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The Value of the Right Distribution in Stochastic Programming with Application to a Newsvendor Problem

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Abstract In this paper we introduce the concepts of the *Value of the Right Distribution* (VRP), the *Performance Bound* (PB) and the *Worst-Case Performance Bound* (WPB), which allow us to quantify how much we lose if we guess the wrong distribution of the uncertain parameters affecting a stochastic optimization problem. In order to show how they apply, we introduce a cost-based variant of the classical Newsvendor problem and model it as a two-stage stochastic programming model. For this problem, we first provide optimal solutions in closed form for different probability distributions and then compute, both analytically and computationally, the VRP measure and the corresponding performance bounds PB and WPB. Finally, systematic numerical results are provided.

Keywords stochastic programming · value of the right distribution · worst-case analysis · newsvendor problem

1 Introduction

Many real life decision problems are affected by uncertainty: there are several situations in which we are asked to take decisions even if some of the parameters are unknown. One of the main paradigm to deal with problems with uncertain data, both in the single period and multi-period decision-making process, is given by *Stochastic Programming* (SP) (see [1,2,3,4]). A basic assumption in SP is that the

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probability laws of the uncertain parameters are known and only their realizations are unknown when the optimization is carried out. The probability distributions are often selected on the basis of a given time series of past observations. Doing that, it is implicitly assumed that the future will be similar, or it can be predicted from the past. However, this assumption can be not realistic in some cases, for example for new products or services, and for failure of automatized processes. In fact, different applications have shown that the choice of the appropriate probability model is crucial for the quality of the solution (see for example [5] for a robust optimization approach to address the problem of optimally controlling a supply chain subject to stochastic demand in discrete time and [6] where the value of knowing in advance the probability distribution of rental demand for a new bike-sharing service has been evaluated). Even the model class itself can be chosen erroneously: on the basis of the available information (e.g. the support or low-order moments) there exist multiple distributions that could represent the real phenomenon, according to the available information. This fact is usually called *model ambiguity* (see [7,8]). One way to deal with ambiguity is to investigate the *stability* of the optimal solution in SP with respect to continuity properties of the solution on model parameters (see [9ömisch1991,10,11]). Another way is given by the *Minimax Stochastic Optimization* pioneered in [12], also known with the name *Distributionally Robust Optimization* (DRO). DRO can be regarded as a natural generalization of Stochastic Programming and Robust Optimization [13]. In DRO optimal decisions are sought for the worst-case probability distribution within a family of possible distributions defined by certain properties (like the support, mean, covariance matrix, and upper bounds on its directional deviations). A growing literature in this direction both from theoretical and applied point of view can be found in [14,15,16,17,18] and many others.

In the SP literature, several bounds and approximations have been proposed to approximate the objective function value and to cope with the typical SP computational complexity. The standard measure of the expected gain from solving a stochastic model rather than its deterministic counterpart is given by the *Value of the Stochastic Solution* (*VSS*) (see [19]), computed by comparing the solution values of the stochastic and expected-value variant of the problem. A high *VSS* indicates that stochastic programming models are necessary despite the computational efforts involved. Easy-to-compute bounds have been also proposed in literature by solving small size sub-problems instead of the big one associated with the large discrete scenario tree model, representing a discretization of the underlying random process (see for instance [20,21,22,23,24,25]). In [26] multistage approximations obtained by reducing the number of stages in the original problem have been proposed and the benefit of including an additional stage in the approximation computed. All these methodologies measure the quality of the approximating solution in terms of objective function values. Another approach, proposed in the SP literature to assess the value of a given solution, is to approximate its relative gap to the optimum value of the stochastic problem. To assess solution quality, a Monte Carlo sampling-based procedure was proposed in [27] and [28] allowing to estimate an upper bound on the optimality gap. A study on the structure of the first-stage solution associated to the expected value formulation in terms of basis/out-of-basis variables and reduced costs has been proposed in [29] and in [30], as indicator for excluding/retaining decision variables in the corresponding stochastic model.

In the same spirit of the bounds and metrics described above, the goal of this paper is to evaluate in the SP setting, how much we lose if we guess the wrong distribution assuming to know only the support. In other words we evaluate the objective function value increase when the decision-maker a priori selects the most likely probability distribution (that we will call *guessed probability distribution*) which differs with the true probability distribution (that we will call *right probability distribution*). This applies in the case of new products or services for which the time series are not available. For example, consider the case of the demand of a new product which has to be observed on a daily basis. We assume a probability distribution of the demand and solve to optimality the corresponding SP model. Then, the first-stage solution of the SP model is applied on each day for a certain number of days. The total cost we pay in the long run is equal to the expected cost of the observed probability distribution, computed by applying the optimal solution obtained on the basis of the guessed distribution. Our aim is to evaluate the increase in the total cost obtained in this way with respect to the one we have if the right distribution is used to solve to optimality the SP model. Since the right distribution is not known in advance, we compute this increase with respect to several probability distributions that are all candidates to be the true probability distribution. If the maximum increase is small, this means that the guessed probability distribution can be used safely. Otherwise, we know in advance the maximum increase we will have. On this purpose we introduce the *Value of the Right Distribution* (VRD), a new measure of ambiguity in SP, which allows us to quantify the increase in the objective function value when the guessed probability distribution does not match the right one. VRD can be seen as a generalization of VSS in the following sense: when solving the expected-value problem we are in a certain way assuming a guessed distribution, that is a singleton. Then we observe the true distribution, and measure the loss we incurred for not having guessed the right one.

Besides VRD, we introduce the *Performance Bound* (PB) of the guessed distribution with respect to the right one, and the *Worst-case Performance Bound* (WPB). The new measure (VRD) and the bounds (PB and WPB) apply not only when two or more probability distributions may be mismatched, but also when the estimate of the type of the distribution is correct, but its parameters may be wrong (more typically the standard deviation).

A similar study has been addressed in [31] questioning how relevant is to capture different properties of the uncertainty (e.g. means, support, correlations or variances) for the specific decision model, which can lead to better models of the uncertainty as well as to more effective data collection/analysis efforts. Their results numerically show on a case of maritime transportation, that some properties have very little influence on the final decisions (e.g. the correlation properties between the random variables) while others (e.g. the mean values) can lead to noticeable increases in the expected cost if incorrectly estimated.

To investigate the VRD measure and the PB and WPB bounds, we adopt a cost–based variant of the well known *Newsvendor Problem* (see [1]): we first provide optimal solutions in closed form for several probability distributions having the same support and then compute, both analytically and computationally, the VRP measure and the corresponding PB and WPB bounds. A worst-case analysis is carried out, showing the maximum increase in the objective function value that can be obtained guessing a probability distribution different than the right one.

Finally, a computational study shows, in a systematic way with respect to the unit costs, how the expected objective function value varies when the probability distribution varies.

The paper is organized as follows. In Section 2 we define the Value of the Right Distribution VRD and the bounds PB and WPB. In Section 3 we introduce a cost–based variant of the Newsvendor Problem. In Section 4 we show how to compute the bounds for this problem. In Section 5 we show, by means of an extensive experimental campaign, the interest of the proposed VRP, PB and WPB. Conclusions follow.

