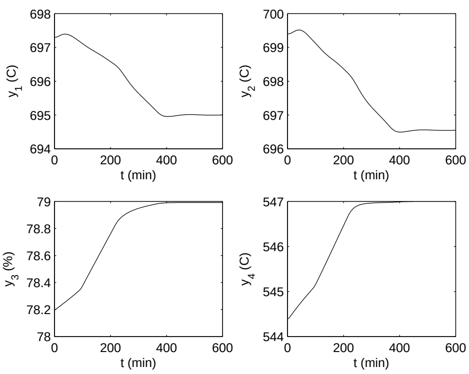

Fig. 2. Evolution of the outputs.

the optimal cost obtained with the solution provided by the proposed controller, that is $V(\lambda)$,

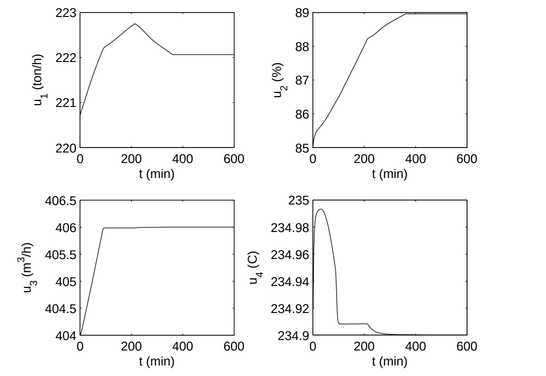

Fig. 3. Evolution of the inputs.

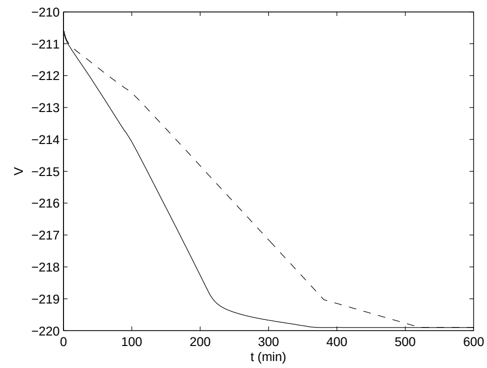

Fig. 4. Comparison of the cost $V(\lambda)$ (in solid line) with the cost $V_g(0) = \hat{V}$ (in dashed line).

and the optimal cost provided by the initial feasible solution, that is $V_g(0) = \hat{V}$. See that, as stated in the theorem, the value of $V(\lambda)$ is smaller than the values of $\hat{V}$. This means that the proposed algorithm provides a better solution, in the sense that the optimal cost is smaller.

Moreover, the performance of the proposed strategy has been compared to the one provided by $V_g(0) = \hat{V}$. The performance index used for this comparison has been:

$$\Phi = \frac{1}{T}\sum_{k=0}^{T} \|x(k) - x_{ss}\|_Q^2 + \|\Delta u(k)\|_R^2$$

The obtained performance have been $\Phi(V(\lambda)) = 14.4894$ and $\Phi(\hat{V}) = 31.5921$, showing that the proposed controller provide better performance than $\hat{V}$.

9. CONCLUSION

A new MPC that accounts for economic objectives and is suitable for industrial application was presented in this work. Based on the inclusion of the gradient of the economic cost, the optimization control problem remains a QP problem. Furthermore, the proposed controller ensures stability and feasibility under any change of the economic function, and has an extended domain of attraction derived from the appropriate use of a decoupled linear model.

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