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A Multistage Risk-averse Stochastic Programming Model for Personal Savings Accrual: the Evidence from Lithuania

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Abstract In this paper we consider the problem of choosing the optimal pension fund in the second pillar of Lithuanian pension system by providing some guidelines to individuals with defined contribution pension plans. A multistage risk-averse stochastic optimization model is proposed that can be used to plan a long-term pension accrual under two different cases: minimum and maximum accumulation plans as possible options in the system. The investment strategy of personal savings is based on the optimal solutions over possible scenario realizations generated for a particular participant. The concept of the risk-averse decision-maker is implemented by choosing the Conditional Value at Risk (CVaR) as the risk measure defined by a nested formulation that guarantees the time consistency in the multistage model. The paper focuses on three important decision-making moments corresponding to the duration of periods to be modelled. The first period is a short-term accumulation, while the second period is a long-term accumulation with possibly high deviation of objective function value. The third period is designed to implement the concept of Target Date Fund in the second pillar pension scheme as the subsequent need to protect against potential losses at risky pension funds. The experimental findings of this research provide insights for individuals as decision-makers to select pension funds, as well as for policy-makers by revealing the vulnerability of pension system.

Keywords Pension system modeling · Multistage stochastic integer programming · Alpha-stable distribution · Time consistency · CVaR · Target date funds

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1 Introduction

The pension system in most European countries has changed significantly over the last two decades, notably caused by financial problems related to an aging population and declining fertility rate. Countries have set up mandatory or quasi-mandatory pension plans, either public or private, based on so-called defined contribution (DC) plans, as a means to diversify retirement income sources across providers or for financing existing pension forms to maintain fiscal sustainability. Reforms vary across countries. The final scheme is determined by the structure of the current pension system itself (OECD, 2013).

Lithuania, like all post-Soviet countries, has inherited a state social security system based on a Pay-As-You-Go (PAYG) pension scheme. In 2003, Lithuania turned to an Anglo-Saxon model by adding two pillars to the current system, comprising of funded schemes and supplementary pension provision. The PAYG pension scheme was replaced by a multi-pillar pension system with three pillars (tiers) (Bitinas and Maccioni, 2014). Contrary to III^{rd} pension pillar, II^{nd} pillar has provoked particular interest since participation has been progressing notwithstanding the fact that employees can choose whether to join. The II^{nd} pillar was made up of individual DC transfers (Pension Funds Online, 2016). In 2013, new pension system regulations were established. Three options were given for participants: remain at the current participation level, increase participation and thereby accrue contribution rate with governmental subsidy, or stop further participation by choosing a PAYG pension scheme or voluntarily building up pension savings in III^{rd} pillar pension funds. Gudaitis and Maccioni (2014) reveal that participation in the II^{nd} pillar brings more savings to a person than cancellation, even if a fund's real interest rate of return is close to zero. The effect was studied by estimating the nominal annual cash flows due to contributions and pensions under three options for each participant, depending on financial and demographic variables. In a similar study that set out to measure the benefits of participation in the II^{nd} pillar, Gudaitis and Medaiskis (2013); Medaiskis and Gudaitis (2013) compared the accumulated pension capital savings in private funds with reduced values of a I^{st} pillar pension, based on a PAYG pension scheme. The comparative study indicates that the accumulated pension capital in all pension fund' groups, classified by investment in equities, exceeds the present value of the lost part of social insurance pensions for men, but not for women, which is explained by women's longer life expectancy. The most significant gain was observed for participants in pension funds that invest a small part in equities. In view of all this, one can see that in the case of Lithuania, relatively few studies provide guidelines to individuals with DC pension plans on how to manage personal savings before retirement. Other researchers draw our attention to the assessment of financial performance of pension funds by collating their historical behavior over a certain time period. The central bank of the Republic of Lithuania (Bank of Lithuania, 2015, 2014, 2013), as a supervisor of financial institutions, annually releases an analysis of the activities of all pension funds for the accumulation of a portion of the state social insurance contribution, supplementary voluntary accumulation of pensions, and collective investment undertakings operating in Lithuania. Those reviews reveal important facts about funds' activity during the previous year. The managers of pension funds (currently, 21 pension funds managed by six companies are operating in Lithuania), unlike the Bank of Lithuania, publish ex-post evaluation reports throughout the life cycle of funds. The managers also give recommendations regarding the selection of a certain pension fund they own, mainly based on the age of a person, which is not a systematic choice of a pension fund by the person concerned.

Two important themes emerge from the studies of the multi-pillar pension system by Lithuanian researchers: (a) the performance analysis of pension funds, and (b) the sustainability of a pension system itself. In the first case, several authors (Jurevičiene and Samoškaite, 2012; Gavrilova, 2011; Šutiene et al, 2014) have considered the comparative analysis of funds by introducing risk measures for returns; some analysts (Novickyte et al, 2016) have attempted to evaluate the pension funds using multi-criteria methods combining a few funds' actions as defining indicators of the whole; other authors (Rabikauskaite and Novickyte, 2015; Armonaite, 2012) assessed the real value of pension funds eroded by inflation and inferred that over 11 years, the funds have generated positive returns and managed to outrun inflation on average but not separately, which poses concerns about the preservation of future personal savings influenced by differences in fund management. The other group of studies (Bitinas and Maccioni, 2014; Bitinas, 2011; Bitinas and Maccioni, 2013; Medaiskis and Jankauskiene, 2014) mainly investigate the impact of both population aging and low fertility rates that raise social expenses on the sustainability of pension system. Considering all the evidence, it seems that demographic risk produces a long-term challenge to ensuring the financial sustainability of pension schemes and providing an adequate income in retirement, as well as the welfare of society. Several recent studies investigating the sustainability of the pension system have been carried out by drawing up similar features with certain indicators by which different national pension systems are compared between three Baltic States (Rajevska, 2015; Volskis, 2012), Central and Eastern Europe countries (Mladen, 2012; Aidukaite, 2011) or emerging economies (Velculescu, 2011). Collectively, these studies outline the approach that a pension system's objectives, such as adequacy of retirement benefits, financial sustainability and affordability, diversification of retirement income sources across providers (public and private) and financing forms (PAYG and funded), need to be managed continuously (OECD, 2013).

While several recent studies have considered the importance of pension scheme management and its evaluation for a particular country (Gokçen and Yalçin, 2015; Thomas et al, 2014; Jackowicz and Kowalewski, 2012; Laun and Wallenius, 2015; Dupačová and Polívka, 2009), there is still very little scientific evidence about optimal savings management for individuals with DC pension plans. Rational management of pension funds helps ensure the prosperity on which well-being and enhanced trust of pension scheme ultimately depend, but equally important is the individual choice of particular pension funds with acceptable level of risk. Up until now, studies (Skučiene, 2011; Rabikauskaite and Novickyte, 2015; Chybalski, 2015) have confirmed that pension system participants do not focus on funds' investment performance while choosing them, but are exposed to the influence of management companies' advertising campaigns or make decisions according to friends' and colleagues' opinions rather than by assessing their indicators. Likewise, the relevance is corroborated by statistical data published by the Bank of Lithuania (Bank of Lithuania, 2015, 2014, 2013): near 65 percent of the working-age population accumulated their pensions in private funds; 78 percent of participants accumulated their pensions in a fund managed by one of the three pension accumulation companies, i.e., most participants chose the funds managed by "Swedbank investiciju valdymas" (38.85%) and "SEB investiciju valdymas" (22.06%); then followed "Aviva Lietuva" (17.46%). As of 31 December 2015, compared to the end of 2014, the Herfindahl-Hirschman index increased by 4.23 points and was at 2323.3 points. The value of the index above 1,800 usually indicates too high market concentration. On average per annum, 3.48 percent of all those who invest in II^{nd} pillar funds decided to change their pension accumulation company or pension fund, indicating that individuals select the particular fund like most members and are not willing to change their choice over time. The role of private pension funds is now growing in importance since Lithuania currently is exposed to the risk of population aging, declining birth rates and growing emigration. Research needs in providing some guidelines to individuals with DC pension plans on how to manage pension savings are linked to the fact that the performance analysis of an individual fund has indicated that not all funds exceed the inflation rate, which raises concern about the accumulation of participants' personal savings and retirement benefits that will be affected by differences in fund management (Rabikauskaite and Novickyte, 2015).

In this paper, we make a contribution in the modelling field concerning optimal investment for a pension saver who wishes to maximize the expected utility of retirement benefits. For relevant studies in this field, see the references (Konicz et al, 2015; Konicz and Mulvey, 2015; Homem-de Mello and Pagnoncelli, 2016; Liutvinavičius and Sakalauskas, 2011; Consiglio et al, 2007). Because of the advantages of stochastic programming methodology (Consigli and Dempster, 1998), we present a multistage risk-averse stochastic optimization model that can be used to plan a long-term pension accrual. The long-term optimal strategy is based on the optimal solution over possible scenarios realizations generated for a particular participant. By determining the probability distribution for pension funds' returns, which is assumed to follow mixed α-stable distribution in this paper, the scenarios are generated, while the accumulated value of pension saving is influenced by inflation rate modelled by implementing a two-regimes switching autoregressive model. The optimization model focuses on three important moments: the selection of pension fund at the beginning of career, after one year, and after forty years. The first period is short; the second and third periods are long-term accumulations with possibly high value deviations. The investment plan associated with an optimal trade-off between the Net Present Value of profits and the risk of a negative impact due to the realization of bad scenarios is determined and expressed as the convex combination of expected profits and the Conditional Value at Risk $CVaR$ (Rockafellar and Uryasev, 2000, 2002). When considering risk in multi-stage stochastic optimization, the recent literature shows that time consistency is a basic requirement to get suitable optimal decisions (Shapiro, 2012; Ruszczynski and Shapiro, 2006; Philpott and de Matos, 2012; Rudloff et al, 2014; Homem-de Mello and Pagnoncelli, 2016). Rudloff, Street, and Valladão (2014) found that a policy is said to be time consistent if and only if planned decisions are actually going to be implemented. To ensure the time consistency in a multi-stage framework, the $CVaR$ risk measure has to be suitably defined. This requirement will imply a high impact on the complexity of the problem, but it will guarantee a reliable solution.