2 The Value of the Right Distribution and the Performance Bounds in Stochastic Programming

The following mathematical model represents a general formulation of a stochastic program in which a decision maker needs to determine x in order to minimize the expected total cost (see e.g. [1]):

$$\min_{x\in\mathcal{X}} \mathbb{E}_{\boldsymbol{\xi}} z\left(x,\boldsymbol{\xi}\right) = \min_{x\in\mathcal{X}} \left\{ f_1(x) + \mathbb{E}_{\boldsymbol{\xi}}\left[h_2\left(x,\boldsymbol{\xi}\right)\right]\right\}, \tag{1}$$

where x is a first-stage decision vector restricted to the set $\mathcal{X} \subseteq \mathbb{R}^n_+$, with $\mathbb{R}^n_+$ is the set of non negative real vectors of dimension n, and $\mathbb{E}_{\boldsymbol{\xi}}$ stands for the expectation with respect to a random vector $\boldsymbol{\xi}$, defined on some probability space $(\varXi, \mathcal{A}, p)$ with support $\varXi$ and given probability distribution p on the $\sigma-$algebra $\mathcal{A}$. The function h_2 is the value function of another optimization problem defined as

$$h_2\left(x,\xi\right) = \min_{y\in\mathcal{Y}(x,\xi)} f_2\left(y;x,\xi\right), \tag{2}$$

which is used to reflect the costs associated to adapt to the information revealed through a realization ξ of the random vector $\boldsymbol{\xi}$. The term $\mathbb{E}_{\boldsymbol{\xi}}\left[h_2\left(x,\boldsymbol{\xi}\right)\right]$ in (1) is referred to as the recourse function. Let us denote with $\boldsymbol{\xi}_{\mathcal{G}}$ and $\boldsymbol{\xi}_{\mathcal{R}}$ the *guessed* and *right* probability distributions of the random process $\boldsymbol{\xi}$, respectively.
Let $x^*_{\mathcal{G}}$ be the solution obtained by solving problem (1) using the probability distribution $\boldsymbol{\xi}_{\mathcal{G}}$, and

$$RP_{\mathcal{G}} := \mathbb{E}_{\boldsymbol{\xi}_{\mathcal{G}}} z(x^*_{\mathcal{G}}, \boldsymbol{\xi}_{\mathcal{G}}), \tag{3}$$

be the optimal value of the associated objective function. We have the same for the solution $x^*_{\mathcal{R}}$ using the probability distribution $\boldsymbol{\xi}_{\mathcal{R}}$. Let us denote with

$$RP_{\mathcal{R}} := \mathbb{E}_{\boldsymbol{\xi}_{\mathcal{R}}} z(x^*_{\mathcal{R}}, \boldsymbol{\xi}_{\mathcal{R}}), \tag{4}$$

its optimal objective function value. Let the *Expectation of the Guessed Distribution* ($E\mathcal{G}D$)

$$E\mathcal{G}D := \mathbb{E}_{\boldsymbol{\xi}_{\mathcal{R}}} z(x^*_{\mathcal{G}}, \boldsymbol{\xi}_{\mathcal{R}}), \tag{5}$$

be the objective function value of the solution $x^*_{\mathcal{G}}$ computed on the basis of the right distribution $\boldsymbol{\xi}_{\mathcal{R}}$.

Definition 1 The Value of the Right Distribution (VRD) is

$$VRD := E\mathcal{G}D - RP_{\mathcal{R}}. \tag{6}$$

Note that VRD is always non–negative since $E\mathcal{G}D \geq RP_{\mathcal{R}}$ being $E\mathcal{G}D$ based on an out-of-sample solution $x^*_{\mathcal{G}}$. When the solution $x^*_{\mathcal{G}}$ is optimal under the right probability distribution $\boldsymbol{\xi}_{\mathcal{R}}$ we have $VRD = 0$. The greater is VRD, the greater is the objective function value increase of using the guessed distribution with respect to the right one in problem (1).

Let us now define the Performance Bound (PB) of $E\mathcal{G}D$ with respect to $RP_{\mathcal{R}}$ as follows:

Definition 2 The Performance Bound (PB) of using the guessed distribution $\boldsymbol{\xi}_{\mathcal{G}}$ with respect to the right distribution $\boldsymbol{\xi}_{\mathcal{R}}$ in problem (1) is:

$$PB(\boldsymbol{\xi}_{\mathcal{G}}, \boldsymbol{\xi}_{\mathcal{R}}) := \frac{E\mathcal{G}D}{RP_{\mathcal{R}}}. \tag{7}$$

Note that this bound is always not lower than 1. It is equal to 1 when using the guessed distribution we have the same objective function value than using the right distribution in problem (1), corresponding to the case $VRD = 0$. The greater is PB, the greater is the objective function value increase of using the guessed distribution with respect to using the right one.

Finally, let us define the Worst-case Performance Bound of using the guessed distribution with respect to using the right one in problem (1). Let I be a given instance of the problem, i.e. a particular value for each deterministic parameter of the problem. Notice that the instance I does not include the values of stochastic parameters $\boldsymbol{\xi}$ and their probability distributions p. Then, we select a guessed probability distribution $\boldsymbol{\xi}_{\mathcal{G}}$, a right probability distribution $\boldsymbol{\xi}_{\mathcal{R}}$ and the corresponding parameters (i.e., support, mean, ...). We denote with $E\mathcal{G}D^I := \mathbb{E}_{\boldsymbol{\xi}_{\mathcal{R}}} z^I(x^*_{\mathcal{G}}, \boldsymbol{\xi}_{\mathcal{R}})$ and $RP^I_{\mathcal{R}} := \mathbb{E}_{\boldsymbol{\xi}_{\mathcal{R}}} z^I(x^*_{\mathcal{R}}, \boldsymbol{\xi}_{\mathcal{R}})$ the objective function values in problem (1) of the solutions $x^*_{\mathcal{G}}$ and $x^*_{\mathcal{R}}$ obtained by solving problem (1) in instance I, using the guessed and the right distributions, respectively. Note that an instance is defined by the data parameters of the problem, but not by the guessed and right probability distributions. Therefore, the Worst-case Performance Bound can be computed for any pair of probability distributions and for any value of the corresponding parameters. For the sake of simplicity, in the following, when we refer to a probability distribution, we mean the probability distribution together with a given value of the corresponding parameters.

Definition 3 The Worst-case Performance Bound (WPB) of using the guessed distribution $\boldsymbol{\xi}_{\mathcal{G}}$ with respect to the right distribution $\boldsymbol{\xi}_{\mathcal{R}}$ in problem (1) is:

$$WPB := \inf \left\{ \gamma \geq 1 \,|\, \frac{E\mathcal{G}D^I}{RP^I_{\mathcal{R}}} \leq \gamma \quad \forall I \right\}. \tag{8}$$

We say that γ is *tight* if, for any $\gamma' < \gamma$ there exists an instance I' such that $\frac{E\mathcal{G}D^{I'}}{RP^{I'}_{\mathcal{R}}} > \gamma'$ (see [32]). For the sake of simplicity, in the following, we will omit the reference to the instance I.

3 A cost-based variant of the Newsvendor Problem

In this section we introduce a cost–based variant of the *Newsvendor Problem*, a classical problem in SP (see [1]). This problem is simply enough to allow us to show in details the application of the previous definitions.