The fact that Lithuanian pension system's participants are very passive and not willing to change the choice they made while entering pension system affirms that a certain risk-based approach should be implemented, which would result in less loss of value in pension savings prior to retirement. Target Date Fund (TDF), also known as life-cycle, dynamic-risk or age-based fund, is designed to reduce exposure to stocks making asset allocation more conservative as the target date approaches. The studies argue TDF's superiority because it ensures fiscal stability (Blake et al, 2014), the behavioural aspects of participants who do not review their investment allocations regularly (Cobbe, 2012), and the risk profile modelled or controlled by introducing glide-paths (Hammond, 2015; Yoon, 2010). The multi-pillar pension system is newly established in Lithuania and requires ongoing research. In the paper, the different duration before retirement is explored to evaluate the effect of TDF, since the targeting date and threshold are not yet fully established.

The remaining parts of the paper proceed as follows. Section 2 introduces risk-neutral and risk-averse multistage stochastic programming models. Section 3 presents $\alpha-$stable and mixed $\alpha-$stable distributions, while Section 4 describes the data used and outlines the scenario generation procedure. Section 5 presents the numerical results: subsection 5.1 discusses their robustness, subsection 5.2 explains first stage optimal solution and transition between selected funds in the second stage, subsection 5.3 presents the particular case study when the selection of conservative pension funds is forbidden at the first and second stages, and subsections 5.4 and 5.5 investigate the effect of target date duration. Section 6 concludes the paper.

2 Stochastic programming models

The model focuses on three important decision-making moments: selection of pension fund at the beginning of a citizen's career (first stage decision variable), re-evaluation of the decision after one year (second stage decision variable), and then after forty years (third stage decision variable). The first period is shorter and of low risk, while the second and third periods are long-term accumulations with possibly high deviation values. The objective of the model is maximization of the revenues according to the best fund choices in the three stages in life.

To proceed with numerical computations, it is useful to have a discretization of the uncertain parameters. This is obtained by considering a finite number of realizations of the random process over the number of periods to be considered.
The information structure can be described in the form of a scenario tree $\mathscr{T}$ where at each stage t, there is a discrete number of nodes $N^{(t)}$ where a specific realization of the stochastic parameters takes place. There are H levels (stages) in the tree that correspond to specific time periods. The final $N^{(H)}$ nodes are called the leaves. Let $\mathscr{N}$ be the set of ordered nodes of the tree and $\mathscr{N}^{(t)}$ be the set of ordered nodes of the tree at stage $t = 1, \ldots, H$. Each node at stage t, except the root, is connected to a unique node at stage $t-1$ called ancestor and to nodes at stage $t+1$ called successors. For each node n at stage t, we denote its ancestor with $a(n)$ and with $\pi_{a(n),n}$ the conditional probability of the random process at node n given its history up to the ancestor node $a(n)$. Let $\mathscr{B}(n)$ the set of successors (children nodes) of node n. A scenario is a path through nodes from the root node to a leaf node. We indicate with π_s the probability of scenario s passing through nodes $n^1, n^2, \ldots, n^H$ (where n^t, $t = 1, \ldots, H$ is the generic node at stage t) defined as $\pi_s := \pi_{n^1,n^2} \cdot \pi_{n^2,n^3} \cdot \ldots \cdot \pi_{n^{H-1},n^H}$. We also indicate with p_n the probability of node n (at stage t): if node n at stage t is reachable through node n^1 at stage 1, node n^2 at stage 2, ..., node n^{t-1} at stage $t-1$, then $p_n := \pi_{n^1,n^2} \cdot \pi_{n^2,n^3} \cdot \ldots \cdot \pi_{n^{t-1},n^t}$. Moreover, $\sum_{n \in \mathscr{N}^{(t)}} p_n = 1$.

2.1 A risk-neutral multistage stochastic programming model

In this section we consider a risk-neutral multistage stochastic optimization model (Birge and Louveaux, 2011; Maggioni and Wallace, 2012; Maggioni et al, 2014, 2016; Maggioni and Pflug, 2016), that can be used to plan a long-term pension accrual.

Let assume the following notation.
Sets:

$$
\begin{aligned}
\mathscr{I} = \{i : i = 1, \ldots, I\} &\text{ , set of available funds at stages 1 and 2;} \\
\mathscr{K} = \{i : i = 1, \ldots, K\} \subset \mathscr{I} &\text{ , set of available funds at stage 3;} \\
\mathscr{S} = \{s : s = 1, \ldots, S\} &\text{ , set of scenarios;} \\
\mathscr{N} &\text{ , set of nodes of the scenario tree;} \\
\mathscr{N}^{(t)} &\text{ , set of nodes of the scenario tree at stage } t = 1, \ldots, H; \\
\mathscr{B}(n) &\text{ , set of children nodes of node } n.
\end{aligned}
$$

Parameters:

γ , fixed parameter dependent on accrual plan (see, Section 4.1);
$f_{n,i}^{(t)}$, returns of fund i in node n at stage t;
$r_n^{(t)}$, inflation rate in node n at stage t;
$c_i^{(t)}$, costs of fund i (transactions, yearly fees) in stage t;
$e_i^{(t)}$, extra cost paid at stage t if the fund is changed with respect to stage $t-1$;
p_n , probability of node $n \in \mathscr{N}$;
$\pi_{a(n),n}$, conditional probability of the random process at node n, given its history up to the ancestor node $a(n)$;
$g_n^{(t)}$, increment of salary at node n in stage t;
$S_n^{(t)}$, salary at node n in stage t;
$q^{(t)}$, number of years in the period t between stage t and $t+1$;
$A_{n,i}^{(t)}$, expected accumulated sum during the previous period at stage t of fund i in node n.

Decision variables:

$x_i^{(1)} \in \{0;1\}$, first-stage decision variable: $x_i^{(1)} = 1$ if fund i is selected at the first stage, 0 otherwise;
$x_{n,i}^{(t)} \in \{0;1\}$, decision variable at stage $t = 2,3$: $x_{n,i}^{(t)} = 1$ if fund i is selected at node n at stage t, 0 otherwise;
$y_{n,i}^{(t)} \in \{0;1\}$, decision variable at stage $t = 2,3$: $y_{n,i}^{(t)} = 1$ if fund i selected at node n at stage t is different from the one selected at stage $t-1$, 0 otherwise.

Notice that the expected accumulated sum $A_{n,i}^{(t)}$ for each fund $i = 1, \ldots, I$ during period $t-1$ is the solution of the following system of difference equations:

$$z(j) = (1 + f_{n,i}^{(t)} - r_n^{(t)}) \cdot z(j-1) + S_n^{(t)}(j) \cdot \gamma \tag{1}$$

$$S_n^{(t)}(j) = (1 + g_n^{(t)}) \cdot S_n^{(t)}(j-1), \tag{2}$$

where $z(j)$ denotes the accumulated fund at any moment $j = 1, \ldots, m^{(t)}$ of period t and $S_n^{(t)}(j)$ denotes the salary at moment j. Initial accumulated fund is equal to 0, since no money is transferred to the fund at the moment of entering pension fund system. Initial accumulated fund in stage $t = 2, \ldots, 4$ depends on the decision made in stage $t-1$.

The multistage stochastic risk neutral formulation of the problem considered is:

$$\max F = \max - \sum_{i=1}^{I} c_i^{(1)} x_i^{(1)} + \gamma S^{(1)} +$$
$$\sum_{t=2}^{3} \sum_{n \in \mathscr{N}^{(t)}} p_n \left[\sum_{i=1}^{I} \left(A_{n,i}^{(t)} x_{a(n),i}^{(t-1)} - c_i^{(t)} x_{n,i}^{(t)} - e_i^{(t)} y_{n,i}^{(t)} \right) \right] +$$
$$\sum_{n \in \mathscr{N}^{(4)}} p_n \left[\sum_{i=1}^{K} \left(A_{n,i}^{(4)} x_{a(n),i}^{(3)} \right) \right] \tag{3}$$

s.t.

$$\sum_{i=1}^{I} x_i^{(1)} = 1, \tag{4}$$
$$\sum_{i=1}^{I} x_{n,i}^{(2)} = 1, \qquad n \in \mathscr{N}^{(2)}, \tag{5}$$
$$\sum_{i=1}^{K} x_{n,i}^{(3)} = 1, \qquad n \in \mathscr{N}^{(3)}, \tag{6}$$
$$y_{n,i}^{(t)} + x_{a(n),i}^{(t-1)} \leq 1, \, i = 1, \ldots, I, \; t = 2, 3, \; n \in \mathscr{N}^{(t)}, \tag{7}$$
$$y_{n,i}^{(t)} - x_{n,i}^{(t)} \leq 0, \, i = 1, \ldots, I, \; t = 2, 3, \; n \in \mathscr{N}^{(t)}, \tag{8}$$
$$-x_{a(n),i}^{(t-1)} + x_{n,i}^{(t)} - y_{n,i}^{(t)} \leq 0, \, i = 1, \ldots, I, \; t = 2, 3, \; n \in \mathscr{N}^{(t)}, \tag{9}$$
$$x_i^{(1)} \in \{0; 1\}, \qquad i = 1, \ldots, I, \tag{10}$$
$$x_{n,i}^{(2)}, y_{n,i}^{(2)} \in \{0; 1\}, \qquad i = 1, \ldots, I, \; n \in \mathscr{N}^{(2)}, \tag{11}$$
$$x_{n,i}^{(3)}, y_{n,i}^{(3)} \in \{0; 1\}, \qquad i = 1, \ldots, K, \; n \in \mathscr{N}^{(3)}. \tag{12}$$

The objective function describes the maximization of the profits over the three periods given by the difference between the revenues and the transaction costs and taking into account the extra fee paid for changing the fund during stages. Constraints (4), (5) and (6) require the selection of only one fund in all three stages, while constraints (7)–(9) express that the extra-cost $e_i^{(t)}$ is paid only if the fund i selected at stage t is different from the one selected at stage $t-1$ given by the conditions:

$$y_{ni}^{(t)} = 1 \Longrightarrow x_{ni}^{(t-1)} = 0, \; x_{ni}^{(t)} = 1, \; t = 2, 3, \tag{13}$$
$$y_{ni}^{(t)} = 0 \Longrightarrow x_{ni}^{(t-1)} = 0, \; x_{ni}^{(t)} = 0, \; t = 2, 3, \tag{14}$$
$$y_{ni}^{(t)} = 0 \Longrightarrow x_{ni}^{(t-1)} = 1, \; x_{ni}^{(t)} = 1, \; t = 2, 3, \tag{15}$$
$$y_{ni}^{(t)} = 0 \Longrightarrow x_{ni}^{(t-1)} = 1, \; x_{ni}^{(t)} = 0, \; t = 2, 3, \tag{16}$$
$$x_{ni}^{(t-1)} = 0, \; x_{ni}^{(t)} = 1 \Longrightarrow y_{ni}^{(t)} = 1, \; t = 2, 3, \tag{17}$$
$$x_{ni}^{(t-1)} = 1, \; x_{ni}^{(t)} = 0 \Longrightarrow y_{ni}^{(t)} = 0, \; t = 2, 3, \tag{18}$$
$$x_{ni}^{(t-1)} = 1, \; x_{ni}^{(t)} = 1 \Longrightarrow y_{ni}^{(t)} = 0, \; t = 2, 3, \tag{19}$$
$$x_{ni}^{(t-1)} = 0, \; x_{ni}^{(t)} = 0 \Longrightarrow y_{ni}^{(t)} = 0, \; t = 2, 3. \tag{20}$$

Conditions (13)–(16) are achieved by constraints (7)–(8) while conditions (17)–(20) are reached by constraint (9). Notice that constraints (7)–(9) at stage $t = 3$ are supposed to be enforced only for the $K \leq I$ funds available. Finally, the constraints (10)–(12) define the decision variables of the problem.

2.2 A risk-averse multistage stochastic optimization model

The risk-neutral model defined in Section 2.1 maximizes the expected total profit along the planning horizon determining the optimal decision variables. However, it does not take into account the possibility of a very low profit realization in some scenarios. To take this into account, in this section, we consider a risk-averse strategy that includes a risk measure, namely the Conditional Value at Risk, $CVaR$, which has to be suitably defined to ensure time consistency. Consider the optimal decisions of the risk-averse multistage stochastic problem on the path from the root to a node n, and consider the problem from the subtree from node n. If the optimal value of the subtree problem coincides with that computed on the complete problem, the solution is said to be time consistent; otherwise, it is time inconsistent (see e.g. Philpott and de Matos (2012); Rudloff, Street, and Valladão (2014)). This property is satisfied by the nested multistage $CVaR$ as shown in Ruszczyński (2010). Following Philpott and de Matos (2012) and Rudloff, Street, and Valladão (2014), we include a nested multistage $CVaR$ measure, that iteratively solves a convex combination of performance and risk at the last stage, using it as a performance measure from the previous stage. We introduce now:

$d_n^{(t)}$, auxiliary variable in node n at stage t;
$\eta_{a(n)}^{(t)}$, auxiliary variable at stage t deriving from the ancestor node $a(n)$;
F_n , profit in node n;
$F^{(t)}$, profit at stage t;
$\alpha^{(t)}$, confidence level in the tradeoff between performance and risk at stage t;
$\rho^{(t)}$, weighting factor in the tradeoff between reward and risk at stage t.

At the fourth stage we define:

$$d_m^{(4)} \geq 0,\ d_m^{(4)} \geq \eta_n^{(4)} - F_m,\ n \in \mathscr{N}^{(3)},\ m \in \mathscr{B}(n). \tag{21}$$

Then for each node of the third stage we define:

$$d_m^{(3)} \geq 0,\ d_m^{(3)} \geq \eta_n^{(3)} - K_m,\ n \in \mathscr{N}^{(2)},\ m \in \mathscr{B}(n), \tag{22}$$

where

$$K_n = F_n + (1-\rho^{(4)}) \left(\sum_{m \in \mathscr{B}(n)} \pi_{n,m} F_m \right) + \rho^{(4)} \left(\eta_n^{(4)} - \frac{1}{\alpha^{(4)}} \sum_{m \in \mathscr{B}(n)} \pi_{n,m} d_m^{(4)} \right),\ n \in \mathscr{N}^{(3)}\ .$$

The term $(1-\rho^{(4)}) \left(\sum_{m \in \mathscr{B}(n)} \pi_{n,m} F_m \right) + \rho^{(4)} \left(\eta_n^{(4)} - \frac{1}{\alpha^{(4)}} \sum_{m \in \mathscr{B}(n)} \pi_{n,m} d_m^{(4)} \right)$ represents the objective function at the fourth stage conditioning to reaching node $n \in \mathscr{N}^{(3)}$ at the third stage.

Then for each node of second stage we define

$$d_n^{(2)} \geq 0,\ d_n^{(2)} \geq \eta_{a(n)}^{(2)} - L_n,\ n \in \mathscr{N}^{(2)}, \tag{23}$$

where

$$L_n = F_n + (1-\rho^{(3)}) \left(\sum_{m \in \mathscr{B}(n)} \pi_{n,m} K_m \right) + \rho^{(3)} \left(\eta_n^{(3)} - \frac{1}{\alpha^{(3)}} \sum_{m \in \mathscr{B}(n)} \pi_{n,m} d_m^{(3)} \right), \ n \in \mathscr{N}^{(2)} .$$

The term $(1-\rho^{(3)}) \left(\sum_{m \in \mathscr{B}(n)} \pi_{n,m} K_m \right) + \rho^{(3)} \left(\eta_n^{(3)} - \frac{1}{\alpha^{(3)}} \sum_{m \in \mathscr{B}(n)} \pi_{n,m} d_m^{(3)} \right)$, defines the objective function at the third stage conditioned to reaching node $n \in \mathscr{N}^{(2)}$ at the second stage. Finally the objective function at the first stage is given by

$$F^{(1)} + (1-\rho^{(2)}) \left(\sum_{n \in \mathscr{N}^{(2)}} \pi_{1,n} L_n \right) + \rho^{(2)} \left(\eta^{(2)} - \frac{1}{\alpha^{(2)}} \sum_{n \in \mathscr{N}^{(2)}} \pi_{1,n} d_n^{(2)} \right) . \tag{24}$$

3 α-stable and mixed α-stable distributions

In this section, we recall the notation of both α-stable distribution and mixed α-stable distribution relevant for the current research. Both distributions fit empirical data more adequately than any other and can be used to model extreme deviations in fund returns.

3.1 α-stable distribution

α-stable distribution belongs to the models for heavy tailed data. It is characterized by four parameters: α – index of stability, σ – scale parameter, β – skewness parameter, μ – location parameter. The parameters are restricted to the range $\alpha \in (0,\ 2]$, $\beta \in [-1,\ 1]$, $\sigma \in (0,\ \infty)$, $\mu \in \Re$. In financial applications, parameter α is usually more than 1; this is essential requirement to guarantee that the theoretical mean or expectation will exist. Shortly, the notation $S_\alpha(\sigma, \beta, \mu)$ is used to denote the class of stable laws. Generally, the characteristic function $\phi_X(t)$ of a random variable X, which is distributed by α-stable law, is

$$\phi_X(t) = \begin{cases} \exp\left\{-\sigma^\alpha |t|^\alpha \left(1 - i\beta \left(\tan \frac{\pi\alpha}{2}\right)(\operatorname{sign} t)\right) + i\mu t\right\}, \ \alpha \neq 1; \\ \exp\left\{-\sigma^\alpha |t|^\alpha \left(1 - i\beta \frac{\pi}{2} (\operatorname{sign} \ln |t|)\right) + i\mu t\right\}, \ \ \alpha = 1. \end{cases} \tag{25}$$

The index of stability α determines the rate at which the tails decay. If $\alpha = 2$, the characteristic function in the given equation reduces to the characteristic function of the normal distribution. If $\alpha \leq 1$, then the expectation of random variable cannot be defined (see Samorodnitsky and Taqqu, 2000).

3.2 Mixed α-stable distribution

Mixed α-stable distribution was applied for modelling the financial data (Kabašinskas et al, 2010, 2012; Šutiene et al, 2014). The additional parameter $p \in [0,1]$ is included to model zero daily returns with a certain probability, i.e.

$$X_{mix} = \begin{cases} 0, & p < u\,; \\ S_\alpha(\sigma,\beta,\mu), & p \geqslant u\,; \end{cases} \tag{26}$$

where u is uniform random variable $u \sim U(0,1)$.

The probability density function of a mixed α-stable distribution is given as

$$f_{mix}(x) = p \cdot \delta(x) + (1-p) \cdot f_\alpha(x) \tag{27}$$

where $f_\alpha(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(t) \cdot e^{-ixt}\mathrm{d}t$ is the probability density function of an α-stable distribution expressed through its characteristic function and $\delta(x)$ is the Dirac delta function.