The problem can be described as follows: a supplier replenishes a retailer facing the stochastic demand $\boldsymbol{\xi} \in [0, b] \subset \mathbb{R}^+$ of a single item. The retailer purchases the item at a given unit procurement cost c and tackles unit holding cost h for positive inventory level and unit stock-out cost $v > c$ for negative inventory level, after demand realization. Let $\alpha := \frac{v-c}{v+h}$ be the retailer's cost ratio. The delivery is assumed to be instantaneous (lead–time equal to zero). Backlogging is not allowed. The sequence of the operations is the following: the order quantity is computed before demand realization, these units are shipped and received by the retailer and, at last, the demand that occurs is satisfied. The aim is to determine an order quantity that minimizes the expected total cost, given by the sum of the procurement cost, the holding cost and the stock–out cost.

Notice that the focus of the decision–maker is not on maximizing the profit as in the classical Newsvendor problem (see [1]), but on minimizing the cost of the service while ensuring a certain service level. If the service level is measured in terms of stock-out cost, an optimal solution can be obtained by minimizing the sum of the cost of the service and of the stock–out cost. This variant has several practical applications: for example, the Newsvendor can represent an intermediate node in a supply chain or an user providing a new service.

The problem can be formulated as a two–stage stochastic linear program with recourse. Let us introduce the first–stage and the second–stage decision variables:

- $x \geq 0$: first–stage decision variable corresponding to the *order quantity*. We denote with x^* the optimal order quantity. This decision must be taken before the realization of the stochastic demand $\boldsymbol{\xi}$;
- $y(\boldsymbol{\xi})$: second–stage decision variable representing the inventory level after the realized demand is satisfied; if $y(\boldsymbol{\xi})$ is positive (i.e. $y(\boldsymbol{\xi}) = [y(\boldsymbol{\xi})]_+$), then an inventory cost $h[y(\boldsymbol{\xi})]_+$ will be paid for the amount left in the warehouse. If $y(\boldsymbol{\xi})$ is negative (i.e. $y(\boldsymbol{\xi}) = -[y(\boldsymbol{\xi})]_-$), then a stock-out cost $v[y(\boldsymbol{\xi})]_-$ will be paid and no stock will be stored.

The cost-based variant of the Newsvendor problem can be then formulated as the following two–stage stochastic linear program:

$$\begin{aligned} \min_x \mathbb{E}_{\boldsymbol{\xi}} z(x, \boldsymbol{\xi}) := \min_x \quad & c\,x + \mathbb{E}_{\boldsymbol{\xi}}[h[y(\boldsymbol{\xi})]_+ + v[y(\boldsymbol{\xi})]_-] \\ & x - \boldsymbol{\xi} = y(\boldsymbol{\xi}) = [y(\boldsymbol{\xi})]_+ - [y(\boldsymbol{\xi})]_- \\ & [y(\boldsymbol{\xi})]_+ \geq 0 \\ & [y(\boldsymbol{\xi})]_- \geq 0 \\ & x \geq 0. \end{aligned} \tag{9}$$

Notice that the objective function of model (9), because of the linearity of the Expectation operator $\mathbb{E}_{\boldsymbol{\xi}}[\cdot]$, can equivalently be rewritten as:

$$\min_x \mathbb{E}_{\boldsymbol{\xi}} \left[c\,x + h[y(\boldsymbol{\xi})]_+ + v[y(\boldsymbol{\xi})]_- \right]. \tag{10}$$

From the first constraint in problem (9), we have $x = [y(\boldsymbol{\xi})]_+ - [y(\boldsymbol{\xi})]_- + \boldsymbol{\xi}$, and by replacing it in (10), we have that problem (9) is equivalent to:

$$\begin{aligned} \min_x \ & c\,\mathbb{E}_{\boldsymbol{\xi}}(\boldsymbol{\xi}) + (c+h)\mathbb{E}_{\boldsymbol{\xi}}\left[[y(\boldsymbol{\xi})]_+\right] + (v-c)\mathbb{E}_{\boldsymbol{\xi}}\left[[y(\boldsymbol{\xi})]_-\right] \\ & x - \boldsymbol{\xi} = [y(\boldsymbol{\xi})]_+ - [y(\boldsymbol{\xi})]_- \\ & [y(\boldsymbol{\xi})]_+ \geq 0 \\ & [y(\boldsymbol{\xi})]_- \geq 0 \\ & x \geq 0. \end{aligned} \tag{11}$$

Let $F_{\boldsymbol{\xi}}$ be the cumulative probability distribution (CDF) of the continuous random variable $\boldsymbol{\xi} \in [0, b]$. Then, model (11) can be analytically solved as stated in the following proposition.

Proposition 1 *If $F_{\boldsymbol{\xi}}$ is invertible, an optimal order quantity is*

$$x^* = F_{\boldsymbol{\xi}}^{-1}(\alpha),$$

where $\alpha = \frac{v-c}{v+h}$ is the retailer's cost ratio.

We omit the proof, as it follows the lines of the one in [1] (pages 15-17).

4 The Performance Bound and the Worst-case Performance Bound for the cost-based variant of the Newsvendor Problem: Exponential vs. Uniform

In this section, we compute the Performance Bound (PB) and the Worst-case Performance Bound (WPB) for different demand distributions for the problem described in Section 3. For simplicity and the sake of brevity, we will show how to compute these bounds in the case the guessed distribution is a truncated Exponential and the right distribution is an Uniform, defined on the same support, even if the bounds can be computed analytically in similar way for different pairs of guessed and right probability distributions.

Let assume to know only the support $[0, b]$ of the retailer's demand distribution and that the guessed retailer's demand distribution $\boldsymbol{\xi}_{\mathcal{G}} \in [0, b]$ follows the truncated Exponential distribution $\mathcal{E}(\lambda)$, while the right distribution $\boldsymbol{\xi}_{\mathcal{R}} \in [0, b]$ is the Uniform $\mathcal{U}$. Recall that $\alpha = \frac{v-c}{v+h}$ is the retailer's cost ratio.

We first compute x^* in closed–form for the two probability distributions, i.e. $x^*_{\mathcal{R}\equiv\mathcal{U}}$ and $x^*_{\mathcal{G}\equiv\mathcal{E}(\lambda)}$, respectively.

Consider the truncated Exponential distribution $\mathcal{E}(\lambda)$, having the following probability density function (PFD), $f_{\mathcal{E}(\lambda)}(\boldsymbol{\xi}; \lambda, 0, b)$ and the cumulative distribution function (CDF), $F_{\mathcal{E}(\lambda)}(\boldsymbol{\xi}; \lambda, 0, b)$, in the interval $[0, b]$, with parameter $\lambda > 0$:

$$f_{\mathcal{E}(\lambda)}(\boldsymbol{\xi}; \lambda, 0, b) = \begin{cases} \lambda e^{-\lambda \boldsymbol{\xi}}(1 - e^{-\lambda b})^{-1} & \text{if } 0 \leq \boldsymbol{\xi} \leq b, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$F_{\mathcal{E}(\lambda)}(\boldsymbol{\xi}; \lambda, 0, b) = \begin{cases} (1 - e^{-\lambda \boldsymbol{\xi}})(1 - e^{-\lambda b})^{-1} & \text{if } 0 \leq \boldsymbol{\xi} \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

Its expected value is equal to $\frac{1}{\lambda}-\frac{b}{e^{\lambda b}-1}$. Note that $\lim_{b\to\infty}(\frac{1}{\lambda}-\frac{b}{e^{\lambda b}-1})=\frac{1}{\lambda}$, as in the exponential distribution. Since in our setting, we assume to know only the support $[0,b]$ of the retailer's demand, we choose $\lambda=(\frac{b}{2})^{-1}$ such that the expected value of the guessed distribution $\mathbb{E}(\boldsymbol{\xi}_{\mathcal{G}}\equiv\mathcal{E}(\frac{2}{b}))=\frac{1}{\lambda}=\frac{b}{2}$. In this way also the means of the guessed and right distributions are equal, i.e. $\mathbb{E}(\boldsymbol{\xi}_{\mathcal{G}}\equiv\mathcal{E}(\frac{2}{b}))=\mathbb{E}(\boldsymbol{\xi}_{\mathcal{R}}\equiv\mathcal{U})$. We perform the worst-case analysis in this case. Other choices of λ can be considered.

Proposition 2 *If the retailer's demand is described by a truncated exponential distribution $\mathcal{E}(\lambda)$, in the interval $[0,b]$, then $x^*_{\mathcal{E}(\lambda)}=-\frac{1}{\lambda}\ln(1-\alpha(1-e^{-\lambda b}))$.*

Proof Since $F_{\mathcal{E}(\lambda)}(\boldsymbol{\xi};\lambda,0,b):\mathbb{R}\to[0,1]$ is continuous from the right and strictly increasing, then it is invertible. By setting $F_{\mathcal{E}(\lambda)}(x;\lambda,0,b)=\alpha$ and solving this equation in x, we have the thesis: $x^*_{\mathcal{E}(\lambda)}=-\frac{1}{\lambda}\ln(1-\alpha(1-e^{-\lambda b}))$.

□

Consider now the uniform distribution $\mathcal{U}$, in the interval $[0,b]$, having the following probability density function PDF $f_{\mathcal{U}}$ and the cumulative distribution function CDF $F_{\mathcal{U}}$:

$$f_{\mathcal{U}}(\boldsymbol{\xi};0,b)=\begin{cases}\frac{1}{b} & \text{if } 0\le\boldsymbol{\xi}\le b,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$F_{\mathcal{U}}(\boldsymbol{\xi};0,b)=\begin{cases}\frac{\boldsymbol{\xi}}{b} & \text{if } 0\le\boldsymbol{\xi}\le b,\\ 0 & \text{otherwise.}\end{cases}$$

Proposition 3 *If the retailer's demand is described by the uniform distribution $\mathcal{U}$, in the interval $[0,b]$, then $x^*_{\mathcal{U}}=\alpha b$.*

Proof Since $F_{\mathcal{U}}(\boldsymbol{\xi};0,b):\mathbb{R}\to[0,1]$ is continuous from the right and strictly increasing in the interval $[0,1]$, then it is invertible. By setting $F_{\mathcal{U}}(x;0,b)=\alpha$ and solving this equation in x, we have the thesis: $x^*_{\mathcal{U}}=\alpha b$.

□

We now provide the Performance Bound $PB(\boldsymbol{\xi}_{\mathcal{G}},\boldsymbol{\xi}_{\mathcal{R}})$ and the Worst-case Performance Bound (WPB) when the retailer's guessed demand distribution $\boldsymbol{\xi}_{\mathcal{G}}$ follows a truncated Exponential $\mathcal{E}((\frac{b}{2})^{-1})$ and the right distribution $\boldsymbol{\xi}_{\mathcal{R}}$ follows an Uniform $\mathcal{U}$, both in the interval $[0,b]$. When $\lambda=(\frac{b}{2})^{-1}$, then in Proposition 2, $\ln(1-\alpha(1-e^{-\lambda b}))=\ln(1-\alpha(1-e^{-2}))$. For simplicity of notation let us denote $\delta(\alpha):=\ln(1-\alpha(1-e^{-2}))$.

Proposition 4 *If the guessed distribution $\boldsymbol{\xi}_{\mathcal{G}}$ for the retailer's demand follows a truncated Exponential $\mathcal{E}((\frac{b}{2})^{-1})$ and the right distribution $\boldsymbol{\xi}_{\mathcal{R}}$ follows an Uniform $\mathcal{U}$, both in the interval $[0,b]$, then*

$$PB(\boldsymbol{\xi}_{\mathcal{E}((\frac{b}{2})^{-1})},\boldsymbol{\xi}_{\mathcal{U}})=\frac{\mathbb{E}_{\boldsymbol{\xi}_{\mathcal{U}}}z(x^*_{\mathcal{E}((\frac{b}{2})^{-1})},\boldsymbol{\xi}_{\mathcal{U}})}{\mathbb{E}_{\boldsymbol{\xi}_{\mathcal{U}}}z(x^*_{\mathcal{U}},\boldsymbol{\xi}_{\mathcal{U}})}=\frac{c+\frac{1}{4}(v+h)\delta(\alpha)^2+(v-c)(1+\delta(\alpha))}{c+\alpha(c+h)}.$$

Proof We have:

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\xi}_{\mathcal{U}}}z(x,\boldsymbol{\xi}_{\mathcal{U}})&=c\,\mathbb{E}_{\boldsymbol{\xi}}(\boldsymbol{\xi})+(c+h)\int_0^x(x-\boldsymbol{\xi})dF_{\boldsymbol{\xi}}+(v-c)\int_x^b(\boldsymbol{\xi}-x)dF_{\boldsymbol{\xi}}=\\&=c\frac{b}{2}+(c+h)\int_0^x\frac{(x-\boldsymbol{\xi})}{b}d\boldsymbol{\xi}+(v-c)\int_x^b\frac{(\boldsymbol{\xi}-x)}{b}d\boldsymbol{\xi}\\&=c\frac{b}{2}+(c+h)\frac{x^2}{2b}+(v-c)\frac{(b-x)^2}{2b}.\end{aligned}$$

Consider first the Exponential distribution $\mathcal{E}(\lambda)$. Recall that, thanks to Proposition 2, $x^*_{\mathcal{E}(\lambda)} = -\frac{1}{\lambda}\ln(1-\alpha(1-e^{-\lambda b}))$. If $\lambda = (\frac{b}{2})^{-1}$, then $x^*_{\mathcal{E}((\frac{b}{2})^{-1})} = -\frac{b}{2}\ln(1-\alpha(1-e^{-2}))$ and since $\delta(\alpha) = \ln(1-\alpha(1-e^{-2}))$, then, $x^*_{\mathcal{E}(\lambda)} = -\frac{b}{2}\delta(\alpha)$ and

$$\mathbb{E}_{\boldsymbol{\xi}_{\mathcal{U}}} z(x^*_{\mathcal{E}((\frac{b}{2})^{-1})}, \boldsymbol{\xi}_{\mathcal{U}}) = \frac{b}{2}[c + \frac{1}{4}(v+h)\delta(\alpha)^2 + (v-c)(1+\delta(\alpha))].$$

Consider now the Uniform distribution $\mathcal{U}$ in the interval $[0,b]$. Recall that, thanks to Proposition 3, $x^*_{\mathcal{U}} = \alpha b$. Therefore, we have:

$$\mathbb{E}_{\boldsymbol{\xi}_{\mathcal{U}}} z(x^*_{\mathcal{U}}, \boldsymbol{\xi}_{\mathcal{U}}) = \frac{b}{2}[c + (c+h)(\alpha)^2 + (v-c)(1-\alpha)^2] = \frac{b}{2}[c + \alpha(c+h)].$$