In this case cumulative distribution function or CDF is

$$F_{mix}(x) = (1-p) \cdot F_\alpha(x, \alpha_0, \beta_0, \mu_0, \sigma_0) + p \cdot \varepsilon(x), \tag{28}$$

where $F_\alpha(x, \alpha_0, \beta_0, \mu_0, \sigma_0)$ is CDF of α-stable distribution, where $\varepsilon(x) = \begin{cases} 0, \; x \leqslant 0 \\ 1, \; x > 0 \end{cases}$, is the CDF of the degenerate distribution, the vector of parameters $\alpha_0, \beta_0, \mu_0, \sigma_0$ is estimated from historical returns with zeros ignored (Kabašinskas et al, 2012).

The estimation of α-stable law parameters is complicated because of the lack of closed-form density function in general. While estimating the parameters of mixed α-stable law, the maximum likelihood method is applied (Kabašinskas et al, 2012). It is time-consuming, but the implementation of parallel algorithms can allow us to get results in an adequate time even for long data series.

In our research we use daily fund returns to estimate the parameters of α-stable distribution. However, in the model proposed we use yearly values of fund returns f_i, $i = 1, \ldots, I$. To solve this problem (Samorodnitsky and Taqqu, 2000) suggests if $X_1, X_2, \ldots, X_n$ are i.i.d. $S_\alpha(\sigma, \beta, \mu)$, then

$$\sum_{i=1}^{n} X_i \stackrel{d}{=} \begin{cases} n^{1/\alpha} X_1 + (\mu - \lambda)(n - n^{1/\alpha}), & \alpha \neq 1, \\ nX_1 + \dfrac{2}{\pi}\sigma\beta n \ln n, & \alpha = 1. \end{cases} \tag{29}$$

It is assumed to set up $n = 252$ equal to number of working days in year, $\lambda = \beta\sigma\tan(\pi\alpha/2)$ is a correction constant of Nolan parametrization.

4 Data analysis and scenario generation

In this section, we describe the data at our disposal, their analysis and how scenarios for the stochastic programming model have been generated.

4.1 Payments to the fund

We start by describing the different options that can be chosen for pension accumulation, denoted in Section 2 as fixed parameter γ. Payments to the fund are managed by the pension law of Lithuania (Seimas of the Republic of Lithuania, 2010). Before entering the system, the participant can choose one of the following options.

– Case A (minimum accumulation plan) is characterized as follows:
  - From 2014 to 2020, payments to the fund are given by the 2% from gross salary (this is paid as social security tax and transferred to the fund by SODRA[1]);
  - Starting in 2020, payments to the fund will be equal to 3.5% from gross salary (this is already paid as social security tax and transferred to the fund by SODRA).
– Case B (maximum accumulation plan) is characterized as follows:

[1] The Lithuanian State Social Insurance Fund Board

1. From 2016 to 2020, payments to the fund are 6% (known as 2% +2% +2% , where 2% is from gross salary already paid as social security tax and transferred to the fund by SODRA, 2% is voluntarily paid by the person, and 2% from average Lithuanian gross salary is covered by the government);
2. Starting in 2020, it will be equal to 7.5% (3.5% +2% +2% where 3.5% is from gross salary already paid as social security tax and transferred to the fund by SODRA, 2% is voluntarily paid by the person, and 2% from average gross Lithuanian salary is covered by the government).

Both cases are analysed. The importance of the comparison is influenced by the recent discussion of the Lithuanian government and Parliament. The discussion arose when fund managers reported that some participants in the pension system, because of very small monthly payments to funds (small salary and small γ), will not accumulate the necessary funds to cover living expenses and inflation.

4.2 Returns of pension funds, fees and salaries

Currently, there are twenty-one IInd pillar pension funds (PF) managed by six companies operating in Lithuania (see Table 2). By investment strategy, PFs are arranged into four categories depending on the percentage of stocks in the investment:

- conservative funds (0% stock), denoted as CF;
- small stock funds (less than 30% of stock), denoted as SF;
- medium stock funds (less than 70% of stock), denoted as MF; and
- stock funds (up to 100% stock), denoted as ST.

According to the review of the Bank of Lithuania, the most popular group is medium stock with 51.65% of participants and market capitalization of 51.86% (see Table 1).

Table 1 Funds market structure (2016 first quarter)

Fund type	Notation	Number of funds available	Number of participants		Value of managed assets	
			citizens	%	€ millions	%
Conservative	CF	6	104,237	8.52	219.75	10.22
Small stock	SF	4	290,822	23.78	555.76	25.84
Medium stock	MF	7	631,818	51.65	1,115.44	51.86
Stock	ST	4	196,258	16.05	259.78	12.08
Total		21	1,223,135		2,150.73	

Table 2 shows a descriptive analysis of the 21 pension funds obtained using daily historical data in the period June 2004–April 2016. The data show that all PFs have positive mean returns; in particular, SEB3 and SWED4 have high mean returns, and they are not accompanied by the highest values of standard deviation, making them attractive for risk-averse decision-makers. The less risky behaviour, measured with standard deviation, was observed only for funds SWED1, SEB1, and DANSKE1. The other PFs belonging to the conservative group CF have experienced rather high variations with low mean returns. PFs investing in stocks, SF, MF and ST have experienced the principle that potential return rises with an increase in risk, except for the INVL2 fund, which yielded a very low return with high standard deviation. The negative skewness is observed for all pension funds except INVL1 and SEB1, indicating a greater chance of extremely negative outcomes and non-symmetrical distribution of returns. All PFs managed by DANSKE are distinguished by the highest values of negative skewness. Notice that fund SWED5 has been recently established and consequently its time series is shorter. Nevertheless its empirical characteristics are consistent with other PFs belonging to the same fund type (ST) and its dynamic historical movement captures enough variability in order to model various possible returns. Finally, the distribution of PFs returns is leptokurtic because of high peak and corresponding fat tails. The latter feature, together with relatively small standard deviation, is particularly seen in the performance of conservative funds. The returns of pension funds were fitted to mixed α-stable

Table 2 Empirical characteristics of fund returns f_i, $i = 1, \ldots, 21$.

i	Fund‘s full title	Fund‘s short title	Type	N	Mean	Standard deviation	Skewness	Kurtosis	Median
1	INVL STABILO II 58+	INVL1	CF	2970	0.000182	0.000650	0.089	27.06	0.000130
2	SEB pensija 1	SEB1	CF	2968	0.000101	0.001337	0.124	11.28	0.0
3	Swedbank Pensija 1	SWED1	CF	2964	0.000072	0.000718	-0.625	11.45	0.0
4	Konservatyvaus valdymo Danske pensija	DANSKE1	CF	3084	0.000104	0.000842	-0.963	10.84	0.0
5	DNB pensija 1	DNB1	CF	2966	0.000135	0.000793	-0.807	19.96	0.000118
6	Europensija	AVIVA1	CF	2768	0.000175	0.001191	-0.537	10.01	0.000192
7	Europensija plius	AVIVA2	SF	2970	0.000196	0.002560	-0.343	3.98	0.000272
8	INVL MEZZO II 53+	INVL2	SF	2970	0.000290	0.002864	-0.717	8.83	0.000400
9	DNB pensija 2	DNB2	SF	2965	0.000168	0.001780	-0.384	2.89	0.000244
10	Swedbank Pensija 2	SWED2	SF	2964	0.000143	0.001965	-0.649	5.37	0.000262
11	SEB pensija 2	SEB2	MF	2968	0.000176	0.004418	-0.912	8.43	0.000393
12	Swedbank Pensija 3	SWED3	MF	2964	0.000170	0.003451	-0.448	4.79	0.000308
13	Swedbank Pensija 4	SWED4	MF	2577	0.000111	0.005963	-0.324	5.09	0.000354
14	INVL MEDIO II 47+	INVL3	MF	3050	0.000178	0.003807	-0.241	9.28	0.0
15	DANSKE pensija 50	DANSKE2	MF	3084	0.000184	0.003100	-1.182	10.03	0.000257
16	DNB pensija 3	DNB3	MF	2966	0.000193	0.003285	-0.508	3.04	0.000317
17	Europensija ekstra	AVIVA2	MF	2552	0.000149	0.004769	-0.245	3.58	0.000369
18	SEB pensija 3	SEB3	ST	2520	0.000113	0.008769	-0.539	7.12	0.000622
19	INVL EXTREMO II 16+	INVL4	ST	3050	0.000186	0.005556	-0.401	8.14	0.0
20	Swedbank Pensija 5	SWED5	ST	1233	0.000221	0.008131	-0.381	3.88	0.000607
21	DANSKE pensija 100	DANSKE3	ST	3084	0.000252	0.005377	-0.908	6.27	0.000306

distribution (see Section 3.2). Table 3 shows that the expectation of INVL3 and INVL4 returns cannot be defined since the fitted value α is less than 1 (see properties of α-stable distribution). Because of their "bad" performance, these funds are removed from the following analysis.