Therefore,

$$PB(\boldsymbol{\xi}_{\mathcal{E}((\frac{b}{2})^{-1})}, \boldsymbol{\xi}_{\mathcal{U}}) = \frac{\mathbb{E}_{\boldsymbol{\xi}_{\mathcal{U}}} z(x^*_{\mathcal{E}((\frac{b}{2})^{-1})}, \boldsymbol{\xi}_{\mathcal{U}})}{\mathbb{E}_{\boldsymbol{\xi}_{\mathcal{U}}} z(x^*_{\mathcal{U}}, \boldsymbol{\xi}_{\mathcal{U}})} = \frac{c + \frac{1}{4}(v+h)\delta(\alpha)^2 + (v-c)(1+\delta(\alpha))}{c + \alpha(c+h)}. \tag{12}$$

□

Proposition 5 *In the cost-based variant of the Newsvendor problem, the Exponential distribution $\mathcal{E}((\frac{b}{2})^{-1})$ has a worst-case performance bound γ with respect to the Uniform distribution $\mathcal{U}$, both in the interval $[0,b]$, not greater than* 1.2798 *and the bound is tight.*

Proof We first compute an upper bound on $PB(\boldsymbol{\xi}_{\mathcal{E}((\frac{b}{2})^{-1})}, \boldsymbol{\xi}_{\mathcal{U}})$ for any instance of the problem:

$$\begin{aligned} PB(\boldsymbol{\xi}_{\mathcal{E}((\frac{b}{2})^{-1})}, \boldsymbol{\xi}_{\mathcal{U}}) &= \frac{c + \frac{1}{4}(v+h)\delta(\alpha)^2 + (v-c)(1+\delta(\alpha))}{c+\alpha(c+h)} \\ &\leq \max\left\{\frac{c}{c}, \frac{\frac{1}{4}(v+h)\delta(\alpha)^2 + (v-c)(1+\delta(\alpha))}{\alpha(c+h)}\right\} \\ &\leq \frac{1}{4}\frac{\delta(\alpha)^2}{\alpha(1-\alpha)} + \frac{1+\delta(\alpha)}{1-\alpha} := \beta(\alpha), \end{aligned} \tag{13}$$

as $\alpha = \frac{v-c}{v+h}$ and $1-\alpha = \frac{c+h}{v+h}$ imply $v+h = \frac{c+h}{1-\alpha}$ and $v-c = \alpha(v+h)$.

Figure 1 shows the plot of the upper bound $\beta(\alpha)$ on $PB(\boldsymbol{\xi}_{\mathcal{E}((\frac{b}{2})^{-1})}, \boldsymbol{\xi}_{\mathcal{U}})$ for $0 < \alpha < 1$.

To compute the worst-case performance bound γ, we optimally solve the following non-linear optimization problem:

$$\max_{\alpha\in(0,1)} \beta(\alpha) = \frac{1}{4}\frac{\delta(\alpha)^2}{\alpha(1-\alpha)} + \frac{1+\delta(\alpha)}{1-\alpha}. \tag{14}$$

Solving the following non–linear equation:

$$\beta'(\alpha) = \frac{-4\alpha^2 + \left(2\left(e^2-1\right)\alpha^2 - 3e^2\alpha + \alpha + e^2\right)\ln^2\left(\left(\frac{1}{e^2}-1\right)\alpha+1\right) + 2\left(2\left(e^2-1\right)\alpha^2 - 3e^2\alpha + \alpha + e^2 - 1\right)\alpha\ln\left(\left(\frac{1}{e^2}-1\right)\alpha+1\right)}{4(\alpha-1)^2\alpha^2\left(\left(e^2-1\right)\alpha - e^2\right)} = 0,$$

where the left-hand side is the first derivative $\beta'(\alpha)$ of the objective function of problem (14) with respect to the decision variable α, we get the unique stationary

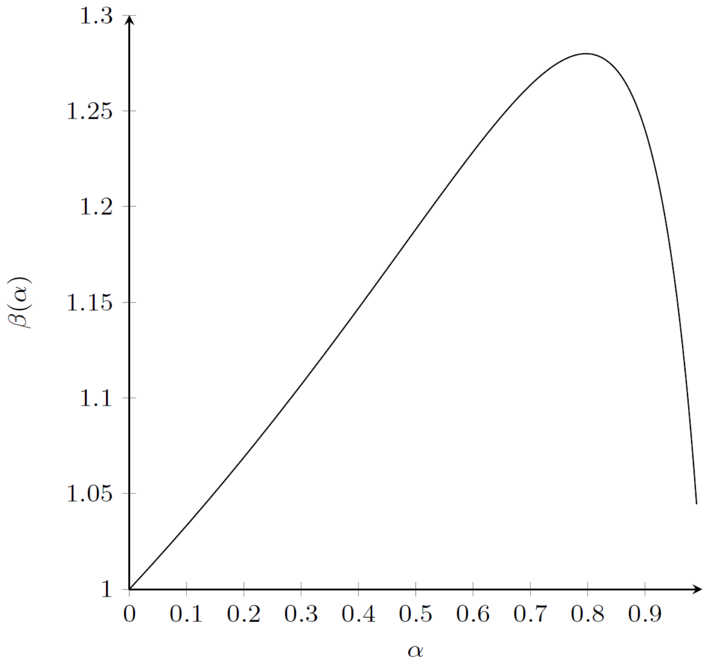

Fig. 1 Upper bound $\beta(\alpha)$ on $PB(\boldsymbol{\xi}_{\mathcal{E}((\frac{b}{2})^{-1})}, \boldsymbol{\xi}_{\mathcal{U}})$ for $0 < \alpha < 1$.

point $\alpha^* = 0.7969$, which is a global maximum since $\beta'(\alpha) > 0$ for $\alpha < 0.7969$ and $\beta'(\alpha) < 0$ for $\alpha > 0.7969$ (see Figure 1). Replacing this value in the objective function, we have $\beta(0.7969)$ =1.2798. Therefore, γ is not greater than 1.2798.

In order to prove that the bound is tight we provide the following instance $I^{'}$ such that $PB^{I^{'}}(\boldsymbol{\xi}_{\mathcal{G}}, \boldsymbol{\xi}_{\mathcal{R}})$=1.2798. The instance $I^{'}$ is obtained setting $c = 0$, while the other data parameters can assume any value. Fixing $c = 0$ in (12) we have that:

$$PB(\boldsymbol{\xi}_{\mathcal{E}((\frac{b}{2})^{-1})}, \boldsymbol{\xi}_{\mathcal{U}}) = \frac{1}{4}\frac{\delta(\alpha)^2}{\alpha(1-\alpha)} + \frac{1+\delta(\alpha)}{1-\alpha}, \tag{15}$$

which corresponds with $\beta(\alpha)$, and the thesis follows by the first part of the proof. □

5 Numerical Results

In this section, we propose numerical results on the Value of the Right Distribution and the Performance Bounds, based on the cost-based variant of the Newsvendor problem presented in Section 3.

Since the demand distribution is unknown, we select some of the most common distributions to carry out our computational experiment: Uniform, Exponential, Normal, Triangular. Moreover, we test several values of the parameters of the Exponential and of the Normal distributions. This is to investigate the case when the estimate of the type of the distribution is correct, but its parameters are wrong. We set the problem parameters as follows:

- procurement cost $c \in \{1, 100, 1000\}$;
- holding cost $h = 1$;
- stock–out cost $v \in \{3/2c, 5c, 10c\}$. Note that, given these values, the retailer cost ratio α is in the intervals $(0, 1/3]$, $(1/3, 2/3]$ and $(2/3, 1]$ for all values of $c \in \{1, 10, 100\}$;
- probability distributions for the integer demand $\xi = 0, 1, \ldots, 200$ truncated over the support [0,200]:
  - Uniform $\mathcal{U}[0, 200]$, having mean 100;
  - Exponential $\mathcal{E}(1/100)$ truncated in the interval $[0, 200]$, having mean 68.04;
  - Exponential $\mathcal{E}(1/10^6)$ truncated in the interval $[0, 200]$, having mean 99.99;
  - Normal $\mathcal{N}(100, 10)$ truncated in the interval $[0, 200]$, having mean 99.50;
  - Normal $\mathcal{N}(100, 50)$ truncated in the interval $[0, 200]$, having mean 99.61;
  - Normal $\mathcal{N}(100, 100)$ truncated in the interval $[0, 200]$, having mean 99.85;
  - Triangular $\mathcal{T}(0, 100, 200)$ in the interval $[0, 200]$ with mode 100, having mean 100.

The probability corresponding to each value of the integer demand $\xi = 0, 1, \ldots, 200$ is numerically computed as $F(\xi + 1) - F(\xi)$, where F is the cumulative distribution function (CDF). The mean of the truncated probability distribution over the support $[0, 200]$ with integer values $\xi = 0, 1, \ldots, 200$ is computed as $\frac{\sum_{\xi=0}^{200} \xi \cdot f(\xi)}{F(200) - F(0)}$, where $f(\xi)$ denotes the probability mass functions (PMF) (see [33] for an introduction to truncated discrete probability distributions). Figure 2 shows the corresponding probability mass functions.

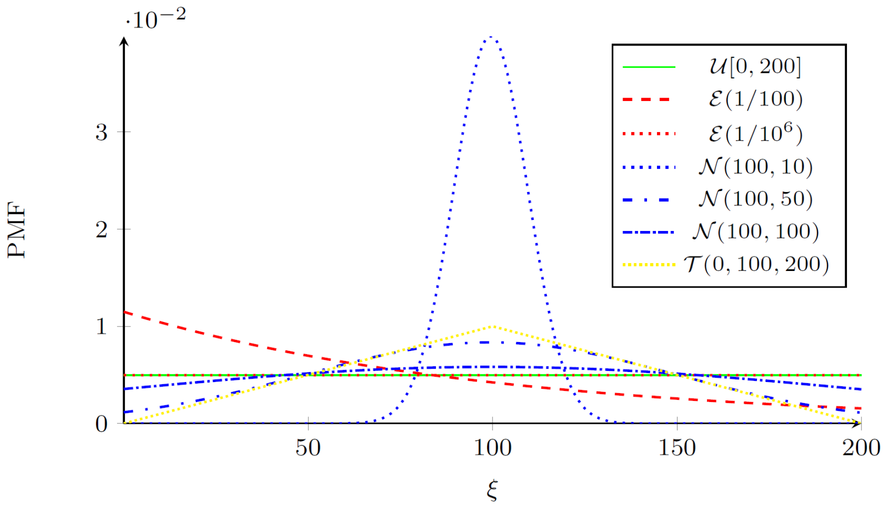

Fig. 2 Probability mass functions for different probability distributions in the interval [0,200].

Results are reported in the following tables. Table 1 shows the optimal order quantities x^*. Columns 4–10 give, for each probability distribution, the corresponding optimal order quantity for different values of procurement cost c and stock-out cost v. Results show that, in general, the optimal order quantity is different with different probability distributions. However, when $\lambda = 1/10^6$, the model based on the Exponential distribution always provides the same optimal solution than the model based on the Uniform distribution. This is due to the shape of the Exponential distribution, that tends to be very similar to the Uniform distribution for such a small value of λ (see Figure 2). Moreover, we can note that, as expected, the optimal order quantity increases with the stock-out cost v and, on average, it is greater than the average value of the support $b/2 = 100$. Only for $v = 1.5c$ it is smaller than it. Finally, we can note that the optimal order quantity is significantly affected by the value of λ in the Exponential distribution and by the standard deviation in the Normal distribution.

c	v	α	$\mathcal{U}$ [0,200]	$\mathcal{E}$ (1/100)	$\mathcal{E}$ $(1/10^6)$	$\mathcal{N}$ (100,10)	$\mathcal{N}$ (100,50)	$\mathcal{N}$ (100,100)	$\mathcal{T}$ (0,100,200)
1	$3/2c$	0.20	40	19	40	91	60	46	63
1	$5c$	0.67	133	86	133	104	120	129	118
1	$10c$	0.81	164	123	164	109	142	158	140
10	$3/2c$	0.31	62	31	62	95	76	67	79
10	$5c$	0.78	157	113	157	107	137	151	134
10	$10c$	0.89	179	147	179	112	157	173	153
100	$3/2c$	0.33	66	33	66	95	79	70	81
100	$5c$	0.80	160	117	160	108	139	154	137
100	$10c$	0.90	180	150	180	112	159	175	155
Average:			126.78	91	126.78	103.67	118.78	124.78	117.78

Table 1 Optimal order quantities x^* for different values of procurement cost c, stock-out cost v and probability distributions.

We now show the results on the Value of the Right Distribution VRD and the corresponding Performance Bound $PB(\boldsymbol{\xi}_{\mathcal{G}}, \boldsymbol{\xi}_{\mathcal{R}})$ for different pairs of guessed and right probability distributions. For the sake of readability, the following tables provide the percentage objective function value increase, computed as $\%PB := (PB-1) \cdot 100$, instead of PB. For the simplicity of notation, we omit the dependence by the probability distributions $(\boldsymbol{\xi}_{\mathcal{G}}, \boldsymbol{\xi}_{\mathcal{R}})$.

Tables 2–3 show the Value of the Right Distribution VRD and the corresponding Percentage Performance Bound $\%PB$, for each of the guessed probability distributions, when the right distribution is $\mathcal{U}[0, 200]$. These tables are organized as follows: the first column provides the procurement cost c, the second the stock-out cost v and the third the corresponding value of α. Columns 4–9 in Table 2 show the VRD, while the same columns in Table 3 give the $\%PB$, for each of the guessed probability distributions. Remember that VRD and $\%PB$ are equal to 0 when the guessed probability distribution performs optimally. When the right distribution is $\mathcal{U}[0, 200]$, this happens for $\mathcal{E}(1/10^6)$, while for the other distributions the VRD and the $\%PB$ are always positive, increase with the procurement cost c and tend to increase with the stock-out cost v. Besides, when the guessed distribution follows the Exponential distribution (see columns 3 and 4 of Table 3), we note that the Worst-case Performance Bound WPB given in Proposition 5 is verified, since for both the cases ($\lambda = 1/100$ and $\lambda = 1/10^6$), the $\%PB$ is always lower than 28%. When the guessed distribution follows the Normal distribution, VRD and $\%PB$ are significantly affected by c and v and by the standard deviation. In particular, VRD and $\%PB$ are very different for different cost combinations when the standard deviation is 10: in fact, in this case $\%PB$ ranges from 2.36% to 60.59%. Moreover, they tend to decrease when the standard deviation increases: the average value of $\%PB$ is 22.94% when the standard deviation is 10, 2.95% when it is 50 and just 0.23% when it is 100. This is due to the shape of the corresponding probability distributions (see Figure 2), that is very different than the Uniform distribution when the standard deviation is 10, while it becomes more and more similar when the standard deviation increases. Finally, the Triangular distribution has a good performance, as the average $\%PB$ is 3.91% and the maximum is 8.30%.