Table 3 Estimates of parameters for mixed alpha-stable distribution and Sharpe ratio for funds $i = 1, \ldots, 21$

i	Fund's short title	Type	$1-p$	α	β	μ	σ	Sharpe ratio*
1	INVL1	CF	0.9396	1.2050	0.2860	0.0001	0.0002	0.280
2	SEB1	CF	0.8633	1.4573	-0.0183	0.0001	0.0006	0.076
3	SWED1	CF	0.5551	1.0379	0.1388	0.0000	0.0002	0.100
4	DANSKE1	CF	0.5095	1.0234	0.1130	0.0001	0.0003	0.124
5	DNB1	CF	0.7579	1.3806	0.1315	0.0001	0.0003	0.170
6	AVIVA1	CF	0.8359	1.4318	0.0146	0.0002	0.0005	0.147
7	AVIVA2	SF	0.9470	1.5948	-0.2102	0.0004	0.0014	0.077
8	INVL2	SF	0.9903	1.5272	-0.1158	0.0004	0.0014	0.101
9	DNB2	SF	0.9017	1.6267	-0.3073	0.0003	0.0010	0.094
10	SWED2	SF	0.9240	1.6207	-0.2351	0.0003	0.0011	0.073
11	SEB2	MF	0.9716	1.5011	-0.2484	0.0006	0.0021	0.040
12	SWED3	MF	0.9550	1.5572	-0.3026	0.0005	0.0018	0.049
13	SWED4	MF	0.9594	1.5468	-0.2718	0.0006	0.0031	0.019
14	INVL3	MF	0.9858	0.6781	0.0841	0.0001	0.0005	0.047
15	DANSKE2	MF	0.6484	1.0152	0.1083	0.0002	0.0013	0.059
16	DNB3	MF	0.9301	1.5989	-0.2957	0.0005	0.0018	0.059
17	AVIVA2	MF	0.9629	1.5896	-0.2507	0.0005	0.0026	0.031
18	SEB3	ST	0.9819	1.5649	-0.2649	0.0008	0.0044	0.013
19	INVL4	ST	0.9948	0.6019	0.0669	0.0001	0.0006	0.033
20	SWED5	ST	0.9703	1.5041	-0.1856	0.0007	0.0042	0.027
21	DANSKE3	ST	0.6650	1.0150	0.1240	0.0002	0.0022	0.047

*to calculate Sharpe ratio we use mean and standard deviation from Table 2 and risk free interest rate equal to 0, according to ECB and Bank of Lithuania

To check the historical dependence of PFs returns, the Kendall rank correlation coefficients were estimated. The estimated values show that only two funds are dependent, while the others experience insignificant correlation. For this reason, the dependence between PFs was not included in the model.

The managers of pension funds cover their operating costs $c_i^{(t)}$ through the fees they charge to the members, as well as a charge $e_i^{(t)}$ for transferring to another provider (see Table 4). Those charges can have a significant effect on the accrual value of savings at retirement, especially annual management charges, which are calculated as a percentage of the fund's value.

Finally, Table 5 reports the main characteristics of Lithuanian historical salary increment g used to generate the possible values of salary scenarios $S_n^{(t)}$ at each node n at stage t.

4.3 Inflation dynamics model

Inflation dynamics are described using a discrete-time Markov-Switching AutoRegressive (MS-AR) model (Goldfeld and Quantd, 2005; Ailliot and Monbet, 2012; Zochowski and Bialowolski, 2011).

For our particular application, MS-AR model has two components: $\{r^{(t)}\} \in \Re$ denotes inflation dynamics in time, $\{R_t\} \in \{1, \ldots, R\}$ represents the regimes corresponding to different states of money market. It is assumed that the regime $\{R_t\}$ is a first order Markov chain which evolution only depends on the value of R_{t-1}. The conditional evolution of $\{r^{(t)}\}$ depends on the values of R_t and $r^{(t-1)} \ldots,\ r^{(t-p)}$. Taken together, the inflation rates $\{r^{(t)}\}$ are modelled by autoregressive process of order $p \geq 0$ with varying coefficients. In general, the model can be formalized by equation

Table 4 Fund management charges c_i and e_i, $i = 1, \ldots, 21$ (2016 first quarter)

i	Fund's short title	Type	Annual management charge, %		Charge for transferring to another PF once per calendar year, %	Charge for transferring to another PF more than once per calendar year, %	
			from assets	from contributions	to PF owned by another PF manager	to PF owned by the same PF manager	to PF owned by another PF manager
1	INVL1	CF	0.65	0.50	0.00	0.00	0.00
2	SEB1	CF	0.65	0.50	0.05	0.00	0.05
3	SWED1	CF	0.65	0.50	0.05	0.00	0.05
4	DANSKE1	CF	0.65	0.00	0.05	0.05	0.05
5	DNB1	CF	0.65	0.50	0.05	0.05	0.05
6	AVIVA1	CF	0.65	0.50	0.05	0.05	0.05
7	AVIVA2	SF	1.00	0.50	0.05	0.05	0.05
8	INVL2	SF	0.99	0.50	0.00	0.00	0.00
9	DNB2	SF	1.00	0.50	0.05	0.05	0.05
10	SWED2	SF	1.00	0.50	0.05	0.00	0.05
11	SEB2	MF	1.00	0.50	0.05	0.00	0.05
12	SWED3	MF	1.00	0.50	0.05	0.00	0.05
13	SWED4	MF	1.00	0.50	0.05	0.00	0.05
14	INVL3	MF	0.99	0.50	0.00	0.00	0.00
15	DANSKE2	MF	1.00	0.50	0.05	0.05	0.05
16	DNB3	MF	1.00	0.50	0.05	0.05	0.05
17	AVIVA2	MF	1.00	0.50	0.05	0.05	0.05
18	SEB3	ST	1.00	0.50	0.05	0.00	0.05
19	INVL4	ST	0.99	0.50	0.00	0.00	0.00
20	SWED5	ST	1.00	0.50	0.05	0.00	0.05
21	DANSKE3	ST	1.00	0.50	0.05	0.05	0.05

Table 5 Empirical statistics of salary increment g in Lithuania

mean	0.00507143
stdev	0.06424648
skewness	-1.78825
kurtosis	3.849164

$$r^{(t)} = a_0^{(R_t)} + a_1^{(R_t)} r^{(t-1)} + \cdots + a_p^{(R_t)} r^{(t-p)} + \sigma^{(R_t)} \varepsilon_t,$$

where $a_i^{(\cdot)},\ i = 0, \ldots, p$ and $\sigma^{(\cdot)}$ denotes parameters to be estimated, ε_t is a sequence of independent and identically distributed standard normal variables. The Markov chain is considered to be homogeneous with transition matrix $T = (p_{s,\check{s}})_{s,\check{s} \in \{1,\ldots,R\}}$, where $p_{s,\check{s}} = \Pr(R_t = \check{s} |\ R_{t-1} = s)$ denotes the transition probabilities.

4.4 Scenario tree generation

Scenario generation is an important part of the modelling process, since wrong misleading data in the scenario tree can lead to a not meaningful solution of the optimization problem. In most

Table 6 Long-term inflation rates with probabilities

Simulated inflation rate at the end of time horizon, %	10.89	8.43	7.19	5.05	3.51	1.87	1.58	1.40	0.88	-1.61
Probability	0.077	0.096	0.061	0.115	0.103	0.105	0.127	0.124	0.1	0.092

practical applications, the distributions of the stochastic parameters are approximated by discrete distributions with a limited number of outcomes.

Scenarios are generated directly from historical data according to these assumptions: the unique source of stochasticity in the first period (one year long) is given by fund returns $f_{n,i}^{(2)}$, $i = 1, \dots, I$, $n \in \mathscr{N}^{(2)}$, which are assumed to follow the fitted distribution using historical PFs returns (see Table 3). Consequently a number of nodes $N^{(2)} = v_1$ related to fund returns is obtained.

In the second period, we consider three sources of stochasticity: salary increment $g_n^{(3)}$, fund return $f_{n,i}^{(3)}$ and inflation rate $r_n^{(3)}$, which influence expected accumulated sum $A_{n,i}^{(3)}$ in nodes $n \in \mathscr{N}^{(3)}$. Notice that earlier studies (Bitinas and Maccioni, 2014; Bitinas, 2011; Bitinas and Maccioni, 2013; Medaiskis and Jankauskiene, 2014) did not include the salary increment in their models. Annual salary increment g is supposed to lie in the interval $[-0.05; 0.05]$ following a Normal distribution $g \sim N(0.00507143; 0.6424648)$ (see Table 5). A number m_2 of nodes with values taken from equally spaced sub-intervals of $[-0.05; 0.05]$ and probabilities $p_i = \Phi(g_i) - \Phi(g_{i-1})$, $i = 1, .., m_2$ where $\Phi(x)$ is CDF of Normal distribution, is considered. Fund returns $f_{n,i}^{(3)}$ are mixed α-stable distributed (see Table 3), and their probabilities can be found from the following equation (30):

$$P(a \leqslant x \leqslant b) = F_{mix}(b) - F_{mix}(a) = (1-p) \cdot S_\alpha(b, \alpha_0, \beta_0, \mu_0, \sigma_0) \\ +p \cdot \varepsilon(b) - (1-p) \cdot S_\alpha(a, \alpha_0, \beta_0, \mu_0, \sigma_0) - p \cdot \varepsilon(a), \quad (30)$$

where $\varepsilon(x)$ and the parameters α_0, β_0, μ_0, and σ_0 are introduced in Section 2.2. More information can be found in published studies (Kabašinskas et al, 2012, 2010). The sample range is divided into a number v_2 of equally spaced intervals. It is assumed that the fund return $f_{n,i}^{(3)}$ for node n is equal to the centre of the corresponding interval with probability obtained from formula (30). Funds and their parameters are listed in Table 3.

The last source of stochasticity is the inflation rate dynamics that are set as a systemic exogenous factor. Using the fitted MS-AR model (see Section 4.3), 20000 replications were simulated. Each replication is considered one realization of plausible inflation dynamics. Since the decision-making model is implemented in the scenario tree, K-means clustering method (Šutiene et al, 2010; Dupačová et al, 2000) is applied to bundle similar replications into clusters that present different scenarios (see Table 6). The required number of scenarios is determined by the clusters' number specified. Thereby, an additional number u_2 of scenarios related to inflation dynamics is obtained. Combining all sources of stochasticity of period two, the total number of nodes in the second period is $N^{(3)} = N^{(2)} m_2 v_2 u_2$.

The scenarios in the third period are only related to changes in fund returns $f_{nj}^{(3)}$ $(j = 1, \dots, k)$ and are modelled in the same way as in the second period, while the salary increment and inflation rate are inherited from the previous period. Then the total number of nodes at the last stage is equal to $N^{(4)} = N^{(3)} \cdot v_3$.