			Guessed distribution						
c	v	α	$\mathcal{E}$ (1/100)	$\mathcal{E}$ ($1/10^6$)	$\mathcal{N}$ (100,10)	$\mathcal{N}$ (100,50)	$\mathcal{N}$ (100,100)	$\mathcal{T}$ (0,100,200)
1	$3/2c$	0.20	22.66	0.00	16.37	2.56	0.25	3.38
1	$5c$	0.67	33.67	0.00	12.99	2.72	0.30	3.58
1	$10c$	0.81	45.90	0.00	82.64	13.19	0.97	15.70
10	$3/2c$	0.31	39.02	0.00	42.52	7.45	0.87	11.08
10	$5c$	0.78	247.25	0.00	319.03	51.49	4.79	67.97
10	$10c$	0.89	250.99	0.00	1114.67	117.28	7.87	164.73
100	$3/2c$	0.33	410.45	0.00	314.67	62.93	5.84	83.88
100	$5c$	0.80	2302.10	0.00	3367.19	548.51	44.55	658.07
100	$10c$	0.90	2273.81	0.00	11588.25	1121.04	67.71	1583.58
Average:			622.87	0.00	1873.15	214.13	14.79	288.00

Table 2 Value of the Right Distribution (VRD) when $\mathcal{U}[0, 200]$ is the Right distribution

			Guessed distribution					
c	v	α	$\mathcal{E}$ (1/100)	$\mathcal{E}$ ($1/10^6$)	$\mathcal{N}$ (100,10)	$\mathcal{N}$ (100,50)	$\mathcal{N}$ (100,100)	$\mathcal{T}$ (0,100,200)
1	$3/2c$	0.20	1.90	0.00	11.67	1.83	0.18	2.41
1	$5c$	0.67	14.39	0.00	5.55	1.16	0.13	1.53
1	$10c$	0.81	17.36	0.00	31.25	4.99	0.37	5.94
10	$3/2c$	0.31	2.90	0.00	3.16	0.55	0.06	0.82
10	$5c$	0.78	13.24	0.00	17.09	2.76	0.26	3.64
10	$10c$	0.89	12.64	0.00	56.15	5.91	0.40	8.30
100	$3/2c$	0.33	3.07	0.00	2.36	0.47	0.04	0.63
100	$5c$	0.80	12.72	0.00	18.60	3.03	0.25	3.63
100	$10c$	0.90	11.89	0.00	60.59	5.86	0.35	8.28
Average:			10.01	0.00	22.94	2.95	0.23	3.91

Table 3 Percentage Performance Bounds ($\%PB$) when $\mathcal{U}[0, 200]$ is the Right distribution

In Tables 4–9, we compute $\%PB$ for the remaining pairs (right vs. guessed) of probability distributions. These tables are organized as Table 3.

The computational results in Table 4 show that, when $\mathcal{E}(1/100)$ is the Right distribution, the best guessed probability distribution is $\mathcal{N}(100, 50)$, followed by $\mathcal{T}(0, 100, 200)$. In the former case the average $\%PB$ is 6.65%, while in the latter it is 6.85%.

The computational results in Table 5 show that, when $\mathcal{E}(1/10^6)$ is the Right distribution, $\mathcal{U}[0, 200]$ is an optimal guessed probability distribution, as $\%PB$ is equal to 0 in all instances. Moreover, $\mathcal{N}(100, 100)$ is very good, having an average $\%PB$ equal to 0.23%.

The computational results in Table 6 show that, when $\mathcal{N}(100, 10)$ is the Right distribution, all guessed probability distributions give a very large $\%PB$ and that the best guessed probability distribution is $\mathcal{T}(0, 100, 200)$, followed by $\mathcal{E}(1/100)$. The former has $\%PB$ equal to 20%, while the latter 22.40%. This is expected due

			Guessed distribution					
c	v	α	$\mathcal{U}$ [0,200]	$\mathcal{E}$ $(1/10^6)$	$\mathcal{N}$ (100,10)	$\mathcal{N}$ (100,50)	$\mathcal{N}$ (100,100)	$\mathcal{T}$ (0,100,200)
1	$3/2c$	0.20	5.22	5.22	50.83	18.28	8.38	20.83
1	$5c$	0.67	14.39	14.39	2.37	7.89	12.22	7.04
1	$10c$	0.81	11.24	11.24	1.51	2.62	8.36	2.11
10	$3/2c$	0.31	6.12	6.12	23.60	12.35	8.12	13.93
10	$5c$	0.78	9.91	9.91	0.23	3.12	7.52	2.40
10	$10c$	0.89	6.68	6.68	10.24	0.68	4.48	0.24
100	$3/2c$	0.33	6.30	6.30	20.52	11.82	7.84	12.80
100	$5c$	0.80	9.31	9.31	0.49	2.59	7.02	2.16
100	$10c$	0.90	5.88	5.88	12.27	0.54	4.13	0.16
Average:			8.34	8.34	13.56	6.65	7.56	6.85

Table 4 Percentage Performance Bounds ($\%PB$) when $\mathcal{E}(1/100)$ is the Right distribution

			Guessed distribution					
c	v	α	$\mathcal{U}$ [0,200]	$\mathcal{E}$ (1/100)	$\mathcal{N}$ (100,10)	$\mathcal{N}$ (100,50)	$\mathcal{N}$ (100,100)	$\mathcal{T}$ (0,100,200)
1	$3/2c$	0.20	0.00	1.90	11.68	1.83	0.18	2.41
1	$5c$	0.67	0.00	14.39	5.55	1.16	0.13	1.53
1	$10c$	0.81	0.00	17.35	31.24	4.99	0.37	5.94
10	$3/2c$	0.31	0.00	2.90	3.16	0.55	0.06	0.82
10	$5c$	0.78	0.00	13.24	17.08	2.76	0.26	3.64
10	$10c$	0.89	0.00	12.64	56.15	5.91	0.40	8.30
100	$3/2c$	0.33	0.00	3.07	2.36	0.47	0.04	0.63
100	$5c$	0.80	0.00	12.71	18.60	3.03	0.25	3.63
100	$10c$	0.90	0.00	11.89	60.58	5.86	0.35	8.28
Average:			0.00	10.01	22.93	2.95	0.23	3.91

Table 5 Percentage Performance Bounds ($\%PB$) when $\mathcal{E}(1/10^6)$ is the Right distribution

			Guessed distribution					
c	v	α	$\mathcal{U}$ [0,200]	$\mathcal{E}$ (1/100)	$\mathcal{E}$ $(1/10^6)$	$\mathcal{N}$ (100,50)	$\mathcal{N}$ (100,100)	$\mathcal{T}$ (0,100,200)
1	$3/2c$	0.20	21.37	31.23	21.37	11.98	18.55	10.57
1	$5c$	0.67	37.26	28.55	37.26	16.18	30.68	13.14
1	$10c$	0.81	77.74	14.22	77.74	43.51	68.40	40.40
10	$3/2c$	0.31	12.44	27.18	12.44	5.83	10.07	4.47
10	$5c$	0.78	42.22	1.75	42.22	23.00	36.46	20.12
10	$10c$	0.89	57.96	28.22	57.96	37.51	52.38	33.80
100	$3/2c$	0.33	10.74	26.46	10.74	4.66	8.84	3.78
100	$5c$	0.80	41.39	3.87	41.39	22.72	36.06	20.94
100	$10c$	0.90	54.30	28.44	54.30	36.20	49.99	32.75
Average:			39.49	21.10	39.49	22.40	34.60	20.00

Table 6 Percentage Performance Bounds ($\%PB$) when $\mathcal{N}(100, 10)$ is the Right distribution

to the different shape of the PMF of the Normal distribution $\mathcal{N}(100, 10)$ compared to the other ones (see Figure 2).

The computational results in Table 7 show that, when $\mathcal{N}(100, 50)$ is the Right distribution, $\mathcal{T}(0, 100, 200)$ is the best possible guessed probability distribution, having an average $\%PB$ equal to 0.08%.

The computational results in Table 8 show that, when $\mathcal{N}(100, 100)$ is the Right distribution, $\mathcal{U}[0, 200]$ and $\mathcal{E}(1/10^6)$ are the best possible guessed probability distributions having an average $\%PB$ equal to 0.23%.

Finally, the computational results in Table 9 show that, when $\mathcal{T}(0, 100, 200)$ is the Right distribution, $\mathcal{N}(100, 50)$ is the best possible guessed probability distribution, having an average $\%PB$ equal to 0.08%.

			Guessed distribution					
c	v	α	$\mathcal{U}$ [0,200]	$\mathcal{E}$ (1/100)	$\mathcal{E}$ (1/10^6)	$\mathcal{N}$ (100,10)	$\mathcal{N}$ (100,100)	$\mathcal{T}$ (0,100,200)
1	$3/2c$	0.20	2.05	7.58	2.05	6.48	1.03	0.06
1	$5c$	0.67	1.86	14.27	1.86	3.12	0.90	0.05
1	$10c$	0.81	5.67	5.79	5.67	18.18	3.08	0.08
10	$3/2c$	0.31	0.93	8.11	0.93	1.75	0.40	0.03
10	$5c$	0.78	3.54	6.08	3.54	9.72	1.81	0.08
10	$10c$	0.89	4.96	1.25	4.96	32.37	2.77	0.18
100	$3/2c$	0.33	0.73	8.12	0.73	1.27	0.35	0.02
100	$5c$	0.80	3.66	5.08	3.66	10.41	1.93	0.04
100	$10c$	0.90	4.48	0.96	4.48	35.15	2.73	0.17
Average:			3.10	6.36	3.10	13.16	1.67	0.08

Table 7 Percentage Performance Bounds ($\%PB$) when $\mathcal{N}(100, 50)$ is the Right distribution

			Guessed distribution					
c	v	α	$\mathcal{U}$ [0,200]	$\mathcal{E}$ (1/100)	$\mathcal{E}$ (1/10^6)	$\mathcal{N}$ (100,10)	$\mathcal{N}$ (100,50)	$\mathcal{T}$ (0,100,200)
1	$3/2c$	0.20	0.15	3.12	0.15	9.96	0.94	1.39
1	$5c$	0.67	0.13	14.08	0.13	4.70	0.59	0.89
1	$10c$	0.81	0.43	13.37	0.43	26.88	2.63	3.36
10	$3/2c$	0.31	0.08	4.14	0.08	2.68	0.27	0.48
10	$5c$	0.78	0.27	10.80	0.27	14.60	1.39	2.08
10	$10c$	0.89	0.39	8.32	0.39	48.51	3.10	4.87
100	$3/2c$	0.33	0.06	4.29	0.06	1.98	0.24	0.36
100	$5c$	0.80	0.29	10.10	0.29	15.84	1.55	2.02
100	$10c$	0.90	0.30	7.67	0.30	52.46	3.07	4.85
Average:			0.23	8.43	0.23	19.73	1.53	2.26

Table 8 Percentage Performance Bounds ($\%PB$) when $\mathcal{N}(100, 100)$ is the Right distribution

			Guessed distribution					
c	v	α	$\mathcal{U}$ [0,200]	$\mathcal{E}$ (1/100)	$\mathcal{E}$ (1/10^6)	$\mathcal{N}$ (100,10)	$\mathcal{N}$ (100,50)	$\mathcal{N}$ (100,100)
1	$3/2c$	0.20	2.91	9.21	2.91	5.41	0.06	1.66
1	$5c$	0.67	2.59	14.89	2.59	2.79	0.03	1.39
1	$10c$	0.81	7.72	4.60	7.72	16.64	0.08	4.55
10	$3/2c$	0.31	1.38	9.42	1.38	1.39	0.05	0.71
10	$5c$	0.78	4.71	5.21	4.71	8.79	0.07	2.64
10	$10c$	0.89	7.22	0.56	7.22	29.66	0.17	4.46
100	$3/2c$	0.33	1.11	9.38	1.11	0.98	0.03	0.62
100	$5c$	0.80	4.86	4.21	4.86	9.38	0.06	2.79
100	$10c$	0.90	6.67	0.35	6.67	32.24	0.20	4.46
Average:			4.35	6.43	4.35	11.92	0.08	2.59

Table 9 Percentage Performance Bounds ($\%PB$) when $\mathcal{T}(0, 100, 200)$ is the Right distribution

In conclusion, we can state that the selection of the guessed distribution is critical to find near-optimal order quantities. Since the right distribution is revealed a posteriori only, the previous analysis allows us also to conclude that, on average, the two best guessed probability distributions are $\mathcal{N}(100, 50)$ and $\mathcal{T}(0, 100, 200)$, as they provide an average $\%PB$, with respect to all right probability distributions, equal to 5.22% and 5.29% respectively, while $\%PB$ is 6.7% for $\mathcal{N}(100, 100)$, 7.93% for $\mathcal{U}[0, 200]$ and $\mathcal{E}(1/10^6)$, 8.91% for $\mathcal{E}(1/100)$ and 14.89% for $\mathcal{N}(100, 10)$.

6 Conclusions

In this paper we introduced the *Value of the Right Distribution* (VRD) in Stochastic Programming, to quantify how much a decision-maker loses if she guesses a wrong distribution. Besides VRD, the Performance Bound (PB) of the guessed distribution with respect to the right one, and the Worst-case Performance Bound (WPB) have been introduced allowing us to investigate the increase of the objective function value when two or more probability distributions are mismatched, and when the selected distribution is correct, but its parameters are wrong. Then, we introduced a cost–based variant of the well known *Newsvendor Problem* for which optimal solutions can be obtained in closed-form for different probability distributions having the same support. This variant has several practical applications: the newsvendor can represent an intermediate node in a supply chain or an user providing a new service interested in minimizing its cost while ensuring a certain service level. For this problem we computed PB and WPB to evaluate the maximum increase in the objective function value that can be obtained guessing a probability distribution different than the right one. We carried out a computational study that shows, in a systematic way in a given set of instances, how the objective function value varies when the guessed and right probability distributions vary. Besides, the results allow us to identify the best guessed probability distribution.

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