5 Numerical results

In this section we discuss the numerical results of risk-neutral and risk-averse solutions for cases A and B (Section 4.1). First, in Section 5.1, we determine the size of the scenario tree. In Section 5.2, we analyse the optimal solution at the first stage. In section 5.3 the particular case study when the selection of conservative pension funds is forbidden at the first and second stages. Finally, in Sections 5.4 and 5.5, we compare cases A and B in terms of total revenues for the optimal accumulation for different combinations of parameters (salary, lengths of third period and risk tolerance level).

All calculations were performed using ©SAS software with Mixed Integer Linear Programming (MILP) solver. The necessity of using SAS was influenced by the combination of a wide range statistical methods used in the study, e.g., fitting of α-stable distribution, estimation of corresponding probabilities and quantiles for scenarios, and optimization.

5.1 Determining the size of the scenario tree

To qualify the results, a tuning of the number of scenarios to obtain stable results is performed by computing in-sample stability (see (Kaut and Wallace, 2007)). The experiment was performed with different duration $t^{(3)}$ of the third period: (a) 3 years, (b) 5 years, (c) 7 years, (d) 10 years, and (e) 15 years, and changing the number of scenarios related to the salary increment at the second period m_2 and number of scenarios related to the fund returns at the second and third periods v_2 and v_3. Numerical results show that in-sample stability is reached with a sufficient precision (having an absolute relative error less that 0.004) already with $N^{(1)} = 1$, $N^{(2)} = 6$, $N^{(3)} = N^{(2)} \cdot u_2 \cdot v_2 \cdot m_2 = 3840$ and $N^{(4)} = N^{(3)} \cdot v_3 = 30720$, where $u_2 = 10$, $v_2 = 8$ and $m_2 = 8$ and $v_3 = 8$. The corresponding branching structure is 1-6-640-8 generates a scenario tree with 30720 scenarios. We declare such a large scenario tree the benchmark for our further investigations.

5.2 First and second stage solutions by changing model parameters

In this section, the optimal first and second stage solutions of the risk-averse multistage stochastic optimization model are discussed. Table 7 shows the number of funds to be recommended for participants based on the different values of $t^{(3)}$, ρ and α at the first stage. The choice $\rho = 0$ corresponds to the risk-neutral case.

First stage solution for investors with low weighting factor ($\rho = 0$ or 0.05) suggests choosing fund number $i = 1$ (INVL1, conservative fund). If the person is risk-avoiding ($\rho = 1$) and in case $\rho = 0.75$, then fund number $i = 4$ (DANSKE1, conservative fund) should be selected. It is clear from Table 7 that CVaR level α has a very weak influence on the solution (with exception of $\rho = 0.75$ and $t^{(3)} = 5, 7, 10, 15$). This can happen if the selected fund has very strong stochastic tail dominance over all other funds (see Eeckhoudt and Hansen, 1992).

In conclusion, it is worth pointing out that the length of the third period influences the choice of investors at the first stage. If there is no target date fund (or $t^{(3)} = 0$), then the investor is advised to choose fund INVL1 in the first stage, no matter what risk tolerance he or she has. However, if $t^{(3)}$ is more than 1 year, CVaR level α and risk tolerance level ρ become more important (see the corresponding lines in Table 7).

Now we investigate how different funds are to be chosen in the second stage depending on investors' risk tolerance level ρ. Since second stage variables are scenario-dependent, we show (Figure 1) only the percentage distribution of selected funds among scenarios in different cases

Table 7 Optimal first stage decision variable representing the number of fund i selected by the multistage stochastic optimization model

	α / ρ	0*	0.05	0.1	0.25	0.5	0.75	1
$t^{(3)} = 0$	0.01	1	1	1	1	1	1	1
	0.05	1	1	1	1	1	1	1
	0.1	1	1	1	1	1	1	1
$t^{(3)} = 1$	0.01	1	1	1	1	1	1	4
	0.05	1	1	1	1	1	1	4
	0.1	1	1	1	1	1	1	4
$t^{(3)} = 3$	0.01	1	1	1	1	1	1	4
	0.05	1	1	1	1	1	1	4
	0.1	1	1	1	1	1	1	4
$t^{(3)} = 5$	0.01	1	1	1	1	1	4	4
	0.05	1	1	1	1	1	1	4
	0.1	1	1	1	1	1	1	4
$t^{(3)} = 7$	0.01	1	1	1	1	1	4	4
	0.05	1	1	1	1	1	4	4
	0.1	1	1	1	1	1	1	4
$t^{(3)} = 10$	0.01	1	1	1	1	1	4	4
	0.05	1	1	1	1	1	4	4
	0.1	1	1	1	1	1	4	4
$t^{(3)} = 15$	0.01	1	1	1	1	1	4	4
	0.05	1	1	1	1	1	4	4
	0.1	1	1	1	1	1	4	4

* the case corresponds to risk-neutral model. Fund #1 corresponds to INVL1 and #4 corresponds to DANSKE1.

of risk tolerance level ρ. We investigate three special cases for risk-neutral ($\rho = 0$), risk-averse ($\rho \in [0.05, 0.95]$) and risk-avoiding ($\rho = 1$) investors. In this comparison, we use fixed $t^{(3)} = 5$, CVaR $\alpha = 0.05$, and accumulation scheme with $\gamma = 0.075$ (maximal accumulation plan, case B).

In the case of risk-neutral investors, selecting fund INVL1 is recommended in the first stage. In the second stage, fund INVL1 is suggested in 66.7% scenarios and fund AVIVA1 dominates in 33.3% of scenarios. It must be noted that other funds are not selected in any scenario as possible options for accumulation in case of risk-neutral investors. In third stage fund INVL1 is suggested in 62.3% scenarios, fund AVIVA1 dominates in 24.8% of scenarios and DANSKE1 is optimal in 12.4%, while other conservative funds altogether are selected in less than 0.6%.

In the case $\rho \in [0.05, 0.95]$ (risk-averse investor), fund INVL1 is optimal in the first stage, but in the second stage, funds INVL1 and DANSKE1 are recommended in equal 50–50 proportions. Fund INVL1 is selected in scenarios with loss, i.e., $f^{(2)}_{n,i} < 0$, while fund DANSKE1 is selected in case of positive fund returns. Other funds are not optimal in any scenario. In third stage fund INVL1 is suggested in 89.5% scenarios while other conservative funds altogether are selected in less than 12% of scenarios.

In the case of risk-avoiding investor, fund DANSKE1 is optimal in the first stage, in the second stage it is optimal in 50% scenarios either, while funds INVL1, DNB1 and AVIVA1 are optimal in 16.7% scenarios. Other funds are not suggested at all. Surprisingly, in third stage fund SEB1 is suggested in 41.7% scenarios, INVL1 in 27.2%, DNB1 in 11.2%, AVIVA1 in 10.2% while SWED1 and DANSKE1 in less than 6% of scenarios.

In general we can say that funds INVL1, DANSKE1 and AVIVA1 are suggested in the most scenarios (depending on the risk tolerance level).

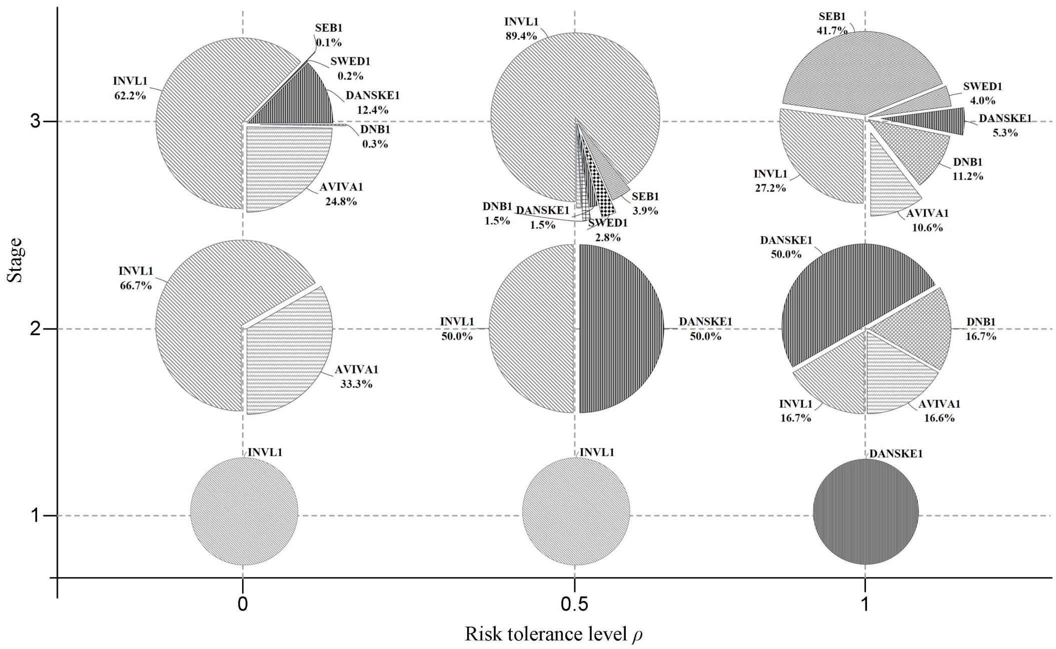

Fig. 1 Selection of fund in first, second and third stages depending on risk tolerance level

5.3 Particular case study imposing no conservative funds

Three experiments were carried out in order to explore the effect of restrictions on pension funds allowed for selection at the first stage, then at the second stage as well:

1. The experiment is run using the model as it was designed (Section 5.2) and allocation among funds is shown in Figure 1;
2. The experiment is run using model by forbidding conservative funds to be selected at the first stage, i.e., $x_i^{(1)}$ were set to 0 for all i that represent conservative funds. The allocation among funds is shown in Figure 2;
3. The experiment is run using model by forbidding conservative funds to be selected both at the first and at the second stage, i.e., $x_i^{(1)}$ and $x_{n,i}^{(2)}$ were set to 0 for all i that represents conservative funds and at all nodes n. The allocation among funds is shown in Figure 3.

The results of such experiment are summarized in Table 8 and Figures 1–3.

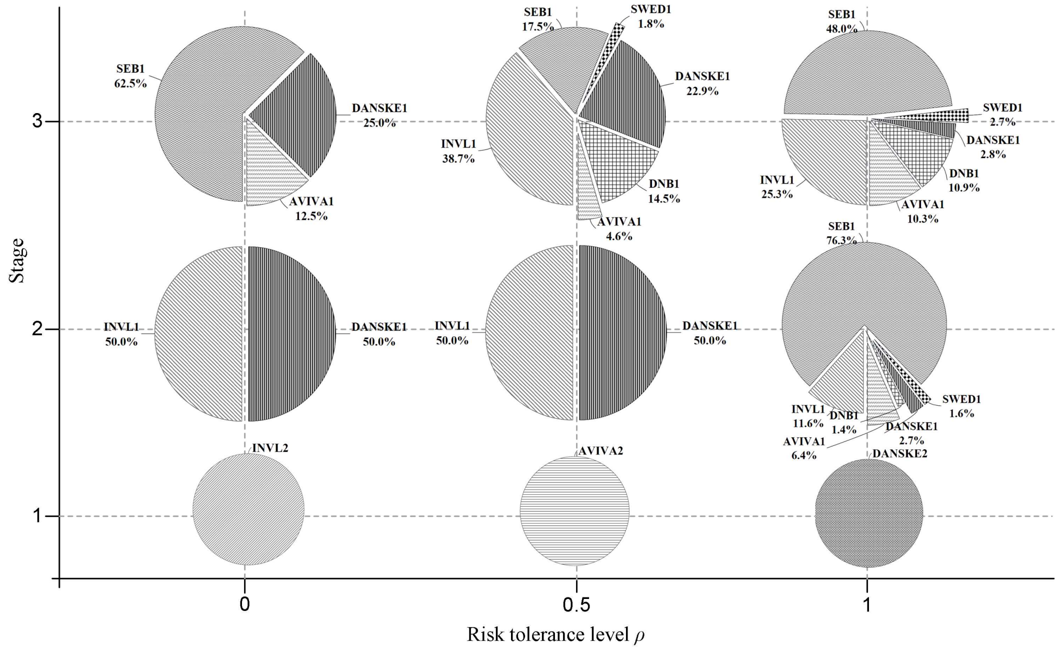

Fig. 2 Selection of fund in first, second and third stages depending on risk tolerance level, when conservative funds are forbidden at the first stage

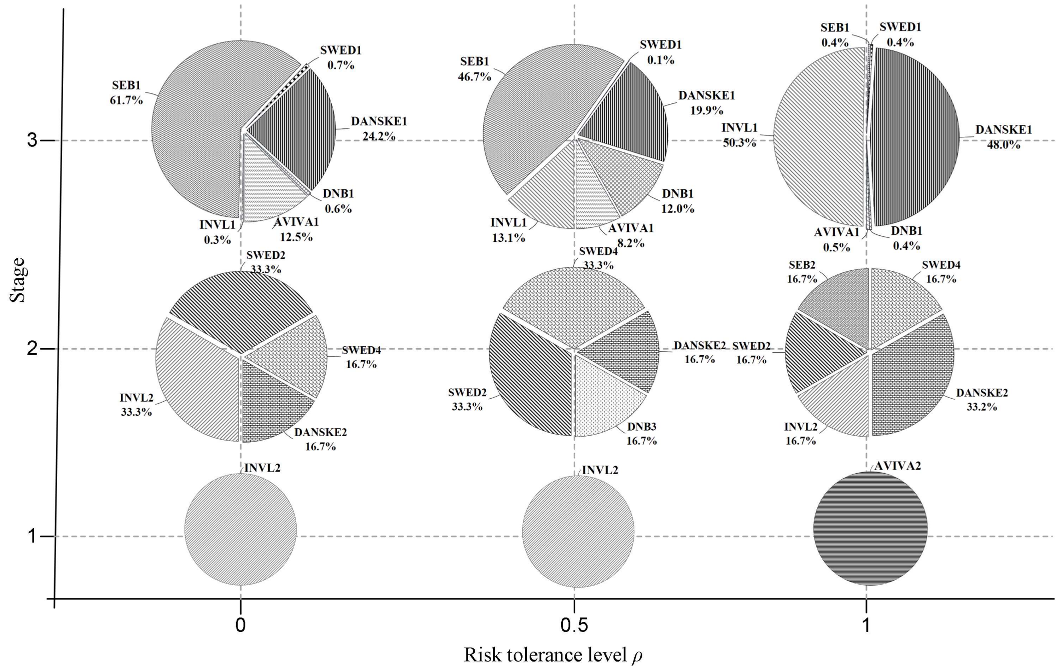

Fig. 3 Selection of fund in first, second and third stages depending on risk tolerance level, when conservative funds are forbidden at first and second stages

Table 8 Value of objective function (24) and of the terms $F^{(1)}$, $(1-\rho^{(2)})\left(\sum_{n\in\mathcal{N}^{(2)}}\pi_{1,n}L_n\right)$ and $\rho^{(2)}\left(\eta^{(2)}-\frac{1}{\alpha^{(2)}}\sum_{n\in\mathcal{N}^{(2)}}\pi_{1,n}d_n^{(2)}\right)$ in eq. (24) for different levels of $\rho=0,0.5,1$.

Risk aversion level		$\rho=0$ risk neutral case	$\rho=0.5$ risk averse case	$\rho=1$ risk avoiding case
No restrictions	$F^{(1)}$	0.883543	0.883543	0.796743
	$(1-\rho)\left(\sum_{n\in\mathcal{N}^{(2)}}\pi_{1,n}L_n\right)$	1724034	215496.5	0
	$\rho\left(\eta^{(2)}-\frac{1}{\alpha}\sum_{n\in\mathcal{N}^{(2)}}\pi_{1,n}d_n^{(2)}\right)$	0	203817.3	-1081.38
	Objective function (24)	1724034.9	419314.7	-1080.59
No conservative funds allowed at the first stage	$F^{(1)}$	0.88295	0.867009	0.796743
	$(1-\rho)\left(\sum_{n\in\mathcal{N}^{(2)}}\pi_{1,n}L_n\right)$	1724034	215496.5	0
	$\rho\left(\eta^{(2)}-\frac{1}{\alpha}\sum_{n\in\mathcal{N}^{(2)}}\pi_{1,n}d_n^{(2)}\right)$	0	203817.3	-1081.38
	Objective function (24)	1724034.8	419314.6	-1080.59
No conservative funds allowed at first and second stages	$F^{(1)}$	0.88295	0.88295	0.796743
	$(1-\rho)\left(\sum_{n\in\mathcal{N}^{(2)}}\pi_{1,n}L_n\right)$	1.199308	-141.826	0
	$\rho\left(\eta^{(2)}-\frac{1}{\alpha}\sum_{n\in\mathcal{N}^{(2)}}\pi_{1,n}d_n^{(2)}\right)$	0	-144.053	-1081.39
	Objective function (24)	2.082258	-284.996	-1080.59

From Table 8 is clearly seen that the introduction of restriction on conservative funds drastically decreases the value of objective function comparing to the case when they are allowed. The decrease is well seen in the third experiment, when conservative funds are forbidden over long period.

When ρ is set to 0 (risk neutral case, Table 8) the objective function drops from 1724034.9 to 2.082. This shows that if the optimal fund is forbidden for one year then the risk neutral investor looses 0.1 units and if it is banned for 47 years then the expected return drops by 1724032.8 units. Similar differences are for $\rho = 0.5$, moreover, in case if the are no conservative funds at all then the person will not benefit from pension accumulation. For risk-avoiding or so called safety-first case (Table 8) there is no decrease of objective function at all, because lot of effort is made to hedge investment from the beginning.

The main conclusion obtained from this experiment is that all II pillar pension funds in Lithuania are dominated by conservative funds during the period selected for the experimental study. There are several reason implied to explain the results of performed experiments:

1. Conservative funds had lower upkeep costs, i.e., annual management charges are by 33% lower compared to the costs for other funds (see Table 4);
2. Conservative funds are much less risky in terms of left tail of their distributions or beta/asymmetry parameter of alpha-stable distribution (see Tables 2 and 3);
3. During period analyzed conservative funds exhibited unbelievable efficiency in terms of Sharpe ratio. The average Sharpe ratio of conservative funds is by 32% higher than the best Sharpe ratio of other funds (see last column of Table 3).

In the next two sections, we will compare pension accumulation plans A and B for different salary sizes, contributions to the funds and length of time period $t^{(3)}$. The investigation is motivated by the fact that recently, fund managers have reported that some participants in the pension system, because of very small monthly contributions to funds, will not be able to accumulate necessary funds to cover living expenses and inflation. The purpose of our numerical results is to see if such a bad scenario can occur.

Risk tolerance level $\rho = 0.5$ is set up in the following calculation to maintain the trade-off between reward and risk.

Table 9 Optimal objective function values (×1000) with minimum accumulation plan (case A) in risk-averse and risk-neutral models for different risk tolerance levels $\alpha^{(t)}$, initial salary S_0 and different lengths of the third period $t^{(3)}$

Risk level	S_0	$t^{(3)}$=0	$t^{(3)}$=3	$t^{(3)}$=5	$t^{(3)}$=7	$t^{(3)}$=10	$t^{(3)}$=15
0.01	350	30.07019	25.68311	21.9715	18.82125	14.96128	10.26522
	500	42.95921	36.69233	31.38735	26.88691	21.37334	14.66426
	750	64.4393	55.03932	47.08152	40.33072	32.05989	21.99563
	1000	85.9182	73.38651	62.7751	53.77638	42.74643	29.32916
	1500	128.8767	110.0775	94.16378	80.66527	64.11948	43.99226
	2000	171.8374	146.7668	125.5505	107.5543	85.49397	58.65706
0.05	350	30.08735	25.7027	21.98966	18.84016	14.97955	10.28294
	500	42.98336	36.71856	31.41381	26.91445	21.39935	14.68944
	750	64.47639	55.07764	47.12045	40.36975	32.09856	22.03347
	1000	85.96854	73.43687	62.82769	53.82886	42.79881	29.37924
	1500	128.9528	110.1552	94.2412	80.74394	64.1955	44.0674
	2000	171.9371	146.8738	125.6541	107.6586	85.59534	58.75646
0.1	350	30.12938	25.78048	22.09081	18.95504	15.1022	10.40071
	500	43.04182	36.82937	31.55812	27.07898	21.57452	14.85842
	750	64.56159	55.24407	47.33849	40.61826	32.36228	22.28712
	1000	86.08219	73.65877	63.11654	54.15729	43.14915	29.71665
	1500	129.1264	110.4886	94.67385	81.23755	64.72324	44.57462
	2000	172.1671	147.3176	126.2312	108.3149	86.29846	59.43357
neutral	350	36.31207	36.19579	35.92854	35.57568	34.98882	34.10737
	500	51.87439	51.70827	51.32648	50.8224	49.98403	48.72482
	750	77.81158	77.56241	76.98973	76.2336	74.97604	73.08723
	1000	103.7488	103.4165	102.653	101.6448	99.96806	97.44964
	1500	155.6232	155.1248	153.9795	152.4672	149.9521	146.1745
	2000	207.4976	206.8331	205.3059	203.2896	199.9361	194.8993

5.4 Numerical results for case A (minimum accumulation plan)

We analyse accrual of a person who is 18 years old, earns the salary S_0 specified in the first column of Table 9 and chooses the minimum accumulation plan (see subsection 4.1). The different durations of the third period are analysed. The aim of this test is to show the influence of the regulation (which is planned to be included in pension law (see Seimas of the Republic of Lithuania, 2010)) to the objective function at the retirement. We set parameter $t^{(3)}$ to 0, 3, 5, 7, 10 and 15 years as duration of accumulation in the conservative fund. The case when the length of the third period is equal to 0 refers to the current version of pension law when there is no regulation for selecting only a conservative fund in the few last years before retirement.

The values of objective function in the risk-averse and risk-neutral models are given in Table 9. For the different risk tolerance levels, ($\alpha^{(t)}$ is constant for all stages $t = 1, 2, 3$). The values of salaries S_0 and the different number of years in the third period $t^{(3)}$, respectively, are given in the first and second columns and first row.

Results show that the selection of fund (third-stage solution $x_{nj}^{(3)}$) is not dependent on the salary of the person. Moreover, in real life, salary increment may be related to fund returns, but in our simulation, we assume that there is no relationship between fund return and other parameters like S_0, $t^{(2)}$ and $g_n^{(t)}$. The companies usually can raise salaries for employees only if the market situation is favorable. Moreover, if there is a crisis, salaries may be reduced, and the companies may dismiss some employees.

Table 9 show the relationship between the duration of the third period and the objective function (in case A) for risk-averse and risk-neutral models. Results reveal that value of objective function decreases in both cases when the length of the third period increases. However, in the

Table 10 Optimal objective function values (×1000) with maximum accumulation plan (B case) for risk-averse and risk-neutral models with different risk tolerance levels, initial salary and different lengths of the third period $t^{(3)}$

Risk level	S_0	$t^{(3)}$=0	$t^{(3)}$=3	$t^{(3)}$=5	$t^{(3)}$=7	$t^{(3)}$=10	$t^{(3)}$=15
0.01	350	70.16669	59.93198	51.26769	43.91622	34.91041	23.95104
	500	100.2371	85.617	73.23948	62.74086	49.87117	34.21537
	750	150.3573	128.425	109.8593	94.11128	74.80785	51.32653
	1000	200.4762	171.2333	146.479	125.4756	99.74374	68.43217
	1500	300.7144	256.8518	219.7158	188.2216	149.6157	102.6469
	2000	400.952	342.4657	292.9576	250.9609	199.4876	136.8678
0.05	350	70.20763	59.97353	51.30726	43.96054	34.9523	23.99241
	500	100.2955	85.6765	73.29896	62.80015	49.93194	34.27439
	750	150.4449	128.5144	109.9482	94.20127	74.89484	51.41321
	1000	200.5931	171.3511	146.5978	125.6011	99.86215	68.55227
	1500	300.8895	257.0285	219.8959	188.4021	149.7905	102.8244
	2000	401.1862	342.7015	293.1941	251.2028	199.7152	137.1017
0.1	350	70.30167	60.15466	51.54481	44.22876	35.23865	24.26895
	500	100.4315	85.935	73.63786	63.18339	50.34122	34.66896
	750	150.6424	128.902	110.4527	94.77463	75.51084	52.0043
	1000	200.8621	171.8703	147.2761	126.3682	100.6833	69.33858
	1500	301.2907	257.807	220.9065	189.5501	151.0227	104.0072
	2000	401.7248	343.7382	294.5437	252.7333	201.363	138.6779
neutral	350	84.72817	84.45684	83.83326	83.00992	81.64058	79.58387
	500	121.0402	120.6526	119.7618	118.5856	116.6294	113.6912
	750	181.5604	180.9789	179.6427	177.8784	174.9441	170.5369
	1000	242.0805	241.3053	239.5236	237.1712	233.2588	227.3825
	1500	363.1207	361.9579	359.2854	355.7568	349.8882	341.0737
	2000	484.161	482.6105	479.0472	474.3424	466.5176	454.765

risk-averse model, the decrease is much more significant than in risk-neutral version (see Table 9). Due to the inclusion of Target Date Funds in pension accumulation schemes, the annual loss increases from approximately 5% to 12% in the risk-averse model, while in the risk-neutral model, the loss is less than 0.5%.

5.5 Numerical results for case B (maximum accumulation plan)

In the following, the person already characterized in Section 5.4 is assumed to have chosen the maximum accumulation plan (see Section 4.1). We analyse different durations of third periods $t^{(3)}$, as was done previously.

Table 10 shows the value of objective function in the risk-averse and risk-neutral models. Different risk tolerance levels $\alpha^{(t)}$, values of salaries S_0 and different number of years in the third period $t^{(3)}$ are respectively given in the first and second columns and first row.

Comparing Table 9 with Table 10, we can see that in almost all cases, the maximum accumulation (case B) gives 7/3 or approximately double the reward in long periods (47 years), while the difference of annual payments between A and B is only 4%. Thus, the results suggest that the maximal accumulation scheme is more profitable since the participant has to pay additional 2% from his or her gross salary, while the additional 2% are given by government of Lithuania[2].

Table 10 show the relationship between the duration of the third period and optimal value of objective function (in case B) for risk-averse and risk-neutral models. As in case A, the results

[2] Approximately 168 €per year

show that the values of objective function are decreasing with the length of third period in both models.

6 Conclusion

In this paper, the possible improvements to be implemented in the IInd pillar of Lithuanian pension system have been investigated. This topic is highly relevant in Lithuania since those changes are under debate by supervisors of financial markets and the government of Lithuania. The pension scheme is modelled using a risk-averse and risk-neutral multistage stochastic programming methodology. The proposed model includes these sources of stochasticity: fund return, salary increment and inflation. This paper makes a contribution to the current literature focusing on decision-making models for individuals to manage their pension savings. In numerical examples, we focus on a participant who is 18 years old. The study covers the minimum and maximum accumulation plans (respectively, Case A and Case B) for the different sizes of initial salary for certain risk tolerance levels and the different duration time until retirement that influences the target date.

Numerical results show that in-sample stability is reached with a sufficient precision (absolute relative error is less than 0.004) with $N^{(2)} = 6$ (second stage), $N^{(3)} = 3840$ (third stage) and $N^{(4)} = 30720$ (last stage) scenarios. Based on such stability results, the following conclusions are valid.

The key finding with respect to fund selection is that in the first stage, the conservative funds (INVL1 and DANSKE1) must be chosen for all possible combinations of model parameters used, while in long-term accumulation, the distribution of recommended funds is parameter dependent. Furthermore, the conservative funds must be also chosen for most scenarios and combinations of model parameters. This can be explained by the tail dominance of selected funds, as the empirical analysis of their returns confirms this assumption because of positive skewness and fat right tails. Moreover, our experiment showed that if conservative funds are forbidden in first and second periods then value of optimal objective function decreases significantly. In both Case A and Case B, switching to conservative pension accumulations must be done as late as possible. In risk-averse cases, such compulsory switching, as an improvement of the pension system, implies approximately 190% loss compared with when such legal regulation, inclusion of Target Date Funds, is not applied. However, in risk-neutral cases, savings will be reduced by approximately 6% in the 15-year period.

In this study, comparing Case A with Case B showed that the accumulation in Case B allows to achieve approximately 7/3 ratio savings in a long period compared to Case A. Moreover, in a minimal accumulation scheme, the participant may not accumulate the necessary funds to cover living expenses over his or her lifetime in retirement. The experimental study reveals such scenarios for parameter combinations $t^{(3)} = 15$ and $S_0 \leq 500$, as well as $t^{(3)} = 10$ and $S_0 = 350$ despite different risk tolerance levels. These findings may be used to identify the areas in which the IInd pillar of the Lithuanian pension system needs to be strengthened by revealing its vulnerability in long-term planning.

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